

Chapter 2

Topological Spaces

A *topological space* (X, τ) is a set X with a *topology* τ , i.e., a collection of subsets of X with the following properties:

1. $X \in \tau, \emptyset \in \tau$;
2. If $A, B \in \tau$, then $A \cap B \in \tau$;
3. For any collection $\{A_\alpha\}_\alpha$, if all $A_\alpha \in \tau$, then $\cup_\alpha A_\alpha \in \tau$.

The sets in τ are called *open sets*, and their complements are called *closed sets*. A *base* of the topology τ is a collection of open sets such that every open set is a union of sets in the base. The coarsest topology has two open sets, the empty set and X , and is called the *trivial topology* (or *indiscrete topology*). The finest topology contains all subsets as open sets, and is called the *discrete topology*.

In a metric space (X, d) define the *open ball* as the set $B(x, r) = \{y \in X : d(x, y) < r\}$, where $x \in X$ (the *center* of the ball), and $r \in \mathbb{R}, r > 0$ (the *radius* of the ball). A subset of X which is the union of (finitely or infinitely many) open balls, is called an *open set*. Equivalently, a subset U of X is called *open* if, given any point $x \in U$, there exists a real number $\epsilon > 0$ such that, for any point $y \in X$ with $d(x, y) < \epsilon$, $y \in U$.

Any metric space is a topological space, the topology (**metric topology**, *topology induced by the metric d*) being the set of all open sets. The metric topology is always T_4 (see below a list of topological spaces). A topological space which can arise in this way from a metric space, is called a **metrizable space**.

A *quasi-pseudo-metric topology* is a topology on X induced similarly by a quasi-semimetric d on X , using the set of open d -balls $B(x, r)$ as the base. In particular, *quasi-metric topology* and *pseudo-metric topology* are the terms used for the case of, respectively, quasi-metric and semimetric d . In general, those topologies are not T_0 .

Given a topological space (X, τ) , a *neighborhood* of a point $x \in X$ is a set containing an open set which in turn contains x . The *closure* of a subset of a topological space is the smallest closed set which contains it. An *open cover* of X is a collection \mathcal{L} of open sets, the union of which is X ; its *subcover* is a cover \mathcal{K}

such that every member of \mathcal{K} is a member of \mathcal{L} ; its *refinement* is a cover \mathcal{K} , where every member of \mathcal{K} is a subset of some member of \mathcal{L} . A collection of subsets of X is called *locally finite* if every point of X has a neighborhood which meets only finitely many of these subsets.

A subset $A \subset X$ is called *dense* if $X = cl(A)$, i.e., it consists of A and its *limit points*; cf. **closed subset of metric space** in Chap. 1. The *density* of a topological space is the least cardinality of its dense subset. A *local base* of a point $x \in X$ is a collection \mathcal{U} of neighborhoods of x such that every neighborhood of x contains some member of \mathcal{U} .

A function from one topological space to another is called *continuous* if the preimage of every open set is open. Roughly, given $x \in X$, all points close to x map to points close to $f(x)$. A function f from one metric space (X, d_X) to another metric space (Y, d_Y) is *continuous* at the point $c \in X$ if, for any positive real number ϵ , there exists a positive real number δ such that all $x \in X$ satisfying $d_X(x, c) < \delta$ will also satisfy $d_Y(f(x), f(c)) < \epsilon$; the function is continuous on an interval I if it is continuous at any point of I .

The following classes of topological spaces (up to T_4) include any metric space.

- **T_0 -space**

A **T_0 -space** (or *Kolmogorov space*) is a topological space in which every two distinct points are *topologically distinguishable*, i.e., have different neighborhoods.

- **T_1 -space**

A **T_1 -space** (or *accessible space*) is a topological space in which every two distinct points are *separated*, i.e., each does not belong to other's closure. T_1 -spaces are always T_0 .

- **T_2 -space**

A **T_2 -space** (or **Hausdorff space**) is a topological space in which every two distinct points are *separated by neighborhoods*, i.e., have disjoint neighborhoods. T_2 -spaces are always T_1 .

A space is T_2 if and only if it is both T_0 and *pre-regular*, i.e., any two *topologically distinguishable* points are separated by neighborhoods.

- **Regular space**

A **regular space** is a topological space in which every neighborhood of a point contains a closed neighborhood of the same point.

- **T_3 -space**

A **T_3 -space** (or *Vietoris space*, *regular Hausdorff space*) is a topological space which is T_1 and **regular**.

- **Completely regular space**

A **completely regular space** (or *Tychonoff space*) is a **Hausdorff space** (X, τ) in which any closed set A and any $x \notin A$ are *functionally separated*.

Two subsets A and B of X are *functionally separated* if there is a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$.

- **Perfectly normal space**

A **perfectly normal space** is a topological space (X, τ) in which any two disjoint closed subsets of X are *functionally separated*.

- **Normal space**

A **normal space** is a topological space in which, for any two disjoint closed sets A and B , there exist two disjoint open sets U and V such that $A \subset U$, and $B \subset V$.

- **T_4 -space**

A **T_4 -space** (or *Tietze space*, *normal Hausdorff space*) is a topological space which is T_1 and **normal**. Any metric space is a perfectly normal T_4 -space.

- **Completely normal space**

A **completely normal space** is a topological space in which any two separated sets have disjoint neighborhoods. It also called a *hereditarily normal space* since it is exactly one in which every subspace with subspace topology is a normal space.

Sets A and B are *separated* in X if each is disjoint from the other's closure.

- **Monotonically normal space**

A **monotonically normal space** is a **completely normal space** in which any two *separated* subsets A and B are *strongly separated*, i.e., there exist open sets U and V with $A \subset U$, $B \subset V$ and $Cl(U) \cap Cl(V) = \emptyset$.

- **T_5 -space**

A **T_5 -space** (or *completely normal Hausdorff space*) is a topological space which is **completely normal** and T_1 . T_5 -spaces are always T_4 .

- **T_6 -space**

A **T_6 -space** (or *perfectly normal Hausdorff space*) is a topological space which is T_1 and **perfectly normal**. T_6 -spaces are always T_5 .

- **Moore space**

A **Moore space** is a **regular space** with a *development*.

A *development* is a sequence $\{\mathcal{U}_n\}_n$ of open covers such that, for every $x \in X$ and every open set A containing x , there exists n such that $St(x, \mathcal{U}_n) = \bigcup \{U \in \mathcal{U}_n : x \in U\} \subset A$, i.e., $\{St(x, \mathcal{U}_n)\}_n$ is a *neighborhood base* at x .

- **Polish space**

A **separable space** is a topological space which has a countable dense subset.

A **Polish space** is a separable space which can be equipped with a complete metric. A *Lusin space* is a topological space such that some weaker topology makes it into a Polish space; every Polish space is Lusin. A *Souslin space* is a continuous image of a Polish space; every Lusin space is Suslin.

- **Lindelöf space**

A **Lindelöf space** is a topological space in which every open cover has a countable subcover.

An *L-space* is a hereditarily Lindelöf space which is not hereditarily separable.

- **First-countable space**

A topological space is called **first-countable** if every point has a countable local base. Any metric space is first-countable.

- **Second-countable space**

A topological space is called **second-countable** if its topology has a countable base. Such space is **quasi-metrizable** and, if and only if it is a T_3 -space, **metrizable**.

Second-countable spaces are **first-countable**, **separable** and **Lindelöf**. The properties **second-countable**, **separable** and **Lindelöf** are equivalent for metric spaces.

The Euclidean space \mathbb{E}^n with its usual topology is second-countable.

- **Baire space**

A **Baire space** is a topological space in which every intersection of countably many dense open sets is dense. Every complete metric space is a Baire space. Every locally compact T_2 -space (hence, every n -**manifold**) is a Baire space.

- **Alexandrov space**

An **Alexandrov space** is a topological space in which every intersection of arbitrarily many open sets is open.

A topological space is called a **P -space** if every G_δ -set (i.e., the intersection of countably many open sets) is open.

A topological space (X, τ) is called a **Q -space** if every subset $A \subset X$ is a G_δ -set.

- **Connected space**

A topological space (X, τ) is called **connected** if it is not the union of a pair of disjoint nonempty open sets. In this case the set X is called a *connected set*.

A connected topological space (X, τ) is called *unicoherent* if the intersection $A \cap B$ is connected for any closed connected sets A, B with $A \cup B = X$.

A topological space (X, τ) is called **locally connected** if every point $x \in X$ has a local base consisting of connected sets.

A topological space (X, τ) is called **path-connected** (or *0-connected*) if for every points $x, y \in X$ there is a *path* γ from x to y , i.e., a continuous function $\gamma : [0, 1] \rightarrow X$ with $\gamma(x) = 0, \gamma(y) = 1$.

A topological space (X, τ) is called **simply connected** (or *1-connected*) if it consists of one piece, and has no circle-shaped “holes” or “handles” or, equivalently, if every continuous curve of X is *contractible*, i.e., can be reduced to one of its points by a *continuous deformation*.

A topological space (X, τ) is called **hyperconnected** (or *irreducible*) if X cannot be written as the union of two proper closed sets.

- **Sober space**

A topological space (X, τ) is called **sober** if every **hyperconnected** closed subset of X is the closure of exactly one point of X . Any sober space is a T_0 -space.

Any T_2 -space is a sober T_1 -space but some sober T_1 -spaces are not T_2 .

- **Paracompact space**

A topological space is called **paracompact** if every open cover of it has an open locally finite refinement. Every **metrizable** space is paracompact.

- **Totally bounded space**

A topological space (X, τ) is called **totally bounded** (or *pre-compact*) if it can be covered by finitely many subsets of any fixed cardinality.

A metric space (X, d) is a **totally bounded metric space** if, for every real number $r > 0$, there exist finitely many open balls of radius r , whose union is equal to X .

- **Compact space**

A topological space (X, τ) is called **compact** if every open cover of X has a finite subcover.

Compact spaces are always **Lindelöf**, **totally bounded**, and **paracompact**. A metric space is compact if and only if it is **complete** and **totally bounded**. A subset of a Euclidean space \mathbb{E}^n is compact if and only if it is closed and bounded. There exist a number of topological properties which are equivalent to compactness in metric spaces, but are nonequivalent in general topological spaces. Thus, a metric space is compact if and only if it is a *sequentially compact space* (every sequence has a convergent subsequence), or a *countably compact space* (every countable open cover has a finite subcover), or a *pseudo-compact space* (every real-valued continuous function on the space is bounded), or a *weakly countably compact space* (i.e., every infinite subset has an accumulation point).

Sometimes, a compact **connected** T_2 -space is called *continuum*; cf. **continuum** in Chap. 1.

- **Locally compact space**

A topological space is called **locally compact** if every point has a local base consisting of compact neighborhoods. The Euclidean spaces \mathbb{E}^n and the spaces \mathbb{Q}_p of *p-adic numbers* are locally compact.

A topological space (X, τ) is called a *k-space* if, for any compact set $Y \subset X$ and $A \subset X$, the set A is closed whenever $A \cap Y$ is closed. The *k-spaces* are precisely quotient images of locally compact spaces.

- **Locally convex space**

A **topological vector space** is a real (complex) vector space V which is a T_2 -space with continuous vector addition and scalar multiplication. It is a **uniform space** (cf. Chap. 3).

A **locally convex space** is a topological vector space whose topology has a base, where each member is a *convex balanced absorbent* set. A subset A of V is called *convex* if, for all $x, y \in A$ and all $t \in [0, 1]$, the point $tx + (1 - t)y \in A$, i.e., every point on the *line segment* connecting x and y belongs to A . A subset A is *balanced* if it contains the line segment between x and $-x$ for every $x \in A$; A is *absorbent* if, for every $x \in V$, there exist $t > 0$ such that $tx \in A$.

The locally convex spaces are precisely vector spaces with topology induced by a family $\{|\cdot|_\alpha\}$ of seminorms such that $x = 0$ if $||x||_\alpha = 0$ for every α .

Any metric space $(V, ||x - y||)$ on a real (complex) vector space V with a **norm metric** $||x - y||$ is a locally convex space; each point of V has a local base consisting of convex sets. Every L_p with $0 < p < 1$ is an example of a vector space which is not locally convex.

- **n-manifold**

Broadly, a *manifold* is a topological space locally homeomorphic to a **topological vector space** over the reals.

But usually, a **topological manifold** is a **second-countable** T_2 -space that is locally homeomorphic to Euclidean space. An **n-manifold** is a topological manifold such that every point has a neighborhood homeomorphic to \mathbb{E}^n .

- **Fréchet space**

A **Fréchet space** is a **locally convex space** (V, τ) which is complete as a **uniform space** and whose topology is defined using a countable set of seminorms $\|\cdot\|_1, \dots, \|\cdot\|_n, \dots$, i.e., a subset $U \subset V$ is *open in* (V, τ) if, for every $u \in U$, there exist $\epsilon > 0$ and $N \geq 1$ with $\{v \in V : \|u - v\|_i < \epsilon \text{ if } i \leq N\} \subset U$.

A Fréchet space is precisely a locally convex **F-space** (cf. Chap. 5). Its topology can be induced by a **translation invariant metric** (Chap. 5) and it is a complete and **metrizable space** with respect to this topology. But this topology may be induced by many such metrics. Every **Banach space** is a Fréchet space.

- **Countably-normed space**

A **countably-normed space** is a **locally convex space** (V, τ) whose topology is defined using a countable set of *compatible norms* $\|\cdot\|_1, \dots, \|\cdot\|_n, \dots$. It means that, if a sequence $\{x_n\}_n$ of elements of V that is fundamental in the norms $\|\cdot\|_i$ and $\|\cdot\|_j$ converges to zero in one of these norms, then it also converges in the other. A countably-normed space is a **metrizable space**, and its metric can be defined by

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}.$$

- **Metrizable space**

A topological space (T, τ) is called **metrizable** if it is homeomorphic to a metric space, i.e., X admits a metric d such that the set of open d -balls $\{B(x, r) : r > 0\}$ forms a neighborhood base at each point $x \in X$. If, moreover, (X, d) is a complete metric space for one of such metrics d , then (X, d) is a *completely metrizable* (or *topologically complete*) space.

Metrizable spaces are always **paracompact** T_2 -spaces (hence, **normal** and **completely regular**), and **first-countable**.

A topological space is called **locally metrizable** if every point in it has a metrizable neighborhood.

A topological space (X, τ) is called **submetrizable** if there exists a metrizable topology τ' on X which is coarser than τ .

A topological space (X, τ) is called **proto-metrizable** if it is paracompact and has an *orthobase*, i.e., a base \mathcal{B} such that, for $\mathcal{B}' \subset \mathcal{B}$, either $\cap \mathcal{B}'$ is open, or \mathcal{B}' is a local base at the unique point in $\cap \mathcal{B}'$. It is not related to the **protometric** in Chap. 1.

Some examples of other direct generalizations of metrizable spaces follow.

A **sequential space** is a quotient image of a metrizable space.

Morita's **M-space** is a topological space (X, τ) from which there exists a continuous map f onto a metrizable topological space (Y, τ') such that f is closed and $f^{-1}(y)$ is countably compact for each $y \in Y$.

Ceder's **M₁-space** is a topological space (X, τ) having a σ -closure-preserving base (metrizable spaces have σ -locally finite bases).

Okuyama's σ -**space** is a topological space (X, τ) having a σ -locally finite *net*, i.e., a collection \mathcal{U} of subsets of X such that, given of a point $x \in U$ with U open, there exists $U' \in \mathcal{U}$ with $x \in U' \subset U$ (a base is a net consisting of open sets). Every compact subset of a σ -space is metrizable.

Michael's **cosmic space** is a topological space (X, τ) having a countable net (equivalently, a Lindelöf σ -space). It is exactly a continuous image of a separable metric space. A T_2 -**space** is called **analytic** if it is a continuous image of a complete separable metric space; it is called a **Lusin space** if, moreover, the image is one-to-one.

- **Quasi-metrizable space**

A topological space (X, τ) is called a **quasi-metrizable space** if X admits a quasi-metric d such that the set of open d -balls $\{B(x, r) : r > 0\}$ forms a neighborhood base at each point $x \in X$.

A more general γ -**space** is a topological space admitting a γ -**metric** d (i.e., a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ with $d(x, z_n) \rightarrow 0$ whenever $d(x, y_n) \rightarrow 0$ and $d(y_n, z_n) \rightarrow 0$) such that the set of open *forward* d -balls $\{B(x, r) : r > 0\}$ forms a neighborhood base at each point $x \in X$.

The *Sorgenfrey line* is the topological space (\mathbb{R}, τ) defined by the base $\{[a, b) : a, b \in \mathbb{R}, a < b\}$. It is not metrizable but it is a first-countable separable and paracompact T_5 -**space**; neither it is second-countable, nor locally compact or locally connected. However, the Sorgenfrey line is quasi-metrizable by the **Sorgenfrey quasi-metric** (cf. Chap. 12) defined as $y - x$ if $y \geq x$, and 1, otherwise.

- **Symmetrizable space**

A topological space (X, τ) is called **symmetrizable** (and τ is called the **distance topology**) if there is a **symmetric** d on X (i.e., a distance $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ with $d(x, y) = 0$ implying $x = y$) such that a subset $U \subset X$ is open if and only if, for each $x \in U$, there exists $\epsilon > 0$ with $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \subset U$. In other words, a subset $H \subset X$ is closed if and only if $d(x, H) = \inf_y \{d(x, y) : y \in H\} > 0$ for each $x \in X \setminus U$. A symmetrizable space is **metrizable** if and only if it is a Morita's **M-space**.

In Topology, the term **semimetrizable space** refers to a topological space (X, τ) admitting a symmetric d such that, for each $x \in X$, the family $\{B(x, \epsilon) : \epsilon > 0\}$ of balls forms a (not necessarily open) neighborhood base at x . In other words, a point x is in the closure of a set H if and only if $d(x, H) = 0$.

A topological space is semimetrizable if and only if it is symmetrizable and **first-countable**. Also, a symmetrizable space is semimetrizable if and only if it is a *Fréchet-Urysohn space* (or *E-space*), i.e., for any subset A and for any point x of its closure, there is a sequence in A converging to x .

- **Hyperspace**

A **hyperspace** of a topological space (X, τ) is a topological space on the set $CL(X)$ of all nonempty closed (or, moreover, compact) subsets of X . The topology of a hyperspace of X is called a *hypertopology*. Examples of such a *hit-and-miss topology* are the *Vietoris topology*, and the *Fell topology*. Examples

of such a *weak hyperspace topology* are the *Hausdorff metric topology*, and the *Wijsman topology*.

- **Discrete topological space**

A topological space (X, τ) is **discrete** if τ is the *discrete topology* (the finest topology on X), i.e., containing all subsets of X as open sets. Equivalently, it does not contain any *limit point*, i.e., it consists only of *isolated points*.

- **Indiscrete topological space**

A topological space (X, τ) is **indiscrete** if τ is the *indiscrete topology* (the coarsest topology on X), i.e., having only two open sets, \emptyset and X .

It can be considered as the semimetric space (X, d) with the **indiscrete semi-metric**: $d(x, y) = 0$ for any $x, y \in X$.

- **Extended topology**

Consider a set X and a map $cl : P(X) \rightarrow P(X)$, where $P(X)$ is the set of all subsets of X . The set $cl(A)$ (for $A \subset X$), its dual set $int(A) = X \setminus cl(X \setminus A)$ and the map $N : X \rightarrow P(X)$ with $N(x) = \{A \subset X : x \in int(A)\}$ are called the *closure*, *interior* and *neighborhood map*, respectively.

So, $x \in cl(A)$ is equivalent to $X \setminus A \in P(X) \setminus N(x)$. A subset $A \subset X$ is *closed* if $A = cl(A)$ and *open* if $A = int(A)$. Consider the following possible properties of cl ; they are meant to hold for all $A, B \in P(X)$.

1. $cl(\emptyset) = \emptyset$;
2. $A \subseteq B$ implies $cl(A) \subseteq cl(B)$ (*isotony*);
3. $A \subseteq cl(A)$ (*enlarging*);
4. $cl(A \cup B) = cl(A) \cup cl(B)$ (*linearity*, and, in fact, 4 implies 2);
5. $cl(cl(A)) = cl(A)$ (*idempotency*).

The pair (X, cl) satisfying 1 is called an **extended topology** if 2 holds, a **Brissaud space** (Brissaud, 1974) if 3 holds, a **neighborhood space** (Hammer, 1964) if 2 and 3 hold, a **Smyth space** (Smyth, 1995) if 4 holds, a **pre-topology** (Čech, 1966) if 3 and 4 hold, and a **closure space** (Soltan, 1984) if 2, 3 and 5 hold. (X, cl) is the usual topology, in closure terms, if 1, 3, 4 and 5 hold.

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