Preface

It would not be an exaggeration to say that the history of probability theory is the story of limit theorems, notably the law of large numbers, the law of small numbers and the central limit theorem. The names of Jakob Bernoulli, Abraham de Moivre, Pierre-Simon Laplace and Siméon Denis Poisson are forever linked to their pioneering limit theorems for the Bernoulli scheme and its simplest generalizations.

After these classical works, probability theory remained largely the theory of limit theorems. Invaluable contributions were made to this theory in the late nineteenth and early twentieth centuries by the outstanding Russian mathematicians P.L. Chebyshev, A.A. Markov and A.M. Lyapunov.

Limit theorems continued to be the most popular topic of probability theory in the first half of the twentieth century, the foundations of which were established by E. Borel, A.N. Kolmogorov, A.Ya. Khintchine, P. Lévy and W. Feller. It became clear in the twentieth century that the problems concerning Bernoulli trials have natural analogues which constitute a more general theory of sums of independent random variables. The monograph by B.V. Gnedenko and A.N. Kolmogorov, published in 1949, is the "final chord" in the area of research concerning the weak convergence of distribution functions related to sums of independent random variables.

The second half of the twentieth century was marked by outstanding achievements within the framework of limit problems for sums of independent random variables: many old problems were solved and new directions appeared. In particular, necessary and sufficient conditions for the law of the iterated logarithm and the strong law of large numbers with an arbitrary normalization were found, and functional limit theorems (often called invariance principles) were discovered.

Despite the undoubted achievements of the theory of limit theorems, its main problem for almost sure convergence remains unresolved, namely to find necessary and sufficient conditions for

$$\mathrm{P}(S_n > b_n \text{ i. o.}) = 0 \quad \text{and} \quad \mathrm{P}(S_n > b_n \text{ i.o.}) = 1$$

as well as for

$$\mathrm{P}(S_n < b_n \text{ i.o.}) = 0 \text{ and } \mathrm{P}(S_n < b_n \text{ i.o.}) = 1,$$

where $\{S_n\}$ is a sequence of cumulative sums of independent identically distributed random variables, $\{b_n\}$ is a sequence of real numbers, and "i. o." abbreviates "infinitely often". Note that the strong law of large numbers and the law of the iterated logarithm mentioned above can obviously be embedded into this setting.

The natural logic of development of limit theorems in probability theory has led to many generalizations of classical results. The first of these generalizations deals with a continuous index t instead of a discrete index n. The wealth of problems and results in this direction are not discussed here because it is virtually impossible to do so within the framework of a single monograph.

At the same time there appeared other generalizations. One of them, closely related to the theory of stochastic processes, concerns limit theorems for sums of random elements taking values in abstract linear spaces. This line of research also cannot be described in a few pages; even a simple summary of various problems and related results requires a significant amount of space.

Another generalization arose by replacing the rule of "accumulation" of random variables. The classical theory uses summation $\sum$ for this, while multiplication $\prod$ or maximum max can also be used. Each of these "substitutions" led to an independent theory with many publications. Of special interest is the case of the operation $\prod$ for random matrices instead of random variables.

Yet another generalization occurs if we drop one of the key assumptions of the classical theory, namely the condition that the terms are infinitesimal negligible. Many dozens of publications related to this topic have appeared in the past few decades.

A special place is occupied by a generalization where independent random variables are replaced by those with different schemes of dependence. This generalization has produced an almost endless stream of results and publications; hundreds of pages in the literature are devoted to various problems for dependent random variables. We will not discuss limit theorems for dependent random variables here, since even a brief listing of results would take too much space.

Instead, we will focus on a different area of limit theorems, which arises if the indices of the random variables belong to a more general space than $\mathbf{N}$ or $\mathbf{R}$, namely we study limit theorems for random variables that depend on several discrete arguments. The set of indices we are interested in is denoted by $\mathbf{N}^d$ where d is the dimension of the indices. In this case, we say that random variables depend on a *multiple index* or *multi-index*. The members of the space are denoted by $\mathbf{k}, \mathbf{m}, \mathbf{n}, \ldots$.

Thus we consider *multi-indexed* independent random variables and study limit theorems for them. The main focus is on almost sure convergence. Any family of random variables that depend on the indices belonging to the space $\mathbf{N}^d$ is called a *random field*.

The convergence of random fields can be defined in various ways depending on what is meant by "$\mathbf{n} \to \infty$" in the case of $d > 1$. We are mostly interested in the following two modes of convergence:

$$(n_1, \ldots, n_d) \to \infty \iff \min(n_1, \ldots, n_d) \to \infty$$

or

$$(n_1, \ldots, n_d) \to \infty \iff \max(n_1, \ldots, n_d) \to \infty$$

The first is called min-convergence, while the second is called max-convergence. In a sense, min-convergence and max-convergence are respectively the weakest and strongest modes of convergence.

A brief survey of the development of the theory of limit theorems for multi-indexed sums of random variables is given in Chap. 1.

Contents of the Book

Following a "telegraphic style", we briefly discuss, chapter by chapter, the contents of the book.

Chapter 1 is an informal introduction to the theory of limit theorems for multi-indexed sums of independent random variables. We describe what this theory is and what the main differences are between the results for $d = 1$ and $d > 1$. The chapter also contains a short history of the development of the subject over the last 50 years.

Chapter 2 contains a number of inequalities for distributions and moments of multi-indexed sums of independent random variables. There is a fundamental difference between the cases $d = 1$ and $d > 1$, which is caused by the absence of a complete ordering of the space $\mathbf{N}^d$ if $d > 1$. We describe several ways to overcome this obstacle in the proof of maximal inequalities for multi-indexed sums. The results of Chap. 2 are used throughout the book.

A classical problem concerning the set of limit distributions in the case $d > 1$ is solved in Chap. 3. We find necessary and sufficient conditions for weak convergence for both modes of convergence, min and max. Although these conditions do not differ in form, there is actually a big difference between them. We exhibit this difference in Chap. 4 when discussing the law of large numbers for both modes of convergence.

Chapter 5 studies almost sure convergence of multi-indexed series of independent random variables. An unexpected aspect of the case $d > 1$ is that the general term of a convergent series does not necessarily approach 0. In the case $d > 1$, in contrast to $d = 1$, the criterion of almost sure convergence of a series of independent terms is expressed in terms of convergence of four numerical series. In some situations, however, this result can be reduced to the classical form by using only three series.

Chapter 6 is closely related to the preceding chapter and discusses the almost sure boundedness of multi-indexed series. We note a phenomenon that occurs if $d > 1$: a convergent series is not necessarily bounded.

Chapter 7 is also related to Chap. 5 and studies the rate of convergence of multi-indexed series. Here we have a new phenomenon for $d > 1$: if a series converges, then its tails are not necessarily convergent. The converse is also true, namely the tails of a divergent series may converge.

The strong law of large numbers for multi-indexed sums of independent random variables is studied in Chap. 8. One of the unexpected obstacles is the lack of an analogue of Kronecker's lemma in the case $d > 1$, which requires new approaches and methods.

Chapter 9 continues the investigation of the strong law of large numbers under an additional assumption that the random variables are identically distributed. The general results of Chap. 8 are used. The peculiarity of the case $d > 1$ is that the strong law of large numbers may hold for some "exotic" normalizations $b(\mathbf{n})$ that have no analogues for $d = 1$. If, for example, $d = 2$, then one of those normalizations is $b(m, n) = m\sqrt{n}$.

Chapter 10 is devoted to yet another classical result, the law of the iterated logarithm. Necessary and sufficient conditions for the law of the iterated logarithm are found in the case of identically distributed random variables. We face a challenging problem when studying the law of the iterated logarithm for non-identically distributed random variables, namely the normalization is not universal for non-identically distributed random variables even if the second moments are finite; it strongly depends on the structure of the $\mathtt{var}\,[S(\mathbf{n})]$.

We study the asymptotic behaviour of renewal functions and processes constructed from random walks with multi-dimensional time in Chap. 11. A new approach to the definition of renewal processes is presented. New methods for developing the results are also introduced.

The existence of moments of supremums of weighted multi-indexed sums of independent identically distributed random variables is established in Chap. 12. Some applications of these results are exhibited: we deal with new forms of the strong law of large numbers and the law of the iterated logarithm that have no analogues in the case $d = 1$.

We study so-called complete convergence in Chap. 13. The results on complete convergence are traditionally used to describe the rate of convergence in the law of large numbers and to express necessary and sufficient conditions for the strong law of large numbers and the law of the iterated logarithm. Several other applications are also known.

Some related references are given at the end of each chapter.

Appendix A discusses min-, max- and other modes of convergence in the space $\mathbf{N}^d$. In particular, all necessary notations can be found in Appendix. Various other results, definitions and notation for the space $\mathbf{N}^d$ are also given. Specific features of the case $d > 1$ and the difference between this case and $d = 1$ are discussed. In general, Appendix A is a very dense introduction to elementary analysis in the space of multi-indices $\mathbf{N}^d$.

I would greatly appreciate feedback from the readers, who are invited to contact me via e-mail address klesov@matan.kpi.ua.

Kyiv, May 2014 Oleg Klesov

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