A Unified (P)DAE Modeling Approach for Flow Networks

Lennart Jansen and Caren Tischendorf

Abstract We present a unified modeling approach for different types of flow networks, for instance electric circuits, water and gas supplying networks. In all cases the flow network is described by the pressures at the nodes of the network and the flows through the branches of the network. It is shown that the mass balance equations at each node are independent of the type of flow medium and can be described by the use of incidence matrices reflecting the network topology. Additionally, various types of net element models are presented. Finally, all network describing equations are summarized for some prototype networks which differ by the various net element models. They yield in pure linear/nonlinear equation systems, differential-algebraic systems or partial differential equation systems. All of them may have serious rank changes in the model functions if switching elements belong to the network. The model descriptions presented here keep all the network structure information and can be exploited for the analysis, numerical simulation and optimization of such networks.

Keywords Modeling • Flow network • Partial differential algebraic equation • Circuit • Water network • Gas network

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1 Modeling of Flow Networks

In flow networks, a certain medium is flowing through branches that are connected by nodes. The medium can be very different, for example, gas, water and electric currents. Pressure or potential differences between nodes cause a flow through the connecting branches of the network. In order to describe the flows through the

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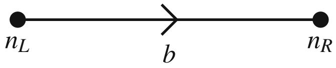

Fig. 1 Network branch b directing from left node n_L to right node n_R

branches and the corresponding pressures or potentials at each node, we consider the network as an oriented graph $G = (N, B)$ with $N = \{n_1, n_2, \ldots, n_N\}$ forming the set of all nodes and $B = \{b_1, b_2, \ldots, b_B\}$ being the set of all branches. We equip each branch with a certain direction. It allows us to distinguish between the two nodes of each branch: a left node n_l and a right node n_r. Each branch directs from the left to the right node (see Fig. 1).

Remark 1 The direction of a branch from the left to the right node does not mean that the medium is always flowing from the left to the right node. Later on, we see that the branch direction tells us how we have to interpret the sign of the flow values. If the flow has a positive sign then it flows from left to right. If the flow has a negative sign then it flows from right to left.

The assignment of the left and right nodes of each branch to the global node numbers may easily described by the incidence matrices $A_L, A_R \in \mathbb{R}^{n_N \times n_B}$ defined by

$$(A_L)_{ij} = \begin{cases} -1 & \text{if node } n_i \text{ is the left node of branch } b_j, \\ 0 & \text{else,} \end{cases}$$

$$(A_R)_{ij} = \begin{cases} +1 & \text{if node } n_i \text{ is the right node of branch } b_j, \\ 0 & \text{else.} \end{cases}$$

For later use, we additionally introduce the incidence matrix

$$A := A_L + A_R \in \mathbb{R}^{n_N \times n_B}.$$

Obviously, we have

$$(A)_{ij} = \begin{cases} -1 & \text{if node } n_i \text{ is the left node of branch } b_j, \\ +1 & \text{if node } n_i \text{ is the right node of branch } b_j, \\ 0 & \text{else.} \end{cases} \tag{1}$$

Remark 2 We exclude in our considerations networks with self-loops. With other words, the network is not allowed to have branches with left and right node being the same node. It ensures the correctness of Eq. (1).

Remark 3 The definition of the incidence matrices differs in the various communities analyzing flow networks. For water supplying networks, one usually uses

$$A_{\text{water}} = A^\top \in \mathbb{R}^{n_B \times n_N},$$

see e.g. [13, 31, 34] but in electric circuit analysis one commonly uses

$$A_{\text{circuit}} = -A \in \mathbb{R}^{n_N \times n_B},$$

see e.g. [9, 12, 16].

1.1 Network Topology Describing Equations

Next, we describe the mass flow balance of each flow network. Independently of the kind of the flow medium, the sum of all flows entering one node equals the sum of all flows leaving this node. It reflects the law of conservation of mass at each node. In circuit analysis, it is known as the Kirchhoff's current law.

Introducing the flow variables $q_L, q_R \in \mathbb{R}^{n_B}$ as vectors with the entries

$$(q_L)_i = \text{flow entering branch } b_i \text{ at the left node,}$$
$$(q_R)_i = \text{flow leaving branch } b_i \text{ at the right node,}$$

we may write the mass flow balance equations as

$$A_L q_L + A_R q_R = 0. \tag{2}$$

The i-the row of this equation system reflects the mass flow balance equation at node n_i. If the network contains nodes connected to flow sources (e.g. to reservoirs) or to flow sinks (e.g. to open hydrants) then we rewrite the mass flow balance equations (2) as

$$A_L q_L + A_R q_R = q_s \tag{3}$$

with

$$(q_s)_j = \text{sum of flows entering/leaving node } n_j \text{ from sources/sinks}$$

for all $j = 1, \ldots, n_N$.

Remark 4 If we may neglect the time delay the flow impulse needs to be transferred from the left to the right node of a branch (it depends on the flow medium and the flow distances) then we may identify q_L with q_R and operate only with

$$q := q_L = q_R.$$

The mass flow balance equations (3) then reduce to

$$Aq = q_s. \tag{4}$$

The time delay is, for example, neglected in standard circuit analysis. However, one can not neglect it, for instance, for the gas flow through longer pipe lines or for the current flow through transmission lines.

Remark 5 The mass flow q is denoted differently for the different flow media. We commonly find the following notations in the literature:

$$\begin{aligned} q_{\text{water}} &= m \text{ or } Q && \text{water flow,} \\ q_{\text{gas}} &= q && \text{gas flow,} \\ q_{\text{circuit}} &= i && \text{current flow,} \\ q_{\text{blood}} &= Q && \text{blood flow.} \end{aligned}$$

As already mentioned, pressure or potential differences between two nodes cause a flow through the branches connecting these nodes. Let $p \in \mathbb{R}^{n_N}$ denote the vector of pressure/potential at the nodes, that means, $p_i \in \mathbb{R}$ describes the pressure/potential at the node n_i for $i = 1, \ldots, n_N$.

Remark 6 The pressure/potential notation differs in the literature depending on the type of flow media:

$$\begin{aligned} p_{\text{water}} &= h \text{ or } H && \text{(scaled) water pressure,} \\ p_{\text{gas}} &= p && \text{gas pressure,} \\ p_{\text{circuit}} &= e && \text{electric potentials,} \\ p_{\text{blood}} &= p && \text{blood pressure.} \end{aligned}$$

The incidence matrix A allows us easily to describe the pressure/potential differences $(\Delta p)_i$ at each branch b_i of the network. We have

$$(\Delta p)_i = p_{R(i)} - p_{L(i)} \qquad \forall i = 1, \ldots, n_B \tag{5}$$

for $L(i)$ and $R(i)$ being the left and right node for the branch b_i. Regarding the entries of A, we see that (5) is equivalent to

$$\Delta p = A^\top p \tag{6}$$

where the i-th component of Δp represents the pressure/potential differences of the i-th branch in the network. Notice that we have

$$\Delta p = A_R^\top p + A_L^\top p = p_R - p_L$$

with

$$p_R := A_R^\top p, \qquad p_L := -A_L^\top p. \tag{7}$$

The i-th component of p_R and p_L represent the pressure $p_{R(i)}$ of the right node $n_{R(i)}$ of pipe i and the pressure $p_{L(i)}$ of the left node $n_{L(i)}$ of pipe i.

We summarize the mass flow balance equation (3) and the pressure difference equation (6).

$$A_R q_R + A_L q_L = q_s, \tag{8}$$

$$\Delta p = A^\top p = A_R^\top p + A_L^\top p = p_R - p_L. \tag{9}$$

They form the network modeling equations describing the network topology. Next, we need the network equations describing the relations between the flows q and the pressures p or pressure differences Δp. These relations are element dependent and, therefore, differ for the various type of flow networks.

1.2 Network Element Modeling

We distinguish between two types of elements: branch elements and node elements. Node elements describe the pressure $\tilde{p}$ at a node $\tilde{n}$. Branch elements describe the relations between the flow $\tilde{q}_L$ and the pressure $\tilde{p}_L$ at the left node of a branch and the flow $\tilde{q}_R$ and the pressure $\tilde{p}_R$ at the right node.

If the node of a node element has the global node number j then we have

$$\tilde{p} = p_j.$$

If the branch of a branch element has the global branch number i then we have

$$\begin{aligned}
\tilde{q}_L &= (q_L)_i, \\
\tilde{q}_R &= (q_R)_i, \\
\tilde{p}_L &= (p_L)_i = (-A_L^\top p)_i, \\
\tilde{p}_R &= (p_R)_i = (A_R^\top p)_i.
\end{aligned}$$

For convenience, we neglect the tilde notation in the following. The reader should have in mind that each scalar p, p_L, p_R, q_L and q_R (Fig. 2) in the following reflects one component of the vectors p, p_L, p_R, q_L and q_R introduced before.

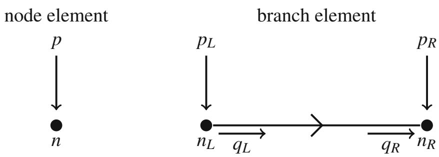

Fig. 2 Types of network elements: node elements and branch elements

Next, we collect some commonly used network element models for water networks, gas networks and electric circuits. The list of elements presented here is not complete but shall exemplify the variety of element models. Furthermore, we want to point out that these element models can also be used for the simulation of other flow networks. For instance, the blood flow can be modelled by circuit element models, see [11, 24].

Electric Circuits

For better understanding, we start with an example. Figure 3 represents a simple dynamic circuit for an induction machine. The rotor circuit part consists of a sinusoidal voltage source p_{s2} and a rotor resistance R_2. The stator circuit part has a constant voltage source p_{s1}, a stator resistance R_1, a leakage inductance L_1 and a magnetizing inductance L_2. For the flows of this lumped model we have

$$q_i = q_{Li} = q_{Ri} \qquad \forall i = 1, \ldots, 5.$$

The flow balance equations (8) are given by

$$\begin{aligned} q_1 - q_2 &= 0 \\ q_2 - q_3 &= 0 \\ q_3 - q_4 - q_5 &= 0 \\ q_5 - q_6 &= 0 \end{aligned}$$

The flow balance equation for the mass node can be neglected since it equals the negative sum of all other flow balance equations. The voltage difference equations (9) are represented by

$$\begin{aligned} (\Delta p)_1 &= p_1 - p_0 \\ (\Delta p)_2 &= p_2 - p_1 \\ (\Delta p)_3 &= p_3 - p_2 \end{aligned}$$

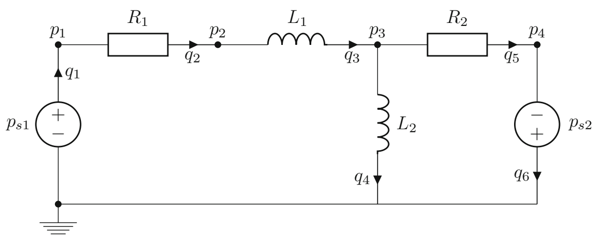

Fig. 3 Lumped inductor circuit with a static voltage source p_{s1} and a rotor voltage source p_{s2}

$$(\Delta p)_4 = p_0 - p_3$$
$$(\Delta p)_5 = p_4 - p_3$$
$$(\Delta p)_6 = p_0 - p_4$$

where the nodal voltage p_0 of the mass node is usually fixed by $p_0 = 0$. The net element models of the two voltage sources are given by

$$(\Delta p)_1 = p_{s1}$$
$$(\Delta p)_6 = p_{s2}.$$

The net element models of the two resistors are given by

$$(\Delta p)_2 = R_1 q_2$$
$$(\Delta p)_5 = R_2 q_5.$$

The net element models of the two inductors are given by

$$(\Delta p)_3 = L_1 q_3$$
$$(\Delta p)_4 = L_2 q_4.$$

In general, we have the following net element descriptions.

Resistor A resistor is a branch element. Ohmic resistors describe a linear relation between the voltage $p_L - p_R$ and the current $q = q_L = q_R$:

$$\Delta p = Rq$$

with the resistance $R > 0$. In general, there is a function f_{re} such that

$$\Delta p = f_{re}(q). \tag{10}$$

Conductor A conductor is a branch element. Ideal conductors describe a linear relation between the current $q = q_L = q_R$ and the voltage Δp:

$$q = G \cdot \Delta p$$

with the conductance $G > 0$. In general, there is a function f_{co} such that

$$q = f_{co}(\Delta p). \tag{11}$$

Capacitor A capacitor is a branch element. Its current $q = q_L = q_R$ is given by the time derivative of the charge. For ideal capacitors, the charge is given by $C \cdot \Delta p$ such that

$$q = C \frac{\mathrm{d}}{\mathrm{d}t} \Delta p$$

with the capacitance $C > 0$. In general, there is a function f_{ca} such that

$$q = \frac{\mathrm{d}}{\mathrm{d}t} f_{ca}(\Delta p). \tag{12}$$

Inductor An inductor is a branch element. Its voltage Δp is given by the time derivative of the magnetic flux. For ideal inductors, the flux can be described as $L \cdot q$ with the current $q = q_L = q_R$ such that

$$\Delta p = L \frac{\mathrm{d}}{\mathrm{d}t} q$$

with the inductance $L > 0$. In general, there is a function f_{in} such that

$$\Delta p = \frac{\mathrm{d}}{\mathrm{d}t} f_{in}(q). \tag{13}$$

Memristor A memristor is a branch element. Its current $q = q_L = q_R$ and its voltage Δp are related by the charge as follows, see [8, 25, 28]:

$$\Delta p = \frac{\mathrm{d}}{\mathrm{d}t} f_{mr}(u), \quad q = \frac{\mathrm{d}}{\mathrm{d}t} u$$

with a given function f_{mr} and an extra variable u reflecting the charge.

Memductor A memductor (also called flux-controlled memristor) is a branch element. Its current $q = q_L = q_R$ and its voltage Δp are related by the flux as follows, see [8, 27]:

$$q = \frac{\mathrm{d}}{\mathrm{d}t} f_{md}(u), \quad \Delta p = \frac{\mathrm{d}}{\mathrm{d}t} u \tag{14}$$

with a given function f_{md} and an extra variable u representing the flux. A modeling discussion of further memelements can be found, for example, in [1, 14, 26].

Voltage source A voltage source is a branch element. It prescribes the branch voltage Δp. An independent voltage source is represented by

$$\Delta p = f_{vs}(t) \tag{15}$$

with a given function f_{vs}. Typically, f_{vs} is either constant or a periodic function of time. Voltage sources can also be controlled ones. If a voltage source is current controlled then we have

$$\Delta p = f_{vs}(q_c) \tag{16}$$

with a given function f_{vs} and the controlling current q_c. If a voltage source is voltage controlled then we have

$$\Delta p = f_{vs}(\Delta p_c) \tag{17}$$

with a given function f_{vs} and the controlling voltage Δp_c.

Current source A current source is a branch element. It prescribes the current $q = q_L = q_R$. An independent current source is represented by

$$q = f_{cs}(t) \tag{18}$$

with a given function f_{cs}. Typically, f_{cs} is either constant or a periodic function of time. Current sources can be also controlled ones. If a current source is current controlled then we have

$$q = f_{cs}(q_c) \tag{19}$$

with a given function f_{cs} and the controlling current q_c. If a current source is voltage controlled then we have

$$q = f_{cs}(\Delta p_c) \tag{20}$$

with a given function f_{cs} and the controlling voltage Δp_c.

Diode A diode is a branch element. The Shockley ideal model description for a diode is given by

$$q = q_S(\exp^{\frac{\Delta p}{p_T}} - 1) =: f_{di}(\Delta p) \tag{21}$$

with q_S being the saturation current and

$$p_T = \frac{kT}{q_e}$$

denoting the thermal voltage, k the Boltzmann constant, T the temperature and q_e the elementary charge. The function f_{di} is monotone but not strongly monotone, see Fig. 4a. Considering real diodes, we have a characteristics as presented in Fig. 4b which can be described by a strongly monotone function f_{di} satisfying

$$q = f_{di}(\Delta p). \tag{22}$$

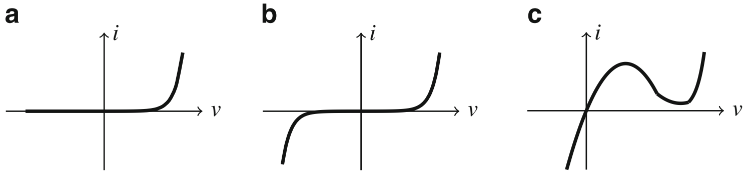

Fig. 4 Diode model functions. **(a)** Ideal Shockley diode. **(b)** Z-diode with a strongly monotone characteristics. **(c)** Tunnel diode

Strong monotonicity properties are crucial for the existence of unique global solutions [20]. However, there are also diodes that have a non-monotone characteristics, for instance tunnel diodes, see Fig. 4c.

Transistor A transistor is a multi-branch element. Typically, equivalent circuits with the basic circuits elements presented before are used to describe their behavior, see for instance [22]. These equivalent circuits become more and more complex. For instance, the BSIM4 model (see [6]) includes more than 800 parameter values. Not all of them have a direct physical interpretation. Therefore, it becomes more and more difficult to tune them for the various frequency regions in which the transistor is operating.

An alternative of such complex equivalent circuits is the use of discretized PDE models that describe the charge carrier movement and the electromagnetic field propagation. A commonly used model are the drift diffusion equations, see for instance [17, 23]. For its embedding into network models see [4, 29, 32, 33].

$$\frac{\mathrm{d}}{\mathrm{d}t}(Wp_c) - \int_{\Omega_S} g(x)(f_{Jn}(u_n, u_{\varphi S}, x) + f_{Jp}(u_p, u_{\varphi S}, x))\,\mathrm{d}x = q_c \tag{23a}$$

$$-\frac{\partial}{\partial t}u_n + \frac{1}{c_q}\nabla \cdot f_{Jn}(u_n, u_{\varphi S}, x) = f_R(u_n, u_p) \tag{23b}$$

$$\frac{\partial}{\partial t}u_p + \frac{1}{c_q}\nabla \cdot f_{Jp}(u_p, u_{\varphi S}, x) = -f_R(u_n, u_p) \tag{23c}$$

$$\nabla \cdot (-c_{\epsilon S}\nabla u_{\varphi S}) = c_q(p - n + c_C) \tag{23d}$$

$$\nabla \cdot (-c_{\epsilon O}\nabla u_{\varphi O}) = 0 \tag{23e}$$

with q_c being the flow entering the node with pressure p_c connected to the contact area Γ_c of the transistor. The matrix W is symmetric and positive definite. The extra variables are the space and time dependent electrostatic potentials $u_{\varphi S}$ and $u_{\varphi O}$ for the substrate and oxide region and the electron and hole densities u_n and u_p in the substrate region. The constants c_q, c_C, $c_{\epsilon S}$ and $c_{\epsilon O}$ are the elementary charge, the doping profile and the dielectric constants of the substrate and the oxide. The functions f_{Jn} and f_{Jp} describe the electron and hole current densities as

$$f_{Jn}(u_n, u_{\varphi S}, x) := q\mu_n(x)(c_{UT}\nabla u_n - u_n\nabla u_{\varphi S}),$$
$$f_{Jp}(u_p, u_{\varphi S}, x) := -q\mu_p(x)(c_{UT}\nabla u_p + u_p\nabla u_{\varphi S})$$

The functions μ_n and μ_p are given functions of x describing the electron and hole mobilities. The function f_R is a given function describing the balance of generation and recombination of electrons and holes. The drift diffusion equations are complete with the Dirichlet boundary conditions

$$u_n(x,\cdot) = g_n(x), \quad u_p(x,\cdot) = g_c(x), \quad u_{\varphi S}(x,\cdot) = g_S(x) + i_S(p_c)$$

at the contact areas Γ_{cS} of the substrate, the Dirichlet boundary conditions

$$u_{\varphi O}(x,\cdot) = g_O(x) + i_O(p_c)$$

at the contact areas Γ_{cO} of the substrate homogeneous Neumann boundary conditions for u_n, u_p and $u_{\varphi S}$ on the isolating boundary Γ_{NS} of the substrate, Neumann boundary conditions for $u_{\varphi S}$ on the isolating boundary Γ_{NO} and the boundary equations

$$f_{Jn}(u_n, u_{\varphi S}, x)\cdot \nu_S(x) = 0, \quad f_{Jp}(u_p, u_{\varphi S}, x)\cdot \nu_S(x) = 0.$$

at the interface Γ_I between oxide and substrate where $\nu_S(x)$ is the outer unit normal of the substrate at point x. The functions i_S and i_O are indicator functions selecting the contacts at the substrate and the oxide. The functions g_n, g_p, g_S and g_O are given functions of the space position x.

Cross Talking Lines Cross talking lines are multi-branch elements. Usually, they describe the mutual influence of two conduction branches and are therefore connected to four nodes of the network (see Fig. 5). If high frequency input signals are applied to circuits then we may observe a cross talk over tight lines. In order to model this cross talk we use the electromagnetic model developed in [3]. The model in [3] arises from the full Maxwell equations spatially discretized with the

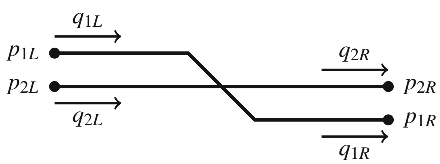

Fig. 5 Two cross talking branches have four contact nodes. Correspondingly, we have four pressure values p_{1L}, p_{1R}, p_{2L} and p_{2R} and four flows q_{1L}, q_{1R}, q_{2L} and q_{2R}

finite integration technique (FIT). The FIT discretization is an established tool to discretize electromagnetic devices which was developed and formulated by Thomas Weiland [10, 35–37].

$$\begin{aligned}(-q_{1L}, q_{1R}, -q_{2L}, q_{2R})^\top - \Lambda^\top C^\top M_\nu C u_A &= 0,\\ \vartheta \tilde{S} M_\epsilon G \frac{\mathrm{d}}{\mathrm{d}t} u_\phi + \tilde{S} M_{\bar{\nu}} u_A &= 0,\\ M_\epsilon \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}}{\mathrm{d}t} u_A + G u_\phi - \Lambda(p_{1L}, p_{1R}, p_{2L}, p_{2R})^\top) + C^\top M_\nu C u_A &\\ + M_\sigma(\frac{\mathrm{d}}{\mathrm{d}t} u_A + G u_\phi - \Lambda(p_{1L}, p_{1R}, p_{2L}, p_{2R})^\top) &= 0.\end{aligned}$$

The discretized vector potential $u_A \in \mathbb{R}^{3n}$ and the discretized scalar potential $u_\phi \in \mathbb{R}^n$ are additional variables for the description of the flow/current relation. The dimension n depends on the refinement of the FIT discretization. The matrices $M_\epsilon, M_\sigma, M_\nu \in \mathbb{R}^{3n \times 3n}$ represent the three material properties for the permittivity, the conductivity and the reluctivity.

The matrix $G \in \mathbb{R}^{3n \times n}$ describes the discretized gradient operator, $\tilde{S} \in \mathbb{R}^{n \times 3n}$ is the discretized divergence operator and $C \in \mathbb{R}^{3n \times 3n}$ reflects discretized rotation operator.

The variable ϑ is used for different types of gauging. The case $\vartheta = 0$ describes the Coulomb gauge condition. The case $\vartheta = 1$ reflects the Lorenz gauge condition.

Finally, the excitation matrix $\Lambda \in \mathbb{R}^{3n \times 4}$ represents the boundary operator. It indicates which points of the discretization grid belong to one of the four contact areas. The transposed excitation matrix $\Lambda^\top \in \mathbb{R}^{4 \times 3n}$ represents a finite approximation of the integral over the four contact areas.

Gas Networks

Gas networks usually consist of pipes, valves, resistors, compressors, preheaters and coolers. We present here some model descriptions for the first three types of elements, while compressors, preheaters and coolers are often modeled by characteristic maps.

Pipe A pipe is a branch element. The flow through the gas pipe can be described on different model levels. As an example we use here the one-dimensional isothermal Euler equations [2, 5, 19] for the description of the compressible flow.

$$\begin{aligned} a\partial_t u_\varrho + \partial_x u_q &= 0\\ \partial_t u_q + \partial_x(u_p + \frac{u_q^2}{a u_\varrho}) &= -\frac{\lambda}{2D}\frac{u_q|u_q|}{a u_\varrho} - a g u_\varrho h' \qquad (24)\\ u_p &= R u_\varrho T(1 + c_\alpha u_p)\end{aligned}$$

with the pipe cross section area a, the pipe diameter D, the temperature T, the gas constant R, the gravity constant g, the pipe slope h' and the pipe friction coefficient λ and a real gas factor c_α. The pressure u_p, the gas density u_ϱ and the gas flow u_q through the pipe are time and position dependent. The network flows q and the node pressures p are related by the boundary conditions

$$\begin{aligned} u_p(x_L, t) &= p_L(t), & u_q(x_L, t) &= q_L(t), \\ u_p(x_R, t) &= p_R(t), & u_q(x_R, t) &= q_R(t). \end{aligned} \tag{25}$$

for all time points t.

Valve A valve is a branch element. It exists in different forms and can be modeled on different model levels. The easiest model describes a valve as a switch with two states *on* and *off*. The flow/pressure equations are then given by

$$\begin{cases} q = q_R = q_L, \quad p_L = p_R & \text{if the valve is } on \\ q = q_R = q_L = 0 & \text{if the valve is } off \end{cases} \tag{26}$$

Usually the state of the valve is controlled from outside and then given as a function of time. However, there are also self-controlled valves, for instance a non-return valve. It allows the flow to move into one direction only. We assume that the flow can move from left to right. Then,

$$\text{the non-return valve switches to } \begin{cases} on & \text{if } p_L > p_R \text{ and } q \geq 0 \\ off & \text{if } q < 0 \end{cases} \tag{27}$$

Resistor A resistor is a branch element. A resistor model is used to describe the hydraulic resistance of a valve. It specifies the pressure loss $\Delta p = p_L - p_R$ by

$$p_L - p_R = \frac{\zeta}{2} u_\varrho v^2 = \frac{\zeta}{2} \frac{q^2}{u_\varrho a^2}$$

with $q = q_L = q_R$, the pressure loss coefficient ζ, the velocity v, the gas density u_ϱ at the left node and the cross-section area a, see [30]. Assuming the gas compression factor to be constant, we have

$$p_L = u_\varrho c^2$$

with the constant sonic velocity c. It results in

$$p_L(p_L - p_R) = bq^2 \tag{28}$$

with the constant $b := \frac{c^2}{2a^2}\zeta$.

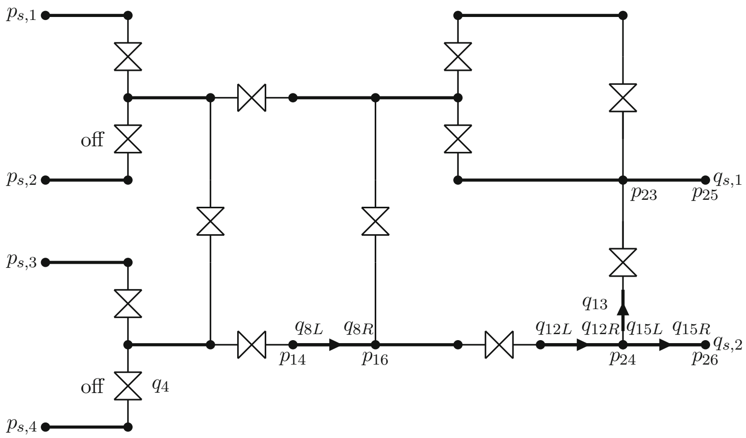

Fig. 6 Gas pipeline network with a gamma pulse characteristics. At the nodes 1–4 the pressure is fixed. The nodes 25 and 26 are sink nodes with a given flow demand. Two valves are closed. All other valves are open

A simple example for a gas pipe network with a gamma pulse characteristics is given in Fig. 6. It consists of 15 pipes and 13 valves. The network has four nodes at the left hand side where the pressure is fixed. At the right hand side we have two demand nodes with the demands $q_{s,1}$ and $q_{s,2}$. The pipe model equations (24) for the pipe between the nodes p_{14} and p_{16} are given by

$$a\partial_t u_{\varrho 8} + \partial_x u_{q8} = 0$$

$$\partial_t u_{q8} + \partial_x \left(u_{p8} + \frac{u_{q8}^2}{a u_{\varrho 8}}\right) = -\frac{\lambda}{2D}\frac{u_{q8}|u_{q8}|}{a u_{\varrho 8}} - a g u_{\varrho 8} h'$$

$$u_{p8} = R u_{\varrho 8} T(1 + c_\alpha u_{p8})$$

with the boundary conditions

$$u_{p8}(x_L, t) = p_{14}(t), \qquad u_{q8}(x_L, t) = q_{8L}(t),$$
$$u_{p8}(x_R, t) = p_{16}(t), \qquad u_{q8}(x_R, t) = q_{8R}(t).$$

Analogously, one can formulate the pipe equations for the other pipes in the network. For the open valve number 13 we get

$$q_{13} = q_{13R} = q_{13L}, \quad p_{23} = p_{24}.$$

Analogously, the equations of the other open valves can be formulated. For the lower closed valve we get

$$q_4 = q_{4R} = q_{4L} = 0$$

Analogously, the flow is zero through the other closed valve. The flow balance equation of the demand node 26 reads

$$q_{15R} = q_{s,2}.$$

Correspondingly, the flow balance equation for the other demand node 25 can be written. The flow balance equation of the node 24 is given by

$$q_{12R} - q_{13} - q_{15L} = 0.$$

Analogously, one can derive the flow balance equations of the remaining nodes.

Water Networks

Water supplying networks usually consist of pipes, valves, pumps, turbines, tanks and reservoirs. As an example we present models for pipes, pumps, tanks and reservoirs. Turbines are often modeled by characteristic maps. Valves are modeled in the same manner as for gas networks.

Pipe A pipe is a branch element. In the absence of shocks the incompressible flow can be described by the quasi-stationary model equations [21]

$$\frac{\mathrm{d}}{\mathrm{d}t} q + c_1(p_R - p_L) + c_2 q|q| = c_3 \tag{29}$$

with $q = q_L = q_R$ and c_1, c_2, c_3 being constants depending on the pipe diameter, length, cross-section area, friction and inclination angle.

Tank A tank is a node element.

$$a_R^\top q_R + a_L^\top q_L = \frac{\mathrm{d}}{\mathrm{d}t} f_c(p) \tag{30}$$

with a_R and a_L being the columns of A_L and A_R belonging to the node of the tank element. Correspondingly, q_R and q_L are the full vectors of left and right network flows, respectively. The function f_c describes the capacity of the tank.

Reservoir A reservoir is a node element. It has an unlimited capacity but constant pressure c_p. Correspondingly, we get the simple relation

$$p = c_p. \tag{31}$$

Pump A pump is a branch element. Its characteristic is usually described by an algebraic relation of the form

$$p_R - p_L = f_{pu}(q) \tag{32}$$

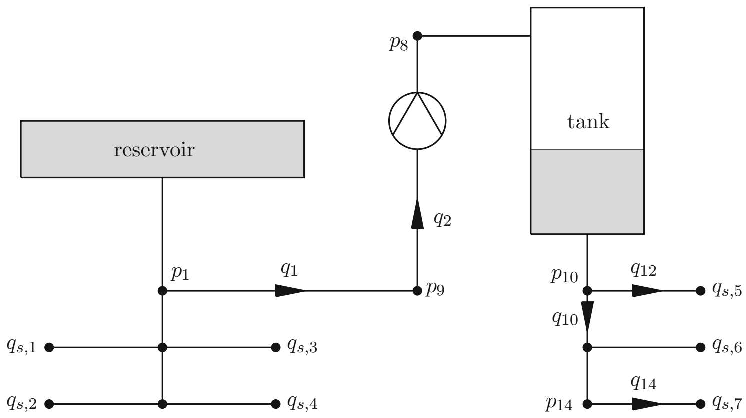

Fig. 7 Water pipeline network with one reservoir, one tank, one pump and 13 pipes. Seven nodes are demand nodes with a flow demand $q_{s,i}$ for $i = 1, \ldots, 7$

with $q = q_L = q_R$. One example is the representation in EPANET [15] as

$$p_R - p_L = aq^b$$

with a certain resistance coefficient a and flow exponent b.

One small example of a water network with a reservoir, one tank, one pump and 13 pipes is given in Fig. 7. The reservoir equation reads

$$p_1 = c_p$$

with the constant pressure c_p. Using the quasi-stationary model equations (29) for the pipes, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} q_1 + c_1(p_1 - p_9) + c_2 q_1 |q_1| = c_3$$

for the pipe between the nodes p_1 and p_9. Analogously, one can describe the other 12 pipes. Taking the EPANET pump model [15], we get

$$p_8 - p_9 = aq_2^b$$

with a certain resistance coefficient a and a flow exponent b. The tank is a node element and may be described by

$$q_{10} + q_{12} = \frac{\mathrm{d}}{\mathrm{d}t} f_c(p_{10})$$

with the function f_c describing the capacity of the tank. The flow balance equation at the demand node with demand $q_{s,7}$ has the simple form

$$q_{14} = q_{s,7}.$$

Analogously, the flow balance equations for the other six demand nodes can be formulated. The flow balance equations for the rest of the nodes are just reflecting that the sum of all inflowing pipe flows equals the sum of all outflowing pipe flows.

2 Model Classes for Flow Networks

Depending on the different element models, the description of flow networks results in different types of equation systems: pure linear/nonlinear equation systems, differential algebraic equation systems, differential algebraic equation systems or partial differential algebraic equation systems.

Next, we want to elaborate some model classes reflecting topological net properties of flow networks.

2.1 Static Networks with Lumped Element Models

Lumped element models are characterized by constant flows on each network branch, i.e. $q_L = q_R = q$. Then, we have four types of net elements:

1. $$\Delta\tilde{p} = \tilde{p}_R - \tilde{p}_L = \tilde{f}_{pq}(\tilde{q}), \qquad (T_{pq})$$
2. $$\tilde{q} = \tilde{f}_{qp}(\Delta\tilde{p}) = \tilde{f}_{qp}(\tilde{p}_R - \tilde{p}_L), \qquad (T_{qp})$$
3. $$\Delta\tilde{p} = \tilde{p}_R - \tilde{p}_L = \tilde{c}_p, \qquad (T_p)$$
4. $$\tilde{q} = \tilde{c}_q \qquad (T_q)$$

with c_p and c_q being constants reflecting constant pressure and flow sources, respectively. The tilde notation is used to stress that we are describing a single element. The subscript in the types of elements shall indicate first the variables to be controlled and second the controlling variables. Summarizing the element equations (T_{pq}), (T_{qp}), (T_q), (T_p) and the mass flow balance equation (4), we obtain the system

$$A_{pq}^{\top} p = f_{pq}(q_{pq}) \qquad (33a)$$

$$q_{qp} = f_{qp}(A_{qp}^{\top} p) \qquad (33b)$$

$$A_p^\top p = c_p \tag{33c}$$

$$q_q = c_q \tag{33d}$$

$$A_{pq}q_{pq} + A_{qp}q_{qp} + A_p q_p + A_q q_q = 0 \tag{33e}$$

with the incidence matrices A_{pq}, A_{qp}, A_p and A_q collecting all columns for network elements of type (T_{pq}), (T_{qp}), (T_p) and (T_q), respectively. Obviously, they are related by

$$A = \begin{pmatrix} A_{pq} & A_{qp} & A_p & A_q \end{pmatrix}$$

Correspondingly, q_{pq}, q_{qp}, q_p and q_q denote the vectors of all flows of type (T_{pq}), (T_{qp}), (T_p) and (T_q), respectively.

For passive networks elements of type (T_{pq}) and (T_{qp}), the functions f_{pq} and f_{qp} are monotone.

Obviously, the system (33) can be reduced to the system

$$A_{qp} f_{qp}(A_{qp}^\top p) + A_{pq}q_{pq} + A_p q_p = -A_q c_q \tag{34a}$$

$$A_{pq}^\top p = f_{pq}(q_{pq}) \tag{34b}$$

$$A_p^\top p = c_p \tag{34c}$$

supplemented by the output equations

$$q_{qp} = f_{qp}(A_{qp}^\top p), \quad q_q = c_q \tag{35}$$

The equation system (34), (35) reflects resistive circuits with constant sources, static water supplying network models and static gas supplying network models.

2.2 Static Networks with Switching Element Models

We consider static networks with lumped element models supplemented with switching net elements of type

$$\tilde{s}(\tilde{p}_R - \tilde{p}_L, \tilde{q})(\tilde{p}_R - \tilde{p}_L) + (1 - \tilde{s}(\tilde{p}_R - \tilde{p}_L, \tilde{q}))\tilde{q} = 0, \tag{T_s}$$

with a switching function $\tilde{s}(\cdot,\cdot)$ having the values 1 or 0 only. For valves controlled from outside, the switching function $\tilde{s}$ is independent of $\Delta\tilde{p}$ and independent of $\tilde{q}$. Summarizing the element equations (T_{pq}), (T_{qp}), (T_q), (T_p), (T_s) and the mass flow balance equation (4), we obtain the system

$$A_{pq}^\top p = f_{pq}(q_{pq}) \tag{36a}$$

$$q_{qp} = f_{qp}(A_{qp}^\top p) \tag{36b}$$

$$A_p^\top p = c_p \tag{36c}$$

$$q_q = c_q \tag{36d}$$

$$S(A_s^\top p, q_s)A_s^\top p + (I - S(A_s^\top p, q_s))q_s = 0 \tag{36e}$$

$$A_{pq}q_{pq} + A_{qp}q_{qp} + A_p q_p + A_q q_q + A_s q_s = 0 \tag{36f}$$

with the incidence matrices A_{pq}, A_{qp}, A_p, A_q, A_s collecting all columns for network elements of type (T_{pq}), (T_{qp}) (T_p), (T_q), (T_s), respectively. Obviously, they are related by

$$A = \begin{pmatrix} A_{pq} & A_{qp} & A_p & A_q & A_s \end{pmatrix}$$

Correspondingly, q_{pq}, q_{qp}, q_p, q_q, q_s denote the vectors of all flows of type (T_{pq}), (T_{qp}), (T_p), (T_q) and (T_s), respectively. The matrix $S(A_s p, q_s)$ is a diagonal matrix with entries 1 or 0 only.

Obviously, the system (36) can be reduced to the system

$$A_{qp} f_{qp}(A_{qp}^\top p) + A_{pq}q_{pq} + A_p q_p + A_s(S(A_s^\top p, q_s)q_s) = -A_q c_q \tag{37a}$$

$$A_{pq}^\top p = f_{pq}(q_{pq}) \tag{37b}$$

$$A_p^\top p = c_p \tag{37c}$$

$$S(A_s^\top p, q_s)A_s^\top p = 0 \tag{37d}$$

supplemented by the output equations

$$q_{qp} = f_{qp}(A_{qp}^\top p), \quad q_q = c_q, \quad q_s = S(A_s^\top p, q_s)q_s \tag{38}$$

The equation system (37), (38) reflects static water supplying network models and static gas supplying network models with valves controlled from outside. The two latter ones are common models for the optimal control of such networks, see e.g. [7, 18].

2.3 Dynamic Networks with Lumped Element Models

As said before, lumped element models are characterized by constant flows on each network branch, i.e. $q_L = q_R = q$. Beside net elements of type (T_{qp}) and (T_{qp}), we have six additional types of elements:

1. $$\tilde{p}_R - \tilde{p}_L = \tilde{f}_{pt}(t), \tag{T_{pt}}$$
2. $$\tilde{q} = \tilde{f}_{qt}(t). \tag{T_{qt}}$$

3. $$\tilde{p}_R - \tilde{p}_L = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_{p\dot{q}}(\tilde{q}), \tag{$T_{p\dot{q}}$}$$

4. $$\tilde{q} = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_{q\dot{p}}(\tilde{p}_R - \tilde{p}_L), \tag{$T_{q\dot{p}}$}$$

5. $$\tilde{p}_R - \tilde{p}_L = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_{pq\dot{u}}(\tilde{u}), \quad \tilde{q} = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}, \tag{$T_{pq\dot{u}}$}$$

6. $$\tilde{q} = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{f}_{qp\dot{u}}(\tilde{u}), \quad \tilde{p}_R - \tilde{p}_L = \frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}. \tag{$T_{qp\dot{u}}$}$$

The tilde notation is used to stress that we are describing a single element. Summarizing the element equations (T_{pq}), (T_{qp}), (T_{qt}), (T_{pt}), ($T_{p\dot{q}}$), ($T_{q\dot{p}}$), ($T_{pq\dot{u}}$), ($T_{qp\dot{u}}$), and the mass flow balance equation (4), we obtain the system

$$A_{pq}^{\top}p = f_{pq}(q_{pq}) \tag{39a}$$

$$q_{qp} = f_{qp}(A_{qp}^{\top}p) \tag{39b}$$

$$A_{pt}^{\top}p = f_{pt}(t) \tag{39c}$$

$$q_{qt} = f_{qt}(t) \tag{39d}$$

$$A_{p\dot{q}}^{\top}p = \frac{\mathrm{d}}{\mathrm{d}t}f_{p\dot{q}}(q_{dpq}) \tag{39e}$$

$$q_{q\dot{p}} = \frac{\mathrm{d}}{\mathrm{d}t}f_{q\dot{p}}(A_{q\dot{p}}^{\top}p) \tag{39f}$$

$$A_{pq\dot{u}}^{\top}p = \frac{\mathrm{d}}{\mathrm{d}t}f_{pq\dot{u}}(u_{pq\dot{u}}), \quad q_{pq\dot{u}} = \frac{\mathrm{d}}{\mathrm{d}t}u_{pq\dot{u}} \tag{39g}$$

$$q_{qp\dot{u}} = \frac{\mathrm{d}}{\mathrm{d}t}f_{qp\dot{u}}(u_{qp\dot{u}}), \quad A_{qp\dot{u}}^{\top}p = \frac{\mathrm{d}}{\mathrm{d}t}u_{qp\dot{u}} \tag{39h}$$

$$A_{pq}q_{pq} + A_{qp}q_{qp} + A_{pt}q_{pt} + A_{qt}q_{qt} \tag{39i}$$

$$+ A_{p\dot{q}}q_{p\dot{q}} + A_{q\dot{p}}q_{q\dot{p}} + A_{pq\dot{u}}q_{pq\dot{u}} + A_{qp\dot{u}}q_{qp\dot{u}} = 0 \tag{39j}$$

with the incidence matrices A_{pq}, A_{qp}, A_{pt}, A_{qt}, $A_{p\dot{q}}$, $A_{q\dot{p}}$, $A_{pq\dot{u}}$, $A_{qp\dot{u}}$ collecting all columns for network elements of type (T_{pq}), (T_{qp}) (T_{pt}), (T_{qt}), ($T_{p\dot{q}}$), ($T_{q\dot{p}}$) ($T_{pq\dot{u}}$) and ($T_{qp\dot{u}}$) respectively. Obviously, they are related by

$$A = \begin{pmatrix} A_{pq} & A_{qp} & A_{pt} & A_{qt} & A_{p\dot{q}} & A_{q\dot{p}} & A_{pq\dot{u}} & A_{qp\dot{u}} \end{pmatrix}$$

Correspondingly, q_{pq}, q_{qp}, q_{pt}, q_{qt}, $q_{p\dot{q}}$, $q_{q\dot{p}}$, $q_{pq\dot{u}}$ and $q_{qp\dot{u}}$ denote the vector of all flows of type (T_{pq}), (T_{qp}), (T_{pt}), (T_{qt}), ($T_{p\dot{q}}$), ($T_{q\dot{p}}$), ($T_{pq\dot{u}}$) and ($T_{qp\dot{u}}$) respectively.

Obviously, the system (39) can be reduced to the system

$$A_{q\dot{p}} \frac{\mathrm{d}}{\mathrm{d}t} f_{q\dot{p}}(A_{q\dot{p}}^\top p) + A_{pq\dot{u}} \frac{\mathrm{d}}{\mathrm{d}t} u_{pq\dot{u}} + A_{qp\dot{u}} \frac{\mathrm{d}}{\mathrm{d}t} f_{qp\dot{u}}(u_{qp\dot{u}})$$
$$+ A_{qp} f_{qp}(A_{qp}^\top p) + A_{p\dot{q}} q_{p\dot{q}} + A_{pq} q_{pq} + A_{pt} q_{pt} = -A_{qt} f_{qt}(t), \tag{40a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{pq\dot{u}}(u_{pq\dot{u}}) = A_{pq\dot{u}}^\top p, \quad \frac{\mathrm{d}}{\mathrm{d}t} u_{qp\dot{u}} = A_{qp\dot{u}}^\top p, \quad \frac{\mathrm{d}}{\mathrm{d}t} f_{p\dot{q}}(q_{p\dot{q}}) = A_{p\dot{q}}^\top p, \tag{40b}$$

$$A_{pq}^\top p = f_{pq}(q_{pq}), \quad A_{pt}^\top p = f_{pt}(t) \tag{40c}$$

supplemented by the output equations

$$q_{q\dot{p}} = \frac{\mathrm{d}}{\mathrm{d}t} f_{q\dot{p}}(A_{q\dot{p}}^\top p), \quad q_{pq\dot{u}} = \frac{\mathrm{d}}{\mathrm{d}t} u_{pq\dot{u}}, \quad q_{qp\dot{u}} = \frac{\mathrm{d}}{\mathrm{d}t} f_{qp\dot{u}}(u_{qp\dot{u}}), \tag{41a}$$

$$q_{qp} = f_{qp}(A_{qp}^\top p), \qquad q_{qt} = f_{qt}(t) \tag{41b}$$

For passive networks elements of type (T_{pq}), (T_{qp}), ($T_{p\dot{q}}$), ($T_{q\dot{p}}$), ($T_{pq\dot{u}}$) and ($T_{qp\dot{u}}$), the functions f_{pq}, f_{qp} $f_{p\dot{q}}$, $f_{q\dot{p}}$, $f_{pq\dot{u}}$ and $f_{qp\dot{u}}$ are monotone.

In absence of elements of type ($T_{pq\dot{u}}$) and ($T_{pq\dot{u}}$) (for instance lumped circuit models without memristors and memductors), the system (40), (41) reduces to

$$A_{q\dot{p}} \frac{\mathrm{d}}{\mathrm{d}t} f_{q\dot{p}}(A_{q\dot{p}}^\top p) + A_{qp} f_{qp}(A_{qp}^\top p) + A_{p\dot{q}} q_{p\dot{q}} + A_{pq} q_{pq} + A_{pt} q_{pt} = -A_{qt} f_{qt}(t), \tag{42a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_{p\dot{q}}(q_{p\dot{q}}) = A_{p\dot{q}}^\top p, \quad A_{pq}^\top p = f_{pq}(q_{pq}), \quad A_{pt}^\top p = f_{pt}(t) \tag{42b}$$

supplemented by the output equations

$$q_{qp} = f_{qp}(A_{qp}^\top p), \quad q_{qt} = f_{qt}(t), \quad q_{q\dot{p}} = \frac{\mathrm{d}}{\mathrm{d}t} f_{q\dot{p}}(A_{q\dot{p}}^\top p). \tag{43a}$$

2.4 Dynamic Networks with Switching Element Models

We consider static networks with lumped element models supplemented with switching net elements of type (T_s) and type

$$\tilde{s}(t)(\tilde{p}_R - \tilde{p}_L) + (1 - \tilde{s}(t))\tilde{q} = 0, \tag{T_{st}}$$

with a switching function $\tilde{s}(t)$ having the values 1 or 0 only. Elements of type (T_{st}) are valves controlled from outside, for instance controlled by a dispatcher. Similarly to the derived net equations in the sections before, we obtain the equation system

$$A_{q\dot{p}}\frac{\mathrm{d}}{\mathrm{d}t}f_{q\dot{p}}(A_{q\dot{p}}^{\top}p)+A_{pq\dot{u}}\frac{\mathrm{d}}{\mathrm{d}t}u_{pq\dot{u}}+A_{qp\dot{u}}\frac{\mathrm{d}}{\mathrm{d}t}f_{qp\dot{u}}(u_{qp\dot{u}})+A_{qp}f_{qp}(A_{qp}^{\top}p)+A_{p\dot{q}}q_{p\dot{q}}$$

$$+A_{pq}q_{pq}+A_{pt}q_{pt}+A_s(S(A_s^{\top}p,q_s)q_s)+A_{st}(S(t)q_{st})=-A_{qt}f_{qt}(t), \tag{44a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}f_{pq\dot{u}}(u_{pq\dot{u}})=A_{pq\dot{u}}^{\top}p,\quad \frac{\mathrm{d}}{\mathrm{d}t}u_{qp\dot{u}}=A_{qp\dot{u}}^{\top}p,\quad \frac{\mathrm{d}}{\mathrm{d}t}f_{p\dot{q}}(q_{p\dot{q}})=A_{p\dot{q}}^{\top}p, \tag{44b}$$

$$A_{pq}^{\top}p=f_{pq}(q_{pq}),\ \ A_{pt}^{\top}p=f_{pt}(t),\ \ S(A_s^{\top}p,q_s)A_s^{\top}p=0,\ \ S(t)A_{st}^{\top}p=0 \tag{44c}$$

supplemented by the output equations

$$q_{q\dot{p}}=\frac{\mathrm{d}}{\mathrm{d}t}f_{q\dot{p}}(A_{q\dot{p}}^{\top}p),\quad q_{pq\dot{u}}=\frac{\mathrm{d}}{\mathrm{d}t}u_{pq\dot{u}},\quad q_{qp\dot{u}}=\frac{\mathrm{d}}{\mathrm{d}t}f_{qp\dot{u}}(u_{qp\dot{u}}), \tag{45a}$$

$$q_{qp}=f_{qp}(A_{qp}^{\top}p),\quad q_{qt}=f_{qt}(t),\quad q_s=S(A_s^{\top}p,q_s)q_s,\quad q_{st}=S(t)q_{st}. \tag{45b}$$

The differential algebraic system (44) has the particular difficulty that the usual rank constancy required in the various index concepts for analyzing differential algebraic equations is not fulfilled anymore. Depending on the type of switching elements, the rank changes can be state or time dependent.

2.5 Dynamic Networks with Distributed Element Models

As we have seen in Sect. 1.2, some net elements are described by distributed models. Depending on the kind of model, they can be hyperbolic (e.g. supersonic flow models), parabolic or elliptic (e.g. subsonic flow models) or a mix of these types (e.g. drift-diffusion flow models). As an example we take here an abstraction from the one-dimensional gas pipe model description in section "Gas Networks". It results in elements of type

$$\begin{gathered}\partial_t\tilde{u}_p+c\,\partial_x\tilde{u}_q=0,\quad \partial_t\tilde{u}_q+\partial_x\tilde{f}(\tilde{u}_p,\tilde{u}_q)=\tilde{g}(\tilde{u}_p,\tilde{u}_q)\\ \tilde{u}_p(x_L,\cdot)=\tilde{p}_L,\ \ \tilde{u}_p(x_R,\cdot)=\tilde{p}_R,\ \ \tilde{u}_q(x_L,\cdot)=\tilde{q}_L,\ \ \tilde{u}_q(x_R,\cdot)=\tilde{q}_R.\end{gathered} \tag{$T_{\partial_t\partial_x}$}$$

with a function $f(u_p,u_q)$ being positive for $u_p>0$ and a function $g(u_p,u_q)$ that is decreasing with respect to u_q and increasing with respect to u_p. Collecting them for all network branches and coupling them with the mass balance equation yields the system

$$\partial_t u_p+C\,\partial_x u_q=0,\quad \partial_t u_q+\partial_x f(u_p,u_q)=g(u_p,u_q),\quad A_Lq_L+A_Rq_R=0, \tag{46a}$$

$$u_p(x_L,\cdot)=-A_L^{\top}p,\ \ u_p(x_R,\cdot)=A_R^{\top}p,\ \ u_q(x_L,\cdot)=q_L,\ \ u_q(x_R,\cdot)=q_R. \tag{46b}$$

Here u_p and u_q denote the vectors of the pressure and the flow of all branches. The evaluation at point x_L/x_R means the evaluation for each branch at their left/right node. C is a constant diagonal matrix. The system (46a)–(46b) can be reduced to the system

$$\partial_t u_p + C\partial_x u_q = 0, \quad \partial_t u_q + \partial_x f(u_p, u_q) = g(u_p, u_q), \tag{47a}$$

$$A_L u_q(x_L, \cdot) + A_R u_q(x_R, \cdot) = 0, \;\; u_p(x_L, \cdot) = -A_L^\top p, \;\; u_p(x_R, \cdot) = A_R^\top p \tag{47b}$$

with the output equations

$$q_L = u_q(x_L, \cdot), \quad q_R = u_q(x_R, \cdot). \tag{48}$$

Conclusions
Flow networks can be described by network graphs with node variables p reflecting the pressure at each node and edge variables q reflecting the flow at each edge. Depending on the kind of net element models, the resulting systems are linear/nonlinear equations, differential algebraic equations or partial differential algebraic equations.

Typical kinds of network models (static/dynamic with lumped/switching/distributed/discretized net element models) have been presented with its inner structure. The common framework described here allows a rigorous network analysis independently of the type of flow media but in dependence of the kind of net element models.

Additionally, it allows the transfer of numerical and optimization methods developed for one kind of network to another kind of network (e.g. circuit methods to methods for water/gas networks) as long as they contain the same type of net element models.

Finally, we particularly want to note that the analytical treatment, numerical simulation and optimal control of network models with switched or distributed net element descriptions requires new developments in the numerical analysis and optimal control of switched differential algebraic systems or partial differential algebraic systems with the structures presented here.

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