

Chapter 2

Mathematical Preliminaries

Abstract We briefly introduce some mathematical concepts and associated important theorems that will be used in the subsequent chapters to study the semantics of probabilistic processes. The main topics covered in this chapter include the Knaster–Tarski fixed-point theorem, continuous functions over complete lattices, induction and coinduction proof principles, compact sets in topological spaces, the separation theorem, the Banach fixed-point theorem, the π - λ theorem and the duality theorem in linear programming. Most of the theorems are stated without proofs because they can be found in many textbooks.

Keywords Lattice · Induction · Coinduction · Topology · Metric · Probability · Linear programming

2.1 Lattice Theory

A very useful tool to study the semantics of formal languages is the Knaster–Tarski fixed-point theorem, which will be used many times in subsequent chapters. We begin with some basic notions from lattice theory.

Definition 2.1 A set X with a binary relation $\sqsubseteq$ is called a *partially ordered set*, if the following holds for all $x, y, z \in X$:

1. $x \sqsubseteq x$ (reflexivity);
2. If $x \sqsubseteq y$ and $y \sqsubseteq x$, then $x = y$ (antisymmetry);
3. If $x \sqsubseteq y$ and $y \sqsubseteq z$, then $x \sqsubseteq z$ (transitivity).

An element $x \in X$ is called an *upper bound* for a subset $Y \subseteq X$, if $y \sqsubseteq x$ for all $y \in Y$. Dually, x is a *lower bound* for Y if $x \sqsubseteq y$ for all $y \in Y$. Note that a subset might not have any upper bound or lower bound.

Example 2.1 Consider the set $X = \{0, 1, 2\}$ with the binary relation $\sqsubseteq$ defined by $\sqsubseteq = Id_X \cup \{(0, 1), (0, 2)\}$, where Id_X is the identity relation $\{(0, 0), (1, 1), (2, 2)\}$. It constitutes a partially ordered set. The subset $\{1, 2\}$ has no upper bound, as Fig. 2.1a shows.

Often we can use *Hasse diagrams* to describe finite partially ordered sets. For a partially ordered set $(X, \sqsubseteq)$, we represent each element of X as a vertex and draw

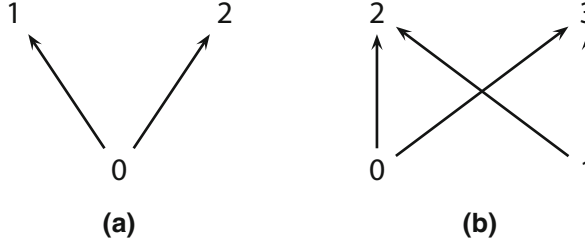


Fig. 2.1 Partially ordered sets

an arrow that goes upward from x to y if y is an immediate successor of x , i.e. $x \sqsubset y$ and there is no z with $x \sqsubset z \sqsubset y$, where $\sqsubset$ is obtained from $\sqsubseteq$ by removing elements (x, x) for all x . For instance, the partially ordered set given in the above example can be depicted by the diagram in Fig. 2.1a.

If Y has an upper bound that is also an element of Y , then it is said to be the *greatest element* in Y . We can dually define the *least element*. In the presence of a least element we speak of a *pointed* partially ordered set. If the set of upper bounds for Y has a least element x , then x is called the *supremum* or *join* of Y , written $\bigsqcup Y$. Dually we have *infimum* or *meet* and write $\bigsqcap Y$. Note that a subset might not have supremum even if it has upper bounds; so is the case for infimum.

Example 2.2 Consider the set $X = \{0, 1, 2, 3\}$ with the binary relation $\sqsubseteq$ defined by $\sqsubseteq = Id_X \cup \{(0, 2), (0, 3), (1, 2), (1, 3)\}$. It forms the partially ordered set depicted in Fig. 2.1b. The subset $\{0, 1\}$ has two upper bounds, namely 2 and 3, but it has no supremum because the two upper bounds are incomparable in the sense that neither $2 \sqsubseteq 3$ nor $3 \sqsubseteq 2$ holds.

We call X a *lattice*, if suprema and infima exist for all the subsets of X with two elements, i.e. if $\bigsqcup\{x, y\}$ and $\bigsqcap\{x, y\}$ exist for any $x, y \in X$. We call X a *complete lattice* if suprema and infima exist for all the subsets of X .

Remark 2.1 A few characteristics of complete lattices are worth mentioning because they are useful in many applications.

1. It suffices to define complete lattices in terms of either suprema or infima only. For example, a candidate definition is to say X is a complete lattice if suprema exist for all subsets of X . Then the infimum of any subset Y also exists because

$$\bigsqcap Y = \bigsqcup \{x \in X \mid \forall y \in Y : x \sqsubseteq y\}$$

2. Let X be a complete lattice. By definition, X itself has a supremum, which is the greatest element of X , written $\top$, and an infimum, which is the least element of X , written $\perp$. It can be checked that the empty set $\emptyset$ has supremum $\perp$ and infimum $\top$.

3. If $(X, \sqsubseteq)$ is a complete lattice, then so is $(X, \sqsupseteq)$, where $\sqsupseteq$ is the inverse relation of $\sqsubseteq$, i.e. $x \sqsupseteq y$ iff $y \sqsubseteq x$ for any elements $x, y \in X$.

Example 2.3 The set $X = \{1, 2, 3, 5, 6, 10, 15, 30\}$ of all divisors of 30, partially ordered by divisibility, constitutes a complete lattice.

Example 2.4 The set of all natural numbers $\mathbb{N}$, with the usual relation $\leq$ on natural numbers, forms a lattice which is not a complete lattice.

Example 2.5 Let X be any set. Its powerset $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$ with the inclusion relation $\subseteq$ forms a complete lattice whose join and meet are set union and intersection, respectively.

Given a function $f : X \rightarrow Y$ and a set $Z \subseteq X$, we write $f(Z)$ for the image of Z under f , i.e. the set $\{f(z) \mid z \in Z\}$. Given a partially ordered set X and a function $f : X \rightarrow X$, we say $x \in X$ is a *fixed point* (resp. *prefixed point*, *postfixed point*) of f if $x = f(x)$ (resp. $f(x) \sqsubseteq x$, $x \sqsubseteq f(x)$).

Definition 2.2 Let X and Y be partially ordered sets. A function $f : X \rightarrow Y$ is called *monotone*, if $x \sqsubseteq y$ implies $f(x) \sqsubseteq f(y)$ for all $x, y \in X$.

Theorem 2.1 (Knaster–Tarski Fixed-Point Theorem) *If X is a complete lattice, then every monotone function f from X to X has a fixed point. The least of these is given by*

$$lfp(f) = \bigcap \{x \in X \mid f(x) \sqsubseteq x\},$$

and the greatest by

$$gfp(f) = \bigcup \{x \in X \mid x \sqsubseteq f(x)\}.$$

Proof Let $X' = \{x \in X \mid f(x) \sqsubseteq x\}$ and $x^* = \bigcap X'$. For each $x \in X'$ we have $x^* \sqsubseteq x$ and then by monotonicity $f(x^*) \sqsubseteq f(x) \sqsubseteq x$. Taking the infimum over x we get, $f(x^*) \sqsubseteq \bigcap f(X') \sqsubseteq \bigcap X' = x^*$, thus $x^* \in X'$ and is the least prefixed point. On the other hand, $x \in X'$ implies $f(x) \in X'$ by monotonicity. Applying this to x^* yields $f(x^*) \in X'$ which implies $x^* \sqsubseteq f(x^*)$. Therefore, we obtain that $x^* = f(x^*)$. In fact, x^* is the least fixed point because we have just shown that it is the least prefixed point.

The case for the greatest fixed point is dual and thus omitted. $\square$

Definition 2.3 Given a complete lattice X , the function $f : X \rightarrow X$ is *continuous* if it preserves increasing chains, i.e. for all sequences $x_0 \sqsubseteq x_1 \sqsubseteq \dots$ we have

$$f\left(\bigcup_{n \geq 0} x_n\right) = \bigcup_{n \geq 0} f(x_n).$$

Dually, f is *cocontinuous* if it preserves decreasing chains.

Notice that both continuity and cocontinuity imply monotonicity. For example, if f is continuous and $x \sqsubseteq y$, then from the increasing sequence $x \sqsubseteq y \sqsubseteq y \sqsubseteq \dots$

we obtain that $f(\bigsqcup\{x, y\}) = f(y) = \bigsqcup\{f(x), f(y)\}$, which means $f(x) \sqsubseteq f(y)$. With continuity and cocontinuity we can construct, in a more tractable way, the least and greatest fixed point, respectively.

Proposition 2.1 *Let X be a complete lattice.*

1. *Every continuous function f on X has a least fixed point, given by $\bigsqcup_{n \geq 0} f^n(\perp)$, where $\perp$ is the bottom element of the lattice, and $f^n(\perp)$ is the n th iteration of f on $\perp$: $f^0(\perp) := \perp$ and $f^{n+1}(\perp) := f(f^n(\perp))$ for $n \geq 0$.*
2. *Every cocontinuous function f on X has a greatest fixed point, given by $\bigsqcap_{n \geq 0} f^n(\top)$, where $\top$ is the top element of the lattice.*

Proof We only prove the first clause, since the second one is dual.

We notice that $\perp \sqsubseteq f(\perp)$, and then monotonicity of f yields an increasing sequence:

$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \dots$$

By continuity of f we have, $f(\bigsqcup_{n \geq 0} f^n(\perp)) = \bigsqcup_{n \geq 0} f^{n+1}(\perp)$ and the latter is equal to $\bigsqcup_{n \geq 0} f^n(\perp)$.

And in fact that limit is the least fixed point: for if some other x were also a fixed point, then we have $\perp \sqsubseteq x$ and moreover $f^n(\perp) \sqsubseteq x$ for all n by induction. So x is an upper bound of all $f^n(\perp)$. $\square$

Let $(X, \sqsubseteq)$ be a partially ordered set. We say X is a *complete partially ordered set* (CPO) if it has suprema for all increasing chains. A CPO with *bottom* is a CPO with a least element $\perp$. The least fixed point of a continuous function f on the CPO can be characterised in the same way as in Proposition 2.1 (1), namely $\bigsqcup_{n \geq 0} f^n(\perp)$.

2.2 Induction and Coinduction

We observe that Theorem 2.1 provides two proof principles: the *induction principle* says that, to show $\text{lfp}(f) \sqsubseteq x$ it is enough to prove $f(x) \sqsubseteq x$; the *coinduction principle* says that, to show $x \sqsubseteq \text{gfp}(f)$ it suffices to prove $x \sqsubseteq f(x)$. Those are two important proof principles in concurrency theory. In this section, we briefly introduce inductive and coinductive definitions by rules and their associated proof techniques. For more detailed accounts, we refer the reader to [1, 2].

Given a set X , a *ground rule* on X is a pair $(S, x) \in \mathcal{P}(X) \times X$, meaning that, from the premises S we can derive the conclusion x . Usually, we write the rule (S, x) as

$$\frac{}{x} \text{ if } S = \emptyset, \text{ and as } \frac{x_1, \dots, x_n}{x} \text{ if } S = \{x_1, \dots, x_n\}. \quad (2.1)$$

A rule without any premise, i.e. in the form $(\emptyset, x)$, is said to be an *axiom*.

Quite often we define a set of objectives by rules. For example, let Var be a set of variables ranged over by x . In the lambda calculus the set Λ of all lambda terms can be expressed by a BNF (Backus–Naur Form) grammar:

$$M, N ::= x \mid \lambda x.M \mid MN$$

which says that A is the least set satisfying the following three rules:

$$\frac{}{x} \qquad \frac{M}{\lambda x.M} \qquad \frac{M \quad N}{MN}$$

A set of ground rules R determines an operator $\hat{R} : X \rightarrow X$, which maps a set S to the set

$$\hat{R}(S) = \{x \mid \exists S' \subseteq S : (S', x) \in R\}$$

which includes all the elements derivable from those in S by using the rules in R .

Definition 2.4 A set S is *forward-closed* under R iff $\hat{R}(S) \subseteq S$; a set S is *backward-closed* under R iff $S \subseteq \hat{R}(S)$.

In other words, if S is forward-closed, then

$$\forall x \in X : (\exists S' \subseteq S : (S', x) \in R) \Rightarrow x \in S$$

If S is backward-closed, then

$$\forall x \in X : x \in S \Rightarrow (\exists S' \subseteq S : (S', x) \in R)$$

Given a set of rules R on X , we let

$$S_f = \bigcap \{S \subseteq X \mid \hat{R}(S) \subseteq S\} \quad \text{and} \quad S_b = \bigcup \{S \subseteq X \mid S \subseteq \hat{R}(S)\}.$$

So S_f is the intersection of all forward-closed sets and S_b is the union of all backward-closed sets. Note that $\hat{R}$ is monotone on the complete lattice $(\mathcal{P}(X), \subseteq)$. It follows from Theorem 2.1 that S_f is in fact the least fixed point of $\hat{R}$ and S_b is the greatest fixed point of $\hat{R}$. Usually, an object is defined *inductively* (resp. *coinductively*) if it is the *least* (resp. *greatest*) fixed point of a function. So the set S_f (resp. S_b) is inductively (resp. coinductively) defined by the set of rules R .

Example 2.6 The set of finite lists with elements from a set A is the set **List** inductively defined by the following two rules, i.e. the least set closed forward under these rules.

$$\frac{}{nil \in \text{List}} \qquad \frac{a \in A \quad l \in \text{List}}{a \cdot l \in \text{List}} \qquad (2.2)$$

In contrast, the set of all finite or infinite lists is coinductively defined by the same rules, i.e. the greatest set closed backward under the two rules above; the set of all infinite lists is the set coinductively defined by the second rule above, i.e. the greatest set closed backward under the second rule.

The two rules in (2.2) are not ground rules because they are not in the form shown in (2.1). However, we can convert them into ground rules. Take X to be the set of all

(finite or infinite) strings with elements from $A \cup \{nil\}$. The ground rules determined by the two rules in (2.2) are

$$\frac{}{nil} \text{ and, for each } l \in X \text{ and } a \in A, \frac{l}{a \cdot l}.$$

Let R be a set of ground rules on X . In general $\hat{R}$ is not necessarily continuous or cocontinuous on the complete lattice $(\mathcal{P}(X), \subseteq)$. For example, consider the rule

$$\frac{x_1 \dots x_n \dots}{x}$$

whose premise is an infinite set. For any $n \geq 1$, let $X_n = \{x_1, \dots, x_n\}$. Then we have $x \in \hat{R}(\bigcup_n X_n)$, but $x \notin \bigcup_n \hat{R}(X_n)$. Thus, $\hat{R}$ is not continuous.

Below we impose some conditions on R to recover continuity and cocontinuity.

Definition 2.5 A set of rules R is *finite in the premises* (FP), if in each rule $(S, x) \in R$ the premise set S is finite. A set of rules R is *finite in the conclusions* (FC), if for each x , the set $\{S \mid (S, x) \in R\}$ is finite; that is, there are finitely many rules whose conclusion is x , though each premise set S may be infinite.

Proposition 2.2 Let R be a set of ground rules.

1. If R is FP, then $\hat{R}$ is continuous;
2. If R is FC, then $\hat{R}$ is cocontinuous.

Proof The two statements are dual, so we only prove the second one.

Let $S_0 \supseteq S_1 \supseteq \dots \supseteq S_n \supseteq \dots$ be a decreasing chain. We need to show that

$$\bigcap_{n \geq 0} \hat{R}(S_n) = \hat{R}\left(\bigcap_{n \geq 0} S_n\right).$$

One direction is easy. Since $\bigcap_{n \geq 0} S_n \subseteq S_n$ and $\hat{R}$ is monotone, we have that

$$\hat{R}\left(\bigcap_{n \geq 0} S_n\right) \subseteq \bigcap_{n \geq 0} \hat{R}(S_n).$$

The converse inclusion follows from the condition that R is FC. Let x be any element in $\bigcap_{n \geq 0} \hat{R}(S_n)$. Then $x \in \hat{R}(S_n)$ for each n , which means that for each $n \geq 0$ there is some $S'_n \subseteq S_n$ with $(S'_n, x) \in R$. Since R is FC, there exists some $k \geq 0$ such that $S'_n = S'_k$ for all $n \geq k$. Moreover, $S'_k \subseteq S_k \subseteq S_n$ for all $n \leq k$. Therefore, $S'_k \subseteq \bigcap_{n \geq 0} S_n$ and we have that $x \in \hat{R}(\bigcap_{n \geq 0} S_n)$. Thus, we have proved that

$$\bigcap_{n \geq 0} \hat{R}(S_n) \subseteq \hat{R}\left(\bigcap_{n \geq 0} S_n\right).$$

□

Corollary 2.1 Let R be a set of ground rules on X .

1. If R is FP, then $\text{lfp}(\hat{R}) = \bigcup_{n \geq 0} \hat{R}^n(\emptyset)$;
2. If R is FC, then $\text{gfp}(\hat{R}) = \bigcap_{n \geq 0} \hat{R}^n(X)$.

Proof Combine Propositions 2.1 and 2.2. □

Intuitively, the set $\hat{R}^0(\emptyset) = \emptyset$; the set $\hat{R}^1(\emptyset) = \hat{R}(\emptyset)$ consisting of all the conclusions of instances of axioms. In general, the set $\hat{R}^{n+1}(\emptyset)$ contains all objects which immediately follow by ground rules with premises in $\hat{R}^n(\emptyset)$. The above corollary states that if R is FP, each element in the set $\text{lfp}(\hat{R})$ can be derived via a derivation tree of finite depth whose leaves are instances of axioms; if R is FC, elements in $\text{gfp}(\hat{R})$ can always be destructed as conclusions of ground rules whose premises can also be destructed similarly.

2.3 Topological Spaces

In this section, we review some fundamental concepts in general topology such as continuous functions, compact sets and some other related properties. They will be used in Chap. 6.

Definition 2.6 Let X be a nonempty set. A collection $\mathcal{T}$ of subsets of X is a *topology* on X iff $\mathcal{T}$ satisfies the following axioms.

1. X and $\emptyset$ belong to $\mathcal{T}$;
2. The union of any number of sets in $\mathcal{T}$ belongs to $\mathcal{T}$;
3. The intersection of any two sets in $\mathcal{T}$ belongs to $\mathcal{T}$.

The members of $\mathcal{T}$ are called *open sets*, and the pair $(X, \mathcal{T})$ is called a *topological space*.

Example 2.7 The collection of all open intervals in the real line $\mathbb{R}$ forms a topology, which is called the *usual topology* on $\mathbb{R}$.

Let $(X, \mathcal{T})$ be a topological space. A point $x \in X$ is an *accumulation point* or *limit point* of a subset Y of X iff every open set Z containing x must also contain a point of Y different from x , that is,

$$Z \text{ open, } x \in Z \text{ implies } (Z \setminus \{x\}) \cap Y \neq \emptyset.$$

Definition 2.7 A subset Y of a topological space $(X, \mathcal{T})$ is *closed* iff Y contains each of its limit points.

Definition 2.8 Let Y be a subset of a topological space $(X, \mathcal{T})$. The *closure* of Y is the set of all limit points of Y . We say Y is *dense* if the closure of Y is X , i.e. every point of X is a limit point of Y .

It is immediate that Y coincides with its closure if and only if Y is closed.

Definition 2.9 A topological space $(X, \mathcal{T})$ is said to be *separable* if it contains a countable dense subset; that is, if there exists a finite or denumerable subset Y of X such that the closure of Y is the entire space.

Definition 2.10 Let $(X, \mathcal{T})$ and $(X^*, \mathcal{T}^*)$ be topological spaces. A function f from X into X^* is *continuous* iff the inverse image $f^{-1}(Y)$ of every open subset Y of X^* is an open subset of X , i.e.

$$Y \in \mathcal{T}^* \quad \text{implies} \quad f^{-1}(Y) \in \mathcal{T}.$$

Example 2.8 The projection mappings from the plane $\mathbb{R}^2$ into the line $\mathbb{R}$ are continuous with respect to the usual topologies. For example, consider the projection $\pi : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $\pi(\langle x, y \rangle) = y$. The inverse of any open interval (a, b) is an infinite open strip parallel to the x -axis.

Continuous functions can also be characterised by their behaviour with respect to closed sets.

Theorem 2.2 A function $f : X \rightarrow Y$ is continuous iff the inverse image of every closed subset of Y is a closed subset of X . $\square$

Let $\mathcal{Y} = \{Y_i\}$ be a class of subsets of X such that $Y \subseteq \cup_i Y_i$ for some $Y \subseteq X$. Then $\mathcal{Y}$ is called a *cover* of Y , and an *open cover* if each Y_i is open. Furthermore, if a finite subclass of $\mathcal{Y}$ is also a cover of Y , i.e. if

$$\exists Y_{i_1}, \dots, Y_{i_m} \in \mathcal{Y}, \quad \text{such that} \quad Y \subseteq Y_{i_1} \cup \dots \cup Y_{i_m}$$

then $\mathcal{Y}$ is said to contain a *finite subcover* of Y .

Example 2.9 The classical Heine–Borel theorem says that every open cover of a closed and bounded interval $[a, b]$ on the real line contains a finite subcover.

Definition 2.11 A subset Y of a topological space X is *compact* if every open cover of Y contains a finite subcover.

Example 2.10 By the Heine–Borel theorem, every closed and bounded interval $[a, b]$ on the real line $\mathbb{R}$ is compact.

Theorem 2.3 Continuous images of compact sets are compact. $\square$

A set $\{X_i\}$ of sets is said to have the *finite intersection property* if every finite subset $\{X_{i_1}, \dots, X_{i_m}\}$ has a nonempty intersection, i.e. $X_{i_1} \cap \dots \cap X_{i_m} \neq \emptyset$.

Example 2.11 Consider the following class of open intervals:

$$X = \left\{ (0, 1), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{2^2}\right), \dots \right\}$$

Clearly, it has the finite intersection property. Observe however that X has an empty intersection.

Theorem 2.4 A topological space X is compact iff every class $\{X_i\}$ of closed subsets of X that satisfies the finite intersection property has, itself, a nonempty intersection. $\square$

2.4 Metric Spaces

The main subject in probabilistic concurrency theory is the quantitative analysis of system behaviour. So we meet metric spaces from time to time. This section gives some background knowledge on them.

Let $\mathbb{R}_{\geq 0}$ be the set of all nonnegative real numbers.

Definition 2.12 A *metric space* is a pair (X, d) consisting of a set X and a distance function $d : X \times X \rightarrow \mathbb{R}_{\geq 0}$ satisfying:

1. For all $x, y \in X$, $d(x, y) = 0$, iff $x = y$ (isolation);
2. For all $x, y \in X$, $d(x, y) = d(y, x)$ (symmetry);
3. For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

If we replace the first clause with $\forall x \in X : d(x, x) = 0$, we obtain the definition of *pseudometric space*.

A metric d is c -bounded if $\forall x, y \in X : d(x, y) \leq c$, where c is a positive real number.

Example 2.12 Let X be a set. The *discrete metric* $d : X \times X \rightarrow [0, 1]$ is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Example 2.13 Let $\mathbb{R}^n$ denote the product set of n copies of the set $\mathbb{R}$ of real numbers, i.e. it consists of all n -tuples $\langle a_1, a_2, \dots, a_n \rangle$ of real numbers. The function d is defined by

$$d(x, y) = \sqrt{(a_1 - b_1)^2 + \dots + (a_n - b_n)^2}$$

where $x = \langle a_1, \dots, a_n \rangle$ and $y = \langle b_1, \dots, b_n \rangle$, is a metric, called the *Euclidean metric* on $\mathbb{R}^n$. This metric space $(\mathbb{R}^n, d)$ is called *Euclidean n -space*.

Let (X, d) be a metric space. For any point $x \in X$ in the space and any real number $\varepsilon > 0$, we let $S(x, \varepsilon)$ denote the set of points within a distance of ε from x :

$$S(x, \varepsilon) := \{y \mid d(x, y) < \varepsilon\}.$$

We call $S(x, \varepsilon)$ the *open sphere* with centre x and radius ε .

Theorem 2.5 Let (X, d) be a metric space. The collection of open spheres in X generates a topology, called the *metric topology*, whose open sets are those open spheres in X . $\square$

Definition 2.13 A sequence (x_n) in a metric space (X, d) is *convergent* to $x \in X$, if for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that $d(x_n, x) < \varepsilon$ whenever $n > N$.

Definition 2.14 A sequence (x_n) in a metric space (X, d) is called a *Cauchy sequence* if for an arbitrary $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that $d(x_m, x_n) < \varepsilon$ whenever $m, n > N$.

Definition 2.15 A metric space is *complete* if every Cauchy sequence is convergent. For example, the space of real numbers with the usual metric is complete.

Example 2.14 Let X be a nonempty set and F denote the collection of functions from X to the interval $[0, 1]$. A metric is defined on F as follows:

$$d(f, g) := \sup_{x \in X} |f(x) - g(x)|.$$

In fact, (F, d) is a complete metric space. Let (f_n) be a Cauchy sequence in F . Then for every $x \in X$, the sequence $(f_n(x))$ is Cauchy; and since $[0, 1]$ is complete, the sequence converges to some $a_x \in [0, 1]$. Let f be the function defined by $f(x) = a_x$. Thus (f_n) converges to f .

Example 2.15 Similar to Example 2.14, it can be seen that Euclidean n -space $\mathbb{R}^n$ is complete.

Let Y be a subset of a metric space (X, d) and let $\varepsilon > 0$. A finite set of points $Z = \{z_1, z_2, \dots, z_m\}$ is called an ε -*net* for Y , if for every point $y \in Y$ there exists an $z \in Z$ with $d(y, z) < \varepsilon$.

Definition 2.16 A subset Y of a metric space X is *totally bounded* if Y possesses an ε -net for every $\varepsilon > 0$.

Theorem 2.6 Let (X, d) be a complete metric space. Then $Y \subseteq X$ is compact iff Y is closed and totally bounded. $\square$

Definition 2.17 (Convex Set) A set $X \subseteq \mathbb{R}^n$ is *convex* if for every two points $x, y \in X$ the whole segment between x and y is also contained in X . In other words, for every $p \in [0, 1]$, the point $px + (1 - p)y$ belongs to X . We write $\uparrow X$ for the *convex closure* of X , the smallest convex set containing X .

Given two n -dimensional vectors $x = \langle a_1, \dots, a_n \rangle$ and $y = \langle b_1, \dots, b_n \rangle$, we use the usual definition of dot-product $x \cdot y = \sum_{i=1}^n a_i \cdot b_i$.

A basic result about convex set is the separability of disjoint convex sets by a hyperplane [3].

Theorem 2.7 (Separation Theorem) Let $X, Y \subseteq \mathbb{R}^n$ be convex sets with $X \cap Y = \emptyset$. Then there is a hyperplane whose normal is $h \in \mathbb{R}^n$ and a number $c \in \mathbb{R}$ such that

$$\text{for all } x \in X \text{ and } y \in Y, \text{ we have } h \cdot x \leq c \leq h \cdot y$$

or

$$\text{for all } x \in X \text{ and } y \in Y, \text{ we have } h \cdot x \geq c \geq h \cdot y.$$

If X and Y are closed and at least one of them is bounded, they can be separated strictly, i.e. in such a way that

$$\text{for all } x \in X \text{ and } y \in Y, \text{ we have } h \cdot x < c < h \cdot y$$

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