

Chapter 2

Plane Bending of a Curved Rod

Abstract We present a combination of the asymptotic, direct, and numerical methods on the example plane problem of finite deformations of a thin curved strip. The study includes both static and dynamic analyses; the inhomogeneity of the strip is taken into account. The method of asymptotic splitting allows for a consistent dimensional reduction of the equations of the original two-dimensional continuous problem into a one-dimensional formulation of the reduced theory and a problem in the cross section. It also provides a consistent way to recover the distributions of stresses, strains, and displacements in two dimensions. The direct approach to a material line extends the results to the geometrically nonlinear range. Several analytical solutions and the implementation of a finite element scheme are demonstrated with the *Mathematica* computer system. We investigate the convergence of solutions of various problems in the original (two-dimensional) and reduced (one-dimensional) models with respect to the thickness. This justifies the analytical conclusion that the classical Kirchhoff theory of rods remains asymptotically accurate irrespective from both the curvature of the strip and the variation of the material properties over the thickness. This chapter quotes extensively from Vetyukov (Acta Mech., 223(2):371–385, 2012) with kind permission from Springer Science and Business Media.

2.1 Asymptotic Analysis

2.1.1 Linear Plane Problem of Elasticity for a Curved Strip

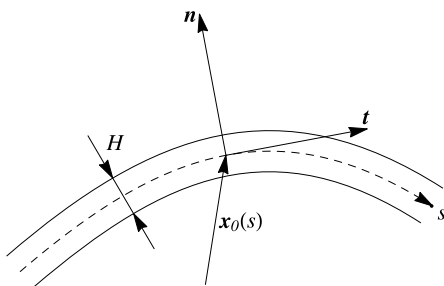
We consider a rod as a two-dimensional strip with the constant thickness H ; positions of the points of the middle line are parametrically defined by the vector $\mathbf{x}_0(s)$, see Fig. 2.1. This results in an orthonormal basis

$$\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{k}\}; \quad \mathbf{t} = \mathbf{x}'_0, \quad \mathbf{n} = \mathbf{k} \times \mathbf{t}, \quad \mathbf{n}' = \alpha \mathbf{t}, \quad \mathbf{t}' = -\alpha \mathbf{n}; \quad (2.1)$$

here, $\mathbf{k}$ is the unit out-of-plane vector, $\mathbf{t}$ is the tangent vector ($\mathbf{t} \cdot \mathbf{t} = 1$ because s is the arc coordinate for the middle line), $\mathbf{n}$ is the unit normal vector and $\alpha(s)$ is the

Electronic supplementary material Supplementary material is available in the online version of this chapter at http://dx.doi.org/10.1007/978-3-7091-1777-4_2.

Fig. 2.1 Geometry of a curved strip with a given middle line $\mathbf{x}_0(s)$ and thickness H (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)



curvature. The position vector of a point in the strip

$$\mathbf{x}(s, n) = \mathbf{x}_0(s) + n\mathbf{n} \quad (2.2)$$

is defined by the arc coordinate s and by the thickness coordinate n . This expression determines the two-dimensional differential operator:

$$\nabla = \mathbf{n}\partial_n + (1 + \alpha n)^{-1}\mathbf{t}\partial_s, \quad \nabla\mathbf{x} = \mathbf{t}\mathbf{t} + \mathbf{n}\mathbf{n} = \mathbf{I}_2; \quad (2.3)$$

the gradient of the position vector is the two-dimensional identity tensor $\mathbf{I}_2$.

Relations (1.82), (1.95), and (1.98) in the linear plane stress problem at hand may be summarized into the following system of equations for the in-plane tensors of stresses $\boldsymbol{\tau}$, strains $\boldsymbol{\epsilon}$ and displacements $\mathbf{u}$:

$$\begin{aligned} \nabla \cdot \boldsymbol{\tau} + \mathbf{f} &= \rho \ddot{\mathbf{u}}, \quad \mathbf{n} \cdot \boldsymbol{\tau}|_{n=\pm H/2} = 0, \\ \boldsymbol{\epsilon} &= \nabla \mathbf{u}^S, \\ \Delta \boldsymbol{\epsilon} + \nabla \nabla \operatorname{tr} \boldsymbol{\epsilon} &= 2(\nabla \nabla \cdot \boldsymbol{\epsilon})^S, \\ \boldsymbol{\tau} &= {}^4\mathbf{C} \cdot \boldsymbol{\epsilon}, \quad {}^4\mathbf{C} = {}^4\mathbf{C}(n); \end{aligned} \quad (2.4)$$

external forces $\mathbf{f} = f_n\mathbf{n} + f_t\mathbf{t}$ are considered inside the domain, and the boundary is free from traction forces; see discussion after (1.133). The material may be generally anisotropic, and the fourth-rank tensor of elastic properties at plane stress ${}^4\mathbf{C}$ as well as the density ρ may vary over the thickness of the strip in the case of a composite or a functionally graded structure [92]. The condition of compatibility (2.4)₃ is equivalent to (1.98) and will play an important role in the subsequent analysis.

A general exact solution of the formulated plane problem of the theory of elasticity cannot be written in a closed form. But for thin strips one can reduce the problem to an asymptotically equivalent one-dimensional formulation of a rod in the plane, in which all unknowns are functions of the arc coordinate s .

2.1.2 Asymptotic Splitting of Two-Dimensional Equations

Equations of balance with the small parameter. Similar to (1.129), in the expression for the position vector of a point (2.2) we introduce a formal small parameter λ , which indicates that the size of the curved middle line of the strip is much larger than its thickness:

$$\mathbf{x}(s, n) = \lambda^{-1} \mathbf{x}_0 + n\mathbf{n}. \quad (2.5)$$

With this new definition of $\mathbf{x}$, the magnitudes of n and s in (2.5) have formally the same asymptotic order. For curved structures it is generally easier to introduce the small parameter into the geometric expression of the position vector and then to proceed to the differential operator:

$$\nabla \mathbf{x} = \mathbf{I}_2 \Rightarrow \nabla = n\partial_n + \lambda(1 + \lambda\alpha n)^{-1} \mathbf{t}\partial_s. \quad (2.6)$$

The new expression for ∇ means that we seek rod-like solutions, which vary along the length of the strip much slower than over the thickness. This formal procedure corresponds to the traditional approach with non-dimensional variables. One may introduce a new coordinate along the thickness $\tilde{n} = H^{-1}n$, $-1/2 \leq \tilde{n} \leq 1/2$, and rewrite the original operator (2.3) with $\partial_n = H^{-1}\partial_{\tilde{n}}$. Then we say that the thickness is small and replace H by λH . Finally, we transform the expression back to the dimensional coordinate n . This is equivalent to substituting λn instead of n in (2.3) from the very beginning and results in an expression for ∇ , which differs from (2.6) by the factor λ^{-1} . The reason is that (2.5) corresponds to a lengthy strip instead of a thin one.

Decomposing the stress tensor into components,

$$\boldsymbol{\tau} = \sigma_t \mathbf{t}\mathbf{t} + \sigma_n \mathbf{n}\mathbf{n} + \tau(\mathbf{n}\mathbf{t} + \mathbf{t}\mathbf{n}) + \sigma_z \mathbf{k}\mathbf{k}, \quad (2.7)$$

and using (2.6) we write the balance equations (2.4)₁ with the small parameter:

$$\begin{aligned} \partial_n \sigma_n + \lambda\alpha(\sigma_n - \sigma_t + n\partial_n \sigma_n) + \lambda\partial_s \tau + f_n(1 + \lambda\alpha n) &= 0, \\ \partial_n \tau + \lambda\alpha(2\tau + n\partial_n \tau) + \lambda\partial_s \sigma_t + f_t(1 + \lambda\alpha n) &= 0, \\ \sigma_n|_{n=\pm H/2} = 0, \quad \tau|_{n=\pm H/2} &= 0 \end{aligned} \quad (2.8)$$

(we yet restrict ourselves to the static case).

The unknown field of stresses is sought in the form of a power series in the small parameter:

$$\boldsymbol{\tau} = \lambda^{-2} \overset{0}{\boldsymbol{\tau}} + \lambda^{-1} \overset{1}{\boldsymbol{\tau}} + \lambda^0 \overset{2}{\boldsymbol{\tau}} + \dots \quad (2.9)$$

Our aim is to find those terms in the solution, which dominate as the relative thickness of the strip is decreasing and $\lambda \rightarrow 0$. It means that we are interested in the convergence of the leading order term $\overset{0}{\boldsymbol{\tau}}$ to the exact solution rather than in the convergence of the series itself. The role of the rest of the series is to determine the principal term from the conditions of solvability for the minor terms.

The simple advantage of the dimensionless formal small parameter in comparison to the ratio between the thickness of the strip to its characteristic size is that we set $\lambda = 1$ after the analysis is finished and the terms of interest are determined. The deeper idea is that this formalism allows us to consider not just a thin strip, but rather a special class of “rod” solutions with a particular asymptotic behavior. The leading power λ^{-2} in (2.9) is guessed in advance according to the known solution in Sect. 1.2.4.

It is important to notice that we do not assign any asymptotic order to the curvature of the rod: the thickness is assumed to be small with respect to both the length and the curvature radius. This idea is illustrated by the numerical examples below in Sect. 2.4.

Asymptotic splitting of the equations of equilibrium. At the *first step* of the procedure we substitute the expansion (2.9) in (2.8). Gathering together the leading order terms in the resulting equations and boundary conditions, which have the order of smallness λ^{-2} , results in

$$\begin{aligned} \partial_n \overset{\circ}{\sigma}_n &= 0, \quad \overset{\circ}{\sigma}_n|_{n=\pm H/2} = 0 \Rightarrow \overset{\circ}{\sigma}_n = 0; \\ \partial_n \overset{\circ}{\tau} &= 0, \quad \overset{\circ}{\tau}|_{n=\pm H/2} = 0 \Rightarrow \overset{\circ}{\tau} = 0. \end{aligned} \quad (2.10)$$

The principal term $\overset{\circ}{\sigma}_t$ in the tangential stress component remains yet undetermined.

At the *second step* of the procedure we proceed to the terms of the order λ^{-1} , taking (2.10) into account:

$$\begin{aligned} \partial_n \overset{\dagger}{\sigma}_n - \alpha \overset{\circ}{\sigma}_t &= 0, \quad \overset{\dagger}{\sigma}_n|_{n=\pm H/2} = 0; \\ \partial_n \overset{\dagger}{\tau} + \partial_s \overset{\circ}{\sigma}_t &= 0, \quad \overset{\dagger}{\tau}|_{n=\pm H/2} = 0. \end{aligned} \quad (2.11)$$

Considering the general case of a curved rod with $\alpha \neq 0$, we find

$$\int_{-H/2}^{H/2} \partial_n \overset{\dagger}{\sigma}_n \, dn = 0 \Rightarrow \int_{-H/2}^{H/2} \overset{\circ}{\sigma}_t \, dn = 0; \quad (2.12)$$

for a straight rod the asymptotics looks slightly different. Denoting

$$Q_n(s) = h \int_{-H/2}^{H/2} \overset{\dagger}{\tau} \, dn, \quad M(s) = -h \int_{-H/2}^{H/2} n \overset{\circ}{\sigma}_t \, dn \quad (2.13)$$

(here h is the out-of-plane width of the strip), integrating the expression for Q_n by parts (see (1.138)), using the boundary conditions and applying the relation in the second line of (2.11) we arrive at

$$M' = -Q_n; \quad (\dots)' \equiv \partial_s(\dots). \quad (2.14)$$

The *third step* completes the procedure. We collect the terms of the order λ^0 and integrate them over the thickness. As

$$\int_{-H/2}^{H/2} \partial_n \overset{\circ}{\sigma}_n \, dn = 0, \quad \int_{-H/2}^{H/2} \partial_n \overset{\circ}{\tau} \, dn = 0 \quad (2.15)$$

(these are the conditions of solvability for the minor terms), we obtain

$$\begin{aligned} \int_{-H/2}^{H/2} (f_n + \alpha(\dot{\sigma}_n - \dot{\sigma}_t) + n\alpha\partial_n\dot{\sigma}_n + \partial_s\dot{\tau}) \, dn &= 0, \\ \int_{-H/2}^{H/2} (f_t + 2\alpha\dot{\tau} + n\alpha\partial_n\dot{\tau} + \partial_s\dot{\sigma}_t) \, dn &= 0. \end{aligned} \quad (2.16)$$

Integrating again by parts, using the boundary conditions and denoting

$$q_n = h \int_{-H/2}^{H/2} f_n \, dn, \quad q_t = h \int_{-H/2}^{H/2} f_t \, dn, \quad Q_t = h \int_{-H/2}^{H/2} \dot{\sigma}_t \, dn, \quad (2.17)$$

we arrive at

$$q_n + Q'_n - \alpha Q_t = 0, \quad q_t + Q'_t + \alpha Q_n = 0. \quad (2.18)$$

With the vectors of external force, moment, and transverse force

$$\begin{aligned} \mathbf{q} &= q_n \mathbf{n} + q_t \mathbf{t} = h \int_{-H/2}^{H/2} \mathbf{f} \, dn, \\ \mathbf{M} &= M \mathbf{k}, \quad \mathbf{Q} = Q_n \mathbf{n} + Q_t \mathbf{t}, \end{aligned} \quad (2.19)$$

Eqs. (2.14) and (2.18) are rewritten in the known vectorial form of the equations of equilibrium of forces and moments in the rod [6, 172]:

$$\mathbf{Q}' + \mathbf{q} = 0, \quad \mathbf{M}' + \mathbf{t} \times \mathbf{Q} = 0. \quad (2.20)$$

The distributed external moments do not enter the resulting asymptotic equations of equilibrium for the stress resultants and stress couples, which can now be identified as the force $\mathbf{Q}$ and the bending moment $\mathbf{M}$ in the cross section. The analysis of stresses does not only produce the known balance equations (2.20), but it provides important information for the subsequent analysis of the deformation.

Asymptotic analysis of the field of strains. Components of ${}^4\mathbf{C}$ in the relation of elasticity (2.4)₄ are independent from the small parameter, and the strain tensor must have the form

$$\boldsymbol{\varepsilon}(s, n) = \varepsilon_n \mathbf{n}\mathbf{n} + \varepsilon_t \mathbf{t}\mathbf{t} + \varepsilon_{nt}(\mathbf{n}\mathbf{t} + \mathbf{t}\mathbf{n}) = \lambda^{-2} \overset{0}{\boldsymbol{\varepsilon}} + \lambda^{-1} \overset{1}{\boldsymbol{\varepsilon}} + \lambda^0 \overset{2}{\boldsymbol{\varepsilon}} + \dots \quad (2.21)$$

Equating the coefficients of like powers of λ in the condition of compatibility (2.4)₃, we obtain

$$\begin{aligned} \lambda^{-2}: \quad \partial_n^2 \overset{0}{\varepsilon}_t &= 0 \quad \Rightarrow \quad \overset{0}{\varepsilon}_t = \varepsilon + \kappa n; \\ \lambda^{-1}: \quad \partial_n^2 \overset{1}{\varepsilon}_t &= 2\partial_n \partial_s \overset{0}{\varepsilon}_{nt} + \alpha \partial_n \overset{0}{\varepsilon}_n - 2\alpha \kappa; \\ \lambda^0: \quad &\dots \text{ etc.} \end{aligned} \quad (2.22)$$

The principal term of the axial strain $\overset{0}{\varepsilon}_t$ is linearly distributed over the cross section, which completes the system of equations for the leading order solution.

- The elastic relation (2.4)₄ and the results of the first step of the analysis of stresses (2.10) allow to write $\overset{0}{\varepsilon}_n, \overset{0}{\varepsilon}_{nt}$ and, what is most important, $\overset{0}{\sigma}_t$ as functions of n, ε , and κ :

$$\overset{0}{\tau} = {}^4\mathbf{C}(n) \cdot \overset{0}{\varepsilon}, \quad \overset{0}{\sigma}_n = 0, \quad \overset{0}{\tau}_n = 0, \quad \overset{0}{\varepsilon}_t = \varepsilon + \kappa n \Rightarrow \overset{0}{\varepsilon}_n, \quad \overset{0}{\varepsilon}_{nt}, \quad \overset{0}{\sigma}_t. \quad (2.23)$$

For a homogeneous cross section and a homogeneous material the results will be equivalent to (1.144) with the tangential direction being x and the transverse one being y . A similar procedure in the mechanics of plates is expressed by (4.20). The curvature of the strip does not affect the analysis in the cross section.

- We express ε via κ from (2.12): the only independent parameter in the principal term of strains has the meaning of change of curvature, which is intrinsic to the classical Kirchhoff rod theory with constrained shear and extension; for a straight rod the asymptotics is different and ε remains independent.
- From the second integral in (2.13) we explicitly compute the bending stiffness a , which is independent from the curvature of the strip:

$$M = a\kappa. \quad (2.24)$$

Asymptotics of displacements. The non-contradictory procedure is possible, when the series expansion of the vector of displacements begins with the order λ^{-4} :

$$\mathbf{u}(s, n) = u_t \mathbf{t} + u_n \mathbf{n} = \lambda^{-4} \overset{0}{\mathbf{u}} + \lambda^{-3} \overset{1}{\mathbf{u}} + \lambda^{-2} \overset{2}{\mathbf{u}} + \dots \quad (2.25)$$

From the kinematic relation (2.4)₂ and from (2.22) we obtain

$$\begin{aligned} \lambda^{-2} : \quad & \partial_n \overset{0}{u}_n = 0, \quad \partial_n \overset{0}{u}_t = 0 \Rightarrow \overset{0}{\mathbf{u}} = \mathbf{U}(s); \\ \lambda^{-1} : \quad & \partial_n \overset{1}{u}_n = 0, \quad \alpha \overset{0}{u}_t = \partial_n \overset{1}{u}_t + \partial_s \overset{0}{u}_n, \quad \alpha \overset{0}{u}_n + \partial_s \overset{0}{u}_t = 0; \\ \lambda^0 : \quad & \partial_s \overset{1}{u}_t + \alpha \overset{1}{u}_n = \overset{0}{\varepsilon}_t, \quad \dots \end{aligned} \quad (2.26)$$

The principal term is the displacement of the cross section $\mathbf{U}(s)$, which has an evident meaning for the rod theory. From the second and third lines in (2.26) we conclude that

$$\begin{aligned} \alpha \overset{0}{u}_n + \partial_s \overset{0}{u}_t &= 0, \quad \partial_s (\alpha \overset{0}{u}_t - \partial_s \overset{0}{u}_n) = \kappa \\ \Rightarrow \quad \mathbf{U}' \cdot \mathbf{t} &= 0, \quad (\mathbf{U}' \cdot \mathbf{n})' = \kappa. \end{aligned} \quad (2.27)$$

These are the kinematic relations of the classical theory: the rod is inextensible, and κ is the change of curvature of the line. It remains to say that the first correction term $\overset{1}{\mathbf{u}}$ corresponds to the rotation of the cross section with the angle θ , i.e., the classical Kirchhoff hypothesis of straight normals is fulfilled asymptotically. Ignoring the conditions of compatibility and solving the problem in displacements from the very

beginning, we would lose the clarity of the approach, as the solution would require several additional steps; see the asymptotic analysis in Sect. 4.3.3.

Dynamic effects. Finally we address the dynamic effects owing to the inertial term in the equation of balance (2.4)₁. We consider processes with a particular asymptotic rate over time, which is reflected by a formal change of the time variable:

$$t = \lambda^{-2} \tilde{t}, \quad (\dots)' = \partial_{\tilde{t}}(\dots) = \lambda^2 \partial_{\tilde{t}}(\dots), \quad \ddot{u} = \partial_{\tilde{t}}^2 U + \dots \quad (2.28)$$

After the asymptotic procedure is completed, the formal small parameter λ is set to 1. This allows to introduce the inertial term in the equation of balance of forces in its classical form:

$$\mathcal{Q}' + \mathbf{q} = \ddot{U} h \int_{-H/2}^{H/2} \rho \, dn. \quad (2.29)$$

2.1.3 Concluding Remarks on the Asymptotic Analysis

Finding the leading order solution of the original two-dimensional problem requires the following steps.

1. The problem in the cross section (solution scheme before (2.24)) needs to be treated, which produces the bending stiffness of the one-dimensional model.
2. Equations of the reduced one-dimensional rod model (2.20), (2.24), and (2.27) need to be solved. Generally speaking, the boundary conditions for the reduced model should also be obtained by the asymptotic analysis of two-dimensional fields near the boundary and by matching the two asymptotic expansions inside the domain and in the edge layer, see [65, 99, 137, 157] as well as the discussion of the boundary conditions in the theory of plates in Sect. 4.1.1.
3. With the displacements $\mathbf{U}$ and strains κ we recover the relevant terms in the corresponding fields of the non-reduced two-dimensional model; the principal terms are found according to (2.23).

The formality of the procedure may be considered as its particular advantage: as soon as the “slow” and “fast” variables are selected, the analysis is determined by the equations of the original non-reduced theory with a formal small parameter. The asymptotics is different for a straight rod, when the axial force $\mathcal{Q}_t$ receives its own relation of elasticity.

The main outcome of the presented analysis is that the classical Kirchhoff theory of rods (with the Bernoulli–Euler theory of straight beams as a particular case) is the simplest one, which predicts the leading order solutions of the original continuum problem; a numerical validation is discussed in Sect. 2.4. Achieving higher asymptotic accuracy would require treatment of higher order terms in the expansions of the unknown fields; a Timoshenko-like theory with the effects of shear and extension may be expected as a result. The equation in the second line of (2.22)

for the first correction term in the expansion of $\mathfrak{e}$ includes α , which means that the curvature should enter the new constitutive relations in a non-trivial way. The term with $\partial_s \varepsilon_{nt}^0$ in the second line of (2.22) means that at least for anisotropic materials the constitutive relations of the higher order theory shall include derivatives of the strain measures. This issue is thoroughly discussed by Rajagopal et al. [124], who developed a higher order theory with the help of the variational-asymptotic method. Another serious difficulty, which stands in the way to a theory with higher asymptotic accuracy, is the complexity of the analysis of the edge effect and of the corresponding matching procedure for posing consistent boundary conditions in the reduced formulation. The fundamental role of classical theories of beams and plates is underlined by the fact that the corresponding systems of equations and boundary conditions may be applied for finding solutions according to higher order refined theories by analogy, see Irschik [77], Irschik et al. [80] for details.

2.2 Direct Approach

2.2.1 Nonlinear Theory

We consider the rod as a one-dimensional continuum of particles. Each particle is a rigid body with three in-plane degrees of freedom: two translations and one rotation. Following the strategy, presented in [51] and demonstrated in Sect. 1.2.3, we write the principle of virtual work for a segment of the rod $s_0 \leq s \leq s_1$:

$$\int_{s_0}^{s_1} (\mathbf{q} \cdot \delta \mathbf{x} + m \delta \theta + \delta A^i) ds + (\mathbf{Q} \cdot \delta \mathbf{x} + M \delta \theta) \Big|_{s_0}^{s_1} = 0. \quad (2.30)$$

Here, the distributed external force $\mathbf{q}$ and the moment m are counted per unit arc length in the reference configuration, $\mathbf{Q}$ and M are the force and the moment of interaction with the remaining parts of the rod ($s < s_0$ and $s > s_1$), $\mathbf{x}(s)$ and $\theta(s)$ are the position vector and the angle of rotation of particles, and δA^i is the virtual work of internal forces on the translation $\delta \mathbf{x}$ and rotation $\delta \theta$. An extension to a dynamic formulation with inertial terms is straightforward. It is important to notice that the virtual work of internal interactions $\mathbf{Q}$ and M at the boundary between two subdomains corresponds to the choice of degrees of freedom of particles. This proposition looks almost trivial here, but it will be used with benefit while constructing the model of a shell.

In the reference configuration we have $\mathbf{x} = \mathbf{x}_0$ and $\theta = 0$. Similarly to (1.121), we rewrite the boundary term in (2.30) as an integral; a local variational relation follows from the arbitrariness of $s_{0,1}$:

$$(\mathbf{Q}' + \mathbf{q}) \cdot \delta \mathbf{x} + (M' + m) \delta \theta + \mathbf{Q} \cdot \delta \mathbf{x}' + M \delta \theta' - \delta A^i = 0. \quad (2.31)$$

Now, the internal forces produce no work at the rigid body motion; hence,

$$\delta \mathbf{x} = \delta \mathbf{x}_1 + \delta \theta_1 \mathbf{k} \times \mathbf{x}, \quad \delta \mathbf{x}_1 = \text{const}, \quad \delta \theta_1 = \text{const} \quad \Rightarrow \quad \delta A^i = 0. \quad (2.32)$$

We substitute (2.32) in (2.31), set coefficients at the independent variations $\delta\theta_1$ and $\delta\mathbf{x}_1$ equal to 0 and arrive at the equations of balance of forces and moments:

$$\begin{aligned}\mathbf{Q}' + \mathbf{q} &= 0, \\ M' + \mathbf{x}' \times \mathbf{Q} \cdot \mathbf{k} + m &= 0.\end{aligned}\tag{2.33}$$

In contrast to (2.20), the equations of equilibrium (2.33) allow for the external moments and the axial extension of the rod; writing the static boundary conditions is simple in the considered case. The following terms remain in (2.31):

$$M\delta\theta' + \mathbf{Q} \cdot (\delta\mathbf{x}' - \delta\theta\mathbf{k} \times \mathbf{x}') + \delta A^i = 0.\tag{2.34}$$

Instead of the known model of a Cosserat continuum [3, 133], we will consider the case of constrained shear deformation, see [79, 128]. The axial extension is allowed for the sake of an efficient numerical implementation. The kinematic constraint between the virtual displacements and the rotation of particles, which is similar to (2.27), follows as

$$\delta(\mathbf{n} \cdot \mathbf{x}') = 0, \quad \delta\mathbf{n} = \delta\theta\mathbf{k} \times \mathbf{n} \Rightarrow \delta\theta\mathbf{t} \cdot \mathbf{x}' = \mathbf{n} \cdot \delta\mathbf{x}'; \quad \mathbf{t} = \frac{\mathbf{x}'}{|\mathbf{x}'|};\tag{2.35}$$

here, the normal vector $\mathbf{n}$ is “frozen” into a particle and rotates with it.

For an elastic rod, the work of the internal forces is $\delta A^i = -\delta U$; U is the strain energy per unit “length” s . From (2.34) and (2.35) we show that the strain energy is a function of the deformation of bending κ (change of curvature of the rod) and of the axial extension ε :

$$\delta U = M\delta\theta' + \mathbf{Q} \cdot (\delta\mathbf{x}' - \mathbf{n}\mathbf{n} \cdot \delta\mathbf{x}') = M\delta\kappa + Q_t\delta\varepsilon,\tag{2.36}$$

in which

$$Q_t = \mathbf{Q} \cdot \mathbf{t}, \quad \kappa = \theta', \quad \mathbf{x}' = (1 + \varepsilon)\mathbf{t}, \quad \varepsilon = |\mathbf{x}'| - 1.\tag{2.37}$$

The elastic relations read

$$M = \frac{\partial U}{\partial \kappa}, \quad Q_t = \frac{\partial U}{\partial \varepsilon};\tag{2.38}$$

the transverse force $Q_n = \mathbf{Q} \cdot \mathbf{n}$ is determined by the constraint (2.35). This type of a generalization of the Bernoulli–Euler beam theory has been discussed in the literature [6, 95].

For problems with small local strains (which does not exclude large overall rotations and displacements) a quadratic approximation of the strain energy should be sufficiently accurate:

$$U = \frac{1}{2}a\kappa^2 + \frac{1}{2}b\varepsilon^2 + c\kappa\varepsilon.\tag{2.39}$$

The third coupling term needs to be considered for non-symmetric cross sections as in (2.57) below. The stiffness coefficients a , b , and c cannot be determined in the framework of the direct approach.

2.2.2 Relation to the Asymptotic Model

As it was shown above, the effect of the axial extension is asymptotically negligible for thin curved rods, and the second term in (2.39) can be considered as a penalty, which keeps $\varepsilon \rightarrow 0$ as $b \rightarrow \infty$. Moreover, in [50] it is shown that the results of the direct approach are asymptotically equivalent to the classical model of Kirchhoff when the stiffness b is asymptotically larger than a : the present theory allows for further asymptotic simplification. Nevertheless, the axial extension may be important for the straight rods or for the numerical analysis.

The main idea of the *hybrid formulation* is that the direct approach must be equivalent to the asymptotic model in the linear setting. It means that a is the bending stiffness in (2.24), and the problem in the cross section, which results from the asymptotic study, needs to be solved prior to the analysis of the rod problem. The recovery of strains and stresses in the cross section is also performed on the basis of the linear theory. It is important that the asymptotics determines the minimal set of degrees of freedom of particles, which need to be included in the direct approach, and thus justifies the formulation of the principle of virtual work (2.30).

The above assumption concerning the quadratic approximation of the strain energy might seem to be contradictory: if a beam is bent into a circle, is the corresponding change of curvature κ small enough? The question shall again be considered from the point of view of the asymptotic theory: the beam is a thin structure, whose thickness is approaching zero. Then it is clear that in the vicinity of any point of the beam the deformed state may be represented as “small deformation plus finite rigid body rotation and translation”. It means that the geometrically nonlinear theory can consistently be applied with the quadratic approximation of U as long as the components of the non-reduced strain tensor in the cross section, recovered on the basis of the one-dimensional solution, remain small.

2.2.3 Example: Large Bending of a Beam

Finite bending of a cantilever, which is straight in the reference configuration, has already been considered numerically in the non-reduced setting in Sect. 1.1.4. A semi-analytical solution with the developed rod model is simple. We consider an inextensible rod with the following kinematics:

$$\varepsilon = 0, \quad \mathbf{x}' = \mathbf{t} = \mathbf{i} \cos \theta(s) + \mathbf{j} \sin \theta(s). \quad (2.40)$$

There are no distributed loads, and from the first equilibrium equation (2.33) we see that the internal force equals the dead tip force $\mathbf{F}$ (for a discussion of behavior of external loads at the deformation see (1.100) and Sect. 3.1.2):

$$\mathbf{Q} = \mathbf{F}. \quad (2.41)$$

The moment is simply related to the angle of rotation:

$$M = a\theta', \quad (2.42)$$

and from the second equation of equilibrium (2.33) we write

$$a\theta'' + F_y \cos \theta - F_x \sin \theta = 0. \quad (2.43)$$

We have a kinematic boundary condition (clamping) at $s = 0$. At the other end the moment must vanish:

$$\theta|_{s=0} = 0, \quad \theta'|_{s=L} = 0. \quad (2.44)$$

Linearizing (2.43) with respect to θ , one can easily see the influence of the compressive force on the flexural rigidity of the beam. One speaks about *buckling* when the stiffness vanishes and the linearized problem allows for a non-trivial solution at $F_y = 0$; the minimal eigenvalue is called Euler's critical force:

$$F_* = \frac{\pi^2 a}{4L^2}. \quad (2.45)$$

The nonlinear boundary value problem (2.43), (2.44) can be solved either analytically in terms of elliptic functions or numerically. In *Mathematica* equation (2.43) may be formulated as follows:

$$| \quad \text{eq} = a \theta''[s] + F_y \cos[\theta[s]] - F_x \sin[\theta[s]] == 0;$$

Now we set up the parameters for a square cross section ($H = h$) and compute the critical compressive force:

$$| \quad \begin{aligned} \text{params} &= \{L \rightarrow 1, h \rightarrow 0.01, E \rightarrow 2 \times 10^{11}, a \rightarrow E h h^3 / 12\}; \\ \text{Fcrit} &= \pi^2 a / (4 L^2) /. \text{params} \\ &411.234 \end{aligned}$$

The definition of the bending stiffness in the list of parameters is based on the other values from the same list, and therefore a repeated substitution `/. params` needs to be used here.

The following function numerically solves the boundary value problem and finds $\theta(s)$ for the prescribed components of the tip force:

$$| \quad \begin{aligned} \text{solveProblem}[F_]&:= \text{NDSolve}[\\ &\{(\text{eq} /. \text{params} /. \{F_x \rightarrow F[[1]], F_y \rightarrow F[[2]]\}), \theta[0] == 0, \\ &\theta'[L /. \text{params}] == 0\}, \theta, \{s, 0, L /. \text{params}\}][[1]] \end{aligned}$$

A kind of the *shooting method* is used by `NDSolve`: the value $\theta'(0)$ is determined iteratively such that the condition at $\theta'(L) = 0$ is fulfilled; an initial value problem is solved at each iteration. One can improve the convergence by specifying an ap-

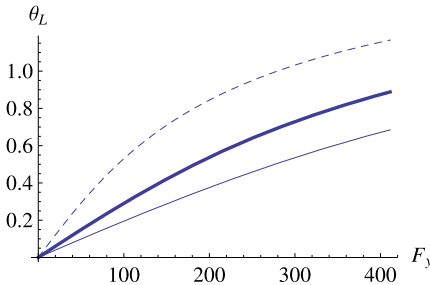
proprate initial approximation for $\theta'(0)$, but the default settings work fine as long as the load is small enough.

Plotting $\theta_L = \theta(L)$ in dependence on the transverse force F_y for given F_x is programmed in the function below; the style of the line is its second argument:

```
plotθL[Fx_, style_] :=
  Plot[θ[L /. params] /. solveProblem[{Fx, Fy}],
    {Fy, 0, Fcrit}, PlotStyle → style]
```

Now we can observe $\theta_L(F_y)$ for $F_x = -F_*/2$ (thin line), 0 (thick line) and $F_*/2$ (dashed line):

```
Show[
  plotθL[0, Thick],
  plotθL[0.5 Fcrit, Thin],
  plotθL[-0.5 Fcrit, Dashed],
  PlotRange → All, AxesLabel → {"F_y", "θ_L"}
]
```



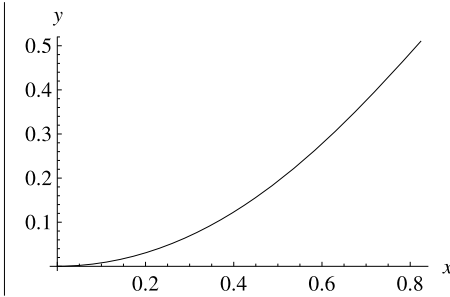
The effects of stiffening and softening may clearly be seen.

A particular deformed configuration is studied at $\mathbf{F} = -200\mathbf{i} + 200\mathbf{j}$, which is equivalent to the distributed tip force in the second example of Sect. 1.1.4. With a given solution for $\theta(s)$, we integrate (2.40) and determine the configuration of the rod $\mathbf{x}(s)$; the ability of `NDSolve` to integrate matrix equations is used here:

```
sol = solveProblem[{-200, 200}];
xsol = NDSolve[
  {
    x'[s] == {Cos[θ[s]], Sin[θ[s]]} /. sol, x[0] == {0, 0}
  },
  x, {s, 0, L /. params}
][[1]];
```

Now we plot the deformed configuration:

```
ParametricPlot[x[s] /. xsol, {s, 0, L /. params},
  AspectRatio -> Automatic, AxesLabel -> {"x", "y"}]
```



The computed transverse position of the end point

```
x[L /. params] /. xsol
{0.82403, 0.510061}
```

is close to the finite element solution of the non-reduced problem (1.108). The solution scheme is easy to extend to the model of an extensible rod by first finding ε as a function of θ from the constitutive relation for Q_t , and additional terms with the stiffness for extension b would enter (2.43). The difference in the tip deflections of the models with and without extension is as low as 3×10^{-6} .

2.3 Finite Element Scheme for Classical Rods in Plane

2.3.1 Formulation of the Finite Element

The rod model, derived in Sect. 2.2 with the help of the principle of virtual work, allows for a simple finite element implementation. The classical rod theory has evident advantages in comparison to the model with independent approximations of translations and rotations: less unknowns and less stiffness coefficients need to be involved. With the present approach we substantially resolve other inherent computational drawbacks of the model with shear, namely (1) the *shear locking* because of the inability of the combined approximation to represent shear-free configurations and (2) the “steep-sided valley” structure of the functional of strain energy, which may lead to a poor convergence of the minimization algorithm in static or to

a stiff system of differential equations in dynamic simulations. This general argumentation is relevant for all other kinds of continua, considered in this book. In what follows we will not substantially differentiate classical rods from unshearable, but extensible ones in the context of numerical modeling. As it was pointed out above in Sect. 2.2.2, for thin structures we may consider b as a high penalty coefficient for constraining axial extensibility; see also the outcome of the comparison of the results of the semi-analytical modeling in the end of the previous section (Sect. 2.2.3).

Similarly to (1.99), the balance equations and the static boundary conditions in the elastic case are equivalent to the following variational formulation:

$$\begin{aligned} U^\Sigma[\mathbf{x}(s)] &= U^{\text{strain}} + U^{\text{ext}} \rightarrow \min, \quad U^{\text{strain}} = \int_0^L U(\varepsilon, \kappa) \, ds, \\ U^{\text{ext}} &= - \int_0^L (\mathbf{q} \cdot \mathbf{x} + m\theta) \, ds - (\mathbf{Q} \cdot \mathbf{x} + M\theta)|_0^L. \end{aligned} \quad (2.46)$$

At a stable equilibrium, the sum of the total strain energy U^{strain} (integrated over the length of the rod L) and of the potential energy of external loads U^{ext} has a minimum. The functional U^Σ is considered on the configurations $\mathbf{x}(s)$, which are kinematically admissible. While ε can be directly computed from (2.37), the second strain measure equals the change of the *material curvature*:

$$\kappa = \mathbf{n}' \cdot \mathbf{t} - \dot{\mathbf{n}}' \cdot \dot{\mathbf{t}} = \Omega - \dot{\Omega}, \quad \Omega = \frac{1}{|\mathbf{x}'|} \mathbf{n} \cdot \mathbf{x}'', \quad \dot{\Omega} = \dot{\mathbf{n}} \cdot \dot{\mathbf{x}}''. \quad (2.47)$$

It should be noted that the material curvature Ω for the deformed state differs from the common geometric curvature by the coefficient $|\mathbf{x}'|$. We may refer to a simple example of thermal expansion of a ring to demonstrate the consistency of this statement. Increasing the radius of a circle, we change its geometric curvature, but $\kappa = 0$ and $\Omega = \dot{\Omega}$: no bending moment is induced in the ring by its axial deformation. For a comprehensive discussion see Gerstmayr and Irschik [61].

The problem of minimization of (2.46) is well suited for the Rayleigh–Ritz method. The approximation of the field $\mathbf{x}(s)$ shall be smooth enough to avoid singularities in κ : the vector of unit normal $\mathbf{n}$ needs to be continuous. The C^1 continuity of $\mathbf{x}(s)$ is a sufficient condition. A two-node finite element with the following approximation can serve to this purpose:

$$\mathbf{x}(\tilde{s}) = \mathbf{x}_i S_1(\tilde{s}) + \partial_{\tilde{s}} \mathbf{x}_i S_2(\tilde{s}) + \mathbf{x}_j S_3(\tilde{s}) + \partial_{\tilde{s}} \mathbf{x}_j S_4(\tilde{s}), \quad -1 \leq \tilde{s} \leq 1. \quad (2.48)$$

The cubic shape functions S_k are such that the position vector and its derivative in the nodes i and j take on the corresponding values:

$$\mathbf{x}(-1) = \mathbf{x}_i, \quad \partial_{\tilde{s}} \mathbf{x}(-1) = \partial_{\tilde{s}} \mathbf{x}_i, \quad \mathbf{x}(1) = \mathbf{x}_j, \quad \partial_{\tilde{s}} \mathbf{x}(1) = \partial_{\tilde{s}} \mathbf{x}_j; \quad (2.49)$$

in each node k we have four scalar degrees of freedom: $\mathbf{x}_k$ and $\partial_{\tilde{s}} \mathbf{x}_k$. Because the local coordinate on the element $\tilde{s}$ is no longer the arc coordinate in the reference

configuration, the coefficient $ds/d\tilde{s}$ will enter the kinematic relations for ε in (2.37) and for κ in (2.47). Let us see in more detail what happens with $\mathbf{x}'$ at the nodes:

$$\mathbf{x}' = \frac{\partial \mathbf{x}}{\partial \tilde{s}} \frac{d\tilde{s}}{ds} = \frac{\partial_{\tilde{s}} \mathbf{x}}{|\partial_{\tilde{s}} \mathbf{x}|}. \quad (2.50)$$

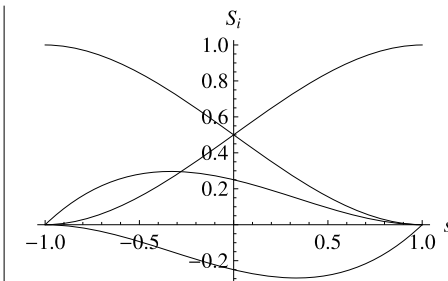
With the conditions (2.49) we have $\partial_{\tilde{s}} \mathbf{x}$ continuous between the elements. The so-called *isoparametric* formulation [26, 76], in which the same interpolation rule is applied for both the reference geometry $\mathring{\mathbf{x}}$ and for the actual state $\mathbf{x}$, provides a C^1 continuous approximation. Smoothness of $\mathbf{x}$ means that ε is approximated continuously. At the same time, κ undergoes jump discontinuities at the nodes, which is evident from (2.47). The finiteness of this jump makes the approximation at hand suitable for the variational formulation (2.46).

It is also worth noting that the element does not make use of rotational degrees of freedom at the nodes and shall be classified as belonging to the so-called absolute nodal coordinate formulation (ANCF), for further details see [45, 61, 140]. The main benefit is that the mass matrix remains constant throughout the simulation, which makes the transient analysis particularly efficient.

2.3.2 Implementation in Mathematica

In the following we omit tilde and denote the local coordinate on the element s , and $(\dots)'$ will have a meaning of $\partial_{\tilde{s}}(\dots)$. Similar to (2.50), certain coefficients will have to be introduced in the relations, which include derivatives or integrals with respect to the “true” arc length in the reference configuration. First we define four cubic shape functions $S_i(s)$ with the desired properties and plot them:

```
shapeFunctions[s_] :=
  {2 - 3 s + s^3, (1 - s)^2 (s + 1), 2 + 3 s - s^3, (s - 1) (s + 1)^2} / 4;
Plot[shapeFunctions[s], {s, -1, 1}, AxesLabel -> {s, Si}]
```



Indeed, the conditions (2.49) are satisfied identically, as the values and the derivatives of the shape functions at the ends of the domain form a unit matrix:

```
| {shapeFunctions[-1], shapeFunctions'[-1],  
|   shapeFunctions[1], shapeFunctions'[1]}  
| {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
```

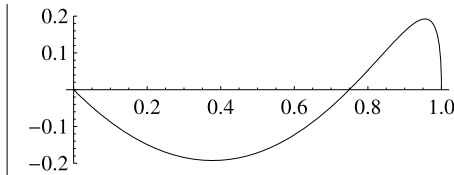
Any polynomial up to the third order can be represented by these shape functions, which guarantees that all simple forms of deformation will be computed exactly.

At the next step we define the position vector on an element as a function of the degrees of freedom of two adjacent nodes and of the local coordinate:

```
| positionVector[{x1_, dx1_, x2_, dx2_}, s_] :=  
|   shapeFunctions[s].{x1, dx1, x2, dx2}
```

To demonstrate the functionality of the defined `positionVector`, we plot the geometry of a finite element with the sample values of the nodal degrees of freedom $x_i = 0$, $x'_i = i - j$, $x_j = i$, and $x'_j = -j$:

```
| ParametricPlot[positionVector[  
|   {{0, 0}, {1, -1}, {1, 0}, {0, -1}}, s], {s, -1, 1}]
```



Degrees of freedom at a node with a given number are functions of the time variable t . The routine below collects them into a nested list:

```
| nodalDOF[i_] := {{x[i][t], y[i][t]}, {dx[i][t], dy[i][t]}}
```

An element with the number i includes nodes i and $i + 1$, and its position vector for given s can be generally formulated:

```
| elementPositionVector[i_, s_] :=  
|   positionVector[Join[nodalDOF[i], nodalDOF[i + 1]], s]
```

The analytical integration of the strain energy of the finite element is hardly possible in the present case, and the *Gaussian quadrature* formula should be applied for the

numerical integration, see Hughes [76]:

$$U_{\text{el}}^{\text{strain}} = \int_{-1}^1 U |\dot{\mathbf{x}}'| \, ds \approx \sum_{i=1}^{n_{\text{int}}} U |\dot{\mathbf{x}}'| \Big|_{s=s_i} w_i. \quad (2.51)$$

Here s_i are the integration points and w_i are their weights, and with n_{int} integration points, all polynomial functions with orders up to $2n_{\text{int}} - 1$ will be integrated exactly. In our case, κ is not a polynomial function of s , and the optimal number of the integration points shall be chosen based on the computational experience. The practice shows that with $n_{\text{int}} = 4$ the element is fully integrated, and $n_{\text{int}} = 3$ works fine when the mesh is not extremely coarse. Lower values shall be used with caution owing to the known hourglass effect: the deformation modes adjust themselves such that U vanishes exactly in the integration points, although the true strain level may be high.

Gaussian quadrature rules are defined in *Mathematica* in an additional package. The examples below feature mostly $n_{\text{int}} = 4$, but a reduced integration rule with two points will also be considered:

```
Needs["NumericalDifferentialEquationAnalysis`"]
gaussianPointsκ =
  gaussianPointse = GaussianQuadratureWeights[4, -1, 1];
gaussianPointsReduced =
  GaussianQuadratureWeights[2, -1, 1];
```

Both integration rules are defined as lists of four and two integration points, and each point is a pair of values $\{s_i, w_i\}$.

For the sake of simplicity we will assume a reference state with the constant curvature $\dot{\Omega}$ and with constant $|\dot{\mathbf{x}}'(s)|$ instead of a full isoparametric formulation (which does not affect the necessary smoothness of the approximation). Our next and most important step is the computation of the strain energy $U_{\text{el}}^{\text{strain}}$ of the finite element. We begin by expressing the strain measures for an element with the number i , whose reference length is l and reference curvature is $\dot{\Omega}$:

```
strainMeasures[i_, l_, Ω0_, s_] := Module[
  {x, dx, ddx},
  x = elementPositionVector[i, s];
  dx = D[x, s] 2 / l; ddx = D[dx, s] 2 / l;
  {
    (dx[[1]] ddx[[2]] - dx[[2]] ddx[[1]]) / (dx.dx) - Ω0,
    Sqrt[dx.dx] - 1
  }
]
```

This function returns expressions for both ε and κ at the point with the local coordinate s , which depend on the nodal degrees of the element. The local coordinate

varies from -1 to 1 , and derivatives with respect to the arc coordinate in the reference configuration are obtained with the coefficient $|\dot{\mathbf{x}}'|^{-1} = 2/l$. Now, we are able to compute the strain energy of an element for given stiffness properties a, b, c and reference geometry $l, \dot{\Omega}$. The function below provides $U_{\text{el}}^{\text{strain}}$ depending on the degrees of freedom of the element i :

```

elementStrainEnergy[i_, l_, Ω0_, a_, b_, c_] := Module[
  {κ, ε, Uκ, Uε, s},
  {κ, ε} = strainMeasures[i, l, Ω0, s];
  Uκ = 1 / 2 a κ2 + c κ ε;
  Uε = 1 / 2 b ε2;
  Sum[gp[[2]] Uκ 1 / 2 /. s → gp[[1]],
    {gp, gaussianPointsκ}] +
  Sum[gp[[2]] Uε 1 / 2 /. s → gp[[1]],
    {gp, gaussianPointsε}]
]

```

Two different integration rules may be used for the parts of the strain energy, which are, respectively, related to bending and to extension. Finally, we assemble the finite element model by computing the total strain energy of n finite elements; L is the total length of the rod:

```

modelStrainEnergy[n_, L_, Ω0_, a_, b_, c_] :=
  Sum[elementStrainEnergy[i, L/n, Ω0, a, b, c], {i, n}];

```

2.3.3 Examples of Simulations

Particular solutions require boundary conditions. For the problem of *bending of a clamped beam*, considered analytically in Sect. 2.2.3, we constrain the position and the vertical component of the directional derivative at the first node:

```

constraints = {x[1][t] → 0, y[1][t] → 0, dy[1][t] → 0};

```

Traditional finite element codes feature simple constraints, which are imposed in the form of penalty terms or Lagrange multipliers. In *Mathematica* we may afford the luxury of simple elimination of the constrained degrees of freedom. The active ones, which may freely be varied at given n are collected in a list (which is traditionally called *vector of degrees of freedom*) by the function below:

```

activeDegreesOfFreedom[n_] := DeleteCases[
  Flatten[Table[nodalDOF[i], {i, n+1}]],
  z_ /; MemberQ[constraints[{All, 1}], z]
]

```

We again address the problem of large bending of a cantilever, considered in Sect. 2.2.3; here the parameters are defined accordingly:

```
params = {L → 1, h → 10-2, E → 2 × 1011, a → E h h3 / 12,
          b → E h2, ρ3 → 7800, ρ → ρ3 h2, F → {-200, 200}};
```

The finite element solutions for the position of the tip are going to be compared against the analytical values, obtained for the extensible model of the beam:

```
exactEndPosition = {0.824027290537, 0.510059838223};
```

The initial (reference) configuration is determined by the following set of substitutions for the nodal degrees of freedom, which are generally formulated by using a pattern for the number of the node:

```
initSubst[n_] := {x[i_] [t] → L (i - 1) / n, y[i_] [t] → 0,
                  dx[i_] [t] → L / n / 2, dy[i_] [t] → 0} /. params
```

The numerical minimization requires an initial approximation for the values of the unconstrained degrees of freedom, which is constructed by the following function:

```
initApprox[n_] := Table[{dof, dof /. initSubst[n]},
                        {dof, activeDegreesOfFreedom[n]}];
```

We are ready to solve the problem (2.46) and to compare the result to the analytically obtained one. In the function below, FindMinimum is invoked for minimizing the total energy, and then the difference from the exact solution is computed:

```
staticError[n_] := Module[{Ustrain, Uext, sol},
  Ustrain =
    modelStrainEnergy[n, L, 0, a, b, 0] /. constraints;
  Uext = -nodalDOF[n + 1][[1]].F;
  sol = FindMinimum[Ustrain + Uext /. params,
                    initApprox[n], MaxIterations → ∞][[2]];
  Norm[exactEndPosition - nodalDOF[n + 1][[1]] /. sol]
]
```

Now we can ascertain that satisfactory accuracy is achieved in the model with 2 finite elements, and with 15 we are almost exact:

```
| {staticError[2], staticError[15]}
| {0.0316394, 8.07333 × 10-6}
```

It should be noted that exact results for small deflections would be obtained with just a single element: the sought solution is then cubic in s and can be exactly represented by the shape functions.

A function for the *mesh convergence study* is programmed below:

```
| staticErrors :=
|   ParallelTable[
|     With[{n = Floor[1.2meshLevel]}, {n, staticError[n]}],
|     {meshLevel, 1, 22}
|   ]
```

Varying the mesh level (which is dynamically indicated to let the user see the progress of evaluation), we exponentially increase the number of elements from 1 to $1.2^{22} \approx 55$. The computation time is reduced with the ability of *Mathematica* to automatically share the work across multiple processors or cores using `ParallelTable`. First we compute the static errors in the model with the full 4-point integration rule for all parts of U^{strain} :

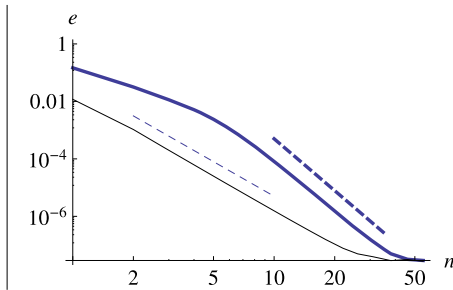
```
| gaussianPoints $\epsilon$  = gaussianPoints $\kappa$ ;
| staticErrorsFull = staticErrors;
```

And for the sake of comparison, we do the same, but using only two integration points for the part of the strain energy $b\epsilon^2/2$:

```
| gaussianPoints $\epsilon$  = gaussianPointsReduced;
| staticErrorsReduced = staticErrors;
```

The results are presented by the solid lines in a logarithmic scale below, and the dashed lines help estimating the rate of convergence:

```
| Show[
|   ListLogLogPlot[staticErrorsFull,
|     Joined → True, PlotStyle → Thick],
|   ListLogLogPlot[staticErrorsReduced, Joined → True],
|   LogLogPlot[5 × 102 n-6,
|     {n, 10, 35}, PlotStyle → {Thick, Dashed}],
|   LogLogPlot[5 × 10-2 n-4, {n, 2, 10}, PlotStyle → Dashed],
|   AxesLabel → {n,  $\epsilon$ }
| ]
```



Numerical round-off effects dominate, as the error drops below 5×10^{-8} . The slope of the thick dashed line answers to the 6th order convergence ($e = kn^{-6}$), which is achieved by the model with full integration at finer meshes. Coarser models suffer from the so-called *membrane locking*: thin structures are very flexible at bending in comparison to the axial rigidity, and the inability of the finite elements to bend without axial deformation in the general curved case leads to additional stiffness at bending. The effect is getting more pronounced when the thickness is decreased. Thus, we repeated the solution for a five times thinner beam, i.e., we used $h = 2 \times 10^{-3}$ and adjusted the magnitude of the force to keep the same level of deformation. The error of the solution shows the same behavior, but the rapid 6th order convergence begins as $n > 15$.

Membrane locking may be avoided by using reduced integration rules for the terms in the strain energy, which lock in the numerical scheme: a steady 4th order convergence can be observed for this case in the figure above, see the thin lines. The explanation is simple. The axial deformation may be small at the two integration points, but its general level still remains high: the element retains some axial stiffness, and the bending behavior is less affected. This or other techniques for avoiding membrane locking may become important for very slender structures, when the threshold in the number of elements, required for achieving the high convergence rate is too high; see discussion in Sect. 4.5.4. Under normal circumstances, the need of using more finite elements to achieve the same accuracy is usually compensated by the robustness of simulations with the full integration applied, especially in non-linear and buckling problems. It remains to add that the membrane locking is usually less severe than the shear one, which is intrinsic to finite element modeling of shear deformable structures.

Our next example is *bending of a half of a circle*. A mesh with 10 finite elements and a full integration rule will be used:

```
n = 10;
gaussianPointsc = gaussianPointsK;
```

At first, we clamp the left end of the rod in the vertical direction:

```
constraints = {x[1][t] → 0, y[1][t] → 0, dx[1][t] → 0};
```

Specifying constant curvature $-\pi/L$, we seek an equilibrium state, and the straight rod configuration is used as a starting approximation:

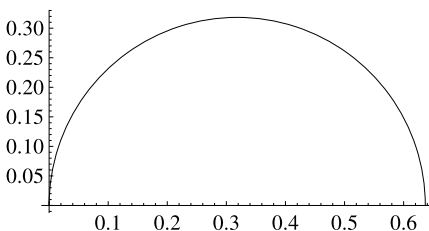
```
U = modelStrainEnergy[n, L, -π / L, a, b, 0] /.
  constraints /. params;
sol = FindMinimum[U, initApprox[n]] [[2]];
```

We will draw deformed configurations by parametrically plotting all finite elements, which is programmed in the function below:

```
plotSol[sol_] := Show[Table[
  ParametricPlot[
    elementPositionVector[i, s] /. constraints /. sol,
    {s, -1, 1}, PlotRange → All],
  {i, n}
]]
```

We can ascertain that the previously obtained solution is actually a semicircle:

```
plotSol[sol]
```



Certain inaccuracy may be expected, as a circular line cannot be exactly represented with cubic shape functions. Indeed, the coordinates of the right end point in the solution slightly deviate from the exact values $\{2L/\pi, 0\}$:

```
(nodalDOF[n + 1] [[1]] /. sol) - {(2 L / π /. params), 0}
{0.000033882, 0.0000397907}
```

A statically indeterminate problem of a *clamped-hinged semicircle* was previously considered in the literature by Simo and Vu-Quoc [142]. Now we additionally constrain the position of the right end:

```
constraints = {x[1][t] → 0, y[1][t] → 0, dx[1][t] → 0,
  x[n + 1][t] → 2 L / π, y[n + 1][t] → 0} /. params;
```

A vertical force is applied in the middle of the rod (n is even, and there is a node in this point), so that the total energy of the system can be computed:

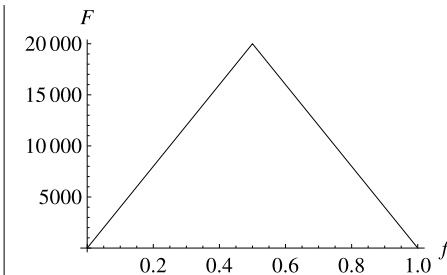
```
ymid = y[n / 2 + 1][t];
U = -F ymid + modelStrainEnergy[n, L, -π / L, a, b, 0] /.
  constraints /. params;
```

The following function finds static equilibrium for a given force value F , which acts downwards, and for a given initial approximation:

```
solveProblem[F_, initApprox_] := FindMinimum[
  U /. Fy → -F, initApprox, MaxIterations → ∞][[2]]
```

We are wishing to track the snap-through of the structure as the load exceeds a certain critical value, and then to follow the equilibrium path when the load is slowly released. It is convenient to introduce a loading parameter f , which varies from 0 to 1, such that the force depends on it non-monotonously:

```
FF[f_] := 40 000 (0.5 - Abs[f - 0.5])
Plot[FF[f], {f, 0, 1}, AxesLabel → {f, F}]
```



Below, we let the loading parameter increase with small steps. For each step we solve a static problem with the solution from the previous step used as an initial approximation i.e. Solution at the first step is again obtained, starting from a straight reference configuration (which might fail to converge to the desired semicircular configuration with other values of parameters or boundary conditions). Dynamic printing of the loading parameter helps to track the progress of the computation, which cannot be parallelized now as the solution at the next step depends on the previous one:

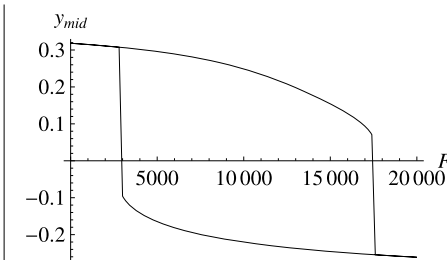
```

ia = initApprox[n];
PrintTemporary[Dynamic[f]];
data = Table[
  sol = solveProblem[FF[f], ia];
  ia = Table[{dof, dof /. sol},
    {dof, activeDegreesOfFreedom[n]}};
  {FF[f], ymid /. sol, sol},
  {f, 0, 1, 0.005}
];

```

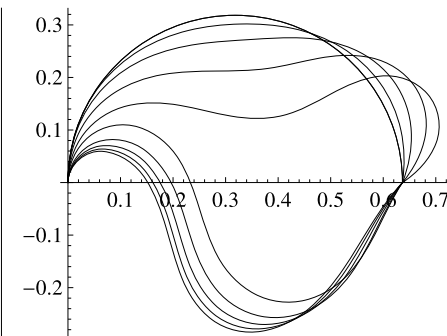
We plot the force-displacement characteristic for the middle point of the rod:

```
ListLinePlot[data[[All, {1, 2}]], AxesLabel → {"F", "ymid"}]
```



Two equilibrium paths can be observed here, and the system “jumps” as the force passes one of the two critical values (snap-through and snap-back) in the corresponding direction. The presented solution is nearly converged: the computed upper critical (snap-through) force value is approximately 17500, and with 20 finite elements the result would be 17000. Finally, we plot intermediate equilibrium configurations (each 20th from the computed set):

```
Show[plotSol /@ data[[;; ;; 20, 3]]]
```



Simo and Vu-Quoc [142] considered a circular arc with the total angle of more than 180° and presented a solution with a negative snap-back force. In other words, the unloaded structure may have three equilibrium configurations. As the middle one is evidently instable, kinematically driven modeling is suggested in [142]. Another consistent way to deal with the jumps of the equilibrium path is to perform a *dynamic simulation*. The following function computes the kinetic energy of a finite element:

```
elementKineticEnergy[i_, l_, ρ_] := Module[
  {v, s},
  v = D[elementPositionVector[i, s], t];
  Sum[ρ v.v / 2 l / 2 /. s → gp[[1]], {gp, gaussianPointsκ}]
]
```

The total kinetic energy may now be assembled:

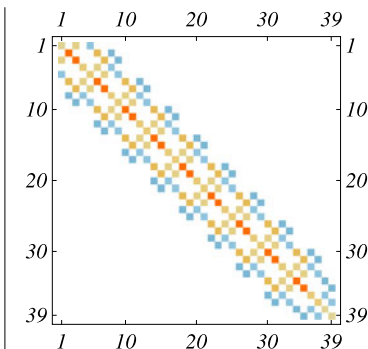
```
T = Sum[elementKineticEnergy[i, L / n, ρ], {i, n}];
```

As it was already mentioned above, the *mass matrix* M (which is the symmetric matrix of coefficients of the quadratic form of kinetic energy) is constant for the finite element formulation at hand:

```
MassMatrix = CoefficientArrays[
  2 T /. params, D[activeDegreesOfFreedom[n], t],
  "Symmetric" → True][[3]];
```

The profile structure of the mass matrix is presented below.

```
MatrixPlot[MassMatrix]
```



Now the equations of motion

$$M\ddot{e} = -\left(\frac{\partial U^\Sigma}{\partial e}\right)^T \quad (2.52)$$

(in which e is the column of variable degrees of freedom) are formulated:

```
dUde = D[U, {activeDegreesOfFreedom[n]}];
eqs = Thread[MassMatrix.
  D[activeDegreesOfFreedom[n], {t, 2}] == -dUde];
```

As an initial condition, we use the first semicircular static configuration at $F = 0$, computed above, and the velocities are set to zero at $t = 0$:

```
initCond = {
  #[[1]] == #[[2]] & /@ data[[1, 3]],
  Thread[D[activeDegreesOfFreedom[n], t] == 0]
} /. t -> 0;
```

(here it is convenient to use a pure function in *Mathematica* to transform the list of variables and their values to a list of equations, see Sect. 6.2.5). We choose the total time of simulation and set the value for an instantly applied force, which is slightly lower than the snap-through force in statics:

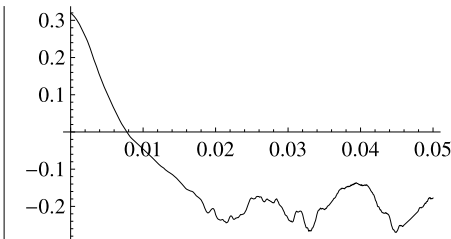
```
tMax = 0.05;
Fy = -16 000;
```

(note that the value of the variable Fy needs to be removed with the command `Clear` if the above static simulations are again to be performed in the current *Mathematica* session). Finally, we integrate the equations of motion over time with the command `NDSolve`; the option `StepMonitor` allows tracking the progress of the simulation, and the overall computation time returned by `Timing` is ca. 2.5 minutes on a computer with Intel i7-3740QM 2.6 GHz CPU:

```
PrintTemporary[Dynamic[progress]];
Timing[
  sol = NDSolve[Flatten[{eqs, initCond}],
    activeDegreesOfFreedom[n], {t, 0, tMax},
    StepMonitor -> (progress = t), MaxSteps -> ∞
  ][[1]]
] [[1]]
154.940193
```

The solution is stored in the form of an `InterpolatingFunction` object, and the vertical position of the point under the load can easily be plotted over the time:

```
| Plot[ymid /. sol, {t, 0, tMax}, PlotRange -> All]
```

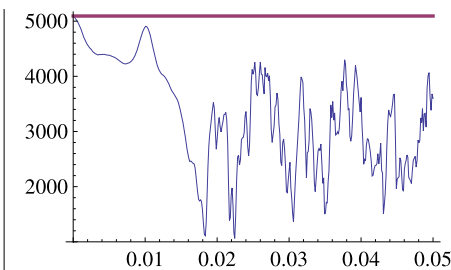


Because of its inertia, the structure quickly passes the upper equilibrium position and keeps oscillating near the lower one. The nonlinear coupling helps transferring the energy to the higher frequency vibration modes. The total energy can be computed for any instant of the time:

```
| totalEnergy = (U /. sol) +  
| (T /. params /. D[sol, t] /. D[constraints, t]);
```

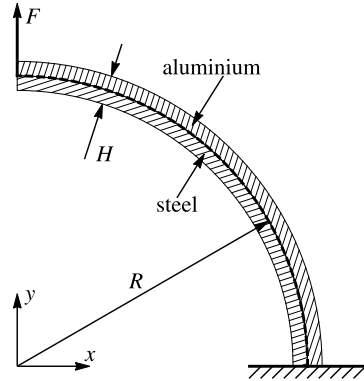
And now we can check that the total energy is preserved in the solution:

```
| Plot[{U /. sol, totalEnergy}, {t, 0, tMax},  
| PlotStyle -> {{}, Thick}, MaxRecursion -> 3]
```



Although the strain energy (thin line) shows high frequency oscillations, the total energy (thick line) remains at the initial level; the option `MaxRecursion` reduces the efforts of *Mathematica* in finding all particularities of the function being plotted. The obtained dynamic solution can easily be animated and studied interactively with the following command (the plotting range needs to be held constant for a smooth animation):

Fig. 2.2 Force bending of a layered curved strip in the form of a quarter of a circle (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)



```

Animate[Show[
  plotSol[
    Thread[sol[[All, 1]] → (sol[[All, 2]] /. t → tt)],
    PlotRange → {{0, 0.7}, {-0.3, 0.3}}
  ], {tt, 0, tMax}]

```

Finally, it remains to notice that the dynamic simulations presented above can be performed on an ordinary laptop computer with the 9th version of *Mathematica* software. In versions 7 and 8, one would need to express the higher order derivatives in the equations of dynamics explicitly to get `NDSolve` doing its work: `MassMatrix.` should be removed in the computation of eqs above, and the right-hand side should read `-Inverse[MassMatrix].dUde`. After this modification, the time integration with *Mathematica 7* runs smoothly and provides identical results, but one needs a powerful computer to be able to solve the problem with *Mathematica 8*: the simulation requires approx. 13 GB RAM and much more CPU time. Possibly, the situation can be improved by finer tuning the settings of the numerical time integration routine. It remains to hope that modeling complicated nonlinear dynamics of elastic systems will be feasible in the future versions of *Mathematica* as smoothly as in the version 9.

2.4 Asymptotic Equivalence of Rod and Non-reduced Two-Dimensional Models: Experimental Validation

The theoretical conclusion on the asymptotic consistency of the classical Kirchhoff theory shall be validated in a series of numerical experiments. Consider a problem of bending of a clamped quarter of a circle by a dead force F , see Fig. 2.2. The radius of the middle line is $R = 1$; material parameters of the two layers of the composite strip, each of them having the thickness $H/2$, are presented in Table 2.1. The out-of-plane width was chosen $h = 1$.

Table 2.1 Material parameters of the layers of the strip (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)

	Layer 1 (inner, steel)	Layer 2 (outer, aluminum)
Young's modulus E	2.1×10^{11}	7×10^{10}
Poisson's ratio ν	0.3	0.34
Density ρ	7800	2700

In the following, we will consider solutions of various problems when the thickness tends to zero:

$$H = H_0/k; \quad (2.53)$$

the reference thickness value is $H_0 = 0.1$ (as in Fig. 2.2), and the factor k will be sequentially increased. Comparing solutions of the problems in the rod formulation and non-reduced two-dimensional solutions, obtained by the finite element method, we validate the asymptotic accuracy of the rod theory as $H \rightarrow 0$. The non-reduced strip model was analyzed in ABAQUS. The force F was uniformly distributed along the corresponding edge of the finite element model. We used quadratic quadrilateral elements (CPS8R) in ABAQUS simulations; the aspect ratio for all elements in the regular mesh was approximately 1, and the number of elements over the thickness varied from 20 (dynamic transient simulations) to 60 (linear static analysis) such that the achieved accuracy was sufficient for the goal of comparison of the rod and non-reduced solutions.

2.4.1 Linear Static Bending

Choosing the arc coordinate s with its origin in the clamped point of the inextensible rod, for a statically determined linear problem we write the bending moment M and the total strain energy U^{strain} using (2.38) and (2.39) for a Legendre transformation of U with $\varepsilon = 0$:

$$M = -FR \cos \frac{s}{R}, \quad U^{\text{strain}} = \int_0^{\pi R/2} \frac{M^2}{2a} ds = \frac{F^2 \pi R^3}{8a}. \quad (2.54)$$

Finding an analytical solution is simple with the help of Castigliano's method. The vertical displacement of the point under the load is found as a derivative of the total strain energy with respect to the force:

$$u_y = \frac{\partial U^{\text{strain}}}{\partial F} = \frac{F \pi R^3}{4a}. \quad (2.55)$$

The bending stiffness a is determined in accordance with the procedure, described before (2.24). The relation between ε and κ , which results from (2.12) is equivalent

to finding the coordinate n_0 of the neutral fiber of the rod:

$$\begin{aligned} \int_{-H/2}^{H/2} E(n)(n - n_0) \, dn = 0 &\Rightarrow n_0 = \frac{H}{4} \frac{E_2 - E_1}{E_2 + E_1} \\ \Rightarrow a = h \int_{-H/2}^{H/2} E(n)(n - n_0)^2 \, dn &= \frac{hH^3}{96} \frac{E_1^2 + 14E_1E_2 + E_2^2}{E_1 + E_2}. \end{aligned} \quad (2.56)$$

Substituting this a in (2.55), we obtain an expression for the vertical displacement, which we will call $u_y^{(1)}$. It may also seem to be reasonable to modify the formula by using the radius of the neutral fiber $R + n_0$ instead of the radius of the geometric middle line R ; the corresponding expression for the vertical displacement will be denoted $u_y^{(2)}$ in the following. And the third rod solution, which we considered, was obtained by decoupling the axial extension from the bending according to the model, presented in Sect. 2.2. We used an expression for the strain energy (2.39) with coupling between κ and ε due to the asymmetry of the material properties; the particular coefficients result from the simple integration in the cross section:

$$U = \frac{1}{48} h (12(E_1 + E_2)\varepsilon^2 + 6(E_2 - E_1)H\varepsilon\kappa + (E_1 + E_2)H^2\kappa^2). \quad (2.57)$$

Writing again the expressions for the bending moment M and for the axial force Q_t , performing Legendre transformation of (2.57) according to (2.38), integrating the strain energy over the length and computing the derivative with respect to F , we obtain an expression for the vertical deflection

$$u_y^{(3)} = \frac{2RF((E_1 + E_2)H^2\pi + 12(E_2 - E_1)HR + 12(E_1 + E_2)\pi R^2)}{h(E_1^2 + 14E_1E_2 + E_2^2)H^3}. \quad (2.58)$$

The finite element solutions u_y^* for particular values of k , obtained in ABAQUS, were considered as reference values for computing the relative errors of the rod models

$$e^{(i)} = |u_y^{(i)} - u_y^*|/u_y^*. \quad (2.59)$$

Commonly, it is supposed that the rod model is stiffer than the non-reduced one because of the imposed internal constraints. But computations show that $u_y^{(1)} > u_y^*$; the opposite effect can be achieved by switching the layers. The two other solutions $u_y^{(2)}$ and $u_y^{(3)}$ generally predict smaller displacements than the non-reduced model.

The dependence of the three relative errors on the thickness factor k in a logarithmic scale is presented in Fig. 2.3. All three errors obviously converge towards zero. No positive effect from the correction of the radius by the offset of the neutral fiber can be observed. The accuracy is significantly improved by taking axial extension into account. Particularities concerning the last set of results at $k = 2^5$ are probably caused by numerical difficulties in the finite element solution. With more than one million finite elements in total, accumulated numerical errors seem to prevent

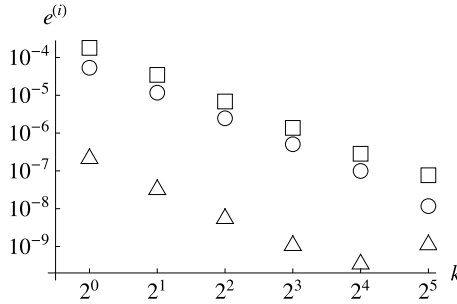


Fig. 2.3 Convergence of the solutions of the linear static rod problem to the non-reduced strip solution; *circle*, $u_y^{(1)}$: only bending, R is the radius of the middle line; *square*, $u_y^{(2)}$: only bending, R is the radius of the neutral fiber; *triangle*, $u_y^{(3)}$: bending and axial extension (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)

ABAQUS from reaching the required precision of the results, and using less finite elements we arrive at solutions, which are not sufficiently converged.

The noticeable outcome of the results in Fig. 2.3 is that although the model with the axial extension is more accurate, the rate of convergence is evidently the same. The problem of developing a general and complete model with higher asymptotic accuracy, which is discussed in the end of Sect. 2.1, remains yet open.

2.4.2 Linear Eigenvalue Problem

For dynamic problems, the correspondence of the eigenfrequencies of the reduced rod model ω_i to the eigenfrequencies of the continuous model ω_i^* is crucial. While the reference values ω_i^* are again computed in ABAQUS, for the rod model we also decided to use a numerical method. The model was discretized with the finite elements, discussed in Sect. 2.3, and the stiffness coefficients a , b , and c were chosen according to (2.57). Linearizing the equations of motion in the vicinity of the undeformed state, we solved the eigenvalue problem. For thicker rods, sufficiently converged values ω_i were obtained with 10 one-dimensional finite elements, while at higher k we went up to 20 (compare with hundreds of thousands of two-dimensional finite elements, needed for thin structures!).

The relative errors

$$e_i = |\omega_i - \omega_i^*| / \omega_i^* \quad (2.60)$$

are presented logarithmically for the first four eigenfrequencies in Fig. 2.4 depending on the thickness factor k (the number of the eigenfrequency is marked to the left of the points at $k = 1$). The reason for the third eigenfrequency to converge faster than the others is yet unclear, but the general convergence of the solutions of the two-dimensional and one-dimensional eigenvalue problems as the thickness tends to zero can clearly be seen.

Fig. 2.4 Convergence of the first four eigenfrequencies of the linear dynamic rod problem to the non-reduced strip solutions (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)

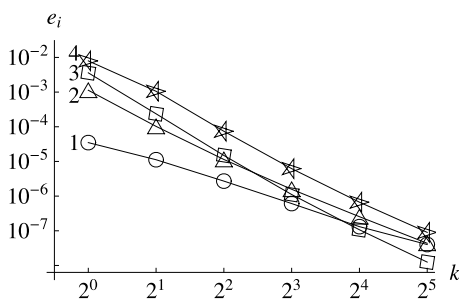
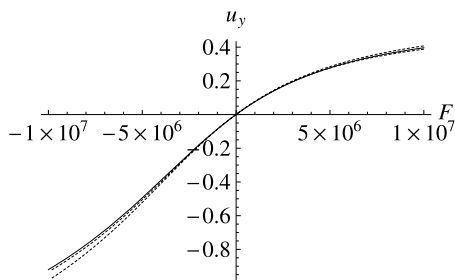


Fig. 2.5 Nonlinear static analysis of the thick strip: force-displacement characteristic in the rod model (*solid line*) and in the non-reduced strip model (*dashed lines*, which answer to the opposite corners on the loaded edge) (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)



2.4.3 Large Static Bending

Consider geometrically nonlinear static deformation. For a thick strip with $k = 1$, the vertical displacement of the loaded end u_y as a function of the dead force F is presented in Fig. 2.5 in the range $-F_0 \leq F \leq F_0$, $F_0 = 10^7$. The solution, obtained with the rod finite element model (4 finite elements are fine enough for the visual presentation), is compared to the displacements of the corner points of the loaded edge of the non-reduced strip model; displacements of the two corners are not the same because of the finite rotation of the edge. Large deformations of a relatively thick strip are accurately described by the rod model.

Convergence of the vertical displacement was studied for the force values, which were scaled according to the thickness factor such that the overall deformation remains in the same range:

$$F = F_0/k^3. \quad (2.61)$$

Rod solutions (circles in Fig. 2.6) converge to the mean displacement of the loaded edge of the strip model (triangles). Numerical issues, encountered by ABAQUS at solving the nonlinear problem with a very detailed mesh, prevent us from getting finer results for the last point $k = 2^5$.

Fig. 2.6 Convergence of the tip displacements in the nonlinear static problem: rod model (*circles*) and non-reduced strip model (*triangles*) (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)

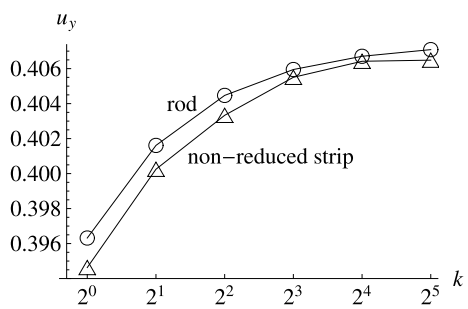


Table 2.2 Relative errors of the first four eigenfrequencies of the rod model of the pre-deformed strip depending on the thickness factor (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)

k	e ₁	e ₂	e ₃	e ₄
2 ⁰	0.01505	0.01191	0.03239	0.00517
2 ¹	0.0085	0.00147	0.00606	0.0175
2 ²	0.00449	0.00016	0.00021	0.0035
2 ³	0.00231	0.0005	0.00066	0.00038
2 ⁴	0.00115	0.00026	0.0005	0.00015
2 ⁵	0.02724	0.01085	0.00301	0.00127

2.4.4 Eigenfrequencies of a Pre-deformed Structure

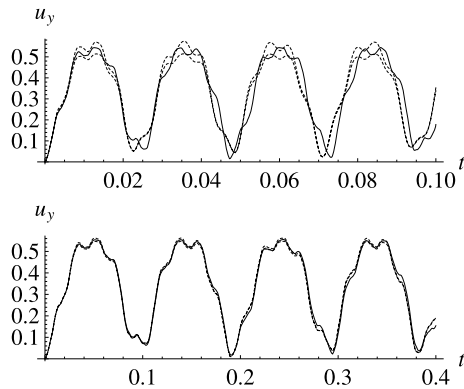
Numerical estimation of the convergence of the dynamic properties of the reduced and continuous models at nonlinear behavior is a non-trivial problem, which is largely affected by the chosen criteria and by the accuracy of the available solutions. We begin with the analysis of the eigenfrequencies of a rod, pre-deformed by the force F ; the force itself was supposed not to bring additional inertia to the structure. The computed relative errors (2.60) for the first four eigenfrequencies depending on the thickness factor are presented in Table 2.2. Although numerical effects make the situation less clear than in the fully linear case (see Fig. 2.4), convergence of the two solutions can again be seen up to the value $k = 2^5$.

2.4.5 Transient Analysis

Finally, we addressed the full dynamic simulation of large vibrations of the strip, instantly loaded by an inertialess tip force. Time histories of the displacement of the loaded end for the original thick structure and for a thinner one (with $k = 4$) are shown in Fig. 2.7. The force level was again scaled according to (2.61) and a four times longer time domain was used for the thin strip to keep the two solutions similar.

Displacements of the two corners of the loaded edge in the strip model (which differ because of the finite rotation of the tip of the rod) are shown by the dashed

Fig. 2.7 Time history of the vertical tip displacement in the dynamic simulation of the thick strip with $H = H_0$ (*above*) and of the thin strip with $H = H_0/4$ (*below*): rod solution (*solid line*) and non-reduced solution (*dashed lines* for the two corners of the loaded edge) (Adapted from Vetyukov [160] with kind permission from Springer Science and Business Media)



lines, and the solid line corresponds to the rod solution. The system of equations (2.52) was again integrated over time in *Mathematica* using the routine `NDSolve`; 6 finite elements were used in the rod model, and ca. 2×10^4 finite elements were needed in ABAQUS to obtain accurate solutions for the thin strip. The classical rod model behaves well when $H = H_0$, and the results are getting more accurate for slender structures.

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