

# Chapter 2

## Momentum Transfer

**Abstract** The beginning of this chapter presents the governing equations for the motion of fluid flows: equation of continuity, equation of motion, and energy equation. In the following sections, dynamic similitudes and information required for designing pipeline systems are introduced.

**Keywords** Bernoulli equation • Equation of continuity • Equation of motion • Friction coefficient • Laminar flow • Navier–Stoke’s equation • Pressure drop • Reynolds number • Transition to turbulence • Turbulent flow

### 2.1 Conservation Laws

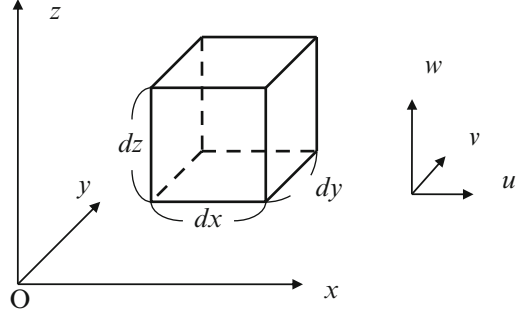
The motion of a fluid is governed by conservation laws such as the equations of continuity, motion, and energy. These equations are collectively called the governing equations. Each equation will be described briefly in this section. Derivation of the governing equations is beyond the scope of this text, and the details are available in specialized texts including Schlichting [1], Bird et al. [2], and Poirier and Geiger [3].

#### 2.1.1 Equation of Continuity

Figure 2.1 shows a fluid element in the Cartesian coordinate system. The conservation of mass for the fluid element is described by the equation of continuity. This equation is expressed for a Newtonian fluid as

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (2.1)$$

**Fig. 2.1** Fluid particle in the Cartesian coordinate system



where  $\rho$  is the density of the fluid,  $t$  is time, and  $u$ ,  $v$ , and  $w$  are the velocity components in the  $x$ ,  $y$ , and  $z$  directions, respectively.

When the flow is steady,  $\partial\rho/\partial t = 0$  and Eq. (2.1) becomes

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (2.2)$$

For a steady incompressible flow ( $\rho = \text{constant}$ ), Eq. (2.2) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.3)$$

**Exercise 2.1** Derive the equation of continuity, Eq. (2.1).

**Answer to Exercise 2.1** Consider a fluid element at a position,  $P(x, y, z)$ , as shown in Fig. 2.1. The mass of the fluid entering the element per unit time is expressed as

$$\rho u dy dz + \rho v dz dx + \rho w dx dy \quad (2.4)$$

The outgoing mass of the fluid per unit time is given by

$$\begin{aligned} \left(u + \frac{\partial u}{\partial x} dx\right) \left(\rho + \frac{\partial \rho}{\partial x} dx\right) dy dz &+ \left(v + \frac{\partial v}{\partial y} dy\right) \left(\rho + \frac{\partial \rho}{\partial y} dy\right) dz dx \\ &+ \left(w + \frac{\partial w}{\partial z} dz\right) \left(\rho + \frac{\partial \rho}{\partial z} dz\right) dx dy \end{aligned} \quad (2.5)$$

The mass accumulated in the element per unit time becomes

$$\frac{\partial}{\partial t}(\rho dx dy dz) \quad (2.6)$$

The mass balance is expressed by

$$(\text{Accumulated mass}) = (\text{entering mass}) - (\text{outgoing mass}) \quad (2.7)$$

Substituting Eqs. (2.4)–(2.6) into Eq. (2.7) yields

$$\begin{aligned} \frac{\partial}{\partial t}(\rho dx dy dz) = & \left[ -\rho \frac{\partial u}{\partial x} - u \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial u}{\partial x} dx \right] dx dy dz \\ & + \left[ -\rho \frac{\partial v}{\partial y} - v \frac{\partial \rho}{\partial y} - \frac{\partial \rho}{\partial y} \frac{\partial v}{\partial y} dy \right] dx dy dz \\ & + \left[ -\rho \frac{\partial w}{\partial z} - w \frac{\partial \rho}{\partial z} - \frac{\partial \rho}{\partial z} \frac{\partial w}{\partial z} dz \right] dx dy dz \end{aligned} \quad (2.8)$$

The third terms within the parenthesis on the right-hand side of Eq. (2.8) are negligibly small compared to the other terms and, hence, Eq. (2.8) reduces after dividing both sides by  $dx dy dz$  to

$$\frac{\partial \rho}{\partial t} = -\rho \frac{\partial u}{\partial x} - u \frac{\partial \rho}{\partial x} - \rho \frac{\partial v}{\partial y} - v \frac{\partial \rho}{\partial y} - \rho \frac{\partial w}{\partial z} - w \frac{\partial \rho}{\partial z} \quad (2.9)$$

The equation of continuity, Eq. (2.1), can readily be derived from this equation.

Batchelor [4] defined a streamline as a line in the fluid whose tangent is everywhere parallel to the velocity vector instantaneously. A stream tube is also defined as the surface formed instantaneously by all the streamlines at time  $t$ .

Streeter and Wylie [5] defined a streamline as a continuous line drawn through the fluid so that it has the direction of the velocity vector at every point. A stream tube is defined as the tube made by all the streamlines passing through a small, closed curve.

Irrespective of the definition, there is no through flow at the surface of a stream tube. Figure 2.2 shows a stream tube thus defined. The mass of a fluid entering the stream tube at a position 1 in time  $dt$  is expressed by

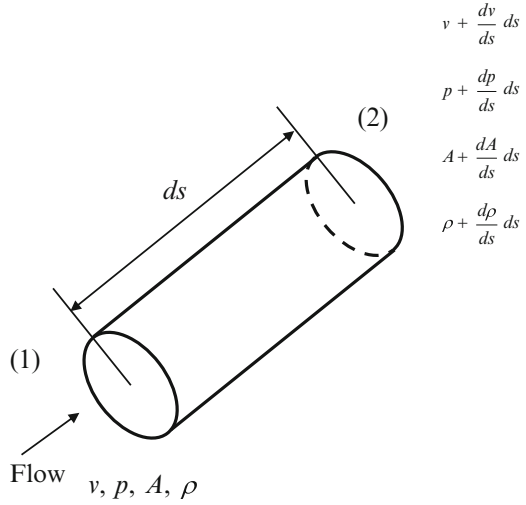
$$\rho v A dt \quad (2.10)$$

On the other hand, the mass of fluid flowing out of the tube at a position 2 is given by

$$\left( \rho + \frac{\partial \rho}{\partial s} ds \right) \left( v + \frac{\partial v}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right) dt \quad (2.11)$$

The mass of fluid accumulated in the tube in time  $dt$  becomes

**Fig. 2.2** Notations of velocity,  $v$ , pressure,  $p$ , cross-sectional area,  $A$ , and density,  $\rho$ , at the inlet and exit of a stream tube



$$\frac{\partial(\rho A ds)}{\partial t} dt \quad (2.12)$$

The mass balance in the stream tube is expressed by

$$\frac{\partial(\rho A ds)}{\partial t} dt = \rho v A dt - \left( \rho + \frac{\partial \rho}{\partial s} ds \right) \left( v + \frac{\partial v}{\partial s} ds \right) \left( A + \frac{\partial A}{\partial s} ds \right) dt \quad (2.13)$$

Rearranging Eq. (2.13) yields

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho v A)}{\partial s} = 0 \quad (2.14)$$

This equation is called the equation of continuity.

If a flow is steady ( $\partial \rho A / \partial t = 0$ ), Eq. (2.14) becomes

$$\rho v A = \text{const.} \quad (2.15)$$

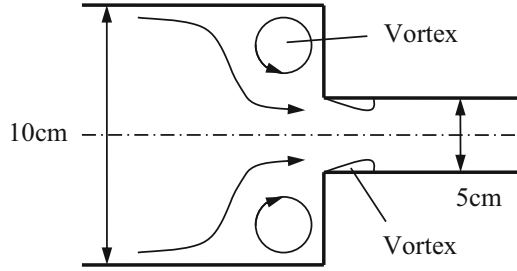
As the density of an incompressible fluid is constant, Eq. (2.14) reduces to

$$\frac{\partial A}{\partial t} + \frac{\partial(v A)}{\partial s} = 0 \quad (2.16)$$

If a flow is steady ( $\partial A / \partial t = 0$ ), Eq. (2.16) is further simplified to

$$v A = \text{const.} \quad (2.17)$$

**Fig. 2.3** Flow in an abrupt contraction



This relationship means that the flow rate  $Q (=vA)$  is kept constant in the flow direction in the stream tube when the flow is incompressible and steady.

In the case of real pipe flows, the no-slip condition holds on the pipe wall. It is therefore necessary to replace  $v$  in Eq. (2.17) by the cross-sectional mean velocity,  $v_m$ . The equation of continuity for compressible steady flow in a real pipe is therefore

$$\rho v_m A = \text{const.} \quad (2.18)$$

where  $\rho$  is the density of the fluid and  $A$  is the cross-sectional area of the pipe.

For an incompressible flow ( $\rho = \text{constant}$ ), Eq. (2.18) reduces to

$$v_m A = Q = \text{const.} \quad (2.19)$$

The flow rate,  $Q$ , therefore is kept constant in the flow direction in real pipes.

**Exercise 2.2** The diameter of a circular pipe,  $D$ , varies suddenly from 10.0 to 5.0 cm, as shown in Fig. 2.3. The cross-sectional mean velocity of a steady water flow in the larger pipe,  $v_{m1}$ , is 20 cm/s. Determine the cross-sectional mean velocity in the smaller pipe,  $v_{m2}$ .

**Answer to Exercise 2.2** The equation of continuity, Eq. (2.18), is rewritten as

$$v_{m1} A_1 = v_{m2} A_2 \quad (2.20)$$

$$A_1 = \pi D_1^2 / 4 \quad (2.21)$$

$$A_2 = \pi D_2^2 / 4 \quad (2.22)$$

Combining Eqs. (2.20)–(2.22) gives

$$v_{m2} = v_{m1} (D_1 / D_2)^2 \quad (2.23)$$

Substituting  $v_{m1} = 20$  cm/s,  $D_1 = 10.0$  cm, and  $D_2 = 5.0$  cm into Eq. (2.23) yields

$$v_{m2} = 20 \times (10.0/5.0)^2 = 80 \text{ cm/s} \quad (2.24)$$

### 2.1.2 Navier–Stoke’s Equation

The conservation of momentum for a compressible fluid flow was derived separately by Navier and Stokes. This equation therefore is usually called Navier–Stoke’s equation. In the Cartesian coordinate system the components of Navier–Stoke’s equation in the three directions for an incompressible fluid flow are expressed as follows:

*x* component

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + F_x \quad (2.25)$$

*y* component

$$\rho \left( \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + F_y \quad (2.26)$$

*z* component

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + F_z \quad (2.27)$$

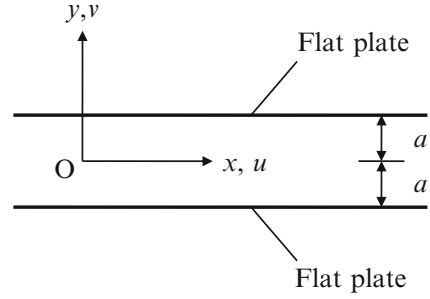
where  $p$  is the static pressure,  $\mu$  is the dynamic viscosity of fluid, and  $F_x$ ,  $F_y$ , and  $F_z$  are the external forces acting on the fluid element in the  $x$ ,  $y$ , and  $z$  directions, respectively.

**Exercise 2.3** Show that the conservation of momentum reduces to the following equation for steady incompressible laminar flow between two parallel plates depicted in Fig. 2.4. The flow is assumed to be fully developed and steady and external forces are negligible.

$$-\frac{dp}{dx} + \mu \frac{d^2 u}{dy^2} = 0 \quad (2.28)$$

**Solution to Exercise 2.3** Since the flow is fully developed, the following relationships hold:

**Fig. 2.4** Flow between two parallel flat plates



$$v = w = 0 \quad (2.29)$$

where  $v$  and  $w$  are the velocity components in the  $y$  and  $z$  directions, respectively.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} = 0 \quad (2.30)$$

Equation (2.30) implies that  $u$  is a sole function of  $y$ .

In addition, as the flow is steady and there is no external force,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = \frac{\partial w}{\partial t} = 0 \quad (2.31)$$

$$F_x = F_y = F_z = 0 \quad (2.32)$$

Substituting these relationships into Eqs. (2.25)–(2.27) yields

$x$  component

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0 \quad (2.33)$$

$y$  component

$$\frac{\partial p}{\partial y} = 0 \quad (2.34)$$

$z$  component

$$\frac{\partial p}{\partial z} = 0 \quad (2.35)$$

Equations (2.34) and (2.35) collectively imply that the pressure,  $p$ , is a sole function of  $x$ .

Since  $u$  is a sole function of  $y$  and  $p$  is a sole function of  $x$ , Eq. (2.33) reduces to the following ordinary differential equation:

$$-\frac{dp}{dx} + \mu \frac{d^2u}{dy^2} = 0 \quad (2.36)$$

In addition, the two terms in Eq. (2.36),  $-dp/dx$  and  $\mu d^2u/d^2y$ , must be constant. The above equation may also be rewritten as

$$\mu \frac{d^2u}{dy^2} = \frac{dp}{dx} \quad (2.37)$$

**Exercise 2.4** Solve Eq. (2.36) subject to the following boundary conditions:

$$u = 0 \text{ at } y = \pm a \quad (2.38)$$

$$\frac{du}{dy} = 0 \text{ at } y = 0 \quad (2.39)$$

where  $2a$  is the separation between the parallel passage.

**Answer to Exercise 2.4** As mentioned above, the pressure gradient term,  $-dp/dx$ , is constant. Integrating Eq. (2.36) with respect to the vertical distance,  $y$ , yields

$$\mu \frac{du}{dy} = \frac{dp}{dx} y + C_1 \quad (2.40)$$

where  $C_1$  is the integration constant. Equation (2.39) gives

$$C_1 = 0 \quad (2.41)$$

Further integration of Eq. (2.40) gives

$$\mu u = \frac{dp}{dx} \frac{y^2}{2} + C_2 \quad (2.42)$$

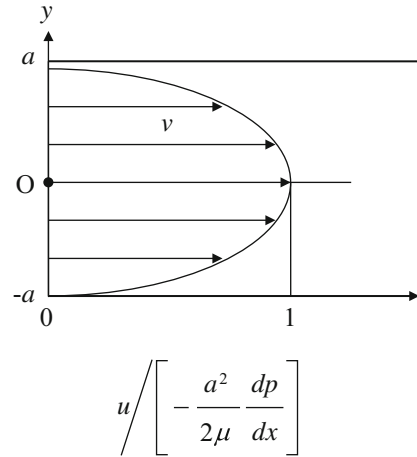
The integration constant,  $C_2$ , can be determined from Eq. (2.38) to be

$$C_2 = -\frac{dp}{dx} \frac{a^2}{2} \quad (2.43)$$

Substituting Eq. (2.43) into Eq. (2.42) yields

$$u = -\frac{a^2}{2\mu} \frac{dp}{dx} \left[ 1 - \left( \frac{y}{a} \right)^2 \right] \quad (2.44)$$

**Fig. 2.5** Parabolic velocity distribution in a passage between two parallel flat plates



Equation (2.44) is rewritten as

$$\frac{u}{\frac{a^2}{2\mu} \frac{dp}{dx}} = 1 - \left(\frac{y}{a}\right)^2 \quad (2.45)$$

Figure 2.5 shows that the velocity distribution is parabolic in shape.

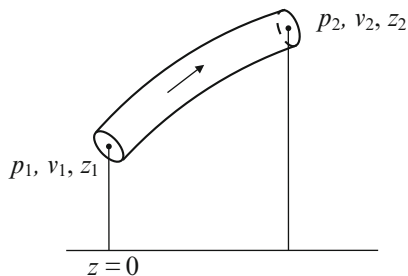
## 2.2 Inviscid Flow and Bernoulli Equation

An inviscid fluid is a medium in which viscous effects can be considered negligible. An example is the outer flow region beyond the boundary layer region above a flat plate. The following equation holds for an incompressible, inviscid, steady flow in a stream tube (see Fig. 2.6). A stream tube is an imaginary tube considered in a fluid flow through whose wall the fluid does not cross.

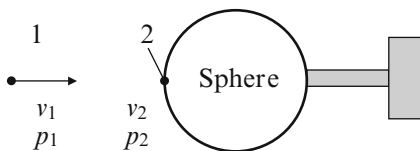
$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 \quad (2.46)$$

where  $p_1$  and  $v_1$  are the pressure and velocity at a location 1. The vertical distance from that position to a reference position is denoted by  $z_1$ , and  $\rho$  and  $g$  are the density and the acceleration due to gravity, respectively. The corresponding values at location 2 are denoted by  $p_2$ ,  $v_2$ , and  $z_2$ .

**Fig. 2.6** Flow in a stream tube



**Fig. 2.7** Pressure on the forward stagnation point of a sphere fixed in uniform approaching flow



In the real pipeline systems Eq. (2.46) is modified as follows:

$$\frac{p_1}{\rho g} + \frac{v_{m1}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_{m2}^2}{2g} + z_2 + h_L \quad (2.47)$$

where  $v_{m1}$  and  $v_{m2}$  are the cross-sectional mean velocities at the locations 1 and 2, respectively, and  $h_L$  is the hydrodynamic loss. The method of evaluating  $h_L$  will be presented in a subsequent section.

Figure 2.7 shows a solid sphere immersed in a fluid flowing at velocity  $V$ . Location 1 is upstream of the sphere and location 2 is at the forward stagnation point of the sphere. Assuming  $z_1 = z_2$ , the Bernoulli equation becomes

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} \quad (2.48)$$

Substituting  $p_1 = p_s$ ,  $v_1 = V$ ,  $p_2 = p_t$ , and  $v_2 = 0$  into Eq. (2.48) gives

$$p_t = p_s + \frac{1}{2} \rho V^2 \quad (2.49)$$

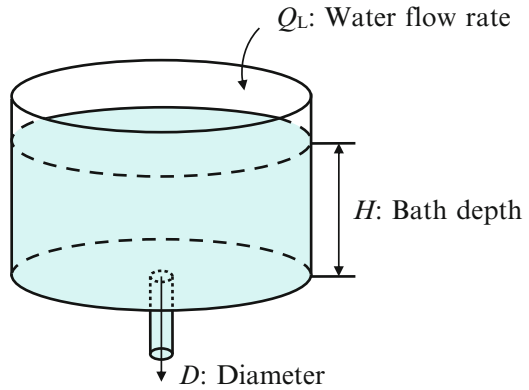
where  $p_t$ ,  $p_s$ , and  $\rho V^2/2$  are the total pressure, static pressure, and dynamic pressure,  $p_d$ , respectively.

**Exercise 2.5** Water flows at a velocity of 3.0 m/s. Calculate the dynamic pressure,  $p_d$ , by assuming that the density of water,  $\rho$ , is 998 kg/m<sup>3</sup>.

**Answer to Exercise 2.5** The dynamic pressure,  $p_d$ , is given by

$$p_d = \rho V^2/2 = 998 \times 3.0 \times 3.0/2 = 4.49 \text{ kPa} \quad (2.50)$$

**Fig. 2.8** Liquid discharge from a tank



**Exercise 2.6** Water is poured into a very large cylindrical vessel at a flow rate,  $Q_L$ , of  $2.0 \times 10^{-2} \text{ m}^3/\text{s}$  (see Fig. 2.8). The water issues through a hole at the bottom of the vessel. The diameter of the hole,  $D$ , is 5.0 cm. Calculate the depth of water,  $H$ , in the vessel at steady state by assuming that the hydrodynamic loss is negligible.

**Answer to Exercise 2.6** Consider the flow between location 1 on the water surface and the location 2 at the exit of the bottom hole. The Bernoulli law without hydrodynamic loss is expressed by

$$\frac{p_1}{\rho g} + \frac{v_{m1}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_{m2}^2}{2g} + z_2 \quad (2.51)$$

Since the vessel is large, the falling velocity of the water surface,  $v_{m1}$ , is negligibly small,  $z_1$  is equivalent to  $H$ , and  $z_2 = 0$ . The pressures,  $p_1$  and  $p_2$ , are equal to the atmospheric pressure,  $p_0$ . Equation (2.51) reduces to

$$H = v_{m2}^2 / (2g) \quad (2.52)$$

The cross-sectional mean velocity,  $v_{m2}$ , is

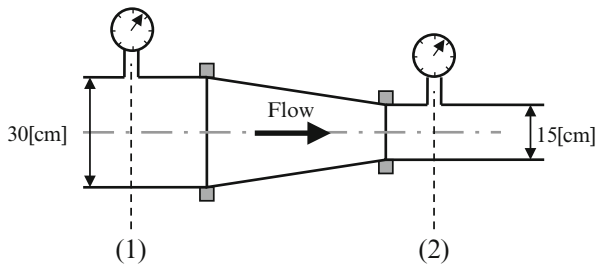
$$\begin{aligned} v_{m2} &= Q_L / (\pi D^2 / 4) = 2.0 \times 10^{-2} \times 4 / [3.14 \times (5.0 \times 10^{-2})^2] \\ &= 10.2 \text{ m/s} \end{aligned} \quad (2.53)$$

Substituting Eq. (2.53) into Eq. (2.52) yields

$$H = (10.2)^2 / (2 \times 9.80) = 5.31 \text{ m} \quad (2.54)$$

**Exercise 2.7** Water flows in a horizontal circular pipe from location 1 to location 2, as shown in Fig. 2.9. The cross-sectional mean velocity at location 1 is 3.0 m/s, and the density of water is  $1,000 \text{ kg/m}^3$ . Calculate the difference between the gauge

**Fig. 2.9** Flow in a converging pipe



pressure at location 1,  $p_1$ , and that at location 2,  $p_2$ . The hydrodynamic loss is negligible.

**Answer to Exercise 2.7** The cross-sectional mean velocity at location 2,  $v_{m2}$ , can be calculated from the equation of continuity, thus:

$$v_{m2} = v_{m1}(A_1/A_2) = v_{m1}(D_1/D_2)^2 = 3.0 \times (0.30/0.15)^2 = 3.0 \times 4 = 12 \text{ m/s} \quad (2.55)$$

The Bernoulli equation without hydrodynamic loss is

$$\frac{p_1}{\rho g} + \frac{v_{m1}^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{v_{m2}^2}{2g} + z_2 \quad (2.56)$$

Since the pipe is placed horizontally,

$$z_1 = z_2 \quad (2.57)$$

Substituting Eq. (2.57) into Eq. (2.56) gives

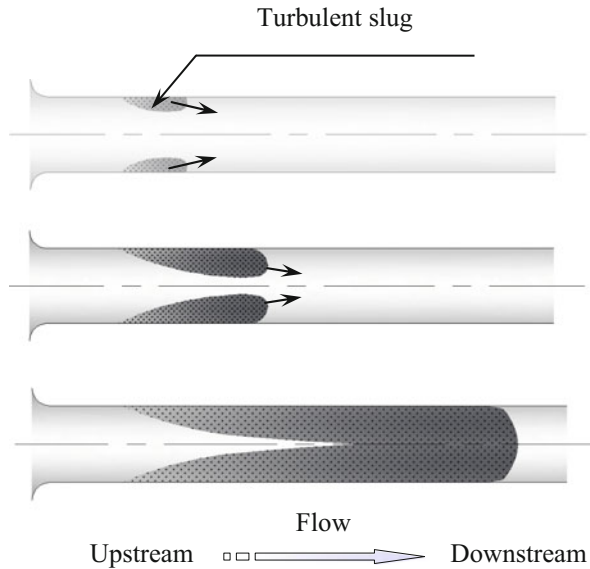
$$p_1 - p_2 = \rho(v_{m2}^2 - v_{m1}^2)/2 = 1,000 \times (144 - 9)/2 = 6.75 \times 10^4 \text{ Pa} \quad (2.58)$$

## 2.3 Flow Regime

### 2.3.1 Laminar–Turbulent Transition

More than 100 years ago, Reynolds systematically investigated flows in a circular pipe and observed that two types of patterns (laminar and turbulent flows) appear with respect to the pipe diameter,  $D$ , cross-sectional mean velocity,  $v_m$ , and the kinematic viscosity of fluid,  $\nu_f (= \mu/\rho)$ . Finally, he derived the following dimensionless number to discriminate the two flow patterns:

**Fig. 2.10** Generation and propagation of a turbulent slug in a circular pipe



$$Re = v_m D / \nu_f \quad (2.59)$$

Later, this number was named the Reynolds number in honor of the excellent work of Reynolds. The kinematic viscosity is a measure of the momentum transfer rate in a fluid flow. The Reynolds number in effect is a measure of the ratio of inertial force to viscous force in the fluid flow.

Turbulent flow is characterized by many vortices of different sizes. These vortices are believed to be generated due to the bursting phenomenon. The bursting phenomenon is characterized by two typical motions called the sweep and ejection. The critical Reynolds number for transition from laminar flow to turbulent flow in a smooth circular pipe is

$$Re_c = 2,320 \quad (2.60)$$

Even if the disturbance in an incoming flow is very strong, the flow in the pipe is maintained laminar for  $Re < 2,320$ . This critical Reynolds number therefore is called the lower critical Reynolds number.

On the other hand, when the disturbance is very weak, laminar flow is maintained until the Reynolds number reaches another critical value greater than 2,320. This critical value is dependent on the magnitude of the disturbance and called the upper critical Reynolds number.

The lower critical Reynolds number in a noncircular pipe is different from that in a circular pipe and dependent on the shape of the cross section.

In the transition of Reynolds number range a turbulent slug appears almost periodically near the entrance of the pipe and then propagates in the downstream direction, as shown in Fig. 2.10.

**Exercise 2.8** Water with density,  $\rho$ , of  $997 \text{ kg/m}^3$  and kinematic viscosity,  $\nu_f$ , of  $1.00 \times 10^{-6} \text{ m}^2/\text{s}$  flows in a circular pipe  $5.0 \text{ cm}$  in diameter,  $D$ , at a flow rate,  $Q_w$ , of  $2.0 \times 10^{-2} \text{ m}^3/\text{min}$ . Calculate the Reynolds number,  $Re$ .

**Answer to Exercise 2.8** The cross-sectional mean velocity,  $v_m$ , is given by

$$v_m = 4Q_w/(\pi D^2) = 4 \times (2.0 \times 10^{-2}/60)/(3.14 \times 0.050 \times 0.050) = 0.170 \text{ m/s} \quad (2.61)$$

Accordingly, the Reynolds number is given by

$$Re = v_m D/\nu_f = 0.170 \times 0.050/(1.00 \times 10^{-6}) = 8.5 \times 10^3 \quad (2.62)$$

### 2.3.2 Reverse Transition

A turbulent flow subjected to high spatial acceleration may become laminar. This phenomenon is called the reverse transition or relaminarization. This situation is observed, for example, in a nozzle with very high contraction ratio.

### 2.3.3 Instantaneous Velocity

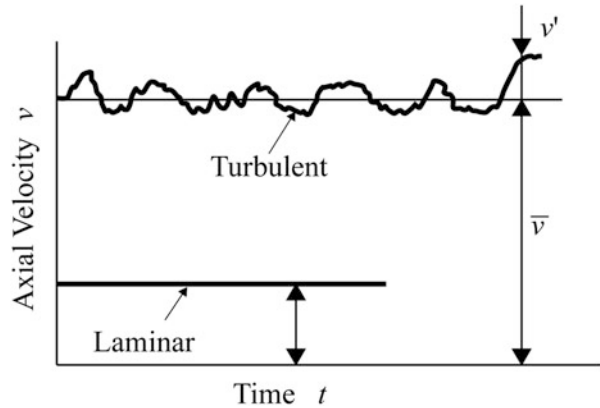
Velocities typical of steady laminar and turbulent flows are schematically shown in Fig. 2.11. The velocity at a point in a laminar flow is independent of time, while that in a turbulent flow fluctuates around a mean value of  $\bar{v}$ . This is because of the many vortices that are present in the turbulent flow. As the flow is steady,  $\bar{v}$  is constant. The deviation of the instantaneous velocity,  $v$ , from its mean value,  $\bar{v}$ , is designated the turbulent or fluctuating component,  $v'$ .

## 2.4 Flow in a Pipe

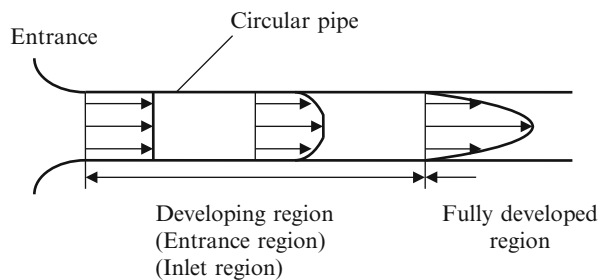
### 2.4.1 Flow Development

The flow velocity distribution in a pipe varies in the axial direction until the boundary layer developing on the pipe wall reaches the centerline of the pipe (see Fig. 2.12). The axial regions upstream and downstream of this position are called the developing region and the fully developed region, respectively. The developing region is sometimes called the entrance region or the inlet region.

**Fig. 2.11** Velocity signals in laminar and turbulent flows



**Fig. 2.12** Flow field near the entrance of a circular pipe



The entrance length,  $L_e$ , is introduced to quantitatively determine the extent of the developing region. It is defined as the distance from the entrance of the pipe to the axial position at which the centerline velocity,  $v_{cl}$ , becomes  $0.99v_{cl,fd}$ , where  $v_{cl,fd}$  is the centerline velocity in the fully developed region. The entrance length of a flow in a circular pipe is given by

$$L_e = 0.05DRe \quad (\text{laminar}) \quad (2.63)$$

$$L_e = 50D \quad (\text{turbulent}) \quad (2.64)$$

where  $Re$  is the Reynolds number expressed by Eq. (2.59). When the flow is turbulent, Eq. (2.64) is approximately valid for noncircular pipes such as a rectangular pipe. In the latter case  $D$  is interpreted as the hydraulic diameter defined subsequently in Sect. 2.4.5.

**Exercise 2.9** Silicone oil of density  $936 \text{ kg/m}^3$  and kinematic viscosity  $100 \text{ mm}^2/\text{s}$  is flowing in a horizontal circular pipe of inner diameter,  $D$ , of  $1.0 \text{ cm}$ . The flow rate,  $Q_L$ , is  $1.5 \times 10^{-3} \text{ m}^3/\text{min}$ . Estimate the entrance length,  $L_e$ .

**Answer to Exercise 2.9** The cross-sectional mean velocity,  $v_m$ , is given by

$$\begin{aligned} v_m &= 4Q_L/(\pi D^2) = 4 \times (1.5 \times 10^{-3}/60) / \left[ 3.14 \times (1.0 \times 10^{-2})^2 \right] \\ &= 0.318 \text{ m/s} \end{aligned} \quad (2.65)$$

The Reynolds number is calculated from Eq. (2.59) to give

$$Re = v_m D / \nu_f = 0.318 \times 1.0 \times 10^{-2} / (100 \times 10^{-6}) = 31.8 \quad (2.66)$$

This Reynolds number is much smaller than the critical Reynolds number of 2,320. Accordingly, the flow is laminar. Equation (2.63) therefore yields

$$L_e = 0.05 D Re = 0.05 \times 1.0 \times 10^{-2} \times 31.8 = 0.016 \text{ m} \quad (2.67)$$

## 2.4.2 Velocity Distribution

### 2.4.2.1 Laminar Pipe Flow

The velocity distribution of laminar flow in a circular pipe is expressed by

$$\frac{v}{v_{cl}} = 1 - \left( \frac{r}{R} \right)^2 \text{ (laminar)} \quad (2.68)$$

where  $v$  is the axial velocity component,  $v_{cl}$  is the centerline value of  $v$ ,  $r$  is the radial distance, and  $R$  is the radius of the pipe (see Fig. 2.13).

The relationship between the cross-sectional mean velocity,  $v_m$ , and the centerline velocity,  $v_{cl}$ , is given by

$$v_m = v_{cl}/2 \quad (2.69)$$

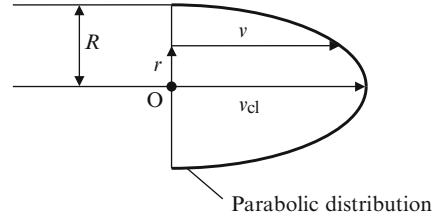
### 2.4.2.2 Turbulent Pipe Flow

The following velocity distribution holds when flow in a circular pipe is turbulent, as shown in Fig. 2.14;

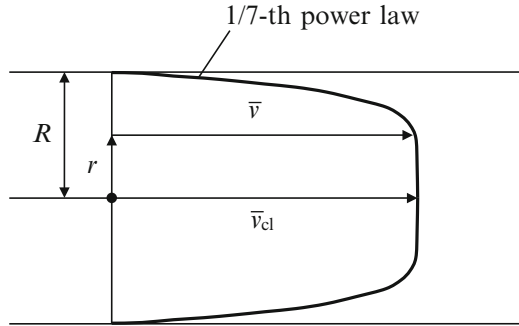
$$\frac{\bar{v}}{\bar{v}_{cl}} = \left( 1 - \frac{r}{R} \right)^{1/7} \text{ (turbulent)} \quad (2.70)$$

This velocity distribution is termed the 1/7th power law and it is valid for  $3,000 < Re < 10^5$ .

**Fig. 2.13** Parabolic velocity distribution in a circular pipe



**Fig. 2.14** 1/7th-Power law velocity distribution in a circular pipe



The following  $1/n$ th power law is valid over a wider Reynolds number range than the  $1/7$ th power law:

$$\frac{\bar{v}}{\bar{v}_{cl}} = \left(1 - \frac{r}{R}\right)^{1/n} \text{ (turbulent)} \quad (2.71)$$

$$n = 2.0 \log_{10}(Re/10) \quad (3,000 < Re < 10^6) \quad (2.72)$$

These two velocity distributions can satisfactorily approximate the measured distributions in the central part of a circular pipe. However, the accuracy of approximation is insufficient near the pipe wall. Accordingly, the following logarithmic velocity distributions have been proposed (see Fig. 2.15):

$$\frac{\bar{v}}{v_*} = y_* \quad (y_* \leq 5) \quad (2.73a)$$

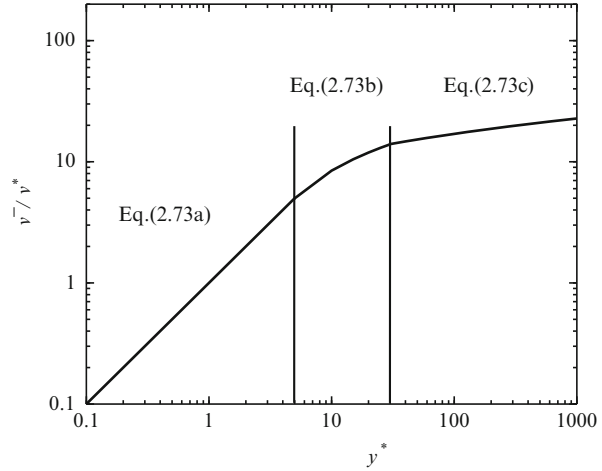
$$\frac{\bar{v}}{v_*} = 11.5 \log_{10} y_* - 3.05 \quad (5 < y_* < 70) \quad (2.73b)$$

$$\frac{\bar{v}}{v_*} = 5.75 \log_{10} y_* + 5.5 \quad (70 \leq y_*) \quad (2.73c)$$

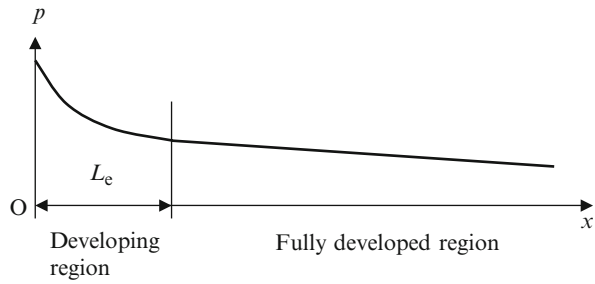
$$v_* = (\tau_w/\rho)^{1/2} \quad (2.74)$$

$$y_* = \frac{yv_*}{v_f} \quad (2.75)$$

**Fig. 2.15** Logarithmic velocity distribution in a circular pipe



**Fig. 2.16** Pressure distribution near the entrance of a circular pipe



where  $v^*$  is the friction (or shear) velocity,  $\tau_w$  is the wall shear stress, and  $y$  ( $=R - r$ ) is the distance from the wall.

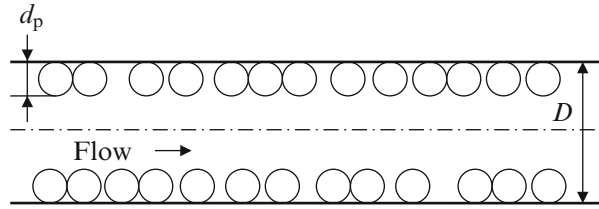
### 2.4.3 Pressure Drop

The static pressure in a circular pipe first decreases rapidly in the developing region with axial distance from the entrance of the pipe and then decreases linearly with axial distance in the fully developed region (see Fig. 2.16). The pressure drop,  $\Delta p$ , in the fully developed region is expressed by

$$\Delta p = \lambda \frac{L}{D} \frac{\rho v_m^2}{2} \quad (2.76)$$

where  $\lambda$  is the friction coefficient,  $L$  is the pipe length,  $D$  is the pipe diameter,  $\rho$  is the density of the fluid, and  $v_m$  is the cross-sectional mean velocity.

**Fig. 2.17** Sand roughness in a circular pipe



#### 2.4.4 Friction Coefficient, $\lambda$

It is of practical significance to estimate the pressure drop in circular and noncircular pipes. Expressions are introduced in this chapter for friction coefficients in the fully developed region of laminar and turbulent pipe flows.

##### 2.4.4.1 Pipe Roughness

The inner wall of a circular pipe used over a prolonged time for flow delivery is known to be quite rough. The wall roughness affects the pressure drop significantly for turbulent pipe flows. It is therefore necessary to distinguish between smooth and rough pipes. The following dimensionless number is employed to judge whether the wall of a circular pipe is smooth or rough:

$$v^* d_p / \nu_f = 70 \quad (2.77)$$

where

$$v^* = (\tau_w / \rho)^{1/2} \quad (2.78)$$

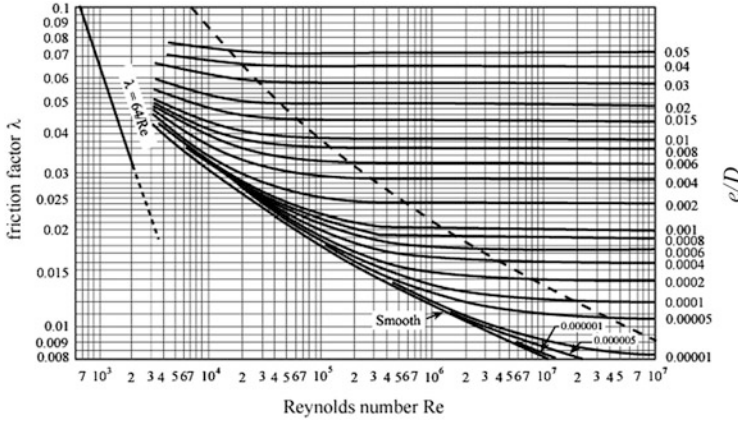
is the friction velocity,  $d_p$  is the roughness height,  $\nu_f$  is the kinematic viscosity of fluid,  $\tau_w$  is the wall shear stress, and  $\rho$  is the density of fluid (see Fig. 2.17). The wall of a pipe is smooth for  $v^* d_p / \nu_f \leq 70$ , while it is rough for  $v^* d_p / \nu_f > 70$ .

##### 2.4.4.2 Friction Coefficient in a Smooth Circular Pipe

###### Laminar Pipe Flow

When a flow in a circular pipe is laminar, the friction coefficient,  $\lambda$ , is given by

$$\lambda = 64 / Re \quad (2.79)$$



**Fig. 2.18** Moody diagram

### Turbulent Pipe Flow

Many empirical equations have been proposed for turbulent flow in a circular pipe. Some of these equations are listed below:

$$\text{Blasius : } \lambda = 0.3164Re^{-1/4} \quad (2,320 < Re < 10^5) \quad (2.80)$$

$$\text{Nikuradse : } \lambda = 0.0032 + 0.221Re^{-0.237} \quad (10^5 < Re < 10^8) \quad (2.81)$$

$$\text{Prandtl-Karman-Nikuradse : } 1/\lambda = [2\log_{10}(Re\lambda^{1/2}) - 0.8]^2 \quad (2.82)$$

#### 2.4.4.3 Friction Coefficient in a Rough Circular Pipe

Colebrook proposed the following empirical equation for turbulent flow in rough pipes:

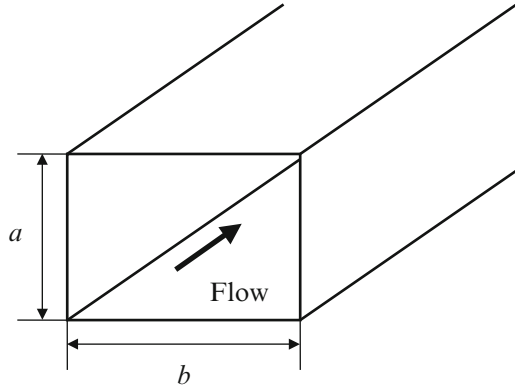
$$1/\lambda = [2.0\log_{10}\{D/(2e)\} + 1.74]^2 \quad (2.83)$$

where  $D$  is the pipe diameter and  $e$  is the height of the roughness element. The friction coefficient,  $\lambda$ , in rough pipes is given in Fig. 2.18, known as the Moody diagram.

#### 2.4.4.4 Friction Coefficient in a Rectangular Duct

The friction coefficient of laminar flow in a smooth rectangular duct (see Fig. 2.19) is expressed by

**Fig. 2.19** Flow in a rectangular duct



$$\lambda Re = 96(1 - 1.3553a^* + 1.9467\alpha^{*2} - 1.7012\alpha^{*3} + 0.9564\alpha^{*4} - 0.2537\alpha^{*5}) \quad (2.84)$$

$$a^* = a/b \quad (2.85)$$

#### 2.4.4.5 Friction Coefficient for Laminar Flow in Smooth Concentric Annuli (Fig. 2.20)

$$\lambda Re = 64(1 - r^*)^2 / [1 + r^{*2} - 2r_m^{*2}] \quad (2.86)$$

$$r^* = d/D \quad (2.87)$$

$$r_m^* = [(1 - r^{*2}) / \{2 \ln(1/r^*)\}]^{1/2} \quad (2.88)$$

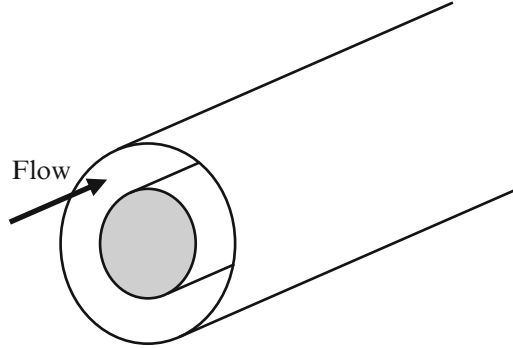
$$Re = v_m D_h / \nu \quad (2.89)$$

$$D_h = D - d \quad (2.90)$$

where  $d$  and  $D$  denote the inner and outer diameters, respectively, and  $D_h$  is the hydraulic diameter defined in the next section.

**Exercise 2.10** Silicone oil with kinematic viscosity  $\nu_f = 100 \text{ mm}^2/\text{s}$  flows in a smooth pipe of diameter  $D = 10 \text{ mm}$  (see Fig. 2.21). The cross-sectional mean velocity,  $v_m$ , is  $20 \text{ mm/s}$  and the critical Reynolds number,  $Re_c$ , is 2,320. Calculate the frictional coefficient,  $\lambda$ .

**Fig. 2.20** Flow in a concentric annuli (outer and inner diameters are denoted by  $D$  and  $d$ , respectively)



**Fig. 2.21** Flow in a circular pipe



**Answer to Exercise 2.10** The Reynolds number,  $Re$ , is given by

$$Re = v_m D / \nu_f = 20 \times 10^{-3} \times 10 \times 10^{-3} / (100 \times 10^{-6}) = 2.0 \quad (2.91)$$

Since the flow is laminar, the friction coefficient,  $\lambda$ , can be calculated from Eq. (2.79) as follows:

$$\lambda = 64 / Re = 64 / 2.0 = 32 \quad (2.92)$$

**Exercise 2.11** Water with kinematic viscosity  $\nu_f = 0.892 \text{ mm}^2/\text{s}$  flows in a smooth pipe of diameter  $D = 50 \text{ mm}$ . The cross-sectional mean velocity,  $v_m$ , is  $1.50 \text{ m/s}$ . Calculate the friction coefficient,  $\lambda$ .

**Answer to Exercise 2.11** The Reynolds number,  $Re$ , is given by

$$Re = v_m D / \nu_f = 1.50 \times 50 \times 10^{-3} / (0.892 \times 10^{-6}) = 8.41 \times 10^4 \quad (2.93)$$

The friction coefficient,  $\lambda$ , is calculated from the Blasius equation to give

$$\lambda = 0.3164 Re^{-1/4} = 0.3164 (8.41 \times 10^4)^{-1/4} = 0.0186 \quad (2.94)$$

**Exercise 2.12** Compare the friction coefficients calculated from the correlations proposed by Blasius, Nikuradse, and Prandtl–Karman–Nikuradse for Reynolds number,  $Re$ , of  $1.0 \times 10^5$ .

**Answer to Exercise 2.12** Blasius:

$$\lambda = 0.3164Re^{-1/4} = 0.3164 \times (1.0 \times 10^5)^{-1/4} = 0.0178 \quad (2.95)$$

Nikuradse:

$$\lambda = 0.0032 + 0.221Re^{-0.237} = 0.0032 + 0.221 \times (1.0 \times 10^5)^{-0.237} = 0.0176 \quad (2.96)$$

Prandtl–Karman–Nikuradse:

The root of Eq. (2.82) can be numerically obtained to give

$$\lambda = 0.0180 \quad (2.97)$$

### 2.4.5 Hydraulic Diameter, $D_h$

Compared to flows in a circular pipe, information on the friction coefficient in noncircular pipes is limited. In order to apply the friction coefficient expressions proposed originally for circular pipes to noncircular pipes, the pipe diameter is replaced by the following hydraulic diameter,  $D_h$ , defined thus:

$$D_h = \frac{4A}{P} \quad (2.98)$$

where  $A$  and  $P$  are the cross-sectional area and the peripheral length (or wetted perimeter) of the noncircular pipe, respectively (see Fig. 2.22).

**Exercise 2.13** Obtain the hydraulic diameter for a rectangular pipe shown in Fig. 2.23. The side lengths of the pipe are denoted by  $a$  and  $b$ , respectively.

**Answer to Exercise 2.13** The cross-sectional area,  $A$ , and the peripheral length,  $P$ , are respectively given as follows:

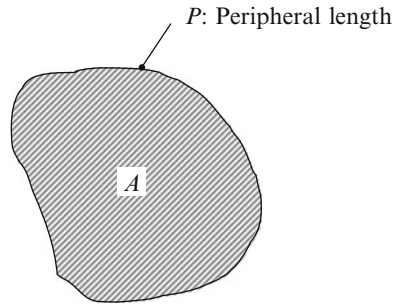
$$A = ab \quad (2.99)$$

$$P = 2(a + b) \quad (2.100)$$

Substituting Eqs. (2.99) and (2.100) into Eq. (2.98) yields

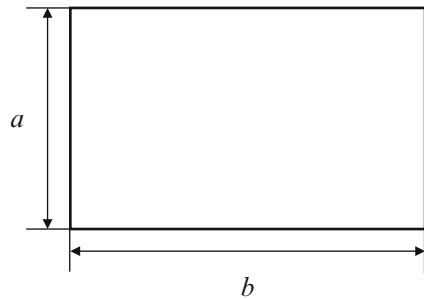
$$D_h = \frac{4A}{P} = \frac{4ab}{2(a + b)} = \frac{2ab}{a + b} \quad (2.101)$$

**Fig. 2.22** Cross section of a non-circular duct



*A*: Cross-sectional area

**Fig. 2.23** Cross section of a rectangular duct



**Exercise 2.14** Obtain the hydraulic diameter for a triangular pipe (see Fig. 2.24). The length of each side of the pipe is  $a$ .

**Answer to Exercise 2.14**

$$A = \frac{\sqrt{3}a^2}{4} \quad (2.102)$$

$$P = 3a \quad (2.103)$$

$$D_h = \frac{4A}{P} = \frac{4\sqrt{3}a^2/4}{3a} = \frac{\sqrt{3}a}{3} \quad (2.104)$$

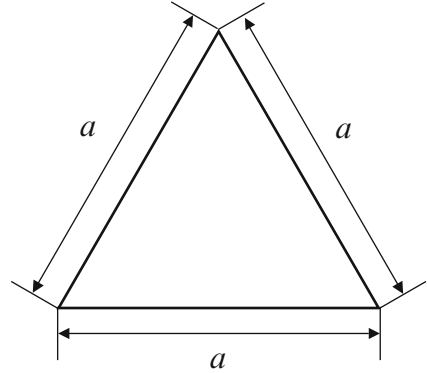
**Exercise 2.15** Obtain the hydraulic diameter of a passage between two parallel plates. The clearance of the passage is  $a$ .

**Answer to Exercise 2.15** Taking the limit of  $b$  ( $b \rightarrow \infty$ ) in Eq. (2.101) gives  $D_h = 2a$ .

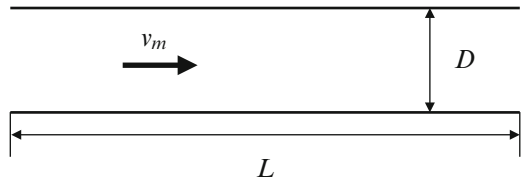
**Exercise 2.16** Verify that  $D_h = D$  for a circular pipe.

**Answer to Exercise 2.16** Substituting  $A = \pi D^2/4$  and  $P = \pi D$  into Eq. (2.98) yields  $D_h = D$ .

**Fig. 2.24** Cross section of a triangular duct



**Fig. 2.25** Flow in a circular pipe ( $D$ : diameter,  $L$ : length)



**Exercise 2.17** Water flows in a horizontal circular pipe of inner diameter,  $D$ , of 45 mm and length,  $L$ , of 250 m, as shown in Fig. 2.25. The temperature of the water is 20 °C and the flow rate,  $Q_L$ , is  $1.23 \times 10^{-3} \text{ m}^3/\text{s}$ . Obtain the following quantities by considering that the hydrodynamic loss is caused only by friction:

- Cross-sectional mean velocity,  $v_m$ .
- Reynolds number,  $Re$ .
- Determine whether the flow is laminar or turbulent. The critical Reynolds number is 2,320.
- Frictional coefficient,  $\lambda$  (see Fig. 2.18).
- Pressure loss,  $\Delta p$ .

**Answer to Exercise 2.17**

$$(a) \quad v_m = Q/A = 1.23 \times 10^{-3} / [\pi \times (0.045)^2 / 4] = 0.774 \text{ m/s} \quad (2.105)$$

$$(b) \quad Re = v_m D / \nu_f = 0.774 \times 0.045 / (1.00 \times 10^{-6}) = 3.48 \times 10^4 \quad (2.106)$$

(c) Turbulent since  $Re > 2,320$

$$(d) \quad \lambda = 0.3164 Re^{-1/4} = 0.3164 (3.48 \times 10^4)^{-1/4} = 0.0232 \quad (2.107)$$

$$(e) \quad \Delta p = \lambda (L/D) \rho v_m^2 / 2 = 0.0232 \times (250/0.045) \times 997 \times (0.774)^2 / 2 = 3.85 \times 10^4 \text{ Pa} \quad (2.108)$$

### 2.4.6 Pipe Elements and Loss Coefficient, $\zeta$

#### 2.4.6.1 Pipe Elements

There are many types of pipe elements in pipeline systems. The common elements in real pipeline systems are briefly described below:

*Sudden expansion:* The cross-sectional area of the pipe increases suddenly, as shown in Fig. 2.26.

*Sudden contraction:* The cross-sectional area of the pipe decreases suddenly (see Fig. 2.27).

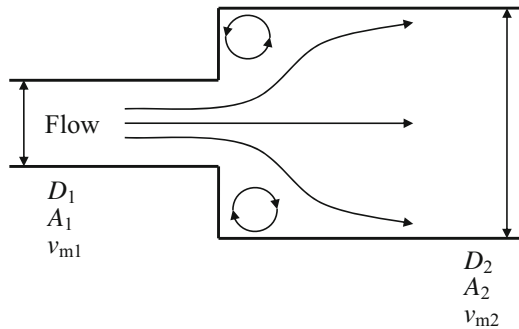
*Diffuser:* The cross-sectional area of the pipe increases gradually in subsonic flow (see Fig. 2.28).

*Nozzle:* The cross-sectional area of the pipe decreases gradually in subsonic flow (see Fig. 2.29).

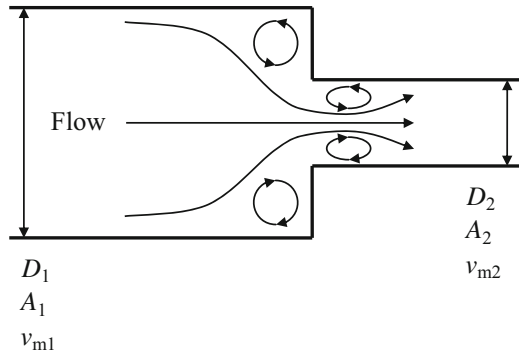
*Bend:* The pipe bends gradually (see Fig. 2.30).

*Elbow:* The pipe bends suddenly (see Fig. 2.31).

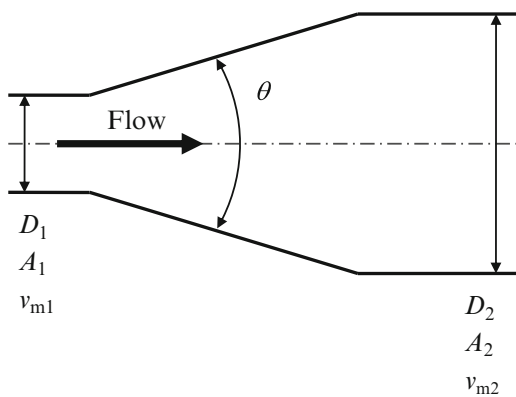
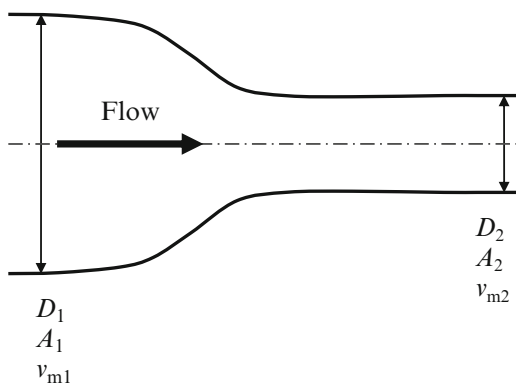
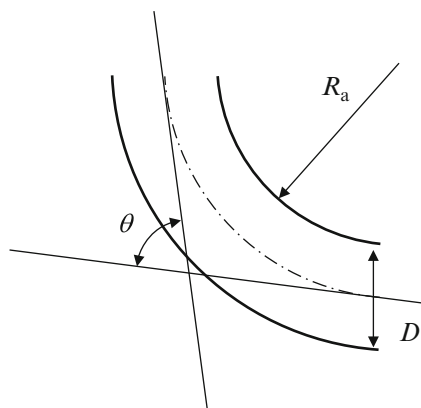
*Orifice:* The cross-sectional area of pipe is locally decreased using a hollow plate (see Fig. 2.32).

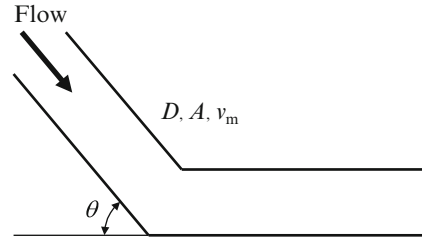
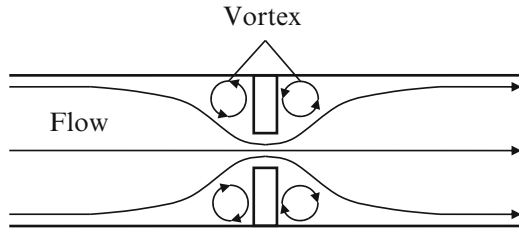
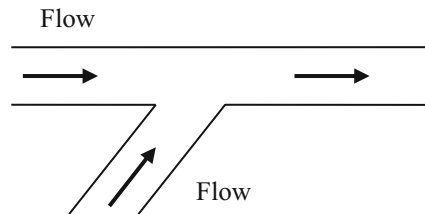
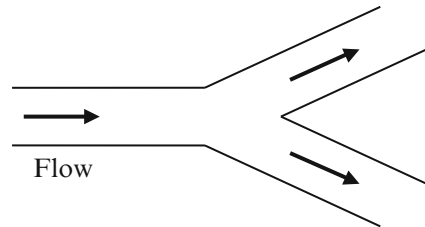


**Fig. 2.26** Flow in a sudden expansion



**Fig. 2.27** Flow in a sudden contraction

**Fig. 2.28** Flow in a diffuser**Fig. 2.29** Flow in a nozzle**Fig. 2.30** Flow in a bend

**Fig. 2.31** Flow in an elbow**Fig. 2.32** Flow through a plate orifice**Fig. 2.33** Flow in a combining junction**Fig. 2.34** Flow in a dividing junction

*Combining junction:* More than two pipes are connected to a single pipe (see Fig. 2.33).

*Dividing junction:* A single pipe is divided into plural pipes (see Fig. 2.34).

#### 2.4.6.2 Definition of Loss Coefficient $\zeta$

The loss coefficient,  $\zeta$ , is defined in the following manner:

$$\Delta p = \rho g h_L \quad (2.109)$$

where  $\Delta p$  is the pressure drop,  $\rho$  is the density of fluid,  $g$  is the acceleration due to gravity, and  $h_L$  is the head loss.

$$h_L = \zeta v_m^2 / (2g) \quad (2.110)$$

### 2.4.6.3 Loss Coefficient of Pipe Elements

#### Sudden Expansion

$$h_L = \zeta v_{m1}^2 / (2g) \quad (2.111)$$

$$\zeta = \xi(1 - m)^2 \quad (2.112)$$

$$m = A_1 / A_2 \quad (2.113)$$

where  $v_{m1}$  is the cross-sectional mean velocity in the smaller pipe upstream of the sudden expansion,  $m$  is the expansion ratio,  $\xi$  is the coefficient and usually assumed to be unity, and  $A_1$  and  $A_2$  are the cross-sectional areas of the upstream and downstream pipes, respectively.

**Exercise 2.18** The smaller and larger diameters of a sudden expansion,  $D_1$  and  $D_2$ , are 5.0 and 10.0 cm, respectively. Estimate the loss coefficient of the sudden expansion by assuming that the coefficient,  $\xi$ , in Eq. (2.112) is unity.

**Answer to Exercise 2.18** The area ratio,  $m$ , is expressed by

$$m = A_1 / A_2 = (D_1 / D_2)^2 = (5.0 / 10.0)^2 = 1/4 = 0.25 \quad (2.114)$$

The loss coefficient is calculated from Eq. (2.113) to give

$$\zeta = \xi(1 - m)^2 = 1 \times (1 - 0.25)^2 = 0.5625 \quad (2.115)$$

#### Sudden Contraction

$$h_L = \zeta v_{m2}^2 / (2g) \quad (2.116)$$

$$\zeta = (1/C_a - 1)^2 \quad (2.117)$$

where  $v_{m2}$  is the cross-sectional mean velocity in the downstream (smaller) pipe and  $C_a$  is the contraction coefficient. Merriman's equation is commonly used for evaluating  $C_a$ , thus [6]:

$$C_a = 0.582 + \frac{0.048}{1.1 - D_2/D_1} \quad (2.118)$$

Diffuser

$$h_L = \xi(v_{m1} - v_{m2})^2/(2g) \quad (2.119)$$

$$\xi = 3.50(\tan \theta/2)^{1.22} \quad (2.120)$$

where  $\theta$  is the angle of expansion ( $^\circ$ ).

Nozzle

$$h_L = 0.04v_{m2}^2/(2g) \quad (2.121)$$

The loss coefficient of a nozzle is very small compared to that of sudden contraction. This means that the generation of vortex (or vortices) downstream of the corner (or corners) significantly affects the loss coefficient.

Bend

$$h_L = \zeta v_m^2/(2g) \quad (2.122)$$

$$\zeta = \left[ 0.131 + \frac{1.84}{[D/(2R_a)]^{3.5}} \right] \frac{\theta}{90} \quad (2.123)$$

where  $R_a$  is the radius of curvature of the bend and  $\theta$  is the deflection angle ( $^\circ$ ), as shown in Fig. 2.30.

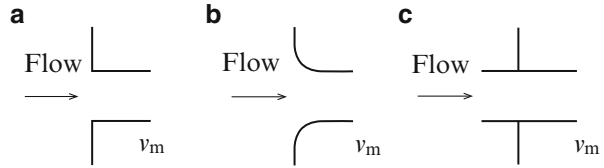
Elbow

$$h_L = \zeta v_m^2/(2g) \quad (2.124)$$

$$\zeta = 0.946\sin^2(\theta/2) + 2.05\sin^4(\theta/2) \quad (2.125)$$

where  $\theta$  denotes the deflection angle ( $^\circ$ ), as shown in Fig. 2.31.

**Fig. 2.35** Flow entering various types of entrances (a) Sharp corner (b) Round corner (c) Borda mouthpiece



### Orifice

$$h_L = \zeta v_m^2 / (2g) \quad (2.126)$$

$$\zeta = [1 / (C_a m) - 1]^2 \quad (2.127)$$

$$m = A_o / A \quad (2.128)$$

where  $A_o$  is the cross-sectional area of the orifice plate, and  $A$  is the cross-sectional area of the pipe. The contraction coefficient,  $C_a$ , is given, for example, by JIS (Japanese Industrial Standard). An orifice is widely used as a flow meter. The accuracy is quite excellent in spite of the simple design.

### Pipe Entrance

Three pipe entrance configurations that are typical are shown in Fig. 2.35. The loss coefficient values for configurations (a), (b), and (c) are approximately 0.50, 0.05, and 0.56, respectively. It should be noted that the loss coefficient of the pipe entrance depends strongly on the shape of the entrance. The loss coefficient is closely associated with the generation of vortices around the corners.

### Pipe Exit

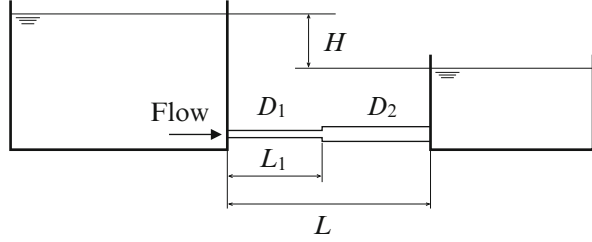
The velocity head of fluid issuing out of the pipe exit is regarded as an energy loss, since the fluid is expanding into a seemingly unbounded infinite medium,

$$h_L = v_m^2 / (2g) \quad (2.129)$$

### Combining Junction and Dividing Junction

Figure 2.33 shows that more than two pipes merge into a single pipe. This configuration is called the combining junction. On the other hand, Fig. 2.34 illustrates a single pipe dividing into more than two pipes. Such a structure is called

**Fig. 2.36** Pipe line connecting two large reservoirs



the dividing junction. Vortices generated around the junction causes an energy loss. The details of the loss coefficients of the two types of junctions are available in the references listed at the end of this book.

**Exercise 2.19** There is a horizontal, circular pipe connecting two big reservoirs, 1 and 2 (see Fig. 2.36). The length of the pipe is 100 m. The pipe diameter is suddenly increased from 300 to 600 mm at the midpoint between the two reservoirs. The loss coefficient at the pipe entrance,  $\zeta_1$ , is 0.50. The frictional coefficients,  $\lambda_1$  and  $\lambda_2$ , in the smaller and larger pipes are 0.030 and 0.020, respectively. The water flow rate,  $Q$ , is  $18 \text{ m}^3/\text{min}$ . The loss coefficient at sudden expansion,  $\zeta_{se}$ , is expressed by

$$\zeta_{se} = (1 - m)^2 \quad (2.130)$$

$$m = A_1/A_2 \quad (2.131)$$

where  $m$  is the area ratio. Obtain the water surface height difference,  $H$ .

**Answer to Exercise 2.19** The liquid velocities,  $v_{s1}$  and  $v_{s2}$ , on the two reservoir surfaces are negligibly small since the reservoirs are very large. The cross-sectional mean velocities in the smaller and larger pipes,  $v_{m1}$  and  $v_{m2}$ , are calculated as

$$v_{m1} = Q/[\pi d_1^2/4] = (18/60)/[3.14 \times (0.300)^2/4] = 4.24 \text{ m/s} \quad (2.132)$$

$$v_{m2} = Q/[\pi d_2^2/4] = (18/60)/[3.14 \times (0.600)^2/4] = 1.06 \text{ m/s} \quad (2.133)$$

The Bernoulli equation results in

$$\frac{p_{BS1}}{\rho g} + \frac{v_{mBS1}^2}{2g} + z_{BS1} = \frac{p_{BS2}}{\rho g} + \frac{v_{mBS2}^2}{2g} + z_{BS2} + h_L \quad (2.134)$$

where  $p_{BS1} = p_{BS2} = p_0$ ,  $v_{mBS1} = v_{mBS2} = 0$ ,  $z_{BS1} - z_{BS2} = H$ , and  $h_L$  is the hydrodynamic loss.

The water surface difference,  $H$ , therefore is

$$H = h_L \quad (2.135)$$

where  $h_L$  is expressed by

$$h_L = \zeta_i \frac{v_{m1}^2}{2g} + \lambda_1 \frac{l_1}{d_1} \frac{v_{m1}^2}{2g} + \zeta_{se} \frac{v_{m1}^2}{2g} + \lambda_2 \frac{l_2}{d_2} \frac{v_{m2}^2}{2g} + \zeta_e \frac{v_{m2}^2}{2g} \quad (2.136)$$

and  $\zeta_{se}$  is calculated as

$$\zeta_{se} = (1 - A_1/A_2)^2 = \left[1 - (d_1/d_2)^2\right]^2 = (1 - 1/4)^2 = 0.5625 \quad (2.137)$$

Accordingly,  $H$  is given by

$$\begin{aligned} H &= 0.5 \times (4.24)^2/19.6 + 0.03 \times (50/0.300) \times (4.24)^2/19.6 \\ &\quad + 0.5625 \times (4.24)^2/19.6 + 0.02 \times (50/0.600) \times (1.06)^2/19.6 \\ &\quad + 1.0 \times (1.06)^2/19.6 = 5.72 \text{ m} \end{aligned} \quad (2.138)$$

## 2.5 Boundary Layer Flow on a Flat Plate

In 1904 Prandtl indicated that the flow about a solid body can be divided into two regions: a very thin layer close to the body (boundary layer) where friction is dominant and the remaining region outside this layer, where friction may be neglected [1].

### 2.5.1 Laminar Flow

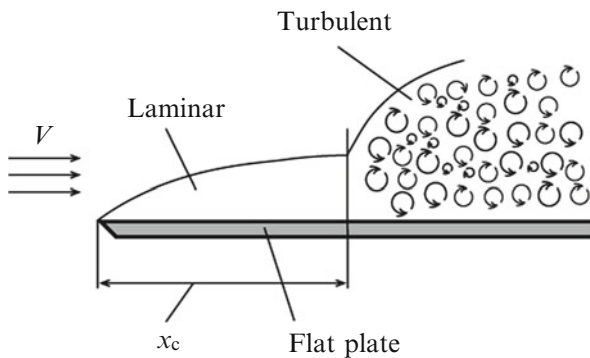
Consider a flat plate placed in a fluid flowing at an approach velocity of  $V$ , as shown in Fig. 2.37. A laminar boundary layer initially develops on the plate. The transition from laminar flow to turbulent flow occurs at a certain distance from the leading edge of the plate. It is possible to induce transition to turbulence near the leading edge by placing a trap wire on the plate in the close vicinity of the leading edge (see Fig. 2.38).

The boundary layer thickness,  $\delta$ , is defined as the distance from the plate to the position at which the axial velocity component,  $u$ , is  $0.99 V$  (see Fig. 2.39). The thickness of laminar boundary layer is expressed by

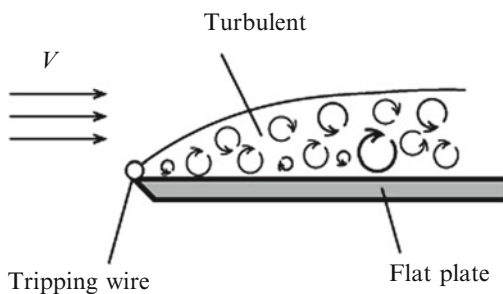
$$\delta = 5.0(\nu_f x/V)^{1/2} = 5.0xRe_x^{-1/2} \quad (2.139)$$

$$Re_x = Vx/\nu_f \quad (2.140)$$

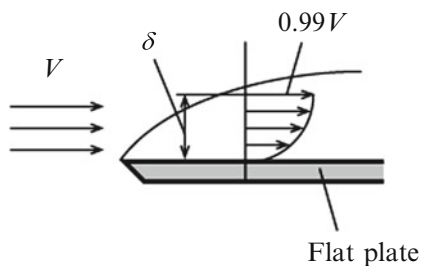
**Fig. 2.37** Boundary layer on a flat plate



**Fig. 2.38** Turbulent boundary layer caused by a tripping wire placed on the leading edge of a flat plate



**Fig. 2.39** Velocity distribution in a flat plate boundary layer and the definition of the boundary layer thickness,  $\delta$



where  $\nu_f$  is the kinematic viscosity of fluid and  $x$  is the distance from the leading edge.

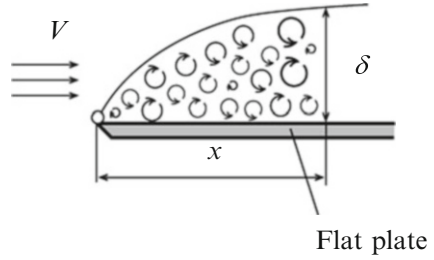
The velocity distribution in a laminar boundary layer is given by the Blasius solution [1]. The wall shear stress,  $\tau_w$ , is expressed as

$$\tau_w = C_f \rho V^2 / 2 \quad (2.141)$$

$$C_f = 0.664 Re_x^{-1/2} \quad (2.142)$$

where  $C_f$  is the skin friction coefficient.

**Fig. 2.40** Boundary layer thickness in a turbulent boundary layer



The mean skin friction coefficient per unit width is given by

$$C_{fm} = 1.328 Re_L^{-1/2} \quad (2.143)$$

$$Re_L = VL/\nu_f \quad (2.144)$$

Equation (2.143) is derived by averaging  $C_f$  between  $x = 0$  and  $x = L$ . The fictional force acting on a flat plate can be obtained by using this skin friction coefficient.

### 2.5.2 Turbulent Flow

The following empirical relations have been proposed by assuming that the flow becomes turbulent from the leading edge of the plate. This situation is realized by placing a trap wire near the leading edge, as mentioned earlier (see Fig. 2.40).

The thickness of a turbulent boundary layer on a flat plate is

$$\delta = 0.37x(Vx/\nu_f)^{-1/5} \quad (2.145)$$

The local and mean friction coefficients are respectively given by

$$C_f = (2.0 \log Re_x - 0.65)^{-2.3} \quad (2.146)$$

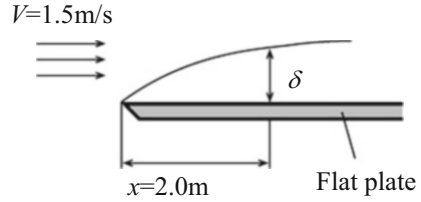
$$C_{fm} = 0.455(\log Re_L)^{-2.58} \quad (2.147)$$

### 2.5.3 Transition Flow

The local Reynolds number,  $Re_x$ , is defined as

$$Re_x = Vx/\nu_f \quad (2.148)$$

**Fig. 2.41** Laminar flat plate boundary layer



The critical Reynolds number for transition to turbulence on a smooth flat plate boundary layer (see Fig. 2.37) is expressed by

$$Re_{xc} = Vx_c/\nu_f = 5 \times 10^5 \quad (2.149)$$

**Exercise 2.20** The velocity of airflow approaching a flat plate,  $V$ , is 1.5 m/s (see Fig. 2.41). Calculate the local Reynolds number,  $Re_x$ , and the thickness of the boundary layer,  $\delta$ , at the axial position  $x = 2.0$  m. The kinematic viscosity of air,  $\nu_f$ , is  $15 \times 10^{-6} \text{ m}^2/\text{s}$ .

**Answer to Exercise 2.20** The local Reynolds number,  $Re_x$ , is calculated as follows:

$$Re_x = Vx/\nu_f = 1.5 \times 2.0/(15 \times 10^{-6}) = 2.0 \times 10^5 \quad (2.150)$$

This local Reynolds number is smaller than  $Re_{xc}$  and, hence, the boundary layer is laminar. The boundary layer thickness,  $\delta$ , is calculated from Eq. (2.139).

$$\begin{aligned} \delta &= 5.0(\nu_f x/V)^{1/2} = 5.0 \times (15 \times 10^{-6} \times 2.0/1.5)^{1/2} \\ &= 2.24 \times 10^{-2} \text{ m} = 22.4 \text{ mm} \end{aligned} \quad (2.151)$$

**Exercise 2.21** The velocity of airflow approaching a flat plate,  $V$ , is 1.5 m/s. Calculate the local Reynolds number,  $Re_x$ , and the thickness of the boundary layer,  $\delta$ , at the axial position of  $x = 6.0$  m. The kinematic viscosity of air,  $\nu_f$ , is  $15 \times 10^{-6} \text{ m}^2/\text{s}$ .

**Answer to Exercise 2.21** The local Reynolds number,  $Re_x$ , is calculated as follows:

$$Re_x = Vx/\nu_f = 1.5 \times 6.0/(15 \times 10^{-6}) = 6.0 \times 10^5 \quad (2.152)$$

This local Reynolds number is greater than  $Re_{xc}$  and, hence, the boundary layer is turbulent there. The boundary layer thickness,  $\delta$ , is given by

$$\begin{aligned}\delta &= 0.37x(Vx/\nu_f)^{-1/5} = 0.37 \times 6.0 \times [1.5 \times 6.0/(15 \times 10^{-6})]^{-1/5} \\ &= 0.155 \text{ m} = 155 \text{ mm}\end{aligned}\quad (2.153)$$

**Exercise 2.22** The velocity of airflow approaching a flat plate is 2.0 m/s. Calculate the local skin friction coefficient at  $x = 1.5$  m and the mean skin friction coefficient between  $x = 0$  and 1.5 m.

**Answer to Exercise 2.22** The local Reynolds number is given by

$$Re_x = Vx/\nu_f = 2.0 \times 1.5/(15 \times 10^{-6}) = 2.0 \times 10^5 \quad (2.154)$$

Since this Reynolds number is smaller than the critical Reynolds number,  $Re_{xc}$ , of  $5 \times 10^5$ , the two coefficients should be calculated from Eqs. (2.142) and (2.143), respectively.

$$C_f = 0.664Re_x^{-1/2} = 0.664 \times (2.0 \times 10^5)^{-1/2} = 1.48 \times 10^{-3} \quad (2.155)$$

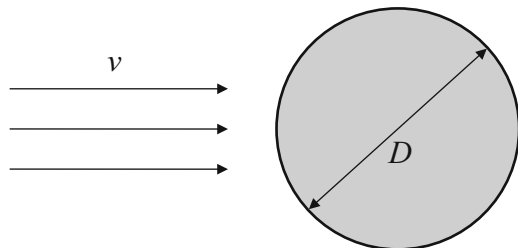
$$Re_L = VL/\nu_f = 2.0 \times 1.5/(15 \times 10^{-6}) = 2.0 \times 10^5 \quad (2.156)$$

$$C_{fm} = 1.328Re_L^{-1/2} = 1.328 \times (2.0 \times 10^5)^{-1/2} = 2.97 \times 10^{-3} \quad (2.157)$$

## 2.6 Flow Around a Solid Body

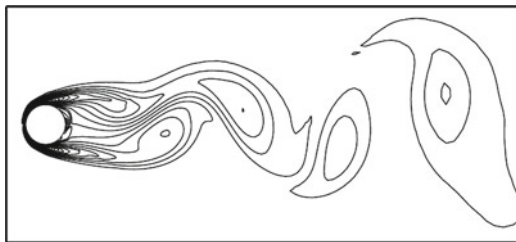
### 2.6.1 Flow Around a Sphere

Figure 2.42 shows a flow approaching a sphere. If the velocity of the flow is very small and, hence, the Reynolds number,  $Re$ , is smaller than unity ( $Re < 1$ ), the fluid flows slowly along the sphere with no flow separation. Such a very slow flow is called creeping flow. When the Reynolds number exceeds a certain critical value, separation occurs and a reverse flow region develops around the rear part of the sphere. With further increase in  $V$ , vortices are formed and shed nearly periodically from the rear part of the sphere and, as a result, the wake behind the sphere becomes turbulent.



**Fig. 2.42** Uniform flow approaching a sphere

**Fig. 2.43** Karman vortex streets formed behind a circular cylinder



$Re=140$

The boundary layer on the sphere remains laminar even in the presence of the vortex shedding. When the Reynolds number attains another critical value, the boundary layer just upstream of the separation point undergoes transition to turbulence. This critical Reynolds number is approximately  $3 \times 10^5$ . With a further increase in the Reynolds number the turbulent boundary layer progressively extends towards the front stagnation point.

### 2.6.2 Flow Around a Circular Cylinder

The relationship between the flow pattern around a circular cylinder placed normal to the flow and the Reynolds number is basically the same as that around a sphere. The vortices successively shedding from the cylinder are specifically called Karman's vortices (see Fig. 2.43).

**Exercise 2.23** The velocity,  $V$ , of water flow approaching a sphere 5.0 cm in diameter,  $D$ , is 0.15 m/s. Calculate the Reynolds number,  $Re$ . The kinematic viscosity of water,  $\nu_f$ , is  $1.00 \times 10^{-6} \text{ m}^2/\text{s}$ .

**Answer to Exercise 2.23** The Reynolds number is

$$Re = VD/\nu_f = 0.15 \times 0.050 / (1.00 \times 10^{-6}) = 7.5 \times 10^3 \quad (2.158)$$

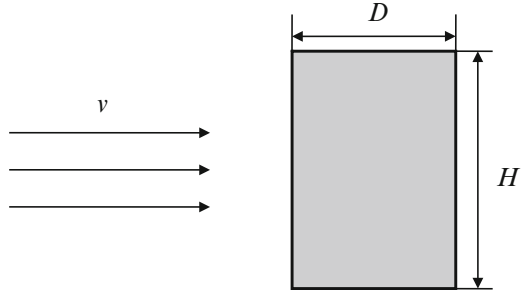
### 2.6.3 Pressure Drag and Drag Coefficient

#### 2.6.3.1 Definition of Drag Coefficient, $C_D$

Precise estimation of drag force acting on a solid body immersed in a fluid flow is of practical importance for design of bodies in external flow including buildings, bridges, cars, ships, and airplanes. The drag force is expressed thus:

$$F_D = C_D A_p \rho V^2 / 2 \quad (2.159)$$

**Fig. 2.44** Uniform flow approaching a cylindrical rod



where  $C_D$  is the drag coefficient,  $A_p$  is the area projected onto a plane normal to the flow,  $\rho$  is the density of fluid, and  $V$  is the approaching flow velocity. The expressions for  $A_p$  and  $C_D$  are (Fig. 2.44)

$$A_p = DH \text{ (cylindrical rod, Fig. 2.44)} \quad (2.160)$$

$$= \pi D^2/4 \text{ (sphere)} \quad (2.161)$$

$$C_D = f(Re, Tu, \lambda_s/D) \quad (2.162)$$

where

$$Tu = v'_{rms}/V \quad (2.163)$$

In the above relations,  $Re$  is the Reynolds number,  $Tu$  is the turbulence intensity,  $\lambda_s/D$  is the relative scale of mean turbulent eddy, and  $v'_{rms}$  is the root-mean-square value of the turbulence component. The effect of  $\lambda_s/D$  is usually very small and the drag coefficient can be expressed by

$$C_D = f(Re, Tu) \quad (2.164)$$

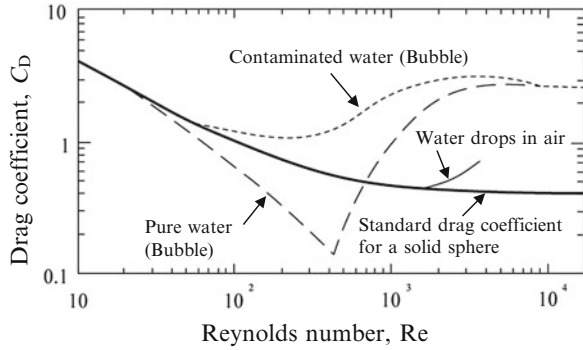
When the turbulence intensity,  $Tu$ , is less than about 0.25 %,  $C_D$  is regarded as a sole function of Reynolds number,  $Re$ , as shown in Fig. 2.45, thus.

$$C_D = f(Re) \quad (2.165)$$

### 2.6.3.2 Drag Coefficient of a Sphere Immersed in a Flow of Very Low Turbulence Intensity

$$C_D = 24/Re \text{ (Stokes' resistance law, } Re < 1) \quad (2.166)$$

**Fig. 2.45** Drag coefficients for a solid sphere, droplet, and a bubble



$$C_D = 10Re^{-1/2} \text{ (Allen's resistance law, } Re = 30 \sim 300) \quad (2.167)$$

$$C_D = 0.44 \text{ (Newton's resistance law, } Re = 300 \sim 10^5) \quad (2.168)$$

### 2.6.3.3 Drag Coefficient of Cylindrical Rod Placed Normal to a Flow of Very Low Turbulence Intensity

$$C_D = 8\pi/[Re(2.002 - \ln Re)] \quad (Re < 0.5) \quad (2.169)$$

$$C_D = (0.707 + 3.42Re^{-1/2})^2 \quad (Re = 5 \sim 40) \quad (2.170)$$

$$C_D = 1.2 \quad (Re = 2 \times 10^4 - 2 \times 10^5) \quad (2.171)$$

**Exercise 2.24** A sphere 6.0 cm in diameter,  $D$ , is fixed in airflow (see Fig. 2.46). The approach velocity,  $V$ , is 4.0 m/s and the kinematic viscosity of air,  $\nu_f$ , is  $15 \times 10^{-6} \text{ m}^2/\text{s}$ . Calculate the Reynolds number,  $Re$ , and then estimate the drag coefficient,  $C_D$ .

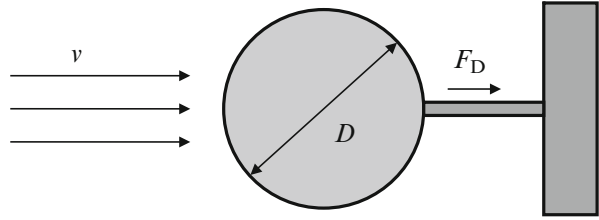
**Answer to Exercise 2.24** The Reynolds number,  $Re$ , is given by

$$Re = VD/\nu_f = 4.0 \times 6.0 \times 10^{-2}/(15 \times 10^{-6}) = 1.6 \times 10^4 \quad (2.172)$$

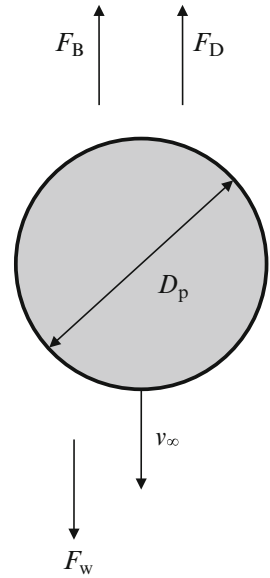
This Reynolds number falls in the regime where Newton's resistance law holds. The drag coefficient therefore is estimated from Eq. (2.168) to be 0.44.

**Exercise 2.25** A sphere of diameter,  $D_p$ , of 8.0 cm and density,  $\rho_p$ , of  $1505 \text{ kg/m}^3$  falls in still water at the terminal velocity (see Fig. 2.47). Obtain the weight,  $F_w$ , of the sphere, the buoyancy force acting on the sphere,  $F_B$ , and the terminal velocity,  $v_\infty$ . The density of water,  $\rho_w$ , is  $998 \text{ kg/m}^3$  and the drag coefficient of the sphere,  $C_D$ , is 0.44.

**Fig. 2.46** Drag force acting on a sphere fixed in a uniform approaching flow



**Fig. 2.47** Sphere falling in still water at a terminal velocity



**Answer to Exercise 2.25**

$$F_w = \rho_p g \pi D_p^3 / 6 = 1505 \times 9.80 \times 3.14 \times (0.080)^3 / 6 = 3.95 \text{ N} \quad (2.173)$$

$$F_B = \rho_w g \pi D_p^3 / 6 = 998 \times 9.80 \times 3.14 \times (0.080)^3 / 6 = 2.63 \text{ N} \quad (2.174)$$

The hydrodynamic drag,  $F_D$ , is expressed by

$$F_D = C_D A_p \rho_w v_\infty^2 / 2 \quad (2.175)$$

The terminal velocity is attained when there is no net force acting on the body in the vertical direction and it falls at a constant steady (terminal) velocity. The force balance on the body is given by

$$F_D = F_w - F_B \quad (2.176)$$

Substituting Eq. (2.175) into Eq. (2.176) yields

$$C_D A_p \rho_w v_\infty^2 / 2 = 1.32 \quad (2.177)$$

$$A_p = \pi D^2 / 4 = \pi \times (0.080)^2 / 4 = 5.02 \times 10^{-3} \text{ m}^2 \quad (2.178)$$

Accordingly, the terminal velocity is given by

$$v_\infty = [1.32 \times 2 / (0.44 \times 5.02 \times 10^{-3} \times 998)]^{1/2} = 1.10 \text{ m/s} \quad (2.179)$$

### 2.6.4 Karman's Vortex Street

The shedding frequency,  $f$ , of Karman's vortex streets formed behind a circular cylinder (see Fig. 2.48) can be correlated in terms of the Strouhal number,  $St$ .

$$St \equiv fD/V = 0.2 \quad (10^2 < Re < 10^5) \quad (2.180)$$

$$Re = VD/\nu_f \quad (2.181)$$

Equation (2.180) indicates that the Strouhal number,  $St$ , is constant over a wide Reynolds number range and that the shedding frequency,  $f$ , is proportional to the approaching flow velocity,  $V$ . This result is very useful for the measurement of fluid flow velocity and flow rate. Specifically, the velocity,  $V$ , can be determined by measuring the shedding frequency,  $f$ . The so-called vortex flow meter works on the basis of this principle. The sound induced by electric wires in a windy day is associated with the shedding of Karman's vortex streets from the wires.

**Exercise 2.26** An electric wire 1.0 cm in diameter,  $D$ , is placed in the atmosphere. The wind velocity,  $V$ , and the kinematic viscosity of air,  $\nu_a$ , are 30 m/s and  $15 \times 10^{-6} \text{ m}^2/\text{s}$ , respectively. Calculate the shedding frequency of Karman's vortex streets,  $f$ .

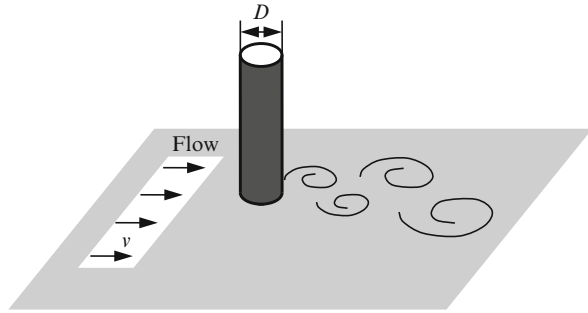
**Answer to Exercise 2.26** The Reynolds number,  $Re$ , is given by

$$Re = VD/\nu_f = 30 \times 1.0 \times 10^{-2} / (15 \times 10^{-6}) = 2.0 \times 10^4 \quad (2.182)$$

This Reynolds number falls in the regime where Karman's vortex streets are formed behind the wire. The shedding frequency is given by

$$f = 0.2V/D = 0.2 \times 30/0.010 = 600 \text{ Hz} \quad (2.183)$$

**Fig. 2.48** Karman's vortex streets formed behind a circular cylinder placed normal to the flow direction



The Strouhal number,  $St$ , has a close relationship with the drag coefficient,  $C_D$ , over a wide Reynolds number range.

$$St = 0.21 / C_D^{0.622} \quad (10^3 < Re < 10^6) \quad (2.184)$$

## 2.7 Dimensional Analysis

Engineering equations must be dimensionally homogeneous. This implies that the units of all terms of any equation must be equal to one another. This fundamental principle is useful for deriving a functional relationship among parameters governing fluid flow phenomena. Two types of methods are used for dimensional analysis: the Lord Rayleigh method and the Buckingham II theorem. The Lord Rayleigh method is valid for flow fields governed by parameters less than or equal to four. The Buckingham II theorem is applicable when parameters are greater than or equal to five. The concepts of these methods will be introduced below with some examples.

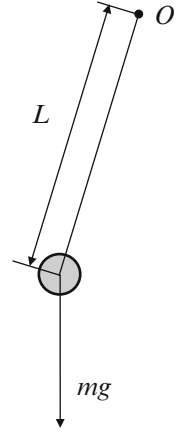
### 2.7.1 Lord Rayleigh Method

This method is useful for flow phenomena governed by quantities less than or equal to four. One of the quantities is expressed as a product of the power of the other quantities.

We consider a pendant as shown in Fig. 2.49, consisting of a small sphere supported by a string. The period,  $t_p$ , is assumed to be affected by the length of the string,  $L$ , and the acceleration due to gravity,  $g$ . The relationship among these quantities is assumed as follows:

$$t_p = kL^\alpha g^\beta \quad (2.185)$$

**Fig. 2.49** Pendant moving under terrestrial condition



The two indices,  $\alpha$  and  $\beta$ , are constant. The dimensions of  $T$ ,  $L$ , and  $g$  are indicated by  $[T]$ ,  $[L]$ , and  $[LT^{-2}]$ , respectively, in which  $[T]$  represents dimension of time and  $[L]$  for length.

The dimensions of the left-hand side and right-hand side of Eq. (2.185) must be equal to each other, thus:

$$[T] = [L]^\alpha [LT^{-2}]^\beta \quad (2.186)$$

This equation is rewritten as

$$[L^0 T^1] = [L^{\alpha+\beta} T^{-2\beta}] \quad (2.187)$$

Using the law of exponents, the following two algebraic equations must be satisfied:

$$\alpha + \beta = 0 \quad (2.188)$$

$$-2\beta = 1 \quad (2.189)$$

Solving these equations yields

$$\alpha = 1/2 \quad (2.190)$$

$$\beta = -1/2 \quad (2.191)$$

Substituting these values into Eq. (2.185) gives

$$t_p = k \left( \frac{L}{g} \right)^{1/2} \quad (2.192)$$

The exact solution is known to be

$$t_p = 2\pi \left( \frac{L}{g} \right)^{1/2} \quad (2.193)$$

It is evident that the Lord Rayleigh method leads to an exact relationship between the three quantities ( $T$ ,  $L$ , and  $g$ ).

**Exercise 2.27** The propagation speed of sound,  $c$ , is assumed to be related to the pressure,  $p$ , density,  $\rho$ , and dynamic viscosity of fluid,  $\mu$ . Derive the relationship between these quantities using the Lord Rayleigh method.

**Answer to Exercise 2.27** The units of the four quantities are expressed as follows:  $c$  (m/s),  $p$  (kg/m s<sup>2</sup>),  $\rho$  (kg/m<sup>3</sup>), and  $\mu$  (kg/m s). The speed of sound,  $c$ , is assumed to be

$$c = k p^\alpha \rho^\beta \mu^\gamma \quad (2.194)$$

where  $k$  is a constant. The relationship for the dimensions of Eq. (2.194) is expressed by

$$[L^1 T^{-1}] = [M L^{-1} T^{-2}]^\alpha [M L^{-3}]^\beta [M L^{-1} T^{-1}]^\gamma \quad (2.195)$$

where  $L$ ,  $T$ , and  $M$  denote the dimensions of length, time, and mass, respectively. The three indices,  $\alpha$ ,  $\beta$ , and  $\gamma$ , must satisfy the following algebraic equations:

$$L : -\alpha - 3\beta - \gamma = 1 \quad (2.196)$$

$$T : -2\alpha - \gamma = -1 \quad (2.197)$$

$$M : \alpha + \beta + \gamma = 0 \quad (2.198)$$

Solving these algebraic equations gives

$$\alpha = \frac{1}{2}, \quad \beta = -\frac{1}{2}, \quad \gamma = 0 \quad (2.199)$$

Substituting these values into Eq. (2.194) yields

$$c = k \left( \frac{p}{\rho} \right)^{1/2} \quad (2.200)$$

### 2.7.2 Buckingham $\Pi$ Theorem

This theorem is applicable for a phenomenon governed by more than five quantities. When the number of quantities is  $n$  and number of dimensions is  $m$ , the following relationship holds among  $k (=n - m)$  dimensionless quantities:

$$f(\Pi_1, \Pi_2, \dots, \Pi_k) = 0 \quad (2.201)$$

or

$$\Pi_1 = f(\Pi_2, \dots, \Pi_k) = 0 \quad (2.202)$$

where  $\Pi_1, \Pi_2, \dots$ , and  $\Pi_k$  are the dimensionless parameters. Unfortunately, it is difficult to determine the functional relationship based on this theorem.

Each dimensionless parameter is determined as a product of  $(m + 1)$  quantities among  $n$  quantities, where any one of  $n$  quantities must be used at least once. The following exercise will help the reader understand the process of obtaining the dimensionless parameters.

**Exercise 2.28** A sphere is fixed in a uniform flow, as shown in Fig. 2.50. The approaching flow velocity, the diameter of the sphere, the density of fluid, and the dynamic viscosity of fluid are denoted by  $V$  (m/s),  $D$  (m),  $\rho$  (kg/m<sup>3</sup>), and  $\mu$  (kg/m s), respectively. Derive a relationship between the hydrodynamic drag,  $F_D$  (N = kgm/s<sup>2</sup>), and these parameters based on the Buckingham  $\Pi$  theorem.

**Answer to Exercise 2.28** The number of quantities,  $n$ , is 5 and number of dimensions,  $m$ , is 3. Thus there are two possible dimensionless parameters,  $\Pi_1$  and  $\Pi_2$ . We choose  $V$  (m/s),  $D$  (m), and  $\rho$  (kg/m<sup>3</sup>) as basic quantities to express

$$\Pi_1 = F_D V^{x_1} D^{y_1} \rho^{z_1} \quad (2.203)$$

$$\Pi_2 = \mu V^{x_2} D^{y_2} \rho^{z_2} \quad (2.204)$$

The following relationship holds concerning the dimensions of Eq. (2.203):

$$[L^0 T^0 M^0] = [MLT^{-2}][LT^{-1}]^{x_1}[L]^{y_1}[ML^{-3}]^{z_1} \quad (2.205)$$

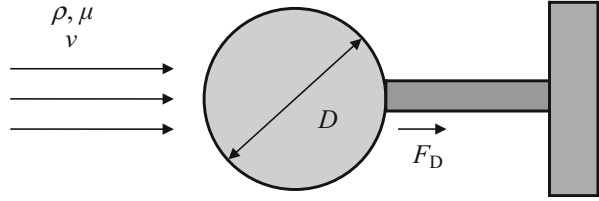
As the dimensions on both the sides must be the same, we have

$$L : 1 + x_1 + y_1 - 3z_1 = 0 \quad (2.206)$$

$$T : -2 - x_1 = 0 \quad (2.207)$$

$$M : 1 + z_1 = 0 \quad (2.208)$$

**Fig. 2.50** A sphere fixed in a uniform approaching flow ( $v$ : approaching flow velocity,  $D$ : diameter,  $\rho$ : density,  $\mu$ : dynamic viscosity)



Solving these algebraic equations yields

$$x_1 = -2, y_1 = -2, z_1 = -1 \quad (2.209)$$

Substituting these values into Eq. (2.203) gives

$$\Pi_1 = \frac{F_D}{v^2 D^2 \rho} \quad (2.210)$$

In a similar manner to the above derivation, the following results can be obtained for Eq. (2.204):

$$[L^0 T^0 M^0] = [ML^{-1} T^{-1}] [LT^{-1}]^{x_2} [L]^{y_2} [ML^{-3}]^{z_2} \quad (2.211)$$

$$L : -1 + x_2 + y_2 - 3z_2 = 0 \quad (2.212)$$

$$T : -1 - x_2 = 0 \quad (2.213)$$

$$M : 1 + z_2 = 0 \quad (2.214)$$

Solving these algebraic equations yields

$$x_2 = -1, y_2 = -1, z_2 = -1 \quad (2.215)$$

Substituting these values into Eq. (2.204) gives

$$\Pi_2 = \frac{\mu}{v D \rho} \quad (2.216)$$

where  $\mu/(v D \rho)$  is the inverse of the Reynolds number,  $Re$ . Accordingly,

$$\Pi_1 = f(\Pi_2) = f(Re) \quad (2.217)$$

Combining Eq. (2.210) and Eq. (2.216) yields

$$F_D = f(Re)\rho D^2 V^2 \quad (2.218)$$

Equation (2.218) can be rewritten as

$$\begin{aligned} F_D = f(Re)\rho D^2 V^2 &= [8f(Re)/\pi](\pi D^2/4)(\rho V^2/2) \\ &= C_D A_p \rho V^2/2 \end{aligned} \quad (2.219)$$

where  $C_D$  is the drag coefficient and  $A_p$  is the area of the sphere projected to a plane perpendicular to the flow direction.

$$C_D = [8f(Re)/\pi] \quad (2.220)$$

$$A_p = \pi D^2/4 \quad (2.221)$$

Equation (2.219) is the well-known definition of the drag coefficient.

**Exercise 2.29** The height of a liquid column lifted up in a capillary tube,  $H$  (m), is dependent on tube diameter,  $D$  (m), the density of liquid,  $\rho$  (kg/m<sup>3</sup>), acceleration due to gravity,  $g$  (m/s<sup>2</sup>), and the surface tension of liquid,  $\sigma$  (N/m = kg/s<sup>2</sup>) (see Fig. 2.51). Express  $H$  as a function of  $D$ ,  $\rho$ ,  $g$ , and  $\sigma$  on the basis of the Buckingham  $\Pi$  theorem.

**Answer to Exercise 2.29** The number of quantities,  $n$ , is 5 and number of dimensions,  $m$ , is 3 ( $L$ ,  $T$ ,  $M$ ). There are two dimensionless parameters,  $\Pi_1$  and  $\Pi_2$ . We choose  $D$  (m/s),  $\rho$  (kg/m<sup>3</sup>), and  $g$  (m/s<sup>2</sup>) as basic quantities to express

$$\Pi_1 = HD^{x_1} \rho^{y_1} g^{z_1} \quad (2.222)$$

$$\Pi_2 = \sigma D^{x_2} \rho^{y_2} g^{z_2} \quad (2.223)$$

The following relationship holds concerning the dimensions of Eq. (2.222):

$$[L^0 T^0 M^0] = [L][L]^{x_1} [ML^{-3}]^{y_1} [LT^{-2}]^{z_1} \quad (2.224)$$

Since the dimensions on both the sides must be the same, we have

$$L : 1 + x_1 - 3y_1 + z_1 = 0 \quad (2.225)$$

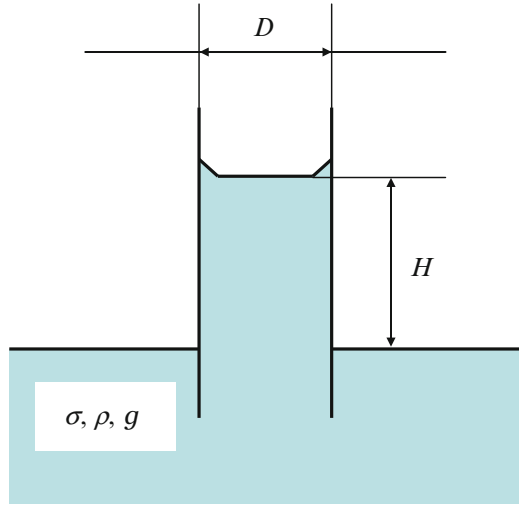
$$T : -2z_1 = 0 \quad (2.226)$$

$$M : y_1 = 0 \quad (2.227)$$

Solving these algebraic equations yields

$$x_1 = -1, y_1 = 0, z_1 = 0 \quad (2.228)$$

**Fig. 2.51** Height of a liquid column lifted up in a capillary tube



Substituting these values into Eq. (2.222) gives

$$\Pi_1 = \frac{H}{D} \quad (2.229)$$

In a similar manner to the above derivation, the following results can be obtained for Eq. (2.223):

$$[L^0 T^0 M^0] = [MT^{-2}][L]^{x_2}[ML^{-3}]^{y_2}[LT^{-2}]^{z_2} \quad (2.230)$$

$$L : x_2 - 3y_2 + z_2 = 0 \quad (2.231)$$

$$T : -2 - 2z_2 = 0 \quad (2.232)$$

$$M : 1 + y_2 = 0 \quad (2.233)$$

The solution of these algebraic equations is given as

$$x_2 = -2, y_2 = -1, z_2 = -1 \quad (2.234)$$

Substituting these values into Eq. (2.223) gives

$$\Pi_2 = \frac{\sigma}{D^2 \rho g} \quad (2.235)$$

where  $\sigma/(D^2\rho g)$  is a form of Bond number. Accordingly,

$$\frac{H}{D} = f\left(\frac{\sigma}{D^2\rho g}\right) \quad (2.236)$$

As shown in a previous section, the exact relationship is given by

$$\frac{H}{D} = \frac{4\sigma \cos \theta_c}{D^2\rho g} = 4 \cos \theta_c \frac{\sigma}{D^2\rho g} \quad (2.237)$$

where  $\theta_c$  is the contact angle ( $^\circ$ ).

As mentioned above, both the Lord Rayleigh method and the Buckingham  $\Pi$  theorem generate dimensionless parameters automatically. However, the number of dimensionless parameters depends solely on the initial choice of the dimensional quantities responsible for the phenomenon. Their adequacy is therefore not guaranteed. Deep insight into the flow phenomenon is a key determinant for adequate dimensionless parameters.

### 2.7.3 Physical Meaning of Dimensionless Numbers

#### 2.7.3.1 Reynolds Number

The Reynolds number,  $Re$ , is defined as

$$Re = \frac{VL}{\nu_f} \quad (2.238)$$

where  $V$  is the representative or characteristic velocity (m/s),  $L$  is the representative or characteristic length of the flow field (m), and  $\nu_f$  is the kinematic viscosity of fluid ( $\text{m}^2/\text{s}$ ). Equation (2.238) can be rewritten as

$$Re = \frac{VL}{\nu_f} \rightarrow \frac{(\rho V^2/2)L^2}{\rho \nu_f \frac{V}{L} L^2} = \frac{(\rho V^2/2)L^2}{\mu \frac{V}{L} L^2} = \frac{(\rho V^2/2)L^2}{\tau L^2} \left( \frac{\text{Inertial force}}{\text{Viscous force}} \right) \quad (2.239)$$

where  $\rho$  is the density of the fluid ( $\text{kg}/\text{m}^3$ ),  $\mu$  is the dynamic viscosity of the fluid ( $\text{kg}/\text{m s}$ ), and  $\tau$  is the shear stress (Pa). Thus, the Reynolds number represents the ratio of the inertial force to the viscous force in the flow field. As  $Re$  increases, the inertial force becomes dominant and, hence, transition to turbulence occurs.

### 2.7.3.2 Mach Number

The Mach number is expressed by

$$M = \frac{V}{c} \quad (2.240)$$

where  $c$  is the speed of sound (m/s). This dimensionless number can be rewritten as

$$M = \left( \frac{\rho V^2/2}{\rho c^2/2} \right)^{1/2} \quad (2.241)$$

where  $\rho V^2/2$  and  $\rho c^2/2$  denote the kinetic and elastic energies, respectively. The ratio of the kinetic energy to the elastic energy of fluid is called the Cauchy number. Accordingly, the Mach number is the root-mean-square value of the Cauchy number.

### 2.7.3.3 Strouhal Number

The Strouhal number is defined as

$$St = \frac{fL}{V} \quad (2.242)$$

where  $f (=1/T)$  is the representative frequency and  $T$  is the representative time.

According to the Navier–Stokes equation in the  $x$  direction Eq. (2.25), described in a previous Sect. 2.1.2, the component terms are expressed by

$$\text{Temporal acceleration : } \partial u / \partial t \quad (2.243)$$

$$\text{Spatial acceleration : } u \partial u / \partial x + v \partial u / \partial y + w \partial u / \partial z \quad (2.244)$$

where the velocity components in the  $x$ ,  $y$ , and  $z$  directions are denoted by  $u$ ,  $v$ , and  $w$ , respectively, and  $t$  is time.

We consider the ratio of  $\partial u / \partial t$  to  $u \partial u / \partial x$  and introduce a representative velocity,  $V$ , and a representative length,  $L$ , in addition to the representative time,  $T$ , described above in order to non-dimensionalize  $u$ ,  $x$ , and  $t$  thus:

$$u' = \frac{u}{V} \quad (2.245)$$

$$x' = \frac{x}{L} \quad (2.246)$$

$$t' = \frac{t}{T} \quad (2.247)$$

The ratio is expressed as

$$\frac{\partial u / \partial t}{u \partial u / \partial x} = \frac{(V/T) \partial u' / \partial t'}{(V^2/L) u \partial u' / \partial x'} = \frac{L}{VT} \frac{\partial u' / \partial t'}{u' \partial u' / \partial x'} \quad (2.248)$$

As the magnitude of  $\partial u' / \partial t' / (u' \partial u' / \partial x')$  is on the order of unity,  $O(1)$ , the value of  $\partial u / \partial t / (u \partial u / \partial x)$  is governed by  $L/(VT)$ . Here,  $L/(VT)$  is rewritten as

$$\frac{L}{VT} = \frac{fL}{V} \equiv St \left( \frac{\text{Temporal acceleration}}{\text{Spatial acceleration}} \right) \quad (2.249)$$

Consequently, the Strouhal number represents the ratio of the temporal acceleration to the spatial acceleration.

#### 2.7.3.4 Weber Number

$$We = \frac{\rho L V^2}{\sigma} \rightarrow \frac{(\rho V^2/2) L^2}{\sigma L} \left( \frac{\text{Inertial force}}{\text{Surface tension force}} \right) \quad (2.250)$$

where  $\rho$  is the density of fluid ( $\text{kg/m}^3$ ),  $L$  is the representative length (m),  $V$  is the representative velocity (m/s), and  $\sigma$  is the surface tension of fluid (N/m).

#### 2.7.3.5 Froude Number

This dimensionless parameter was originally proposed by Froude in ship engineering.

$$Fr = \frac{V}{(gL)^{1/2}} \rightarrow \left[ \frac{(\rho V^2/2) L^2}{\rho g L^3} \right]^{1/2} \left( \frac{\text{Inertial force}}{\text{Buoyancy force}} \right)^{1/2} \quad (2.251)$$

where  $g$  is the acceleration due to gravity.

#### 2.7.3.6 Knudsen Number

$$Kn = \frac{\lambda_m}{L} \quad (2.252)$$

where  $\lambda_m$  is the mean free path.

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