Chapter 2
Reviews of Uncertainty Relations

In this chapter, we provide a brief overview of various uncertainty relations. First, we review historical uncertainty relations: Heisenberg's gamma-ray microscope and von-Neumann's Doppler speed meter. These uncertainty relations epitomize trade-off relation between error and disturbance in quantum measurement process. Next, we review a different type of uncertainty relations: Kennard-Robertson's inequality and Schrödinger's inequality. These characterize trade-off relations of inherent fluctuations of observables. Finally, we review Arthurs-Goodman's inequality and Ozawa's inequality that based on modern quantum measurement theory.

2.1 Heisenberg's Gamma-Ray Microscope

As described in the Introduction, Heisenberg [1, 2] discussed a thought experiment about the position measurement of a particle by using a γ-ray microscope, and found the following trade-off relation between the error $\varepsilon(x)$ in the measured position x and the disturbance $\eta(p_x)$ in the momentum p_x caused by the measurement process:

$$\varepsilon(x)\eta(p_x) \gtrsim \hbar. \tag{2.1}$$

In this section, we follow Heisenberg's orignal discussion and show the importance of the estimation process.

Let us consider that we measure the position x of a particle. By irradiating the γ-ray on the particle, a photon of the γ-ray is scattered by the particle. The scattered photon passes through a lens, impinges on a screen, and makes a blip on the screen. We measure the position x' of the blip, and infer the position x of the particle by the following relation:

$$x = \frac{L_1}{L_2}x', \tag{2.2}$$

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where L_1 and L_2 are the distance between the lens and the particle, and that between the lens and the screen, respectively. It may seem that by determining x' accurately, we can also determine x accurately. However, because of the wave property of the photon, even if we assume that the position x' of the blip can be determined with an arbitrary accuracy,we cannot estimate the position x of the particle accurately. If the particle shifts by Δx from the focal point P, the difference between the optical path lengths AB and AC is

$$\begin{aligned} AB - AC &= \sqrt{L_1^2 + (a + \Delta x)^2} - \sqrt{L_1^2 + (a - \Delta x)^2} \\ &\simeq \frac{2a\Delta x}{\sqrt{L_1^2 + a^2}} = 2\Delta x \sin\theta, \end{aligned} \tag{2.3}$$

where a is the aperture of the lens, and θ is the angle described in Fig. 2.1. To resolve the shift Δx by determining the position x' of the blip, it is necessary that the difference between the path lengths is larger than the wavelength λ. Therefore, the distinguishable minimal shift of the position is

$$2\Delta x \sin\theta = \lambda \qquad \Leftrightarrow \qquad \Delta x = \frac{\lambda}{2\sin\theta}. \tag{2.4}$$

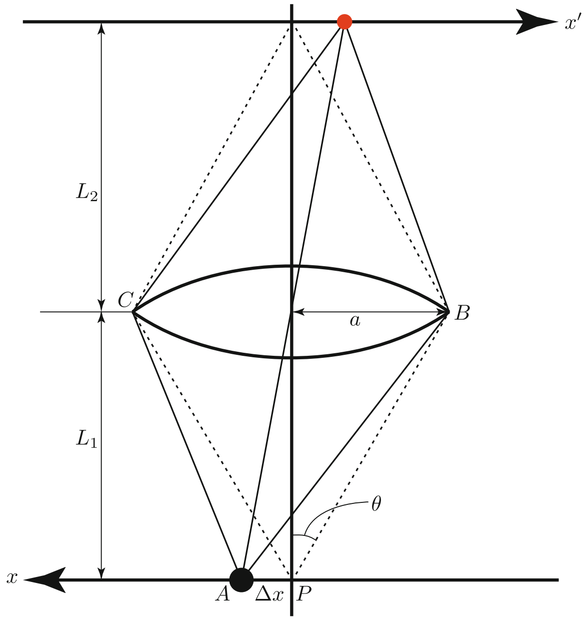

Fig. 2.1 Heisenberg's γ-ray microscope. If the particle shifts its position by $\Delta x \lesssim \lambda/2\sin\theta$, we cannot distinguish the shift

Therefore, the estimated position x involves the error

$$\varepsilon(x) = \frac{\lambda}{2 \sin \theta} \tag{2.5}$$

even if we determine x' accurately.

Next, we consider the disturbance caused by the measurement process. After the scattering of the photon, the momentum of the particle is changed. However, we cannot determine the angle about which direction the photon is scattered. Thus, we cannot estimate the momentum change Δp_x accurately. The uncertainty of the momentum change is given by

$$\eta(p_x) = \frac{2\hbar}{\lambda} \sin \theta. \tag{2.6}$$

Therefore, the error and disturbance satisfy the trade-off relation (2.1).

Heisenberg's uncertainty relation (2.1) is based on a specific model of the position measurement and the semi-classical analysis of the quantum measurement: that is, the particle was assumed to possess definite position and momentum. To rigorously prove the complementarity in quantum measurements, we need to use quantum measurement theory [3, 4]. However, at the time Heisenberg found the trade-off relation, quantum measurement theory was not established yet. Quantum measurement theory was established in the 1970s by Davies and Lewis [3].

2.2 Von Neumann's Doppler Speed Meter

Heisenberg's γ-ray microscope measures the position of a particle and causes the disturbance in the momentum. Von Neumann [5, 6] considered a thought experiment of the momentum of a particle by using a Doppler speed meter,and found the following trade-off relation between the error $\varepsilon(p_x)$ of the measured momentum p_x and the disturbance $\eta(x)$ in the position caused by the measurement process:

$$\eta(x)\varepsilon(p_x) \gtrsim \hbar. \tag{2.7}$$

Note that the roles of x and p_x are exchanged in comparison with Heisenberg's uncertainty relation (2.1). This inequality shows that we cannot measure the momentum without causing disturbance in the position of the particle (Fig. 2.2).

Suppose that we measure the momentum p_x of a particle with mass m. First, we prepare a photon with frequency ω and duration τ that propagates to the particle. If the particle reflects the photon, then the frequency of the reflected photon changes $\delta\omega$ due to the Doppler effect. The frequency change $\delta\omega$ is calculated to be

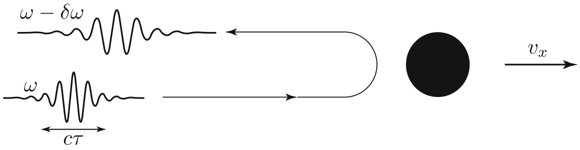

Fig. 2.2 Von Neumann's Doppler speed meter

$$\frac{\delta\omega}{\omega} = \frac{2v_x/c}{1+v_x/c} \simeq \frac{2v_x}{c}, \tag{2.8}$$

where v_x is the velocity of the particle, and c is the speed of light. By measuring the frequency of the reflected photon, we can estimate the velocity v_x and the momentum p_x as

$$v_x = \frac{c\,\delta\omega}{2\omega}, \qquad p_x = \frac{mc\,\delta\omega}{2\omega}. \tag{2.9}$$

However, we can prepare the photon with only a limited accuracy about the frequency. The accuracy of the frequency is $\Delta\omega \sim \tau^{-1}$. Therefore, the error in the estimated momentum p_x is

$$\varepsilon(p_x) = \frac{mc}{2\omega\tau}. \tag{2.10}$$

If we use a photon with the higher frequency and shorter duration, the measured position can be more accurate.

Next, we consider the disturbance in the position of the particle caused by the measurement process. After reflecting the photon, the momentum of the particle changes with $2\hbar\omega/c$. Because the photon emits with duration τ, we cannot find the exact time of the reflection, and the uncertainty of the reflection time is τ. The uncertainty of the position is calculated to be

$$\eta(x) = \frac{2\hbar\omega}{mc}\tau. \tag{2.11}$$

Therefore, we obtain (2.7).

By considering a sequence of measurements, i.e., first, performing the position measurement by using the γ-ray microscope, and then, performing the momentum measurement by using the Doppler speed meter with a high frequency and a short duration, on the post-measurement state, the following inequality can be derived:

$$\varepsilon(x)\varepsilon(p_x) \gtrsim \hbar. \tag{2.12}$$

This inequality shows that we cannot simultaneously perform a measurement of the position and momentum accurately. If the position is measured accurately, the accuracy of the measured momentum decreases, and *vice versa*.

2.3 Kennard-Robertson's Inequality and Schrödinger's Inequality

In 1927, Kennard [7] proved that the inherent fluctuations of the position and momentum are bounded by the Plank constant:

$$\sigma(\hat{x})\sigma(\hat{p}_x) \geq \frac{\hbar}{2}, \tag{2.13}$$

where $\sigma(\hat{x}) := \langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2$, and $\langle\hat{x}\rangle := \mathrm{Tr}[\hat{\rho}\hat{x}]$. Kennard's inequality implies the indeterminacy of the quantum state, that is, the position and momentum cannot be definite simultaneously. In the early days of quantum mechanics, this inequality was erroneously interpreted as a mathematical formulation of the Heisenberg's uncertainty relation. However, $\sigma(\hat{x})$ implies the inherent fluctuation of the observable $\hat{x}$ and depends only on the quantum state $\hat{\rho}$. Kennard's inequality does not concern any trade-off relation between the error and disturbance in the quantum measurement.

Robertson [8] generalized Kennard's inequality for arbitrary observables, and found the following inequality:

$$\sigma(\hat{A})\sigma(\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right|, \tag{2.14}$$

where the square brackets denote the commutator: $[\hat{A}, \hat{B}] := \hat{A}\hat{B} - \hat{B}\hat{A}$. Moreover, Schrödinger [9] generalized Robertson's inequality as

$$\sigma(\hat{A})^2\sigma(\hat{B})^2 - \mathcal{C}_{\mathrm{S}}(\hat{A}, \hat{B})^2 \geq \frac{1}{4}\left|\langle\hat{A}, \hat{B}\rangle\right|^2, \tag{2.15}$$

where $\mathcal{C}_{\mathrm{S}}(\hat{A}, \hat{B})$ is a symmetrized correlation function of the observables defined as

$$\mathcal{C}_{\mathrm{S}}(\hat{A}, \hat{B}) := \frac{1}{2}\langle\{\hat{A}, \hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{2.16}$$

where the curly brackets denote the anti-commutator: $\{\hat{A}, \hat{B}\} := \hat{A}\hat{B} + \hat{B}\hat{A}$. From Schrödinger's inequality, Kennard's inequality and Robertson's inequality are directly derived. Thus, we prove Schrödinger's inequality here.

Let $\mathcal{C}(\hat{A}, \hat{B})$ be a non-symmetrized correlation function of the observables defined as

$$\mathcal{C}(\hat{A}, \hat{B}) := \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{2.17}$$

and $K \in \mathbb{C}^{2\times 2}$ be a Hermitian matrix defined as

$$K := \begin{pmatrix} \sigma(\hat{A})^2 & \mathcal{C}(\hat{A}, \hat{B}) \\ \mathcal{C}(\hat{B}, \hat{A}) & \sigma(\hat{B})^2 \end{pmatrix}. \tag{2.18}$$

For an arbitrary complex vector $\mathbf{z} = (z_1, z_2)^{\mathrm{T}} \in \mathbb{C}^2$, where T denotes the transpose, we have

$$\begin{aligned} \mathbf{z} \cdot K\mathbf{z} &= |z_1|^2 \sigma(\hat{A})^2 + 2\mathrm{Re}\left[z_1^* z_2 \mathcal{C}(\hat{A}, \hat{B})\right] + |z_2|^2 \sigma(\hat{B})^2 \\ &= \langle (z_1 \hat{A} + z_2 \hat{B})^\dagger (z_1 \hat{A} + z_2 \hat{B}) \rangle - \left| \langle z_1 \hat{A} + z_2 \hat{B} \rangle \right|^2 \geq 0. \end{aligned} \tag{2.19}$$

Therefore, K is positive semi-definite, that is, the eigenvalues of K are all non-negative. From the semi-positivity of K, the following inequality is derived:

$$\begin{aligned} \det[K] &= \sigma(\hat{A})^2 \sigma(\hat{B})^2 - \mathcal{C}(\hat{A}, \hat{B})\mathcal{C}(\hat{B}, \hat{A}) \\ &= \sigma(\hat{A})^2 \sigma(\hat{B})^2 - \left\{ \mathcal{C}_{\mathrm{S}}(\hat{A}, \hat{B}) + \frac{1}{2}\langle [\hat{A}, \hat{B}] \rangle \right\} \left\{ \mathcal{C}_{\mathrm{S}}(\hat{A}, \hat{B}) - \frac{1}{2}\langle [\hat{A}, \hat{B}] \rangle \right\} \\ &= \sigma(\hat{A})^2 \sigma(\hat{B})^2 - \mathcal{C}_{\mathrm{S}}(\hat{A}, \hat{B})^2 - \frac{1}{4}\left| \langle [\hat{A}, \hat{B}] \rangle \right|^2 \geq 0. \end{aligned} \tag{2.20}$$

This completes the proof of Schrödinger's inequality.

2.4 Arthurs-Goodman's Inequality

Heisenberg's uncertainty relation and von Neumann's uncertainty relation are based on the semi-classical analysis of quantum measurements. Arthurs and Kelly [10] considered a simultaneous measurement of the position and momentum in fully quantum-mechanical analysis, and Arthurs and Goodman [11] generalized the measurement scheme for two arbitrary non-commuting observables.

To make both observables simultaneously measurable, it is necessary to extend the Hilbert space. This can be done by letting the system interact with another system, called the apparatus. Let us consider that we want to measure observables $\hat{A}$ and $\hat{B}$. Suppose that the initial state of the system is $\hat{\rho}$. First, we prepare the state of the apparatus as $\hat{\rho}_{\mathrm{app}}$, and interact the system and apparatus with the unitary operator $\hat{U}$. After the interaction, we measure the observables $\hat{A}'$ and $\hat{B}'$ of the apparatus. To measure both observables simultaneously, $\hat{A}'$ and $\hat{B}'$ must commute with each other. In order to make the outcomes of the indirect measurement meaningful for $\hat{A}$ and $\hat{B}$, they assumed

$$\langle \hat{A} \rangle := \mathrm{Tr}[\hat{\rho}\hat{A}] = \mathrm{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\mathrm{app}})\hat{U}^\dagger (\hat{I} \otimes \hat{A}')], \tag{2.21a}$$

$$\langle \hat{B} \rangle := \mathrm{Tr}[\hat{\rho}\hat{B}] = \mathrm{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\mathrm{app}})\hat{U}^\dagger (\hat{I} \otimes \hat{B}')] \tag{2.21b}$$

for an arbitrary state $\hat{\rho}$, where $\hat{I}$ is a identity operator. Hereforth, we denote $\hat{I} \otimes \hat{A}'$ as $\hat{A}'$ for simplicity. These conditions are called unbiasedness conditions of the measurement, and measurements that satisfy the unbiasedness conditions are called unbiased measurements.Note that for arbitrary observables $\hat{A}$ and $\hat{B}$, there always exists a set of $\hat{U}$, $\hat{\rho}_{\text{app}}$, $\hat{A}'$ and $\hat{B}'$ that satisfies the unbiasedness condition.The unbiasedness conditions (2.21) imply that the expectation values $\langle \hat{A} \rangle$ and $\langle \hat{B} \rangle$ can directly be estimated from the measurement outcomes.The variances of the measurement outcomes are given by

$$\begin{aligned}\sigma'(\hat{A}') &:= \text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{A}'^2] - \text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{A}']^2 \\ &= \text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{A}'^2] - \langle \hat{A} \rangle^2, \qquad (2.22\text{a}) \\ \sigma'(\hat{B}') &:= \text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{B}'^2] - \text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{B}']^2 \\ &= \text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{B}'^2] - \langle \hat{B} \rangle^2. \qquad (2.22\text{b})\end{aligned}$$

Let $\hat{N}_{\hat{A}}$ be a "noise" operator defined as

$$\hat{N}_{\hat{A}} := \hat{U}^\dagger \hat{A}' \hat{U} - \hat{A}. \tag{2.23}$$

From the unbiasedness condition, the noise operator satisfies

$$\text{Tr}[(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{N}_{\hat{A}}] = 0 \tag{2.24}$$

for an arbitrary state $\hat{\rho}$. From this equation, the following equation can be derived:

$$\text{Tr}_{\text{app}}[(I \otimes \hat{\rho}_{\text{app}})\hat{N}_{\hat{A}}] = \hat{0}, \tag{2.25}$$

where Tr_{app} denotes the partial trace over the apparatus system, and $\hat{0}$ is the null operator. Thus, we have

$$\text{Tr}[\hat{U}(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{U}^\dagger \hat{A}'^2] = \text{Tr}[(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{N}_{\hat{A}}^2] + \text{Tr}[\hat{\rho}\hat{A}^2], \tag{2.26}$$

and

$$\sigma'(\hat{A}')^2 = \text{Tr}[(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{N}_{\hat{A}}^2] + \sigma(\hat{A})^2, \tag{2.27}$$

where $\sigma(\hat{A})^2 := \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$. Therefore, we can find that the variance of the measurement outcome consists of two types of error: inherent fluctuation $\sigma(\hat{A})$, and error in the measurement $\varepsilon_{\text{AG}}(\hat{A})^2 := \text{Tr}[(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{N}_{\hat{A}}^2]$. To clarify the role of the error $\varepsilon_{\text{AG}}(\hat{A})$ in the variance $\sigma'(\hat{A}')$, let us consider the commutation relation of the noise operators. It follows from the fact that $\hat{A}'$ and $\hat{B}'$ commute with each other that

$$[\hat{N}_{\hat{A}}, \hat{N}_{\hat{B}}] + [\hat{N}_{\hat{A}}, \hat{B}] + [\hat{A}, \hat{N}_{\hat{B}}] = -[\hat{A}, \hat{B}]. \tag{2.28}$$

Because of (2.25), we have

$$\mathrm{Tr}\left\{(\hat{\rho}\otimes\hat{\rho}_{\mathrm{app}})[\hat{N}_{\hat{A}},\hat{N}_{\hat{B}}]\right\} = -\langle[\hat{A},\hat{B}]\rangle. \tag{2.29}$$

From the Kennard-Robertson inequality, the following inequality is derived:

$$\varepsilon_{\mathrm{AG}}(\hat{A})\varepsilon_{\mathrm{AG}}(\hat{B}) \geq \frac{1}{2}\left|\mathrm{Tr}\left\{(\hat{\rho}\otimes\hat{\rho}_{\mathrm{app}})[\hat{N}_{\hat{A}},\hat{N}_{\hat{B}}]\right\}\right| = \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|. \tag{2.30}$$

This inequality shows that the product of the errors in the measurement process is also bounded by the commutation relation of the observables. By using the Cauchy-Schwarz inequality and the Kennard-Robertson inequality, the variances of the measurement outcomes satisfy the following trade-off relation:

$$\sigma'(\hat{A}')\sigma'(\hat{B}') \geq \sigma(\hat{A})\sigma(\hat{B}) + \varepsilon_{\mathrm{AG}}(\hat{A})\varepsilon_{\mathrm{AG}}(\hat{B}) \geq \left|\langle[\hat{A},\hat{B}]\rangle\right|. \tag{2.31}$$

The product of the inherent fluctuations and the product of the measurement errors are both bounded from below by $|\langle[\hat{A},\hat{B}]\rangle|/2$. Therefore, the lower bound in (2.31) is doubled.

2.5 Ozawa's Inequality

Arthurs and Goodman derived the trade-off relation between the errors of the observables in the unbiased measurement. By removing the unbiasedness condition of the measurement, Ozawa [12–14] defined the measurement error for an arbitrary measurement as follows:

$$\varepsilon_{\mathrm{Ozawa}}(\hat{A})^2 := \mathrm{Tr}[(\hat{\rho}\otimes\hat{\rho}_{\mathrm{app}})\hat{N}_{\hat{A}}^2]. \tag{2.32}$$

Because the unbiasedness condition is not assumed, the noise operator does not satisfy (2.24) and (2.25). Ozawa proved the following inequality [12–14]:

$$\varepsilon_{\mathrm{Ozawa}}(\hat{A})\varepsilon_{\mathrm{Ozawa}}(\hat{B}) + \varepsilon_{\mathrm{Ozawa}}(\hat{A})\sigma(\hat{B}) + \sigma(\hat{A})\varepsilon_{\mathrm{Ozawa}}(\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right|. \tag{2.33}$$

From (2.28), Ozawa's inequality is derived as follows:

$$\begin{aligned}
\frac{1}{2}\left|\langle[\hat{A},\hat{B}]\rangle\right| &= \frac{1}{2}\left|\langle[\hat{N}_{\hat{A}},\hat{N}_{\hat{B}}]\rangle + \langle[\hat{N}_{\hat{A}},\hat{B}]\rangle + \langle[\hat{A},\hat{N}_{\hat{B}}]\rangle\right| \\
&\leq \frac{1}{2}\left(\left|\langle[\hat{N}_{\hat{A}},\hat{N}_{\hat{B}}]\rangle\right| + \left|\langle[\hat{N}_{\hat{A}},\hat{B}]\rangle\right| + \left|\langle[\hat{A},\hat{N}_{\hat{B}}]\rangle\right|\right) \\
&\leq \sigma'(\hat{N}_{\hat{A}})\sigma'(\hat{N}_{\hat{B}}) + \sigma'(\hat{N}_{\hat{A}})\sigma(\hat{B}) + \sigma(\hat{A})\sigma'(\hat{N}_{\hat{B}}) \\
&\leq \varepsilon_{\mathrm{Ozawa}}(\hat{A})\varepsilon_{\mathrm{Ozawa}}(\hat{B}) + \varepsilon_{\mathrm{Ozawa}}(\hat{A})\sigma(\hat{B}) + \sigma(\hat{A})\varepsilon_{\mathrm{Ozawa}}(\hat{B}).
\end{aligned} \tag{2.34}$$

Ozawa also define "disturbance operator" $\hat{D}_{\hat{B}}$ and disturbance $\eta_{\text{Ozawa}}(\hat{B})$ as

$$\hat{D}_{\hat{B}} := \hat{U}^{\dagger}\hat{B}\hat{U} - \hat{B}, \tag{2.35}$$

$$\eta_{\text{Ozawa}}(\hat{B})^2 := \text{Tr}[(\hat{\rho} \otimes \hat{\rho}_{\text{app}})\hat{D}_{\hat{B}}^2], \tag{2.36}$$

and proved the following inequality [12–14]:

$$\varepsilon_{\text{Ozawa}}(\hat{A})\eta_{\text{Ozawa}}(\hat{B}) + \varepsilon_{\text{Ozawa}}(\hat{A})\sigma(\hat{B}) + \sigma(\hat{A})\eta_{\text{Ozawa}}(\hat{B}) \geq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right|. \tag{2.37}$$

Since $\hat{A}'$ and $\hat{B}$ commute with each other, the following commutation relation can be derived:

$$[\hat{N}_{\hat{A}}, \hat{D}_{\hat{B}}] + [\hat{N}_{\hat{A}}, \hat{B}] + [\hat{A}, \hat{D}_{\hat{B}}] = -[\hat{A}, \hat{B}]. \tag{2.38}$$

Therefore, by following a procedure similar to what was carried out in (2.34), the inequality (2.37) is derived.

From (2.33), the product of the measurement errors is not bounded by the commutation relation of the observables. Because the measurement error is always finite, if the measurement error $\varepsilon_{\text{Ozawa}}(\hat{A})$ vanishes, the product of the errors also vanishes:

$$\varepsilon_{\text{Ozawa}}(\hat{A})\varepsilon_{\text{Ozawa}}(\hat{B}) = 0 \leq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right|. \tag{2.39}$$

A similar situation occurs for the product of the error and disturbance. Because the disturbance is always finite, there exist measurements that satisfy

$$\varepsilon_{\text{Ozawa}}(\hat{A})\eta_{\text{Ozawa}}(\hat{B}) = 0 \leq \frac{1}{2}\left|\langle[\hat{A}, \hat{B}]\rangle\right|. \tag{2.40}$$

Because of them, Ozawa claimed that Heisenberg's uncertainty relation can be violated. However, we are skeptical about this claim. His definition of the error and disturbance does not have correspondence to the accuracy of the estimation.In the following, we show several examples.

Let us consider the case in which the measurement process is a projective measurement, and the measurement outcome is scaled by a factor c. Such a measurement can be constructed by a swapping operator $\hat{U}$ and $\hat{A}' = c\hat{A}$. Although we can fully obtain information about $\hat{A}$ from the measurement outcome, that is, we can estimate $\langle\hat{A}\rangle$ with the same accuracy when we perform the non-scaled PVM measurement of $\hat{A}$, the measurement error does not vanish:

$$\varepsilon_{\text{Ozawa}}(\hat{A}) = (c-1)\sqrt{\langle\hat{A}^2\rangle}. \tag{2.41}$$

If the measurement process is also a projective measurement, and the measurement outcome is shifted by c. In this case, we also have the complete information, but the error is

$$\varepsilon_{\mathrm{Ozawa}}(\hat{A}) = c. \tag{2.42}$$

Such a conclusion does not make sense which indicates a rather serious problem in Ozawa's definition of the error. Next, let us consider the measurement that always gives a fixed value c. Such a measurement can be constructed by $\hat{A}' = c\hat{I}$. According to Ozawa's definition, the error is calculated to be

$$\varepsilon_{\mathrm{Ozawa}}(\hat{A}) = \sqrt{(c - \langle\hat{A}\rangle)^2 + \sigma(\hat{A})^2}. \tag{2.43}$$

From the outcome of the measurement, we obtain no information about the system, that is, we cannot estimate $\langle\hat{A}\rangle$. However, if the measurement outcome c equals to $\langle\hat{A}\rangle$ and $\sigma(\hat{A}) = 0$ by chance, the error vanishes. From those examples, we conclude that the error defined by Ozawa does not imply the obtained information about the system, that is, the error is independent of the accuracy of the estimated value of $\langle\hat{A}\rangle$. Therefore, the definition of the error is independent of the accuracy of the estimation.

Next, we consider the disturbance. Let us consider the case in which the measurement process actually does not obtain information and just rotates the system by a unitary operator $\hat{V}$, that is, the system does not interact with the apparatus. The disturbance is calculated to be

$$\eta_{\mathrm{Ozawa}}(\hat{B}) = \sqrt{\langle\hat{V}^\dagger\hat{B}\hat{V} - B\rangle^2}. \tag{2.44}$$

In this case, the system does not decohere, and we can obtain complete information about the observable $\hat{B}$ by performing a projective measurement corresponding to the spectral decomposition of $\hat{V}\hat{B}\hat{V}^\dagger$ on the post-measurement state $\hat{V}\hat{\rho}\hat{V}^\dagger$ if we know $\hat{V}$. However, the disturbance $\eta_{\mathrm{Ozawa}}(\hat{B})$ does not vanish in general. Next, let us consider the swapping measurement: i.e., first, swapping the states of the system and apparatus, and then performing the projective measurement of $\hat{A}' = \hat{A}$ on the apparatus system. In this case, the disturbance is calculated to be

$$\eta_{\mathrm{Ozawa}}(\hat{B}) = \sigma_{\mathrm{app}}(\hat{B})^2 + \sigma(\hat{B})^2 + (\langle\hat{B}\rangle_{\mathrm{app}} - \langle\hat{B}\rangle)^2, \tag{2.45}$$

where $\langle\hat{B}\rangle_{\mathrm{app}} := \mathrm{Tr}[\hat{\rho}_{\mathrm{app}}\hat{B}]$ and $\sigma_{\mathrm{app}}(\hat{B})^2 := \langle\hat{B}^2\rangle_{\mathrm{app}} - \langle\hat{B}\rangle_{\mathrm{app}}^2$. Because the post-measurement state of the system equals $\hat{\rho}_{\mathrm{app}}$, we cannot obtain information about the original state $\hat{\rho}$ by performing any measurement on the post-measurement state. If the state $\hat{\rho}_{\mathrm{app}}$ is equal to $\hat{\rho}$ by chance and $\sigma(\hat{B}) = 0$, the disturbance vanishes. From those examples, we conclude that the disturbance is independent of the information about the original state $\hat{\rho}$ contained in the post-measurement state, that is, the disturbance does not imply the possibility of the estimation from the post-measurement state. Therefore, Ozawa's definition of the disturbance is independent of the accuracy of the estimation.

The definitions of the error and disturbance by Ozawa ignore the implicit but essential role of the estimation processes in Heisenberg's γ-ray microscope, von Neumann's Doppler speed meter, and the simultaneous unbiased measurement. To quantify error and disturbance in the quantum measurement, it is essential to consider how to estimate from the measurement outcome and the accuracy of the estimation. In those measurements, the estimation processes are rather straightforward and seem to be trivial. To define error and disturbance for an arbitrary measurement, however, we need to invoke quantum estimation theory [15–17].

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