Chapter 2
Firms and Production

2.1 The Theory of the Firm

In a modern society, indispensable roles are played by many different firms. Firms offers a large part of the goods and services which are consumed in the society. The majority of the people live on incomes they receive from firms. Those who run firms and those who are employed by them spend most of their days in offices and factories of the firms and try hard to make their career in the firms. Consumers' taste and industrial technology are much influenced by firms' activities in advertising, marketing, researches and developments. The study on firms should, therefore, be an inter-disciplinary task among economics, sociology, psychology, business administration, engineering and system analysis. Except for the case of a small private firm, a firm is and organization composed by many decision-makers, with partly common, but partly conflicting interests, controlling different means and basing their decisions on partly different information. Even the economic theory of firms, therefore, should be in general based, not on a simple optimization theory, but on the theory of games or the theory of teams.

Fortunately, however, the theory of the firm is expected to play a quite limited role in the neo-classical theory of microeconomics. It is merely an auxiliary device to explain the form of supply functions (curves) of output (the consumables and the intermediate goods) and of demand functions (curves) of input (the service of factors of production and the intermediate goods). For such a limited purpose, the most simple theory of a profit maximizing firm will do. It is true that such a firm is merely a caricature stripped of many important aspects of firms of the real economy. It is also true, however, that such a simple theory can be a starting point of a more general theory of the firm. Whether it is a good starting point or not depends on how approximately it can explain the behavior of firms in the real economy.

The profit to be maximized by a firm is defined as the difference between the revenue it receives from the sale of its outputs and the costs it incurs from the purchase of its inputs. In this chapter, let us assume that our firm is a competitive firm which takes prices prevailing in the market as given, irrespective of the level of

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its outputs and inputs. The rationale for such price-taking behavior will be discussed in Chaps. 4 and 5 in the below. The profit of the firm is, then, simply a linear function of its outputs and inputs. The maximization of the profit with respect to outputs and inputs is, however, subject to technological relations among outputs and inputs. Unless we are in the land of Cockaigne, for example, outputs cannot be increased indefinitely from the given inputs. Generally, there exist complex substitutability and complementarity among outputs and inputs. The problem of the firm is, therefore, that of a constrained maximization of the profit, being subject to the so-called production function which summarizes technological restrictions on the production, i.e., the transformation of inputs into outputs.

2.2 Production Functions

An example of such production functions which have been often used in theoretical as well as empirical studies is the so-called Cobb–Douglas production function.

$$Y = AL^a K^b \tag{2.1}$$

where A, a and b are positive constants and Y, L and K denote, respectively, the output, the input of labor service, and the input of capital service. Since (2.1) can be rewritten in the log-linear form

$$\log Y = \log A + a \log L + b \log K, \tag{2.2}$$

parameters of production function A, a and b can be easily estimated empirically from data of Y, L and K. If $a + b = 1$, the constant returns to scale prevails, while $a + b < (>) 1$ implies the existence of the diminishing (increasing) returns.

More generally, consider an economy of m goods which include the consumables, the intermediate goods and the service of factors of production. Let y_j denote the output of the j-th good if it is positive, and the input of the j-th good if it is negative ($j = 1, \ldots, m$). The production function of a firm can, in general, be written as

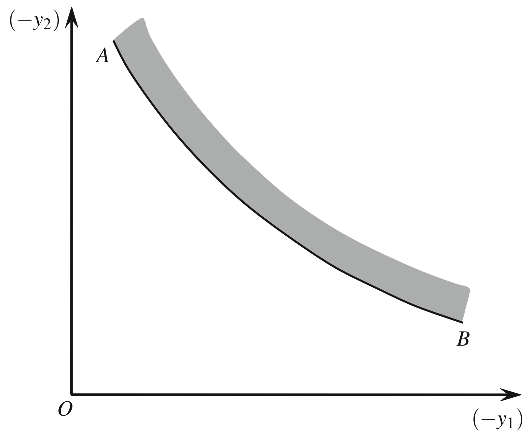
$$f(y_1, \ldots, y_j, \ldots, y_m) = 0 \tag{2.3}$$

in the form of implicit function. If the j-th good is neither output nor input of the firm, consider simply that $y_j = 0$. In the case of Cobb–Douglas production function (2.1), it can be written in the form (2.3) as

$$Y - A(-L)^a(-K)^b(-1)^{a+b} = 0, \tag{2.4}$$

since Y denote the level of output while L and K denote the absolute value of the level of inputs.

Fig. 2.1



Let p_j denote the price of the j-th good ($j = 1, \ldots, m$), which prevails in the market and is taken by the firm as unchanged. The profit of the firm is, then, a simple linear function

$$\pi = \sum_j p_j y_j \tag{2.5}$$

since inputs are denoted by negative y_j's. The firm's problem is to maximize (2.5) subject to (2.3). While the profit (2.5) to be maximized is a simple linear form, the constraint (2.3) is generally non-linear. The nature of the maximization problem depends, therefore, on the form of production function (2.3).

The form of production function generally assumed is usually shown by curves which shows the relation between inputs of two goods, outputs of two goods, and an input of a good and an output of a good. The curve *AB* in Fig. 2.1 is the so-called isoquant which shows the relation between required inputs of the first two goods $(-y_1)$ and $(-y_2)$, when inputs and outputs of all the other goods are given. Usually, it is drawn as convex to the origin. Similarly, the curve *CD* in Fig. 2.2 shows the possible combination of output of the j-th good y_j and that of the k-th good y_k, when inputs and outputs of other goods are given, and is drawn, as usual, concave to the origin. Finally, Fig. 2.3 gives a curve *EF* which shows the possible relation between the input of the j-th good $(-y_j)$ and the output of the k-th good y_k, when inputs and outputs of other goods are given. This curve is usually drawn so that the productivity of the input in terms of the output diminishes as the scale production expands. As a matter of fact the usually assumed forms of these three curves represent the same single assumption made on the production function (2.3).

Fig. 2.2

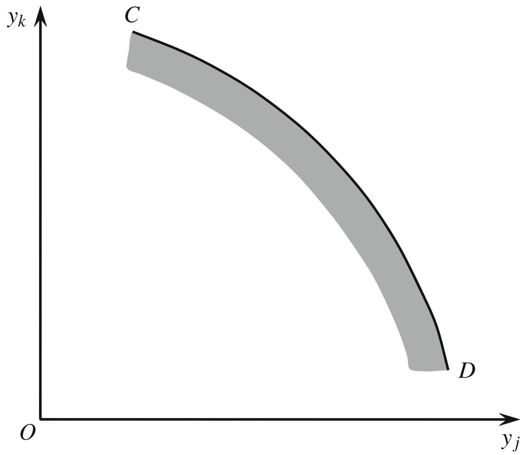
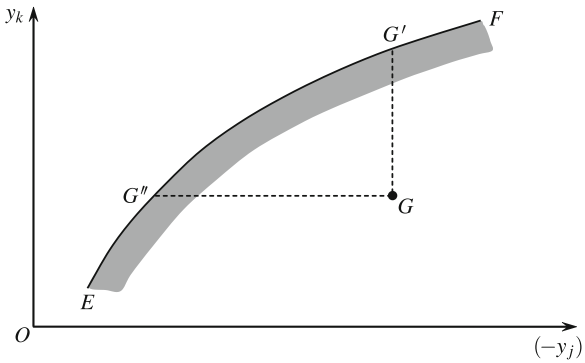


Fig. 2.3

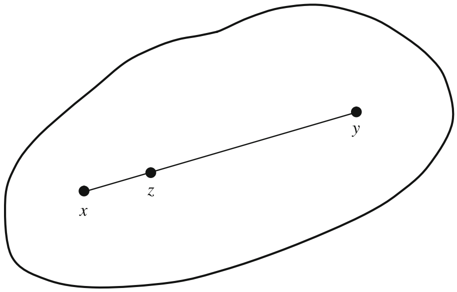


2.3 Feasible Set

To see this, consider the shaded areas in Figs. 2.1–2.3, bounded, respectively, by curves *AB*, *CD* and *EF*. Any point in these areas represents the technologically possible combination, though not necessarily efficient combination, of inputs and outputs, if we assume the free disposal of inputs and outputs. For example, consider the point G in Fig. 2.3. We can reach to this point from the point G' on the curve *EF* by throwing away the output $G'G$. It can also be reached from the point G'' on the curve *EF* by an increase in input $G''G$, which is to be left unused. This point is clearly technologically possible, though not efficient. Points on the curve *EF* are, on the other hand, not only technologically possible, but also efficient. Similarly, curves *AB* and *CD* in Figs. 2.1 and 2.2 represent technologically efficient combinations while points inside of the shaded areas represent possible but inefficient combinations of inputs and outputs.

The shaded areas in Figs. 2.1–2.3 are called feasible sets of combinations of inputs and outputs. In terms of production functions, any point on the boundary

Fig. 2.4



curves *AB*, *CD* and *EF* satisfies the equality form of production function like (2.3) while any point in the feasible sets satisfies the inequality form of production function

$$f(y_1, \ldots, y_j, \ldots, y_m) \leq 0. \tag{2.6}$$

For example, in the case of Cobb–Douglas production function, the inequality form is

$$Y - A(-L)^a(-K)^b(-1)^{a+b} \leq 0, \tag{2.7}$$

which is satisfied by any feasible, i.e., technologically possible, combination of Y, L and K.

The shaded areas in Figs. 2.1–2.3 share a common property. They are all convex sets. As is seen in Fig. 2.4, a set of points is convex if it also contains any point z on the line segment between x and y whenever it contains any points x and y. In other words, whenever it contains any points x and y, a convex set contains also their weighted average with arbitrary positive weights. Now we are ready to state the assumption on production functions, which is usually made in the neoclassical theory of microeconomics. The feasible sets of production functions are convex. In terms of production function, if any two input–output vectors, $y = (y_1, \ldots, y_j, \ldots, y_m)$ and $y' = (y'_1, \ldots, y'_j, \ldots, y'_m)$, satisfy the inequality form of the production function (2.6), then, any positive linear combination of them, i.e.,

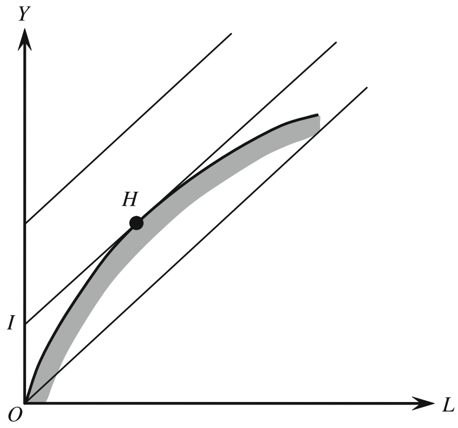
$$\begin{aligned} y'' &= ty + (1-t)y' \\ &= (ty_1 + (1-t)y'_1, \ldots, ty_m + (1-t)y'_m) \end{aligned} \tag{2.8}$$

for any $0 < t < 1$, also satisfies the production function (2.6). This assumption that

$$\text{if} \quad f(y) \leq 0 \quad \text{and} \quad f(y') \leq 0, \quad \text{then} \quad f(y'') \leq 0 \tag{2.9}$$

is called the convexity assumption.

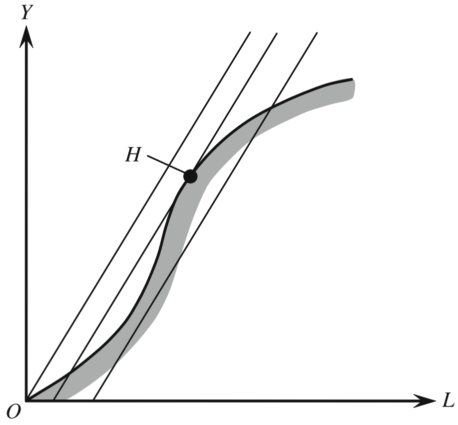
Fig. 2.5



If this convexity assumption is made, the maximization of profit (2.5) under the condition (2.6) becomes a very simple problem, as is seen in Fig. 2.5. Consider a simple case where the firm produces a single product Y and the only variable input is labor L. Figure 2.5 is a reproduction of Fig. 2.3. The equi-profit lines are parallel lines with the slope $\frac{w}{p}$, where p and w denote, respectively, the price of the product and the rate of wage. The profit is maximized at H, where a profit line is tangent to the boundary *EF* of the feasible set and the profit in terms of the product is indicated by *OI*. At H, the marginal productivity of labor, $\frac{dY}{dL}$, is equal to the real wage, $\frac{w}{p}$, so that the so-called first postulate of the classical economics is established. Usually, the condition of the maximization of (2.5) under the constraint (2.6) is that the marginal rate of substitution between any two goods, i.e., the slope of the relevant boundary curves of feasible set, is equal to the price ratio of these two goods. This tangency condition is not generally true, however, when the convexity assumption is not imposed. Consider the difficulty of such a case, which, for example, can be seen in Fig. 2.6.

To assume the convexity so as to make the problem simple is one thing, however, and to justify such an assumption is quite another. The production function is a summary statement of the firm's structure of production. To justify the convexity assumption on production function, therefore, we have to scrutinize the details of the structure of production hidden behind the production function. In view of the fact that the technique of linear programming has been successfully applied to organize the production efficiently in many different firms, we may use the linear programming model of production for this purpose, since the success of linear programming in the production planning of many firms implies that its structure reflects the firms' structure of production fairly well.

Fig. 2.6



2.4 Linear Programming Model

Linear Programming is a mathematical technique to maximize a linear function subject to constraints of linear inequalities. While the profit (2.5) to be maximized is a linear function of inputs and outputs, the constraint of production function (2.6) is a non-linear inequality in our theory of a perfectly competitive firm. To apply linear programming, therefore, we have to decompose the production function into more basic linear production processes which transform inputs into outputs. Let us suppose that there exist n different production processes behind a single production function and denote by x_i (≥ 0) the activity level of the i-th process $(i = 1, \ldots, n)$. If we denote by a_{ij} the amount of the j-th good produced (positive) or consumed (negative) by the unit activity level of the i-th process, i.e., when $x_i = 1$, two basic assumptions of linear programming model of production can be stated as follows.

Divisibility. The amount of the j-th good produced or consumed by the i-th process when its activity level is x_i is simply $a_{ij}x_i$ for any $x_i \geq 0$. In other words, the process can be continuously divided or extended and inputs and outputs are proportional to the activity level of the process.

Additivity. If the i-th process is operated at the level x_i and the i'-th process is operated at the level of x_i', simultaneously, their joint output or joint consumption of the j-th good is simply $a_{ij}x_i + a_{i'j}x_{i'}$ for any $x_i \geq 0$ and $x_{i'} \geq 0$. In other words, processes are independent and there is no external economies or diseconomies among processes.

Decomposition of a firm's production into different process is arbitrary. One may consider that different factories of a firm are independent production processes. Alternatively, a different technology of production may be considered as a different production process. If the firm produces several different products, the production of different goods may be considered as different processes. The point is that the production of a firm should be decomposed into production processes so that we can safely assume divisibility and additivity assumptions. Obviously whether these assumptions are satisfied depends on how we decompose the firm's production into processes. This is why the application of linear programming is sometimes called not the problem of theory but that of art.

Processes transform inputs into outputs. As for inputs, however, we have to distinguish two different types. The first one is that of variable inputs, which a competitive firm can purchase freely whatever amount it likes at a given market price. The second one is that of more or less fixed or limitational inputs, the total amount of which a firm can use are given, at least in the short-run, i.e., within a given period. Examples of variable type inputs are raw materials, energies and labor service of untrained or part-time workers. On the other hand, examples of the second type inputs are land of given characteristics in agriculture, capacity of fixed plants in manufactures, and labor service of managers and trained workers. A firm inherited a given amount of these fixed factors of production from the previous period and has to make a plan of its production in the current period within the fixed supply of these factors.

Suppose there are r different factors of the second type and let s_k (> 0) denote the amount of the fixed supply of the k-th such factor which the firm can use freely $(k = 1, \ldots, r)$. If the i-th process consumes A_{ik} (≥ 0) of the k-th such factor as inputs at its unit activity level, the total consumption of n processes $\sum_i A_{ik} x_i$, from the assumption of divisibility and additivity, when the i-th process is operated at the level of x_i $(i = 1, \ldots, n)$. The condition for the feasibility of activity vectors of processes, $x = (x_1, \ldots, x_n)$, is then

$$\sum_i A_{ik} x_i \leq s_k \quad (k = 1, \ldots, r). \tag{2.10}$$

If we denote by positive a_{ij} the output of the j-th marketable good by the i-th process, and by the negative a_{ij} the first type input of the j-th marketable good by the i-th process, when $x_i = 1$, then, the total output (positive) or input (negative) of the firm is $y_j = \sum_i a_{ij} x_i$, when the i-th process is operated at the level of x_i. In view of (2.5), the profit is

$$\pi = \sum_j p_j \sum_i a_{ij} x_i. \tag{2.11}$$

The firm now maximizes a linear function (2.11) with respect to activity vector $x = (x_1, \ldots, x_n)$ subject to constraints of linear inequalities (2.10), which is a linear programming problem.

2.5 Convexity of Feasible Set

Let us denote by A the $n \times r$ matrix of A_{ik} $(i = 1, \ldots, n,\ k = 1, \ldots, r)$, the feasibility condition (2.10) of activity vector x is rewritten as

$$Ax \leq s \tag{2.12}$$

where $s = (s_1, \ldots, s_r)$ is the given availability vector of the second type inputs. The convexity of the feasible set of activity vectors x of processes is easily seen by the fact that for any x and x'

$$\text{if} \quad Ax \leq s \quad \text{and} \quad Ax' \leq s, \quad \text{then} \quad Ax'' \leq s, \tag{2.13}$$

where $x'' = tx + (1 - t)x'$ for any t such that $0 < t < 1$.

If we denote by a the $n \times m$ matrix of a_{ij} $(i = 1, \ldots, n,\ j = 1, \ldots, m)$, the input–output vector y of the firm corresponding to the activity vector x of processes can be expressed as

$$y = ax. \tag{2.14}$$

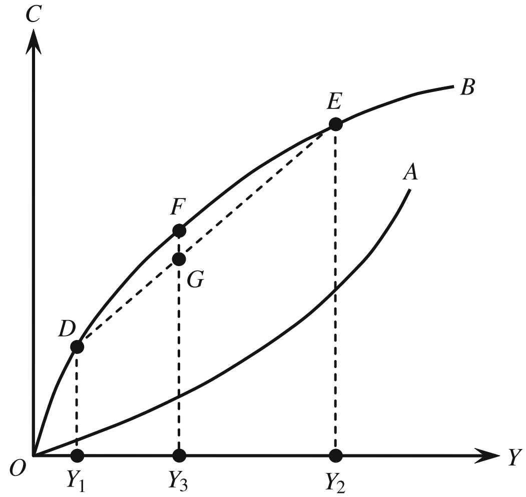
If input–output vectors y and y' of a firm correspond, respectively to activity vectors x and x' of processes, then, $y'' = ty + (1 - t)y'$ corresponds exactly to $x'' = tx + (1 - t)x'$ for any t such that $0 < t < 1$. Suppose any feasible x and x' which satisfy (2.12). The corresponding y and y' satisfy, respectively, $f(y) \leq 0$ and $f(y') \leq 0$. In view of (2.13), then, y'' satisfies $f(y'') \leq 0$. The convexity assumption (2.9) is, therefore, justified.

If linear programming model can be applied to the production of a firm, the feasible set of the firm's input–output vectors of marketable goods can be shown convex. What makes the convexity assumption justifiable is the existence of fixed factors, of which only a given amount can be used in the short-run.

2.6 Marginal Cost

To see the implications of the convexity of the feasible set, let us consider a simplified case of a single product firm. Since the profit is the difference between the revenue from the sale of the single product and the cost of variable, i.e., marketable inputs which are used to produce the product, and prices are taken as given by the firm, the maximization of the profit requires first the minimization of the cost for any given level of output of product Y. The minimum cost C of the firm is then considered a function of Y. Obviously, it is an increasing function. In Fig. 2.7, cost curves are drawn as upward sloping. The problem is whether the curve *OA* or *OB* is the minimum variable cost curve, which is derived from our linear programming

Fig. 2.7



model of production. In other words, whether its slope, i.e., the marginal cost $\frac{dC}{dY}$, is increasing or diminishing with respect to the level of output Y.

Suppose the curve OB is the case. The minimum variable costs necessary for the outputs Y_1, Y_2 and $Y_3 = tY_1 + (1-t)Y_2$, for $0 < t < 1$, are indicated, respectively, as Y_1D, Y_2E and Y_3F. If Y_1 and Y_2 correspond, respectively, to activity vectors x and x' of processes,

$$Y_1D = -\sum_i x_i \sum_j a_{ij} p_j, \tag{2.15}$$

$$Y_2E = -\sum_i x'_i \sum_j a_{ij} p_j \tag{2.16}$$

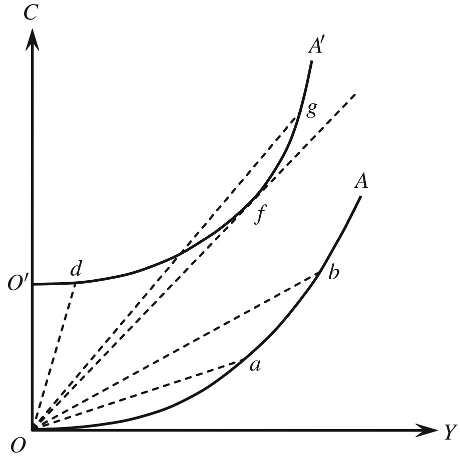
where p_j and a_{ij} (< 0) denote, respectively, the price of the j-th marketable input and the amount of the same input necessary for the unit level of activity of the i-th process. Since the feasible set of the activity vectors is convex, Y_3 can be produced from the activity $x'' = tx + (1-t)x'$ of processes, with the cost $Y_3G = tY_1D + (1-t)Y_2E$, i.e.,

$$Y_3G = -\sum_i x''_i \sum_j a_{ij} p_j \tag{2.17}$$

which is smaller than the minimum cost Y_3F, as is seen in Fig. 2.7. This contradiction implies that our supposition was wrong and that the minimum cost curve is not like the curve OB.

The marginal cost cannot be diminishing. It should be increasing, or at least constant. The constant marginal cost, however, cannot be maintained for a wide

Fig. 2.8



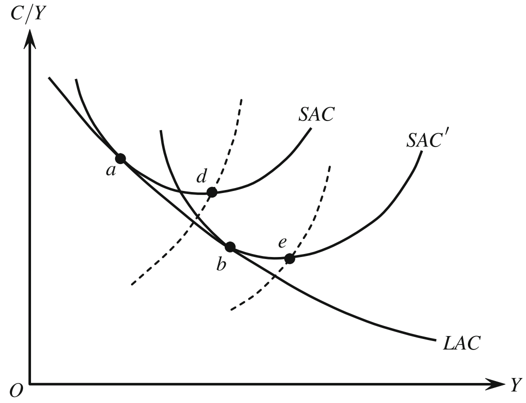
range of output, which can be seen by the consideration of the following simple example. Suppose a firm has two processes to produce a single product from the input of labor service purchasable at the given wage rate. Each process requires the input of a fixed factor, say, the service of machine specific to each process. For a low level of output, the firm uses the process which requires smaller labor input for a unit of output and the marginal cost is constant at the constant labor cost of this process. If the output is increased up to the point where the fixed factor specific to this lower cost process is fully used up, a further increase in output has to be supplied by the use of the process which requires larger labor input for a unit of output. The marginal cost is increased to the level of labor cost of this higher cost process.

The minimum variable cost curve is, therefore, roughly like the curve *OA* in Figs. 2.7 and 2.8, though it may be partly linear. If the number of processes and that of fixed type factors of production are very large, the minimum cost curve can be approximated by the curve *OA* fairly well. If the marginal cost is increasing in this way, the marginal cost curve, which is also upward sloping, is the supply curve of a single product firm, since its profit to be maximized is $pY - C(Y)$, where p is the given price of product, Y is the level of output and $C(Y)$ is the minimum cost function, and the condition for the maximization is $p = \dfrac{dC}{dY}$, from which the supply Y can be read as a function of P.

2.7 Average Cost

While the slope of the minimum cost curve *OA* in Figs. 2.7 and 2.8 gives the marginal cost, the average cost $\dfrac{C(Y)}{Y}$ can be given by the slope of lines (like *Oa*, *Ob* in Fig. 2.8) between points on the curve and the origin. In the case of the curve *OA*,

Fig. 2.9



the average cost is increasing with respect to the level of output, as is seen in Fig. 2.8. According to the empirical studies of average cost, however, it often diminishes first and then increases as the level of output is increased. In other words, the average cost curve is neither simply downward sloping nor simply upward sloping. It is, as it were, a U shaped curve. If there exists the so-called fixed cost which does not vary with the level of output, the U shape of the average cost curve can be easily explained as is seen in Fig. 2.8. Suppose OO' is the fixed cost so that the variable cost curve starts, not from the origin O as the curve OA but from the displaced origin O' as the curve $O'A'$, to show the total cost C as a function of output Y. The average cost $\frac{C}{Y}$ diminishes first and then increases as the changes of the slope of lines Od, Of and Og indicate.

The fixed cost does exist in our linear programming model of production since inputs of fixed factors of production are necessary in addition to variable, marketable inputs to produce the outputs. For example, the salary and wages of those employees who are employed by the long-term contacts are fixed cost to the firm since they do not vary with the current level of output. Similarly, interest cost of bonds issued to finance the purchase of capital assets like plants machines, etc. is also a fixed cost to the firm, which should be paid irrespective of the current level of output. The U shaped average cost curve can thus be explained by the fixed cost related to the fixed limitational inputs and by the rising marginal cost curve, derived from the convexity of feasible set of activity vectors of processes, in our linear programming model.

In the long-run, a firm can change some, at least, of the amount of the fixed supply of the fixed limitational factors (s_k's in Sect. 2.4). If they are increased, the fixed cost OO' in Fig. 2.8, is also increased, but the marginal cost, i.e., the slope of the curve $O'A'$ rises more slowly. In Fig. 2.9, the U shaped short run average cost *SAC* curve is shifted to the right as the firm's stock of fixed factors is increased. *SAC* is shifted to *SAC′* when some of s_k's are increased. It is natural that *SAC′* gives lower

$\frac{C}{Y}$ for larger output, but higher $\frac{C}{Y}$ for smaller output. The long-run average cost *LAC* curve is the envelope of such shifted *SAC* curves and indicates the minimum average cost to produce each level of output when s_k's are changeable. If the input coefficients for marketable inputs of processes (negative a_{ij}'s) are unchanged, and not all s_k's are changeable, *LAC* curve should be eventually upward sloping. If all the s_k's are changeable, however, *LAC* curve is horizontal, since the constant returns to scale prevails when all the s_k's are changed proportionally. Finally, *LAC* curve can always be downward sloping as in Fig. 2.9, if $-a_{ij}$'s decrease as s_k's are increased. Machines with larger capacities may be more efficient in the sense that they require smaller variable inputs for a unit of output.

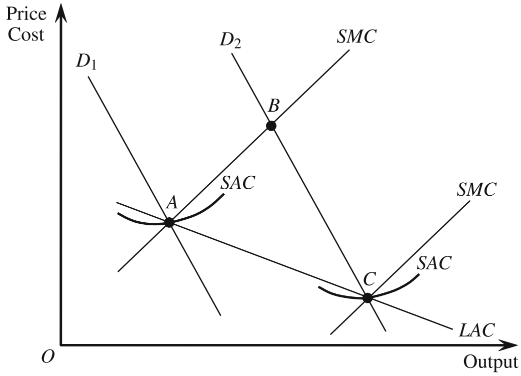
It is to be noted that generally *LAC* curve is tangent to *SAC* curves at the points like a and b which are different from the points of the minimum *SAC* like d and e where marginal cost curves (dotted curves in Fig. 2.9) intersect with *SAC* curves, unless *LAC* curve is horizontal. In other words, generally it is not the locus of the points of minimum short-run average cost. It is, however, not entirely wrong to consider the locus of such points as the long-run average cost curve, as will be seen in next section. What is wrong is to confuse two curves which are relevant, respectively, in two different situations.

2.8 Marshallian Externality

In this section, let us digress on Marshallian external economies and diseconomies. The cost of a single firm is considered as a function not only of the level of its own output, but also of the level of the aggregate output of the industry. When an industry expands, for example, the efficiency of public facilities like roads, sea and air ports, and railways, is likely to be increased, so that the cost of a single firm in the industry is reduced (external economy). Alternatively, an industrial expansion may increase congestion in the use of such facilities and the cost of individual firms may be increased (external diseconomy). In these cases, the cost curve of a firm cannot be drawn unless the level of the aggregate output of the industry is specified. Marshallian device for this problem is the representative firm which is a small replica of an industry with free entry.

Now the long-run average cost (*LAC*) curve is not the envelope but the locus of the minimum point of the short-run average cost (*SAC*) curves. In Fig. 2.10, we measure horizontally the volume of the industrial output or the corresponding output of the representative firm, and vertically, prices and costs. Curves *SAC* are short-run average cost curves of the representative firm, and curves *SMC* are short-run marginal cost curves of the representative firm, which is, however, also the short-run supply curve of the industry. Curves D_1, D_2, etc. are demand curves for the industry. Finally, the curves *LAC* is the long-run average cost curve of the representative firm, which is also the long-run supply curve of the industry.

Fig. 2.10



Suppose the demand curve is D_1, which intersects with *LAC* at the point A. The point A signifies the long-run equilibrium of the industry. The short-run average cost of the representative firm, which corresponds to the industrial output given by the abscissa of the point A, reaches the minimum at the point A, so that the point A is also on the curve *SMC*. Since the normal profit is considered to be included in the cost, the representative firm is earning the normal profit at the point A, and the volume of the industrial output remains unchanged. If the demand curve is shifted to D_2, the equilibrium is shifted first to the point B, at which D_2 intersects with the short-run supply curve *SMC*. Gradually, however, the cost curves of the representative firm are shifted downward by the external economies caused by the expansion of the industrial output due to the entry of new firms and the expansion of individual firms caused by the profit higher than the normal one at the point B. The long-run equilibrium is again established at the point C. Similar consideration can also be made in the case of external diseconomies.

The long-run average cost curve of the representative firm when Marshallian externality exists is thus the locus of the minimum point of the short-run average cost curves, like point A, C, etc., and should not be confused with the long-run average cost curve of a single competitive firm, which is, as was seen in Sect. 2.7, the envelope of the short-run average cost curves. The former curve is the locus of the industrial equilibria while the latter has nothing to do with the equilibrium of the industry.

2.9 Profit Function

Let us return to the consideration of the general case of a multiple product firm. In the short-run, the available quantities of some fixed factors of production are limited, and the feasible set of the firm's production function (2.6) is a convex set, where y_j denotes output of the j-th good, if it is positive, or input as a variable factors of production of the j-th good, if it is negative. Being subject to this condition, the firm maximizes the profit (2.5) where the firm takes the price of the j-th good, p_j, as given in the market, $j = 1, \ldots, m$. Since the maximized profit changes as the given prices are changed, it can be considered as a function of prices,

$$\pi(p) \tag{2.18}$$

where $p = (p_1, \ldots, p_m)$ is the vector of given prices.

We can show that this profit function is a convex function of prices. A convex function of a single variable is shown in Fig. 2.11. It has the positive second order derivative and the minimum point where the first order derivative is zero. More generally, the convex function is defined as follows. A function $f(x)$ of $x = (x_1, \ldots, x_m)$ is convex if and only if

$$tf(x) + (1-t)f(x') \geq f(tx + (1-t)x') \tag{2.19}$$

for any x and x', where t is any positive constant less than one. If $f(x)$ is not linear, and differentiable, furthermore, strict inequality has to hold.

To show that the profit function (2.18) is a convex function of p, consider any price vectors p and p' and their positive linear combination $p'' = tp + (1-t)p'$ for any $0 \leq t \leq 1$. Let y, y' and y'' be, respectively, the profit maximizing input–output vectors corresponding to p, p' and p'', i.e., $\pi(p) = py = \sum_j p_j y_j$, $\pi(p') = p'y' = \sum_j p'_j y'_j, \pi(p'') = p''y'' = \sum_j p''_j y''_j$. Then,

$$\pi(p'') = t \sum_j p_j y''_j + (1-t) \sum_j p'_j y''_j. \tag{2.20}$$

Since y and y' are profit maximizing input–output vectors corresponding, respectively, to p and p',

$$\sum_j p_j y''_j \leq \sum_j p_j y_j = \pi(p) \tag{2.21}$$

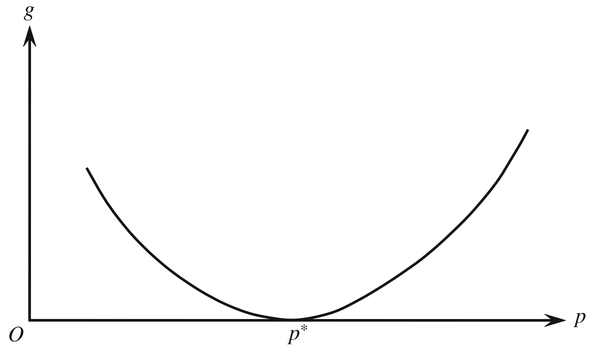
and

$$\sum_j p'_j y''_j \leq \sum_j p'_j y'_j = \pi(p'). \tag{2.22}$$

In view of (2.20), (2.21) and (2.22), therefore,

$$t\pi(p) + (1-t)\pi(p') \geq \pi(p'') = \pi(tp + (1-t)p') \tag{2.23}$$

Fig. 2.11



and the profit function is a convex function. If y and y' maximize profit uniquely at p and p' respectively, the strict inequality holds in (2.23).

An important implication of the convexity of the profit function is that the profit maximizing output or input y_j is derived from the profit function by the differentiation with respect to price p_j. To see this, let $y^* = (y_1^*, \ldots, y_m^*)$ be the profit maximizing input–output vector when the given price vector is $p^* = (p_1^*, \ldots, p_m^*)$. Consider the following function g.

$$g(p) = \pi(p) - \sum_j p_j y_j^*. \tag{2.24}$$

Since the maximum profit is $\pi(p)$ when the price vector is p and y^* maximizes the profit when the price vector is p^*, obviously $g(p) \geq 0$ and $g(p^*) = 0$. As is seen in Fig. 2.11, g reaches the minimum at $p = p^*$. Since $g(p)$ is a convex function, the condition for the maximization is

$$\frac{\partial g(p^*)}{\partial p_j} = \frac{\partial \pi(p^*)}{\partial p_j} - y_j^* = 0 \quad (j = 1, \ldots, m) \tag{2.25}$$

if π is assumed to be differentiable. Since this is true for any p^*, we have the so-called Hotelling's lemma

$$y_j(p) = \frac{\partial \pi(p)}{\partial p_j}. \tag{2.26}$$

2.10 Supply and Demand Function

The profit maximizing output or input function $y_j(p)$ is the firm's supply function of the j-th product or demand function of the j-th variable factor of production. The supply (demand) of the j-th good is an increasing (a diminishing) function of the price of the j-th good. This can be easily seen by comparison of two situations, respectively, with price vector p and price vector p'. If the profit maximizing vectors are, respectively, y and y', we have

$$\sum_j p_j y_j \geq \sum_j p_j y'_j \tag{2.27}$$

and

$$\sum_j p'_j y'_j \geq \sum_j p'_j y_j. \tag{2.28}$$

By subtracting (2.28) from (2.27), then, we have

$$\sum_j (p_j - p'_j)(y_j - y'_j) \geq 0. \tag{2.29}$$

If the difference between p and p' is only for the price of the j-th good,

$$(p_j - p'_j)(y_j - y'_j) \geq 0. \tag{2.30}$$

Since strict inequalities hold in (2.27), (2.28) and therefore in (2.29) and (2.30), if y and y' are unique profit maximizing vectors respectively at p and p', (2.30) implies that the supply of the j-th good ($y_j > 0$) is increasing and the demand for the j-th good ($y_j < 0$) is diminishing with respect to its price p_j. Firm's supply curves are upward sloping and demand curves are downward sloping.

Since the profit maximizing input–output vectors are derived from the convex profit function as we saw in the previous section (see (2.26)), this property of firm's supply and demand functions corresponds to the positive second order derivative of a convex function π with respect to p_j, i.e.,

$$\frac{\partial^2 \pi}{\partial p_j^2} = \frac{\partial y_j}{\partial p_j} > 0. \tag{2.31}$$

The use of the profit function also leads us to the symmetry of the cross price effects,

$$\frac{\partial y_j}{\partial p_k} = \frac{\partial^2 \pi}{\partial p_k \partial p_j} = \frac{\partial y_k}{\partial p_j}. \tag{2.32}$$

We can, therefore, define the substitutability and complementarity among inputs to produce a given output consistently by the cross price effects, i.e., two inputs are substitute if $\dfrac{\partial y_j}{\partial p_k} = \dfrac{\partial y_k}{\partial p_j}$ is negative and complementary if it is positive (since y_j and y_k are negative).

Problems

2.1. Draw curves *AB* and *EF* in Figs. 2.1 and 2.3 for the case of the Cobb–Douglas production function

$$Y = AL^a K^{(1-a)}$$

where Y, L and K denote, respectively, the output, the input of labor, and the input of capital, and A and a are constants such that $A > 0$, and $0 < a < 1$.

2.2. Derive the condition for the maximization of the profit $pY - wL - rK$, where p, w and r denote, respectively, the price of the product, the rate of wage, and the rental price of the capital, taken by firm as constants, being subject to the Cobb–Douglas production function given in Problem 2.1. Show the economic implication of the constant a.

2.3. Consider the so-called *CES* production function

$$Y^{(-b)} = aK^{(-b)} + (1-a)L^{(-b)} \quad (0 < a < 1,\ b > -1)$$

and show that the returns to scale are constant, and that the function approaches the Cobb–Douglas function as b approaches zero (use L'Hospital's rule). Draw curves *AB* and *EF* in Figs. 2.1 and 2.2.

2.4. The absolute value of the slope of the isoquant (curve *AB* in Fig. 2.1) is called the marginal rate of substitution between two inputs. In the case of Cobb–Douglas function and *CES* functions given in Problems 2.1 and 2.3, it is $R(K, L) = -\dfrac{dK}{dL}$, given Y. The elasticity of substitution between two inputs is defined as the elasticity (i.e., the ratio of relative changes) of the ratio of two inputs with respect to the marginal rate of substitution between them, as we move along an isoquant. In our case,

$$s = \frac{d\log(K/L)}{d\log R} = \frac{\dfrac{d(K/L)}{(K/L)}}{\dfrac{dR}{R}}.$$

Calculate s both for Cobb–Douglas and *CES* functions (Express R as a function of K/L). Why is the latter called *CES*?

2.5. Consider the problem of the maximization of the profit (2.11) subject to the conditions (2.10) in Sect. 2.4, by using the method of Lagrangian multipliers. Show the significance of multipliers as implicit or shadow prices of the fixed or limitational factors of production. Demonstrate that the maximized profit is exhaustively distributed to the fixed factors. Discuss the firm's policy to increase the stock (s_k's) of such factors in the long-run, when they can be bought at certain prices in the market.

2.6. Let us define that a set is strictly convex if it contains any point z on the line segment between x and y as an inner point (a point of a set, which does not belong to the boundary of the set), whenever it contains any two points x and y. Demonstrate that the profit maximizing input–output vector is unique, if the feasible set is strictly convex.

2.7. Discuss the difficulty of non-convexity case given in Fig. 2.6 (Compare profits at points H and O).

2.8. In Fig. 2.7, show that the minimum cost function is a convex function of the level of output.

2.9. A function h is defined as a concave function, if $h = -f$ and f is a convex function as defined in Sect. 2.9 (see (2.19)). State the condition of a concave function h directly in terms of h.

2.10. Show that the minimum cost function is a concave function of prices of inputs. Demonstrate the so-called Shephard's lemma that the cost-minimizing input of the j-th variable input to produce the given output is derived from the minimum cost function by the differentiation with respect to its price.

Literature

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