Chapter 2
The First Main Theorem

The value distribution theory with domains in several complex variables was pioneered by Wilhelm Stoll [53a], [53b], [54]. While his presentation may not be familiar or easy to us in modern terminologies, the works which he has contributed, beginning with the integrations over singular analytic subvarieties and the extension of Stokes' theorem, were fundamental. In the 1960s there were many works on the First Main Theorem; these were summarized by W. Stoll (see Stoll [70], in particular its preface and the listed references). The relation to characteristic classes was made explicit first by Bott–Chern [65].[1] In the present chapter we follow Carlson–Griffiths [72], Griffiths–King [73], Noguchi [03b] and Noguchi–Winkelmann–Yamanoi [08] which may be most comprehensive.

2.1 Plurisubharmonic Functions

2.1.1 One Variable

We first investigate subharmonic functions. Let U be an open subset of $\mathbf{C}$. Set

$$d(a;\partial U)=\inf\bigl\{|a-w|;\, w\in\partial U\bigr\}.$$

Definition 2.1.1 A function $\varphi: U\to[-\infty,\infty)$ is said to be *subharmonic* if φ is upper semicontinuous and has the submean property; that is,

(i) (upper semicontinuity) $\overline{\lim}_{z\to a}\varphi(z)\leqq\varphi(a)$, ${}^{\forall}a\in U$:
(ii) (submean property) On an arbitrary disk $\Delta(a;r)\Subset U$

$$\varphi(a)\leqq\frac{1}{2\pi}\int_0^{2\pi}\varphi\bigl(a+re^{i\theta}\bigr)d\theta.$$

[1] Readers may find a number of interesting papers on the theory of holomorphic mappings in Chern, *Selected Papers* (Chern [78]).

J. Noguchi, J. Winkelmann, *Nevanlinna Theory in Several Complex Variables and Diophantine Approximation*, Grundlehren der mathematischen Wissenschaften 350,
DOI 10.1007/978-4-431-54571-2_2,

Remark 2.1.2 (i) If $\varphi : U \to [-\infty, \infty)$ is upper semicontinuous, φ is bounded from above on every compact subset $K \Subset U$.

(ii) The upper semicontinuity of $\varphi : U \to [-\infty, \infty)$ is equivalent to that for every $c \in \mathbf{R}$ the sublevel set $\{z \in U; \varphi(z) < c\}$ is open.

(iii) The function $\varphi : U \to [-\infty, \infty)$ is upper semicontinuous if and only if there is a monotone decreasing sequence of continuous functions $\psi_\nu : U \to \mathbf{R}$, $\nu = 1, 2, \ldots$, such that $\lim_{\nu\to\infty} \psi_\nu(z) = \varphi(z)$.

(iv) It follows from the above Definition 2.1.1 (ii) that

$$\begin{aligned} \varphi(a) &\leqq \frac{1}{\pi r^2}\int_0^r t\,dt \int_0^{2\pi} \varphi\bigl(a + te^{i\theta}\bigr)d\theta \\ &= \frac{1}{r^2}\int_{|\zeta|<r} \varphi(a+\zeta)\frac{i}{2\pi}d\zeta \wedge d\bar{\zeta} < \infty. \end{aligned} \tag{2.1.3}$$

Theorem 2.1.4 (i) *Let φ be a subharmonic function on U. Let $a \in U$ be a point such that $\varphi(a) > -\infty$. Then φ is locally integrable on the connected component of U containing a.*

(ii) *Let φ be a subharmonic function on U. If φ takes the maximum value at $a \in U$, then φ is constant on the connected component of U containing a.*

(iii) *Assume that $\varphi \in C^2(U)$. Then φ is subharmonic if and only if $dd^c\varphi = (i/2\pi)\partial\bar{\partial}\varphi \geqq 0$.*

(iv) *Let $\varphi : U \to [-\infty, \infty)$ be subharmonic, and let λ be a monotone increasing convex function defined on $\mathbf{R}$. Then $\lambda \circ \varphi$ is subharmonic. Here we put $\lambda(-\infty) = \lim_{t\to-\infty} \lambda(t)$.*

(v) *Let $\varphi_\nu : U \to [-\infty, \infty)$, $\nu = 1, 2, \ldots$, be a monotone decreasing sequence of subharmonic functions. Then the limit function $\varphi(z) = \lim_{\nu\to\infty} \varphi_\nu(z)$ is subharmonic, too.*

(vi) *Let $\varphi_\nu : U \to [-\infty, \infty)$, $1 \leqq \nu \leqq l$, be finitely many subharmonic functions. Then $\varphi(z) = \max_{1\leqq\nu\leqq l} \varphi_\nu(z)$ is subharmonic.*

Proof (i) Without loss of generality we may assume that U is connected. Notice that if $\varphi(a) > -\infty$, then φ is integrable on every relatively compact disk $\Delta(a; r) \Subset U$ by (2.1.3). Suppose that there is a point $a \in U$ with $\varphi(a) > -\infty$. Denote by U_0 the set of all points $z \in U$ with a neighborhood W such that the restriction $\varphi|_W$ of φ to W is integrable. Clearly, U_0 is non-empty and open.

We show that U_0 is closed in U. Let $a \in U$ be an accumulation point of U_0. Take a sequence of points $z_\nu \in U_0$, $\nu = 1, 2, \ldots$, convergent to a. One may assume that $\varphi(z_\nu) > -\infty$, $\nu = 1, 2, \ldots$. There are some $r > 0$ and a sufficiently large ν such that $a \in \Delta(z_\nu; r) \Subset U$. By the remark at the beginning, $\varphi|_{\Delta(z_\nu;r)}$ is integrable. Therefore $a \in U_0$. Since U is connected, $U_0 = U$.

(ii) Assume that U is connected and $\varphi(a)$ is the maximum. It follows from (2.1.3) that for every $\Delta(a; r) \Subset U$

$$\int_{\Delta(a;r)} \bigl\{\varphi(\zeta) - \varphi(a)\bigr\}\frac{i}{2\pi}d\zeta \wedge d\bar{\zeta} = 0. \tag{2.1.5}$$

By assumption $\varphi(\zeta)-\varphi(a) \leqq 0$. Suppose that $\varphi(b)-\varphi(a)=\delta_0<0$ at a point $b \in \Delta(a;r)$. The upper semicontinuity of φ implies that $\varphi(\zeta)-\varphi(a)<\frac{\delta_0}{2}$ in a neighborhood of b. Then (2.1.5) does not hold. Hence $\varphi|_{\Delta(a;r)} \equiv \varphi(a)$. Denote by U_1 the set of all points $z \in U$ with a neighborhood W such that $\varphi|_W \equiv \varphi(a)$. By a similar argument to (i) U_1 is open and closed in U. Therefore $U_1=U$.

(iii) About every point $a \in U$ we expand φ to a Taylor series up to degree two:

$$\begin{aligned}\varphi\big(a+\varepsilon e^{i\theta}\big) &= \varphi(a)+\frac{\partial\varphi}{\partial z}(a)\varepsilon e^{i\theta}+\frac{\partial\varphi}{\partial\bar{z}}(a)\varepsilon e^{-i\theta} \\ &\quad + \varepsilon^2\left(\frac{\partial^2\varphi}{\partial z^2}(a)e^{2i\theta}+2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(a)+\frac{\partial^2\varphi}{\partial \bar{z}^2}(a)e^{-2i\theta}\right)\big(1+o(1)\big).\end{aligned}$$

Taking the integration in θ, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(a+\varepsilon e^{i\theta}\big)d\theta=\varphi(a)+\varepsilon^2\big(1+o(1)\big)2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(a).$$

The submean property implies that $\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(a) \geqq 0$.

Conversely, assume that $\frac{\partial^2\varphi}{\partial z\partial\bar{z}} \geqq 0$. It follows from Jensen's formula, Lemma 1.1.5 that about every point $a \in U$

$$\text{(2.1.6)}\quad \frac{1}{2\pi}\int_{|\zeta|=s}\varphi(a+\zeta)d\theta \leqq \frac{1}{2\pi}\int_{|\zeta|=r}\varphi(a+\zeta)d\theta, \quad 0<s<r<d\,(a;\partial U).$$

Let $s \searrow 0$. Then

$$\varphi(a) \leqq \frac{1}{2\pi}\int_{|\zeta|=r}\varphi(a+\zeta)d\theta.$$

(iv) Note that λ is continuous. The remaining part is immediate, for λ is monotone increasing and convex.

(v) The upper semicontinuity of φ follows immediately from the assumption. Since upper semicontinuous functions are bounded from above on every relatively compact subset, φ_ν are uniformly bounded from above on every relatively compact subset. Take an arbitrary disk $\Delta(a;r) \Subset U$. By Fatou's lemma in integration theory we have

$$\begin{aligned}\varphi(a) &= \lim_{\nu\to\infty}\varphi_\nu(a) \leqq \overline{\lim_{\nu\to\infty}}\frac{1}{2\pi}\int_0^{2\pi}\varphi_\nu\big(a+re^{i\theta}\big)d\theta \\ &\leqq \frac{1}{2\pi}\int_0^{2\pi}\overline{\lim_{\nu\to\infty}}\varphi_\nu\big(a+re^{i\theta}\big)d\theta=\frac{1}{2\pi}\int_0^{2\pi}\varphi\big(a+re^{i\theta}\big)d\theta.\end{aligned}$$

(vi) Both the upper-semicontinuity and the submean property are immediate by definition. □

Example 2.1.7 Let $f: U \to \mathbf{C}$ be a holomorphic function. Then $\log|f|$ and $|f|^c$ with $c>0$ are subharmonic. Because a direct computation of partial derivatives of

$\log(|f|^2 + C)$ with $C > 0$ implies the subharmonicity $\log(|f|^2 + C)$. Setting $C = 1/\nu$, $\nu = 1, 2, \ldots$, and taking the limit, we see by Theorem 2.1.4 (v) that $\log|f|^2 = 2\log|f|$ is subharmonic, and so is $\log|f|$. Since the exponential function e^{ct}, $t \in \mathbf{R}$ with $c > 0$ is monotone increasing and convex, Theorem 2.1.4 (iv) implies that $|f|^c$ is subharmonic.

Let $\chi \in C_0^\infty(\mathbf{C})$ be a function such that $\operatorname{Supp}\chi \subset \Delta(1)$, $\chi(z) = \chi(|z|) \geqq 0$ and

$$\int \chi(z)\frac{i}{2}dz \wedge d\bar{z} = 1.$$

Set $\chi_\varepsilon(z) = \chi(\varepsilon^{-1}z)\varepsilon^{-2}$, $\varepsilon > 0$. Then

$$\int \chi_\varepsilon(z)\frac{i}{2}dz \wedge d\bar{z} = 1.$$

Consider a subharmonic function φ on U such that $\varphi \not\equiv -\infty$ on every connected component of U. Put

$$U_\varepsilon = \left\{z \in U; d(z; \partial U) > \varepsilon\right\}.$$

The *smoothing* $\varphi_\varepsilon(z)$ $(z \in U_\varepsilon)$ of φ is defined by

$$\begin{aligned}
(2.1.8)\qquad \varphi_\varepsilon(z) = \varphi * \chi_\varepsilon(z) &= \int_{\mathbf{C}} \varphi(w)\chi_\varepsilon(w - z)\frac{i}{2}dw \wedge d\bar{w} \\
&= \int_{\mathbf{C}} \varphi(z + w)\chi_\varepsilon(w)\frac{i}{2}dw \wedge d\bar{w} \\
&= \int_0^1 \chi(t)t\,dt \int_0^{2\pi} \varphi\big(z + \varepsilon t e^{i\theta}\big)d\theta \\
&\geqq \varphi(z)\int_0^1 2\pi\chi(t)t\,dt = \varphi(z).
\end{aligned}$$

Note that $\varphi_\varepsilon(z)$ is C^∞ on U_ε, and subharmonic. Therefore Theorem 2.1.4 implies

$$\frac{\partial^2}{\partial z\partial\bar{z}}\varphi_\varepsilon(z) \geqq 0.$$

Taking $\varepsilon_1 > \varepsilon_2 > 0$, and $\delta > 0$, we consider the double smoothing $(\varphi_\delta)_{\varepsilon_i} = (\varphi_{\varepsilon_i})_\delta$, $i = 1, 2$. Note that φ_δ is C^∞ and subharmonic. Combining (2.1.6) applied to φ_δ with (2.1.8) applied to $\varphi = \varphi_\delta$, we deduce that $(\varphi_\delta)_{\varepsilon_1} \geqq (\varphi_\delta)_{\varepsilon_2}$. Hence $(\varphi_{\varepsilon_1})_\delta \geqq (\varphi_{\varepsilon_2})_\delta$. Letting $\delta \to 0$, we see that $\varphi_{\varepsilon_1} \geqq \varphi_{\varepsilon_2}$. Thus as $\varepsilon \searrow 0$, $\varphi_\varepsilon(z)$ monotonically decreases, and it follows from (2.1.8) that

$$\varphi(z) \leqq \lim_{\varepsilon\to 0}\varphi_\varepsilon(z).$$

Here we show the equality by making use of the upper semicontinuity.

Suppose that $\varphi(z) = -\infty$. For every $K < 0$ there exists a disk neighborhood $\Delta(z;r) \subset U$ such that $\varphi|_{\Delta(z;r)} < K$. By definition (2.1.8) $\varphi_\varepsilon(z) < K$ for $\varepsilon < r$, and so $\lim_{\varepsilon\to 0} \varphi_\varepsilon(z) = -\infty$.

Suppose that $\varphi(z) > -\infty$. For every $\varepsilon' > 0$ there is a disk $\Delta(z;r) \subset U$ such that $\varphi|_{\Delta(z;r)} < \varphi(z) + \varepsilon'$. By the same reasoning as above, $\varphi_\varepsilon(z) \leqq \varphi(z) + \varepsilon'$ for $\varepsilon < r$. Thus $\lim_{\varepsilon\to 0} \varphi_\varepsilon(z) = \varphi(z)$.

Now we have the convergence, $\varphi_\varepsilon(z) \searrow \varphi(z)$ $(\varepsilon \searrow 0)$. For $\eta \in C_0^\infty(U)$

$$\int \eta(z) dd^c \varphi_\varepsilon(z) = \int \varphi_\varepsilon(z) dd^c \eta(z). \tag{2.1.9}$$

If $\eta \geqq 0$, this integral is non-negative. We set

$$dd^c[\varphi] = \frac{i}{2\pi} \frac{\partial^2}{\partial z \partial \bar{z}} [\varphi] dz \wedge d\bar{z}$$

in the sense of the Schwartz distribution. As $\varepsilon \searrow 0$, (2.1.9) implies

$$\int \eta(z) dd^c \big[\varphi(z)\big] = \int \varphi(z) dd^c \eta(z) \geqq 0, \quad \eta \geqq 0.$$

We see that $dd^c[\varphi]$ is a positive Radon measure. We may also regard $dd^c[\varphi]$ as a differential form with coefficients in Radon measures.

We apply (2.1.6) to the C^∞ subharmonic function φ_ε; for $\Delta(a;r) \Subset U$, $0 < s < r$ and sufficiently small $\varepsilon > 0$ we obtain

$$\frac{1}{2\pi} \int_{|\zeta|=s} \varphi_\varepsilon(a+\zeta) d\theta \leqq \frac{1}{2\pi} \int_{|\zeta|=r} \varphi_\varepsilon(a+\zeta) d\theta.$$

As $\varepsilon \searrow 0$, Lebesgue's monotone convergence theorem implies that

$$\frac{1}{2\pi} \int_{|\zeta|=s} \varphi(a+\zeta) d\theta \leqq \frac{1}{2\pi} \int_{|\zeta|=r} \varphi(a+\zeta) d\theta. \tag{2.1.10}$$

By Theorem 2.1.4 (i) φ is locally integrable on U. Fubini's theorem and (2.1.3) imply that for almost all $s \in (0,r)$ with respect to the Lebesgue measure

$$\frac{1}{2\pi} \int_{|\zeta|=s} \varphi(a+\zeta) d\theta > -\infty. \tag{2.1.11}$$

This with (2.1.10) implies (2.1.11) for all $s \in (0,r]$.

Summarizing the above we have the next theorem.

Theorem 2.1.12 *Let $\varphi : U \to [-\infty, \infty)$ be a subharmonic function on U such that $\varphi \not\equiv -\infty$ on every connected component of U.*

(i) *$dd^c[\varphi]$ is a positive Radon measure.*

(ii) *The smoothing $\varphi_\varepsilon(z)$ is subharmonic; as $\varepsilon \searrow 0$ it is monotone decreasing and converges to $\varphi(z)$.*

(iii) *For $\Delta(a;r) \Subset U$ and any $s \in (0,r)$*

$$-\infty < \frac{1}{2\pi}\int_{|\zeta|=s} \varphi(a+\zeta)d\theta \leqq \frac{1}{2\pi}\int_{|\zeta|=r} \varphi(a+\zeta)d\theta < \infty.$$

Theorem 2.1.13 (i) *The subharmonicity is a local property*; *i.e., if $\varphi : U \to [-\infty, \infty)$ is subharmonic in a neighborhood of every point $a \in U$, then φ is subharmonic in U.*

(ii) *If an upper semicontinuous function $\varphi : U \to [-\infty, \infty)$ satisfies*

$$\varphi(a) \leqq \frac{1}{r^2}\int_{\Delta(a;r)} \varphi(z)\frac{i}{2\pi}dz \wedge d\bar{z}$$

for every disk $\Delta(a;r) \subset U$, then φ is subharmonic.

Proof (i) Take the smoothing $\varphi_\varepsilon(z)$. Let φ be subharmonic in $\Delta(a;r) \subset U$. Then φ_ε, $0 < \varepsilon < r/2$ is subharmonic in $\Delta(a;r/2)$. Therefore $dd^c\varphi_\varepsilon(z) \geqq 0$, and hence by Theorem 2.1.4 (iii) $\varphi_\varepsilon(z)$ is subharmonic in U_ε.

For Definition 2.1.1 (ii), it suffices to show that φ is subharmonic in an arbitrarily fixed U_δ ($\delta > 0$). As $\delta > \varepsilon \searrow 0$, $\varphi_\varepsilon(z) \searrow \varphi(z)$ in U_δ. We infer from Theorem 2.1.4 (v) that φ is subharmonic in U_δ.

(ii) We first assume that φ is of C^2-class. By the same computation as in the proof of Theorem 2.1.4 (iii) we get

$$\int_0^\varepsilon t dt \frac{1}{2\pi}\int_0^{2\pi} \varphi\big(a + te^{i\theta}\big)d\theta = \int_0^\varepsilon \left(t\varphi(a) + t^3\big(1+o(1)\big)2\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(a)\right)dt.$$

It follows that

$$\frac{1}{\varepsilon^2}\int_{\Delta(\varepsilon)} \varphi(a+\zeta)\frac{i}{2\pi}d\zeta \wedge d\bar{\zeta} = \varphi(a) + \varepsilon^2\big(1+o(1)\big)\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(a).$$

This combined with the assumption implies that $\frac{\partial^2\varphi}{\partial z\partial\bar{z}}(a) \geqq 0$. Thus $\varphi(z)$ is subharmonic.

For the general case we may assume that U is connected, and $\varphi \not\equiv -\infty$. By the proof of Theorem 2.1.4 (i) φ is locally integrable in U.

We take the smoothing $\varphi_\varepsilon(z)$ $z \in U_\varepsilon$. Since $\varphi(z)$ is upper semicontinuous, by Remark 2.1.2 (iii) there is a monotone decreasing sequence of continuous functions $\psi_\nu(z)$, $\nu = 1, 2, \ldots$, such that

$$\lim_{\nu\to\infty} \psi_\nu(z) = \varphi(z), \quad z \in U.$$

We are going to show that for every compact subset $K \subset U$

$$\lim_{\varepsilon\to 0}\int_K \big|\varphi_\varepsilon(z) - \varphi(z)\big|\frac{i}{2\pi}dz \wedge d\bar{z} = 0. \tag{2.1.14}$$

Take $W \Subset U$ be an open subset such that $W \Supset K$. Set

$$d(K, \partial W) = \inf\{d(z, \partial W); z \in K\},$$

and

$$\delta_0 = \min\{d(K, \partial W), d(\bar{W}, \partial U)\} > 0.$$

Take any $\varepsilon' > 0$. By Lebesgue's monotone convergence theorem there is a number ν_0 such that for $\|w\| < \delta_0$

$$\begin{aligned} 0 &\leqq \int_K \left(\psi_{\nu_0}(z+w) - \varphi(z+w)\right)\frac{i}{2\pi}dz \wedge d\bar{z} \\ &\leqq \int_{\bar{W}} \left(\psi_{\nu_0}(z) - \varphi(z)\right)\frac{i}{2\pi}dz \wedge d\bar{z} < \varepsilon'. \end{aligned} \tag{2.1.15}$$

Let $0 < \varepsilon < \delta_0$. Then $(\psi_{\nu_0})_\varepsilon(z) \geqq \varphi_\varepsilon(z)$ $(z \in \bar{W})$, and

$$\begin{aligned} &0 \leqq \int_K \left((\psi_{\nu_0})_\varepsilon(z) - \varphi_\varepsilon(z)\right)\frac{i}{2\pi}dz \wedge d\bar{z} \\ &= \int_{z \in K}\left(\int_{w \in \mathbf{C}^m} \left((\psi_{\nu_0})(z+w) - \varphi(z+w)\right)\chi_\varepsilon(w)\frac{i}{2\pi}dw \wedge d\bar{w}\right)\frac{i}{2\pi}dz \wedge d\bar{z} \\ &= \int_{w \in \mathbf{C}^m}\left(\int_{z \in K} \left((\psi_{\nu_0})(z+w) - \varphi(z+w)\right)\frac{i}{2\pi}dz \wedge d\bar{z}\right)\chi_\varepsilon(w)\frac{i}{2\pi}dw \wedge d\bar{w} \\ &\leqq \varepsilon'. \end{aligned} \tag{2.1.16}$$

Since ψ_{ν_0} is uniformly continuous on $\bar{W}$, $(\psi_{\nu_0})_\varepsilon$ uniformly approximates ψ_{ν_0} on K as $\varepsilon \to 0$. Thus there is some $0 < \varepsilon_0 < \delta_0$ such that for every $0 < \varepsilon < \varepsilon_0$

$$\int_K \left|(\psi_{\nu_0})_\varepsilon(z) - \psi_{\nu_0}(z)\right|\frac{i}{2\pi}dz \wedge d\bar{z} < \varepsilon'. \tag{2.1.17}$$

It follows from (2.1.15)–(2.1.17) that for every $0 < \varepsilon < \varepsilon_0$

$$\begin{aligned} \int_K |\varphi_\varepsilon - \varphi|\frac{i}{2\pi}dz \wedge d\bar{z} &\leqq \int_K \left((\psi_{\nu_0})_\varepsilon - \varphi_\varepsilon\right)\frac{i}{2\pi}dz \wedge d\bar{z} \\ &\quad + \int_K \left|(\psi_{\nu_0})_\varepsilon - \psi_{\nu_0}\right|\frac{i}{2\pi}dz \wedge d\bar{z} + \int_K (\psi_{\nu_0} - \varphi)\frac{i}{2\pi}dz \wedge d\bar{z} \\ &< 3\varepsilon'. \end{aligned}$$

Thus (2.1.14) is deduced.

We see by the assumption and Fubini's theorem that for $\Delta(z; r) \subset U_\delta$

$$\varphi_\delta(z) \leqq \frac{1}{r^2}\int_{\Delta(r)} \varphi_\delta(z+\zeta)\frac{i}{2\pi}d\zeta \wedge d\bar{\zeta}.$$

Therefore φ_δ is subharmonic in U_δ. Take $\delta > \varepsilon_1 > \varepsilon_2 > 0$, arbitrarily. For $z \in U_{\delta+\varepsilon_1}$

$$(\varphi_{\varepsilon_1})_\delta(z) = (\varphi_\delta)_{\varepsilon_1}(z) \geqq (\varphi_\delta)_{\varepsilon_2}(z) = (\varphi_{\varepsilon_2})_\delta(z).$$

As $\delta \to 0$, $\varphi_{\varepsilon_1}(z) \geqq \varphi_{\varepsilon_2}(z)$. Put $\psi(z) = \lim_{\varepsilon \to 0} \varphi_\varepsilon(z)$. Since ψ is the limit of a monotone decreasing sequence of subharmonic functions, it is subharmonic. It follows from the upper semicontinuity that for every $z \in U$ and $\varepsilon' > 0$ there is a neighborhood $\Delta(z; r) \subset U$ satisfying

$$\varphi(\zeta) < \varphi(z) + \varepsilon', \quad \zeta \in \Delta(z; r).$$

Hence, for $0 < \varepsilon < r$, $\varphi_\varepsilon(z) \leqq \varphi(z) + \varepsilon'$. We have

$$\varphi(z) - \psi(z) \geqq 0, \quad z \in U.$$

This combined with (2.1.14) implies that for any compact subset $K \subset U$

$$\begin{aligned} 0 &\leqq \int_K \big(\varphi(z) - \psi(z)\big) \frac{i}{2\pi} dz \wedge d\bar{z} \\ &= \lim_{\varepsilon \to 0} \int_K \big(\varphi(z) - \varphi_\varepsilon(z)\big) \frac{i}{2\pi} dz \wedge d\bar{z} \\ &\leqq \lim_{\varepsilon \to 0} \int_K \big|\varphi(z) - \varphi_\varepsilon(z)\big| \frac{i}{2\pi} dz \wedge d\bar{z} = 0. \end{aligned}$$

Therefore $\psi(z) = \varphi(z)$ for almost all $z \in U$ with respect to the Lebesgue measure. For every $\Delta(a; r) \subset U$,

$$\begin{aligned} \varphi(a) &\leqq \frac{1}{r^2} \int_{\Delta(a;r)} \varphi(z) \frac{i}{2\pi} dz \wedge d\bar{z} \\ &= \frac{1}{r^2} \int_{\Delta(a;r)} \psi(z) \frac{i}{2\pi} dz \wedge d\bar{z} \\ &\to \psi(a) \quad (r \to 0). \end{aligned}$$

Hence, $\varphi(a) \leqq \psi(a)$, and $\psi = \varphi$, by which φ is subharmonic. □

Proposition 2.1.18 *A function ϕ on U is subharmonic if and only if there exists a decreasing sequence of C^2 subharmonic functions ϕ_n with $\lim \phi_n = \phi$.*

Proof If there is such a sequence, subharmonicity of ϕ follows from Theorem 2.1.4 (v). The converse is obtained by making use of the smoothing (Theorem 2.1.12 (ii)). □

Proposition 2.1.18 together with Theorem 2.1.4 (iii) implies the following.

Theorem 2.1.19 *Let U, V be open in* **C**.

(i) *If ϕ is subharmonic on U and $f : V \to U$ is holomorphic, then $\phi \circ f$ is subharmonic.*

(ii) *If ϕ is a function on U and $f : V \to U$ is biholomorphic, then ϕ is subharmonic if and only if $\phi \circ f$ is subharmonic.*

Proof (i) Due to Proposition 2.1.18 there is a decreasing sequence of C^2 subharmonic functions ϕ_n $(n = 1, 2, \dots)$ with $\lim \phi_n = \phi$. Subharmonicity of the ϕ_n is equivalent to $dd^c\varphi_n \geqq 0$ (Theorem 2.1.4 (iii)). Hence the functions $\phi_n \circ f$ $(n = 1, 2, \dots)$ form a decreasing sequence of subharmonic functions converging to $\phi \circ f$. Now the subharmonicity of $\phi \circ f$ follows from Theorem 2.1.4 (v).

Statement (ii) is a direct consequence of (i). □

2.1.2 Several Variables

We deal with the case of several complex variables. The notion of plurisubharmonic functions was first introduced by K. Oka [42] VI. We let $U \subset \mathbf{C}^m$ be an open set. Let $z = (z_1, \dots, z_m)$ be the standard coordinate system of $\mathbf{C}^m$. As usual we set

$$\|z\| = \sqrt{\sum |z_j|^2},$$
$$d(z; \partial U) = \inf\left\{\|z - w\|; w \in \partial U\right\}, \quad z \in U,$$
$$U_\varepsilon = \left\{z \in U; d(z; \partial U) > \varepsilon\right\}, \quad \varepsilon > 0.$$

We write $z_j = x_j + iy_j$ $(1 \leqq j \leqq m)$. As in (1.1.1), we define the following differential operators:

$$\frac{\partial\varphi}{\partial z_j} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_j} + \frac{1}{i}\frac{\partial\varphi}{\partial y_j}\right), \qquad \frac{\partial\varphi}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial\varphi}{\partial x_j} - \frac{1}{i}\frac{\partial\varphi}{\partial y_j}\right),$$
$$dz_j = dx_j + i dy_j, \qquad d\bar z_j = dx_j - i dy_j,$$
$$\partial\varphi = \sum_{j=1}^m \frac{\partial\varphi}{\partial z_j} dz_j, \qquad \bar\partial\varphi = \sum_{j=1}^m \frac{\partial\varphi}{\partial \bar z_j} d\bar z_j, \tag{2.1.20}$$
$$d^c\varphi = \frac{i}{4\pi}(\bar\partial\varphi - \partial\varphi) = \frac{1}{4\pi}\sum_{j=1}^m \left(\frac{\partial\varphi}{\partial x_j} dy_j - \frac{\partial\varphi}{\partial y_j} dx_j\right).$$

With this notation we have

$$d\varphi = \partial\varphi + \bar\partial\varphi, \qquad dd^c\varphi = \frac{i}{2\pi}\partial\bar\partial\varphi,$$
$$\partial\bar\partial\varphi = \sum_{j,k=1}^m \frac{\partial^2\varphi}{\partial z_j \partial \bar z_k} dz_j \wedge d\bar z_k.$$

We further introduce the following notation:

$$
\begin{aligned}
B(a;r) &= \{z \in \mathbf{C}^m;\ \|z-a\| < r\}, \quad a \in \mathbf{C}^m,\ r > 0,\\
B(r) &= B(0;r),\\
\alpha &= dd^c\|z\|^2, \qquad \beta = dd^c \log\|z\|^2,\\
\gamma &= d^c \log\|z\|^2 \wedge \beta^{m-1}.
\end{aligned}
\tag{2.1.21}
$$

Let ϕ be a differential form on $\mathbf{C}^m$ and let $\iota : \{\|z\| = r\} \hookrightarrow \mathbf{C}^m$ be the inclusion of the sphere $\{\|z\| = r\}$ into $\mathbf{C}^m$. Associated to the map ι there is the "pull-back" of differential forms. In this way $\iota^*\phi$ is the differential form induced from ϕ on the sphere $\{\|z\| = r\}$. In the present case, $\iota^*(d\|z\|^2) = 0$, and so as differential forms induced over $\{\|z\| = r\}$, $d\|z\|^2 = \partial\|z\|^2 + \bar{\partial}\|z\|^2 = 0$. Therefore as differential forms induced over $\{\|z\| = r\}$,

$$
d\|z\|^2 \wedge d^c\|z\|^2 = 0, \qquad \partial\|z\|^2 \wedge \bar{\partial}\|z\|^2 = 0.
$$

Hence we have, as induced forms on $\{\|z\| = r\}$,

$$
\beta = \frac{1}{r^2}\alpha. \tag{2.1.22}
$$

It follows that

$$
\int_{B(r)} \alpha^m = r^{2m}, \qquad \int_{\|z\|=r} \gamma = 1.
$$

Definition 2.1.23 A function $\varphi : U \to [-\infty, \infty)$ is said to be *plurisubharmonic* if the following conditions are satisfied:

(i) φ is upper semicontinuous.
(ii) For every point $z \in U$ and every vector $v \in \mathbf{C}^m$ the function

$$
\zeta \in \mathbf{C} \to \varphi(z + \zeta v) \in [-\infty, \infty)
$$

is subharmonic where it is defined.

We have the following examples by Example 2.1.7.

Example 2.1.24 If $f : U \to \mathbf{C}$ is a holomorphic function, $\log|f|$ and $|f|^c$ $(c > 0)$ are both plurisubharmonic.

Let φ be a plurisubharmonic function on U, and let $B(a;r) \Subset U$. By making use of the invariance of α with respect to the rotation $z \mapsto e^{i\theta}z$ $(\theta \in [0, 2\pi])$ we have by Definition 2.1.1 (ii) that

$$
\int_{z\in B(r)} \varphi(a+z)\alpha^m(z) = \int_{z\in B(r)} \varphi\bigl(a + e^{i\theta}z\bigr)\alpha^m(z)
$$

(continued)

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_{z\in B(r)} \varphi\bigl(a+e^{i\theta}z\bigr)\alpha^m(z) \\
&= \int_{z\in B(r)} \left(\frac{1}{2\pi}\int_0^{2\pi} \varphi\bigl(a+e^{i\theta}z\bigr)d\theta\right)\alpha^m(z) \\
&\geqq \int_{z\in B(r)} \varphi(a)\alpha^m(z) = r^{2m}\varphi(a).
\end{aligned}
$$

Thus as in (2.1.3) the following is obtained:

$$
\begin{aligned}
(2.1.25)\qquad \varphi(a) &\leqq \frac{1}{r^{2m}}\int_0^r 2mt^{2m-1}dt \int_{\|z\|=t} \varphi(a+z)\gamma(z) \\
&= \frac{1}{r^{2m}}\int_{B(a;r)} \varphi(z)\alpha^m \quad \bigl(B(a;r)\Subset U\bigr).
\end{aligned}
$$

Identifying $\mathbf{C}^m \cong \mathbf{R}^{2m}$, we see that φ is a *subharmonic function*[2] on $U \subset \mathbf{R}^{2m}$. When φ is of C^2-class, we have by definition

$$
dd^c\varphi = \sum_{1\leqq j,k\leqq m} \frac{\partial^2\varphi}{\partial z_j\partial \bar{z}_k}\frac{i}{2\pi}dz_j\wedge d\bar{z}_k.
$$

We write $dd^c\varphi \geqq 0$ if the hermitian matrix $(\frac{\partial^2\varphi}{\partial z_j\partial\bar{z}_k})$ is semi-positive definite.

The next theorem follows from the above and the same arguments as used in the proof of Theorem 2.1.4:

Theorem 2.1.26 (i) *A plurisubharmonic function is subharmonic with identification* $\mathbf{C}^m \cong \mathbf{R}^{2m}$.

(ii) *If φ is a plurisubharmonic function on U and $\varphi(a) > -\infty$ at a point $a\in U$, then φ is locally integrable in the connected component U' of U containing a.*

(iii) *Let φ be a plurisubharmonic function on U. If φ admits the maximum value at $a\in U$, then it is constant on the connected component of U containing a.*

(iv) *Let φ be of C^2-class. Then φ is plurisubharmonic if and only if $dd^c\varphi\geqq 0$.*

(v) *Let $\varphi: U\to[-\infty,\infty)$ be plurisubharmonic and let λ be a monotone increasing convex function defined on* $\mathbf{R}$. *Then $\lambda\circ\varphi$ is plurisubharmonic. Here,* $\lambda(-\infty)=\lim_{t\to-\infty}\lambda(t)$.

(vi) *Let $\varphi_\nu: U\to[-\infty,\infty)$, $\nu=1,2,\ldots$, be monotone decreasing plurisubharmonic functions. Then the limit function $\varphi(z)=\lim_{\nu\to\infty}\varphi_\nu(z)$ is plurisubharmonic.*

(vii) *For finitely many plurisubharmonic functions $\varphi_\nu: U\to[-\infty,\infty)$, $1\leqq \nu\leqq l$, $\varphi(z)=\max_{1\leqq\nu\leqq l}\varphi_\nu(z)$ is plurisubharmonic, too.*

---

[2] In general, a function $\psi: W\to[-\infty,\infty)$ defined on an open subset W of $\mathbf{R}^n$ is said to be subharmonic if ψ is upper semicontinuous and satisfies the submean property in the sense of (2.1.25).

Here we explain the notion of currents, limited to what we will need. Cf. Noguchi–Ochiai [90] (Ochiai–Noguchi [84]) for more detailed treatment. In general a differential form with coefficients in distributions in the sense of Schwartz is called a *current*. We introduce only $(1,1)$ currents that will be needed. We consider only the case where the domain is an open subset U of $\mathbf{C}^m$. A complex-valued measure of the form $\mu = \mu' + i\mu''$ with real-valued Radon measures μ' and μ'' on U is called a *complex Radon measure* on U. Its complex conjugate is defined by $\bar{\mu} = \mu' - i\mu''$. We consider a $(1,1)$ current $T = \sum T_{j\bar{k}} \frac{i}{2} dz_j \wedge d\bar{z}_k$ with coefficients of complex-valued Radon measures $T_{j\bar{k}}$, $1 \leqq j, k \leqq m$. The complex conjugate of T is defined by

$$\bar{T} = \sum \bar{T}_{j\bar{k}} \frac{-i}{2} d\bar{z}_j \wedge dz_k = \sum \bar{T}_{j\bar{k}} \frac{i}{2} dz_k \wedge d\bar{z}_j.$$

If $T = \bar{T}$, i.e., $\bar{T}_{j\bar{k}} = T_{k\bar{j}}$ (hermitian), T is called a *real current*. If for every vector $(\xi_j) \in \mathbf{C}^m$

$$\sum_{j,k} T_{j\bar{k}} \xi_j \bar{\xi}_k$$

is a positive Radon measure, T is called a $(1,1)$ *positive current*, and we write $T \geqq 0$.

For two real $(1,1)$ currents T, S on U we write $T \geqq S$ ($S \leqq T$) if $T - S \geqq 0$.

Take $\chi(z) = \chi(\|z\|) \in C_0^\infty(\mathbf{C}^m)$ so that $\chi(z) \geqq 0$, $\operatorname{Supp} \chi \subset B(1)$, and

$$\int \chi(z) \alpha^m = 1.$$

Set $\chi_\varepsilon(z) = \chi(\varepsilon^{-1} z) \varepsilon^{-2m}$, $\varepsilon > 0$. Let φ be a plurisubharmonic function on U. The *smoothing* $\varphi_\varepsilon(z)$ of φ is defined by

$$\begin{aligned}\varphi_\varepsilon(z) = \varphi * \chi_\varepsilon(z) &= \int_{\mathbf{C}^m} \varphi(w) \chi_\varepsilon(w - z) \alpha^m(w) \\ &= \int_{\mathbf{C}^m} \varphi(z + w) \chi_\varepsilon(w) \alpha^m(w), \qquad z \in U_\varepsilon.\end{aligned}$$

Then $\varphi_\varepsilon(z) \in C^\infty(U_\varepsilon)$ and it is plurisubharmonic. Since $\chi(w) = \chi(\|w\|)$, one gets

$$\begin{aligned}\varphi_\varepsilon(z) &= \int_{\mathbf{C}^m} \varphi(z + \varepsilon w) \chi(w) \alpha^m(w) \\ &= \int_{\mathbf{C}^m} \alpha^m(w) \frac{1}{2\pi} \int_0^{2\pi} d\theta\, \varphi\big(z + \varepsilon e^{i\theta} w\big) \chi(w) \\ &\geqq \varphi(z) \int_{\mathbf{C}^m} \chi(w) \alpha^m = \varphi(z).\end{aligned}$$

Therefore by Theorem 2.1.12 (iii) φ_ε is monotone decreasing as $\varepsilon \searrow 0$. Since φ is upper semicontinuous, in the same way as in the proof of Theorem 2.1.12 (ii), one verifies that $\varphi_\varepsilon(z) \searrow \varphi(z)$.

Since $\sum \frac{\partial^2 \varphi_\varepsilon}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k \geqq 0$ for every vector $(\xi_1, \ldots, \xi_m) \in \mathbf{C}^m$,

$$\sum \frac{\partial^2 [\varphi]}{\partial z_j \partial \bar{z}_k} \xi_j \bar{\xi}_k,$$

where the notation $\frac{\partial^2 [\varphi]}{\partial z_j \partial \bar{z}_k}$ is used in the sense of Schwartz distributions, defines a positive Radon measure, so that

$$dd^c[\varphi] = \sum \frac{\partial^2 [\varphi]}{\partial z_j \partial \bar{z}_k} \frac{i}{2\pi} dz_j \wedge d\bar{z}_k \geqq 0.$$

Hence we have the following.

Theorem 2.1.27 *Let $\varphi : U \to [-\infty, \infty)$ be a plurisubharmonic function such that $\varphi \not\equiv -\infty$ on each connected component of U.*

(i) *$dd^c[\varphi] \geqq 0$ and the coefficients $\frac{\partial^2 [\varphi]}{\partial z_j \partial \bar{z}_k}$ are complex Radon measures which are absolutely continuous with respect to the trace $\sum_{j=1}^m \frac{\partial^2 [\varphi]}{\partial z_j \partial \bar{z}_j}$.*
(ii) *The smoothing $\varphi_\varepsilon(z)$ converges monotone decreasingly to $\varphi(z)$ as $\varepsilon \searrow 0$.*
(iii) *For every $B(a; R) \subset U$ with $0 < s < r < R$,*

$$-\infty < \int_{\|z\|=s} \varphi(a+z)\gamma(z) \leqq \int_{\|z\|=r} \varphi(a+z)\gamma(z) < \infty. \tag{2.1.28}$$

Proof The absolute continuity of (i) follows from the positivity $dd^c[\varphi] \geqq 0$. (ii) was already shown. Only (iii) remains.

First note that by Theorem 2.1.26 (ii) φ is locally integrable. We infer from (2.1.25) and Fubini's theorem that there is a subset $E \subset (0, R)$ of Lebesgue measure zero with finite $\int_{\|z\|=t} \varphi(a+z)\gamma(z)$ for $t \in (0, R) \setminus E$. On the other hand, for every $t \in (0, R)$ and $\vartheta \in [0, 2\pi]$ the $\mathbf{C}^*$-invariance $\gamma(te^{i\vartheta} z) = \gamma(z)$ implies

$$\begin{aligned} \int_{\|z\|=t} \varphi(a+z)\gamma(z) &= \int_{\|z\|=1} \varphi\big(a + te^{i\vartheta} z\big)\gamma(z) \\ &= \int_{\|z\|=1} \int_0^{2\pi} \varphi\big(a + te^{i\theta} z\big) \frac{d\theta}{2\pi} \gamma(z). \end{aligned}$$

It follows from this and Theorem 2.1.12 (iii) that for every $0 < s < r < R$

$$\int_{\|z\|=s} \varphi(a+z)\gamma(z) \leqq \int_{\|z\|=r} \varphi(a+z)\gamma(z) < \infty.$$

Applying this to $0 < t < s$, $t \notin E$, we see that

$$-\infty < \int_{\|z\|=t} \varphi(a+z)\gamma(z) \leqq \int_{\|z\|=s} \varphi(a+z)\gamma(z).$$

Thus (2.1.28) is shown. $\square$

As in Theorem 2.1.13 (i) the following holds.

Theorem 2.1.29 *The plurisubharmonicity is a local property.*

Let $\varphi \not\equiv -\infty$ be a plurisubharmonic function on $\mathbf{C}^m$. In the sense of currents

$$dd^c[\varphi] = \sum \frac{\partial^2[\varphi]}{\partial z_j \partial \bar{z}_k} \frac{i}{2\pi} dz_j \wedge d\bar{z}_k \geqq 0.$$

Then

$$dd^c[\varphi] \wedge \alpha^{m-1} = \left((m-1)! \sum_{j=1}^{m} \frac{\partial^2[\varphi]}{\partial z_j \partial \bar{z}_j} \right) \bigwedge_{j=1}^{m} \frac{i}{2\pi} dz_j \wedge d\bar{z}_j$$

is a volume form with a positive Radon measure as coefficient. Therefore, for a Borel measurable subset $E \subset \mathbf{C}^m$ and a Borel measurable function ψ the integral

$$\int_E \psi\, dd^c[\varphi] \wedge \alpha^{m-1}$$

is defined; in particular, we set

$$n\big(t, dd^c[\varphi]\big) = \frac{1}{t^{2m-2}} \int_{B(t)} dd^c[\varphi] \wedge \alpha^{m-1}, \quad t > 0. \tag{2.1.30}$$

Lemma 2.1.31 *The function $n(t, dd^c[\varphi])$ is left-continuous in $t > 0$ and monotone increasing.*

Proof By the inner regularity of Radon measure, $\int_{B(t)} dd^c[\varphi] \wedge \alpha^{m-1}$ is left-continuous in $t > 0$, and so is $n(t, dd^c[\varphi])$.

Let φ_ε, $\varepsilon > 0$ be the smoothing of φ. By making use of (2.1.22) one gets for $t > s > 0$

$$\begin{aligned}
&\frac{1}{t^{2m-2}} \int_{B(t)} dd^c \varphi_\varepsilon \wedge \alpha^{m-1} - \frac{1}{s^{2m-2}} \int_{B(s)} dd^c \varphi_\varepsilon \wedge \alpha^{m-1} \\
&= \frac{1}{t^{2m-2}} \int_{\|z\|=t} d^c \varphi_\varepsilon \wedge \alpha^{m-1} - \frac{1}{s^{2m-2}} \int_{\|z\|=s} d^c \varphi_\varepsilon \wedge \alpha^{m-1} \\
&= \int_{\|z\|=t} d^c \varphi_\varepsilon \wedge \beta^{m-1} - \int_{\|z\|=s} d^c \varphi_\varepsilon \wedge \beta^{m-1} \\
&= \int_{s<\|z\|<t} dd^c \varphi_\varepsilon \wedge \beta^{m-1}.
\end{aligned}$$

Since $\beta \geqq 0$ (semi-positive), the integral is non-negative. Now we let $\varepsilon \to 0$. Then there is a subset $E \subset (0, \infty)$ of Lebesgue measure zero such that for $0 < s < t$

outside E

$$n\big(t, dd^c[\varphi]\big) - n\big(s, dd^c[\varphi]\big) = \int_{s<\|z\|<t} dd^c[\varphi] \wedge \beta^{m-1} \geqq 0. \tag{2.1.32}$$

For every $0 < s < t$ we take sequences $s_\nu \nearrow s$ and $s < t_\nu \nearrow t$ with $s_\nu \notin E, t_\nu \notin E$, $\nu = 1, 2, \ldots$. Then $n(s_\nu, dd^c[\varphi]) \leqq n(t_\nu, dd^c[\varphi])$. As $\nu \to \infty$, the left-continuity implies

$$n\big(s, dd^c[\varphi]\big) \leqq n\big(t, dd^c[\varphi]\big).$$

□

It follows from Lemma 2.1.31 that at every point $a \in \mathbf{C}^m$ the following limit exists:

$$\mathscr{L}\big(a; dd^c[\varphi]\big) = \lim_{t\to 0} \frac{1}{t^{2m-2}} \int_{B(a;t)} dd^c[\varphi] \wedge \alpha^{m-1}.$$

The limit $\mathscr{L}(a; dd^c[\varphi])$ is called the *Lelong number* of the current $dd^c[\varphi]$ at a, and plays an important role in various aspects of complex analysis. For example, for a given $\delta > 0$ the set $\{a \in \mathbf{C}^m; \mathscr{L}(a; dd^c[\varphi]) \geqq \delta\}$ forms an analytic subset (Y.-T. Siu's Theorem; cf. Siu [74]; Ohsawa [98]; Hörmander [89]).

Lemma 2.1.33 (Jensen's formula) *Let $\varphi \not\equiv -\infty$ be a plurisubharmonic function on $\mathbf{C}^m$. Then for every $0 < s < r$*

$$\begin{aligned} \int_{\|z\|=r} \varphi\gamma - \int_{\|z\|=s} \varphi\gamma &= 2\int_s^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c[\varphi] \wedge \alpha^{m-1} \\ &= 2\int_s^r \frac{dt}{t} \int_{B(t)\setminus\{0\}} dd^c[\varphi] \wedge \beta^{m-1} \\ &\quad + 2\mathscr{L}\big(0; dd^c[\varphi]\big) \log \frac{r}{s}. \end{aligned} \tag{2.1.34}$$

Proof Take the smoothing φ_ε of φ. Since $d\gamma = 0$,

(2.1.35)

$$\begin{aligned} \int_{\|z\|=r} \varphi_\varepsilon\gamma - \int_{\|z\|=s} \varphi_\varepsilon\gamma &= \int_{\{s<\|z\|<r\}} d\varphi_\varepsilon \wedge \gamma \\ &= \int_{\{s<\|z\|<r\}} d\log\|z\|^2 \wedge d^c\varphi_\varepsilon \wedge \big(dd^c \log\|z\|^2\big)^{m-1} \\ &= 2\int_s^r \frac{dt}{t} \int_{\|z\|=t} d^c\varphi_\varepsilon \wedge \big(dd^c\log\|z\|^2\big)^{m-1} \\ &= 2\int_s^r \frac{dt}{t} \int_{\|z\|=t} d^c\varphi_\varepsilon \wedge \frac{\alpha^{m-1}}{t^{2(m-1)}} \end{aligned}$$

(continued)

$$= 2\int_s^r \frac{dt}{t^{2m-1}} \int_{\|z\|=t} d^c\varphi_\varepsilon \wedge \alpha^{m-1}$$

$$= 2\int_s^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c\varphi_\varepsilon \wedge \alpha^{m-1}.$$

As $\varepsilon \searrow 0$, $\varphi_\varepsilon \searrow \varphi$. By the monotone convergence theorem of Lebesgue the first integral of (2.1.35) converges to the integral of φ.

On the other hand, for almost all t with respect to the Lebesgue measure

$$\int_{B(t)} dd^c\varphi_\varepsilon \wedge \alpha^{m-1} \to \int_{B(t)} dd^c\varphi \wedge \alpha^{m-1} \quad (\varepsilon \to 0).$$

It follows from the definition that for $0 < \varepsilon < 1$ and $t \leqq r$

$$0 \leqq \int_{B(t)} dd^c\varphi_\varepsilon \wedge \alpha^{m-1} \leqq \int_{B(r+1)} dd^c[\varphi] \wedge \alpha^{m-1} < \infty.$$

By Lebesgue's bounded convergence theorem we have

$$(2.1.36) \quad 2\int_s^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c\varphi_\varepsilon \wedge \alpha^{m-1} \to 2\int_s^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c[\varphi] \wedge \alpha^{m-1}$$

as $\varepsilon \to 0$. Therefore the first equality of (2.1.34) is inferred.

Letting $s \to 0$ ($s \notin E$) in (2.1.32), we have

$$\frac{1}{t^{2m-2}} \int_{B(t)} dd^c[\varphi] \wedge \alpha^{m-1} = \int_{B(t)\setminus\{0\}} dd^c[\varphi] \wedge \beta^{m-1} + \mathscr{L}\big(0; dd^c[\varphi]\big).$$

This implies the second equality of (2.1.34). □

Corollary 2.1.37 *Jensen's formula* (2.1.34) *holds for Borel measurable functions φ on $\mathbf{C}^m$ if for every $a \in \mathbf{C}^m$ there exist plurisubharmonic functions φ_1, φ_2 in a neighborhood U of a such that φ is written as $\varphi = \varphi_1 - \varphi_2$ on U.*

Proof Let $\{U_j\}_{j=1}^\infty$ be a locally finite open covering of $\mathbf{C}^m$ such that there are plurisubharmonic functions $\varphi_{j1}, \varphi_{j2}$ on U_j satisfying $\varphi = \varphi_{j1} - \varphi_{j2}$. Let $\{\eta_j\}$ be a partition of unity subordinated to $\{U_j\}$. It suffices to show the first equality of (2.1.34). Take a number j_0 so that $U_j \cap \overline{B(r)} = \emptyset$, $j \geqq j_0$. Put $\delta = \min\{d(\operatorname{Supp}\eta_j, \partial U_j); 1 \leqq j \leqq j_0\}$. For $0 < \varepsilon < \delta$ and $z \in \overline{B(r)} \cap \operatorname{Supp}\eta_j$

$$\varphi_\varepsilon(z) = \varphi_{j1\varepsilon}(z) - \varphi_{j2\varepsilon}(z).$$

Thus

$$\varphi_\varepsilon(z) = \sum_{j=1}^{j_0} \eta_j(z)\big(\varphi_{j1\varepsilon}(z) - \varphi_{j2\varepsilon}(z)\big), \quad z \in \overline{B(r)}.$$

By the same computation as in (2.1.35) we have

$$\sum_{j=1}^{j_0}\int_{\|z\|=r}\eta_j\varphi_{j1\varepsilon}\gamma-\sum_{j=1}^{j_0}\int_{\|z\|=s}\eta_j\varphi_{j2\varepsilon}\gamma$$
$$=2\int_s^r\frac{dt}{t^{2m-1}}\int_{B(t)}dd^c\varphi_\varepsilon\wedge\alpha^{m-1}.$$

The monotone convergence theorem of Lebesgue implies that

$$\int_{\{\|z\|=r\}}\eta_j\varphi_{ji\varepsilon}\gamma\to\int_{\{\|z\|=r\}}\eta_j\varphi_{ji}\gamma\quad(\varepsilon\to 0,\ i=1,2).$$

Since the coefficients of $dd^c[\varphi]$ are complex Radon measures, the convergence of (2.1.36) holds. Thus the required formula follows. □

Supplement (Currents) In general, on a differentiable manifold M satisfying the second countability axiom a differential form with coefficients in Schwartz' distributions is called a *current*. In the space of currents, the exterior differential operator d is defined in the sense of derivations of Schwartz' distributions. If M is a complex manifold, the operators $\partial, \bar\partial, d^c$ are defined similarly (cf., e.g., Lelong [68]; Noguchi–Ochiai [90] (Ochiai–Noguchi [84])). Let $(z_1,\dots,z_m)$ be a holomorphic local coordinate system of M ($m=\dim M$). If a current T on M is written as

$$T=\sum_{|I|=p,|J|=q}T_{I\bar J}dz^I\wedge d\bar z^J,$$
$$dz^I=\bigwedge_{i\in I}dz_i,\quad d\bar z^J=\bigwedge_{j\in J}d\bar z_j$$

with multi-index sets $I, J\subset\{1,\dots,m\}$, T is said to be of type (p,q), or called a (p,q) current. The complex conjugate $\bar T$ is defined by

$$\bar T_{I\bar J}(\phi)=\overline{T_{I\bar J}(\bar\phi)},$$
$$\bar T=\sum_{|I|=p,|J|=q}\bar T_{I\bar J}d\bar z^I\wedge dz^J,$$

where ϕ is a test function. A current T of type (p,p) is called a *positive current* if T is real, i.e., $\bar T=T$, and for every C^∞ $(1,0)$ form η_j, $1\leqq j\leqq m-p$

$$T\wedge i\eta_1\wedge\bar\eta_1\wedge\cdots\wedge i\eta_{m-p}\wedge\bar\eta_{m-p}$$

is a positive Radon measure. In this case, we write $T\geqq 0$. For two real (p,p) currents T, T', we write $T\geqq T'$ for $T-T'\geqq 0$.

2.2 Poincaré–Lelong Formula

Let $U \subset \mathbf{C}^m$ be an open subset.

Definition 2.2.1 A closed subset $A \subset U$ is said to be *analytic*, if for every point $a \in A$ there are a neighborhood $W \subset U$ of a and holomorphic functions $g_1, \ldots, g_l$ on W $(l < \infty)$ satisfying

$$A \cap W = \{g_1 = \cdots = g_l = 0\}.$$

When the closedness of A is not assumed, A is called a locally closed analytic subset. In particular, if the above g_j, $1 \leqq j \leqq l$, can be taken so that their differentials at $a \in A$

$$dg_1(a), \ldots, dg_l(a)$$

are linearly independent, a is called a *regular* or *non-singular point* of A; in this case, if W is chosen sufficiently small, $A \cap W$ is a closed complex submanifold of W. A point of A which is not non-singular is called a *singular point*; the subset of all singular points of A is denoted by $S(A)$. Set $R(A) = A \setminus S(A)$. If $S(A) = \emptyset$, A is said to be non-singular.

We describe elementary and useful properties of analytic subsets and of plurisubharmonic functions without proofs, for which cf. Oka [Iw] VII, [50], [51], Grauert–Remmert [84], Gunning–Rossi [65], Hervé [63], Narasimhan [66], Noguchi–Ochiai [90] (Ochiai–Noguchi [84]), Nishino [96], Ohsawa [98], Noguchi [13].

Theorem 2.2.2 *An analytic subset $A \subset U$ satisfies the following*:

(i) $S(A)$ *is an analytic subset and nowhere dense in* A.
(ii) *Let* $R(A) = \bigcup_\lambda A'_\lambda$ *be the decomposition into the connected components. Then the closure* $A_\lambda = \bar{A}'_\lambda$ *is analytic, and* $A = \bigcup_\lambda A_\lambda$ *is a locally finite covering.*
(iii) *If* $A_\lambda \subset U$, $\lambda \in \Lambda$ *are analytic subsets of* U, *then so is the intersection* $\bigcap_{\lambda \in \Lambda} A_\lambda$.

An analytic subset B is said to be *irreducible* if there are no analytic subsets $B_i \subsetneqq B$, $i = 1, 2$, with $B = B_1 \cup B_2$. The above A_λ are known to be irreducible. Each A_λ is called an *irreducible component* of A. Since A'_λ is a locally closed submanifold, its (complex) dimension is denoted by $\dim A'_\lambda$, and one sets

$$\dim A_\lambda = \dim A'_\lambda, \qquad \dim_a A = \max_{a \in A_\lambda} \dim A_\lambda,$$
$$\operatorname{codim}_a A = m - \dim_a A, \qquad \dim A = \max_a \dim_a A,$$
$$\operatorname{codim} A = m - \dim A.$$

In particular, the following holds:

$$\dim S(A) < \dim A.$$

If $\dim_a A = \dim A$ at all points $a \in A$, A is said to be of pure dimension $\dim A$.

Let $a \in A$. Then there is a linear subspace $L \cong \mathbf{C}^l$ passing through a such that a is isolated in $L \cap A$. Moreover, there is a direct decomposition $\mathbf{C}^m \cong \mathbf{C}^{m-l} \times L \ni a = (a_1, a_2)$ by linear subspaces, and there are neighborhoods $U_1 \subset \mathbf{C}^{m-l}$ of a_1, $U_2 \subset L$ of a_2 such that the projection $p : A \cap (U_1 \times U_2) \to U_1$ satisfies the following.

Theorem 2.2.3 *Let the notation be as above.*

(i) $p : A \cap (U_1 \times U_2) \to U_1$ *is surjective, proper, finite, and* $p^{-1}a_1 = \{a\}$.
(ii) *There is a proper analytic subset* $S \subset U_1$ *such that the restriction of* p

$$p|_{A\cap(U_1\times U_2)\setminus p^{-1}S} : A \cap (U_1 \times U_2) \setminus p^{-1}S \to U_1 \setminus S$$

is a finite unramified covering.
(iii) *The collection of such* l*-dimensional linear subspaces* L *of* $\mathbf{C}^m$ *satisfying the above* (i) *and* (ii) *forms an open dense subset in the Grassmann space of* l*-dimensional linear subspaces of* $\mathbf{C}^m$.

If there is one holomorphic function $\phi \not\equiv 0$ in a neighborhood W of every point $a \in A$ satisfying

$$A \cap W = \{\phi = 0\},$$

A is called a *complex hypersurface* or simply *hypersurface*. A hypersurface is an analytic subset of pure dimension $m - 1$.

Let $U_i \subset \mathbf{C}^{n_i}$, $i = 1, 2$, be open sets, and let $X_i \subset U_i$, $i = 1, 2$, be analytic subsets. A map $\varphi : X_1 \to X_2$ is said to be holomorphic if for every point $a \in X_1$ there are a neighborhood W of a in U_1 and holomorphic functions φ_j, $1 \leqq j \leqq n_2$, on W satisfying

$$\varphi(x) = \bigl(\varphi_1(x), \ldots, \varphi_{n_2}(x)\bigr), \quad x \in W \cap X_1.$$

In particular, when $X_2 = \mathbf{C}$, φ is called a holomorphic function. If the inverse $\varphi^{-1} : X_2 \to X_1$ exists and is holomorphic, φ is called a biholomorphic map or a *biholomorphism*, and X_1 is said to be biholomorphic to X_2.

Let X be a Hausdorff topological space. Let X carry an open covering $\{X_\lambda\}_{\lambda\in\Lambda}$ such that there exist analytic subsets $Z_\lambda \subset U_\lambda$ with open subsets $U_\lambda \subset \mathbf{C}^{n_\lambda}$, homeomorphisms $\varphi_\lambda : X_\lambda \to Z_\lambda$, $\lambda \in \Lambda$, and the restrictions

$$\varphi_\mu \circ \varphi_\lambda^{-1}|_{Z_\lambda\cap\varphi_\lambda(X_\lambda\cap X_\mu)} : Z_\lambda \cap \varphi_\lambda(X_\lambda \cap X_\mu) \to Z_\mu \cap \varphi_\mu(X_\lambda \cap X_\mu)$$

are biholomorphisms for all $\lambda, \mu \in \Lambda$. Then X is called a *complex space*.

If we can take $Z_\lambda = U_\lambda$, then X is called a *complex manifold*.

Holomorphic functions on a complex space, holomorphic maps between complex spaces, and analytic subsets are defined using the coordinate charts ϕ_λ; e.g., a function $f : X \to \mathbf{C}$ is *holomorphic*, if all the functions $f|_{X_\lambda} \circ \phi_\lambda^{-1} : Z_\lambda \to \mathbf{C}$ are holomorphic.

Similarly, a function $f : X \to \mathbf{C}$ on a complex manifold X is called *plurisubharmonic* if all the functions $f|_{X_\lambda} \circ \phi_\lambda^{-1} : Z_\lambda \to \mathbf{C}$ are plurisubharmonic. (Note that given a biholomorphic map $\alpha : Z \to Z'$ a function f on Z' is plurisubharmonic if and only if $f \circ \alpha$ is plurisubharmonic; this is proven as in Theorem 2.1.19.)

Theorem 2.2.4 (Remmert) *Let $\phi : X \to Y$ be a proper holomorphic map between complex spaces, and let $A \subset X$ be an analytic subset. Then $\phi(A)$ is an analytic subset of Y and* $\dim \phi(A) \leqq \dim A$.

Theorem 2.2.5 (Remmert) *Let X be a complex space, and let $E \subset X$ be an analytic subset. Let $A \subset X \setminus E$ be an analytic subset such that* $\min\{\dim_a A; a \in A\} > \dim E$. *Then the closure $\bar{A}$ of A in X is an analytic subset of X.*

For the analytic continuation of holomorphic functions we have the following.

Theorem 2.2.6 *Let $U \subset \mathbf{C}^m$ be a domain, and let $E \subsetneqq U$ be a proper analytic subset. Let $f : U \setminus E \to \mathbf{C}$ be a holomorphic function.*

(i) (Riemann extension) *If there is a neighborhood $V \subset U$ of every point $x \in E$ such that $f|_{V \setminus E}$ is bounded, then f is uniquely extended to a holomorphic function on U.*
(ii) (Hartogs extension) *If* $\operatorname{codim} E \geqq 2$, *$f$ is necessarily extended uniquely to a holomorphic function on U.*

We know similar theorems for plurisubharmonic functions (cf. Grauert–Remmert [56]; Noguchi–Ochiai [90] (Ochiai–Noguchi [84])).

Theorem 2.2.7 *Let $U \subset \mathbf{C}^m$ be a domain, and let $E \subset U$ be a proper analytic subset. Let $\psi : U \setminus E \to [-\infty, \infty)$ be a plurisubharmonic function.*

(i) (Riemann type) *If there is a neighborhood $V \subset U$ of every point $x \in E$ such that $\psi|_{V \setminus E}$ is bounded from above, then ψ extends uniquely to a plurisubharmonic function on U.*
(ii) (Hartogs type) *If* $\operatorname{codim} E \geqq 2$, *then ψ necessarily extends uniquely to a plurisubharmonic function on U.*

Since the above two extension theorems are local in nature, the analogous statements hold for arbitrary complex manifolds.

Take a homogeneous coordinate system $[w_0, \ldots, w_n]$ of the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$. A subset $X \subset \mathbf{P}^n(\mathbf{C})$ is said to be *algebraic* if there are finitely many homogeneous polynomials $P_\alpha(w_0, \ldots, w_n)$ satisfying

$$X = \bigcap_\alpha \big\{ P_\alpha(w_0, \ldots, w_n) = 0 \big\}.$$

If a complex space Z is biholomorphic to an algebraic subset of $\mathbf{P}^n(\mathbf{C})$, Z is called a (complex) *projective algebraic variety*. Then an algebraic subset of Z is naturally defined.

Theorem 2.2.8 (Chow) *Every analytic subset of* $\mathbf{P}^n(\mathbf{C})$ *is algebraic.*

The *Zariski topology* is defined on $\mathbf{P}^n(\mathbf{C})$ by taking the algebraic subsets as closed subsets. Similarly, the Zariski topology on a complex space Z (resp. its subset Y) is defined so that analytic subsets X of Z (resp. $X \cap Y$) are closed subsets.

Let U be an open subset of $\mathbf{C}^m$. Let $A \subset U$ be an analytic subset of pure dimension l, and let $\iota_{R(A)} : R(A) \to \mathbf{C}^m$ be the inclusion map. For a compact subset $K \subset U$ the integral

$$\int_{K \cap R(A)} \alpha^l = \int_{K \cap R(A)} \iota_{R(A)}^* \alpha^l$$

is considered as a measure of $K \cap A$. We show that this is finite.

Lemma 2.2.9 *Let the notation be as above.*

(i) *If* $l = \dim A < m$, *the Lebesgue measure of* A *in* $\mathbf{C}^m$ *is zero.*
(ii) $\int_{K \cap R(A)} \alpha^l < \infty$.

Proof (i) We use induction on l. When $l = 0$, A is a discrete subset, and so the measure is zero. Suppose that the statement holds for $\dim A < l$. We decompose $A = R(A) \cup S(A)$. The induction hypothesis implies that the measure of $S(A)$ is zero. Since $R(A)$ is a locally closed submanifold of dimension l ($<m$), its measure is zero, and hence so is the measure of A.

(ii) It suffices to show the claim in a neighborhood W of every point $a \in A$. By making use of a translation and a unitary transformation of coordinates combined with Theorem 2.2.3, we may assume that $a = 0$ and for arbitrary $1 \leqq i_1 < \cdots < i_l \leqq m$ there is a neighborhood W of 0 with projection $W = U_1 \times U_2 \subset \mathbf{C}^l \times \mathbf{C}^{m-l}$ satisfying the following properties:

$$p : (z_1, \ldots, z_m) \in A \cap W \to (z_{i_1}, \ldots, z_{i_l}) \in U_1$$

is a proper finite map, and there is an analytic subset $S \subset U_1$ of $\dim S < l$ such that

$$p|_{A \cap W \setminus p^{-1}S} : (z_1, \ldots, z_m) \in A \cap W \setminus p^{-1}S \to (z_{i_1}, \ldots, z_{i_l}) \in U_1 \setminus S$$

is a unramified covering. Denote by k its covering number. Note that

$$\alpha^l = n \cdots (n - l + 1) \sum_{i_1 < \cdots < i_l} \left(\frac{i}{2\pi}\right)^l dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{i_l}.$$

It is sufficient to show that for every $i_1 < \cdots < i_l$

$$\int_{R(A) \cap W} \left(\frac{i}{2\pi}\right)^l dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{i_l} < \infty.$$

By (i) the Lebesgue measure of $R(A) \cap W \cap p^{-1}S$ is zero in $R(A) \cap W$. Therefore

$$\int_{R(A) \cap W} \left(\frac{i}{2\pi}\right)^l dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{i_l}$$
$$= \int_{R(A) \cap W \setminus p^{-1}S} \left(\frac{i}{2\pi}\right)^l dz_{i_1} \wedge d\bar{z}_{i_1} \wedge \cdots \wedge dz_{i_l} \wedge d\bar{z}_{i_l}$$

(continued)

$$= k\int_{U_1\setminus S}\left(\frac{i}{2\pi}\right)^l dz_{i_1}\wedge d\bar{z}_{i_1}\wedge\cdots\wedge dz_{i_l}\wedge d\bar{z}_{i_l}$$
$$= k\int_{U_1}\left(\frac{i}{2\pi}\right)^l dz_{i_1}\wedge d\bar{z}_{i_1}\wedge\cdots\wedge dz_{i_l}\wedge d\bar{z}_{i_l} < \infty. \qquad \square$$

From now on we write

$$\int_{K\cap A}\alpha^l = \int_{K\cap R(A)}\alpha^l.$$

The following is immediate.

Corollary 2.2.10 *Let $B \subset A$ be an analytic subset of* $\dim B < \dim A$. *Then*

$$\int_B \alpha^l = 0.$$

The following is deduced from the above.

Theorem 2.2.11 *Let $A \subset U$ be an analytic subset of pure dimension l. Let η be a $2l$ form with coefficients which are bounded Borel functions on U with compact supports. Then*

$$\int_A \eta = \int_{R(A)} \iota^*_{R(A)}\eta$$

is defined. Let $B \subset A$ be an analytic subset of $\dim B < \dim A$ *and let χ_B be the characteristic function of the set B. Then*

$$\int_A \chi_B \eta = 0.$$

Lemma 2.2.12 *Let $A \subset U$ be an analytic subset of dimension at most $m-2$. Let φ be a plurisubharmonic function on U such that $\varphi \not\equiv -\infty$ on every connected component of U, and set $dd^c[\varphi] = \sum T_{j\bar{k}}\frac{i}{2\pi}dz_j \wedge d\bar{z}_k$. Then A is of measure zero with respect to Radon measures $T_{j\bar{k}}$.*

Proof One may assume that $0 \in A$, and it suffices to show the lemma in a neighborhood of $0 \in A$. Suppose that $\dim_0 A = l$. It follows from Theorem 2.2.3 (iii) that the coordinate system $(z_1, \ldots, z_m)$ is chosen to satisfy the property: For arbitrary l coordinates $z_{\nu_1}, \ldots, z_{\nu_l}$ and the others $z_{\nu_{l+1}}, \ldots, z_{\nu_m}$ there are neighborhoods U_1 and U_2 of the origins of the $(z_{\nu_1}, \ldots, z_{\nu_l})$-space and the $(z_{\nu_{l+1}}, \ldots, z_{\nu_m})$-space, respectively such that the projection

$$p : (z_j) \in \left\{(z_j) \in A;\ (z_{\nu_1}, \ldots, z_{\nu_l}) \in U_1,\ (z_{\nu_{l+1}}, \ldots, z_{\nu_m}) \in U_2\right\}$$
$$\to (z_{\nu_1}, \ldots, z_{\nu_l}) \in U_1$$

satisfies Theorem 2.2.3 (i), (ii). By Theorem 2.1.27 (i) $T_{j\bar{k}}$ is absolutely continuous with respect to $\sum_j T_{j\bar{j}}$. Therefore it is sufficient to prove that $T_{j\bar{j}}(A \cap (U_1 \times U_2)) = 0$, $1 \leqq j \leqq m$. For instance, let $j = m$. Then there are neighborhoods V_1 of $0 \in \mathbf{C}^{m-1}$ and V_2 of $0 \in \mathbf{C}$ such that

$$q : (z_1, \dots, z_{m-1}, z_m) \in (V_1 \times V_2) \cap A \to (z_1, \dots, z_{m-1}) \in V_1$$

is proper and finite. By φ being plurisubharmonic and the definition we have

$$\begin{aligned} T_{m\bar{m}}\big((V_1 \times V_2) \cap A\big) \\ &= \int_{(V_1 \times V_2) \cap A} \frac{\partial^2[\varphi]}{\partial z_m \partial \bar{z}_m} \bigwedge_{j=1}^{m} \frac{i}{2} dz_j \wedge d\bar{z}_j \\ &= \int_{q(A)} \bigwedge_{j=1}^{m-1} \frac{i}{2} dz_j \wedge d\bar{z}_j \int_{V_2} \frac{\partial^2[\varphi(\cdot, z_m)]}{\partial z_m \partial \bar{z}_m} \frac{i}{2} dz_m \wedge d\bar{z}_m. \end{aligned} \tag{2.2.13}$$

It follows from Fubini's theorem that for $z' = (z_1, \dots, z_{m-1})$

$$z' \in V_1 \to \int_{V_2} \frac{\partial^2[\varphi(z', z_m)]}{\partial z_m \partial \bar{z}_m} \geqq 0$$

is an integrable function on V_1. By Theorem 2.2.4 $q(A)$ is an analytic subset of V_1 of dimension at most l ($\leqq m - 1$). Lemma 2.2.9 (i) implies that $q(A)$ is of Lebesgue measure zero in V_1. Thus (2.2.13) implies that $T_{m\bar{m}}((V_1 \times V_2) \cap A) = 0$. □

Let M be a complex manifold of dimension m. It is clear that Theorem 2.2.11 and Lemma 2.2.12 hold generally on M.

Let $\{A_\lambda\}$ be a locally finite family of hypersurfaces of M. The formal sum

$$\sum_\lambda k_\lambda A_\lambda$$

with integral coefficients $k_\lambda \in \mathbf{Z}$ is called a *divisor*, and the $\mathbf{Z}$-module generated by them is called the *divisor group*. For a given divisor D on M there are uniquely distinct irreducible hypersurfaces D_λ of M and $k_\lambda \in \mathbf{Z} \setminus \{0\}$ such that $D = \sum k_\lambda D_\lambda$ (the *irreducible decomposition*). Each D_λ is called an irreducible component of D. The hypersurface $\operatorname{Supp} D = \bigcup D_\lambda$ is called the *support* of D. If $D = \sum_\lambda k_\lambda A_\lambda$ with $k_\lambda \geqq 0$, D is called an *effective divisor*, and written as $D \geqq 0$. If there is no confusion, the notation D may be used for $\operatorname{Supp} D$. For two divisors D, D' we write $D \geqq D'$ if $D - D' \geqq 0$.

Let $D = \sum k_\lambda D_\lambda$ be the irreducible decomposition of a divisor D. If all $k_\lambda = 1$, then D is called a *reduced divisor*.

Let f be a holomorphic function on M. Then a hypersurface $A = \{f = 0\}$ is defined. Let $A = \bigcup_\lambda A_\lambda$ be the irreducible decomposition. The zero degree m_λ of f at a point x of every $R(A_\lambda) \setminus S(A)$ is naturally defined, independently from the choice of x because $R(A_\lambda) \setminus S(A)$ is connected. Thus a divisor

$$(f) = \sum m_\lambda A_\lambda$$

is determined by f.

A *meromorphic function* f on M is a function locally expressed by the ratio $f = g/h$ of two holomorphic functions g, h with $h \not\equiv 0$. Then the divisor (f) locally expressed by $(g) - (h)$ is defined globally on M. Let $(f) = \sum_\lambda m_\lambda D_\lambda$ be the irreducible decomposition and set

$$(f)_0 = \sum_{m_\lambda > 0} m_\lambda D_\lambda, \qquad (f)_\infty = \sum_{m_\lambda < 0} -m_\lambda D_\lambda.$$

Then $(f)_0$ is called the *zero divisor* of f and $(f)_\infty$ is called the *polar divisor* of f.

Every divisor is locally expressed by the divisor of a meromorphic function; for some special M there exists a global expression.

Theorem 2.2.14 *If M is biholomorphic to $\mathbf{C}^m$ or $B(r)$, then for a divisor D on M, there exists a meromorphic function (f) on M such that $(f) = D$.*

As for Hartogs extension (cf. Theorem 2.2.7 (ii)) we have the following.

Theorem 2.2.15 *Let $E \subset M$ be an analytic subset which has a codimension of at least two everywhere. Then every meromorphic function on $M \setminus E$ extends meromorphically over M.*

Proof This is a local property, so that M is assumed to be an open ball of $\mathbf{C}^m$. Let f be a meromorphic function on $M \setminus E$. Then the support of the polar divisor $(f)_\infty$ of f extends uniquely to an analytic subset of M by Theorem 2.2.5. Therefore, $(f)_\infty$ is an effective divisor on M. By Theorem 2.2.14 there is a holomorphic function g on M such that $(g) = (f)_\infty$, so that gf is holomorphic in $M \setminus E$. Then Theorem 2.2.7 (ii) implies that gf extends holomorphically to a holomorphic function h on M. Thus, $f = h/g$ is meromorphic on M. □

Let $D = \sum k_\lambda A_\lambda$ be a divisor on M. For a $2m - 2$ form η on M whose coefficients are locally bounded Borel-measurable functions and whose support is compact, a current by integration

$$D(\eta) = \int_D \eta = \sum_\lambda k_\lambda \int_{A_\lambda} \eta$$

is defined.

Theorem 2.2.16 (Poincaré–Lelong formula) *Let $f \not\equiv 0$ be a meromorphic function on M and let η be a $2m-2$ form of C^2-class on M with compact support. Then,*

$$\int_{(f)} \eta = \int_M \log|f|^2 dd^c\eta = \int_M dd^c\bigl[\log|f|^2\bigr] \wedge \eta.$$

Here dd^c is taken in the sense of currents. That is, as currents,

$$dd^c\bigl[\log|f|^2\bigr] = (f).$$

Proof The given formula trivially holds outside $\mathrm{Supp}(f)$. It suffices to show it locally in a neighborhood of every point of $\mathrm{Supp}(f)$. Hence one may assume that f is holomorphic. Note that $\log|f|^2$ ($\not\equiv -\infty$) is a plurisubharmonic function. The dimension of the set S of singular points of $\mathrm{Supp}(f)$ is at most $m-2$. It follows from Theorem 2.2.11 and Lemma 2.2.12 that S is a measure-zero set with respect to the currents of both sides of the formula. Therefore it is sufficient to show

$$\int_{(f)\setminus S} \eta = \int_{M\setminus S} \log|f|^2 dd^c\eta \quad (S \cap \mathrm{Supp}\,\eta = \emptyset).$$

Take an arbitrary point $a \in \mathrm{Supp}(f) \setminus S$. Choosing a sufficiently small neighborhood W of a, we have $W \cap S = \emptyset$ and a coordinate system $(w_1, \ldots, w_m)$ in W such that

(i) $\mathrm{Supp}(f) \cap W = \{w_1 = 0\}$,
(ii) $f(w) = (w_1)^k h(w)$ and $h(w) \neq 0$, $^\forall w \in W$,
(iii) $\mathrm{Supp}\,\eta \subset W$.

It is immediate that $\int_M \log|h|^2 dd^c\eta = \int_M dd^c \log|h|^2 \wedge \eta = 0$. We get

(2.2.17)
$$\begin{aligned}\int_M \log|f|^2 dd^c\eta &= \int_M \log|w_1|^{2k} dd^c\eta \\ &= \lim_{\varepsilon\to 0}\left(-2k\log\varepsilon \int_{|w_1|=\varepsilon} d^c\eta - k\int_{|w_1|\geqq\varepsilon} d\log|w_1|^2 \wedge d^c\eta\right).\end{aligned}$$

Since $\int_{|w_1|=\varepsilon} d^c\eta = O(\varepsilon)$, the first term of the right-hand side of (2.2.17) converges to 0. We calculate the second term:

$$\begin{aligned}-k\int_{|w_1|\geqq\varepsilon} d\log|w_1|^2 \wedge d^c\eta &= k\int_{\{|w_1|\geqq\varepsilon\}} d^c\log|w_1|^2 \wedge d\eta \\ &= -k\int_{\{|w_1|\geqq\varepsilon\}} d\bigl(d^c\log|w_1|^2 \wedge \eta\bigr) \\ &= k\int_{\{|w_1|=\varepsilon\}} d^c\log|w_1|^2 \wedge \eta \\ &= k\int_{\{|w_1|=\varepsilon\}} \frac{1}{2\pi} d(\arg w_1) \wedge \eta\end{aligned}$$

(continued)

$$\to k \int_{\{w_1=0\}} \eta \quad (\varepsilon \to 0).$$

Hence, this combined with (2.2.17) implies the required identity. □

It follows from Theorem 2.2.16 that the integration over D defines naturally a current with coefficients in Radon measures (Lelong [68]; Noguchi–Ochiai [90] (Ochiai–Noguchi [84])).

Let $M = \mathbf{C}^m$. Let D be an effective divisor on $\mathbf{C}^m$ with the irreducible decomposition $D = \sum_\lambda k_\lambda D_\lambda$. For $1 \leqq k \leqq \infty$ the *truncated counting functions* to level k are defined by

$$(2.2.18)\qquad \begin{aligned} n_k(t, D) &= \frac{1}{t^{2m-2}} \int_{B(t)\cap(\sum_\lambda \min\{k,k_\lambda\}D_\lambda)} \alpha^{m-1}, \\ N_k(r, D) &= \int_1^r \frac{n_k(t, D)}{t} dt, \quad r > 1, \\ n(t, D) &= n_\infty(t, D), \qquad N(r, D) = N_\infty(r, D). \end{aligned}$$

In particular, $n(t, D)$ and $N(r, D)$ are simply called the *counting functions*.

Theorem 2.2.19 *Let D be an effective divisor on $\mathbf{C}^m$. Then $n_k(t, D)$ is increasing in $t > 0$.*

Proof If we reconsider D to be $\sum_\lambda \min\{k, k_\lambda\}D_\lambda$ in (2.2.18), we may take $k = \infty$. By Theorem 2.2.14 there is a holomorphic function f on $\mathbf{C}^m$ such that $(f) = D$ and $dd^c[\log |f|^2] = D$. Therefore we see by Lemma 2.1.31 that $n(t, D)$ is increasing in $t > 0$. □

2.3 The First Main Theorem

2.3.1 Meromorphic Mappings, Divisors and Line Bundles

Let M and N be complex spaces (cf. Convention (xvi)). A *meromorphic mapping* $f : M \to N$ from M into N is a correspondence such that for a point $x \in M$ a subset $f(x) \subset N$ is assigned and the graph $\Gamma(f) = \{(x, f(x)); x \in M\} \subset M \times N$ forms an irreducible analytic subset and satisfies the following:

(i) The first projection $p : \Gamma(f) \to M$ is proper.
(ii) There is a nowhere dense analytic subset $S \subset M$ such that the restriction $p|_{\Gamma(f)\setminus p^{-1}S} : \Gamma(f) \setminus p^{-1}S \to M \setminus S$ is biholomorphic.

Therefore the restriction $f|_{M\setminus S} : M \setminus S \to N$ is a holomorphic mapping. The set $I(f)$ of points $x \in M$ with $f(x)$ containing more than one point is called the *indeterminacy locus* of f.

Theorem 2.3.1 *Let $f : M \to N$ be a meromorphic mapping. Assume that M is non-singular. Then $I(f)$ is an analytic subset of codimension $\geqq 2$ and f is holomorphic on $M \setminus I(f)$.*

Proof Note that for every $x \in M$, $p^{-1}x$ is connected. Set

$$Z = \left\{z \in \Gamma(f); \dim_z p^{-1}p(x) \geqq 1\right\}.$$

Then Z is an analytic subset, $\dim Z < \dim \Gamma(f) = \dim M$, and $p(Z) = I(f)$. By definition $p(Z) < \dim Z$, and hence $\operatorname{codim} I(f) \geqq 2$. Let $x \notin I(f)$. Then $p^{-1}(x) = \{(x, y)\} \in \Gamma(f)$ is a point set. By Theorem 2.2.3 there are a holomorphic local coordinate neighborhood $(V, (y_1, \ldots, y_n)), |y_j| < 1$ of $y \in N$, a neighborhood U of $x \in M$, and a proper analytic subset $S \subset U$ such that $f(U) \subset V$, $f|(U \setminus S)$ is represented by holomorphic functions $f_j(x) \in \Delta(1)$. It follows from Theorem 2.2.6 (i) that $f_j(x)$ are holomorphic functions on U. Thus $f|U$ is a holomorphic mapping. □

Remark 2.3.2 The above theorem holds if M is a normal complex space. For a holomorphic function which is defined outside an analytic subset Z of codimension $\geqq 2$ in a normal complex space extends holomorphically over Z; the same extension holds for plurisubharmonic functions in a normal complex space (Grauert–Remmert [56]).

Corollary 2.3.3 *If M is a Riemann surface, then a meromorphic mapping $f : M \to N$ is necessarily holomorphic.*

Remark 2.3.4 There is a one-to-one correspondence between non-constant meromorphic functions and non-constant meromorphic mappings to $\mathbf{P}^1(\mathbf{C})$.

Definition 2.3.5 A meromorphic mapping $f : M \to N$ is said to be *analytically degenerate*, if the image $f(M)$ is contained in a proper analytic subset of N; otherwise, f is said to be *analytically non-degenerate*. When N is contained in an projective algebraic manifold, we similarly define f to be *algebraically (non-)degenerate* by using algebraic subsets in place of analytic subsets.

Remark 2.3.6 If N is projective algebraic, by Theorem 2.2.8 the analytic degeneracy is the same as the algebraic degeneracy. If N is an open subset of a projective algebraic variety, the two notions are different.

Let $f : M \to N$ be a meromorphic mapping. Let $A \subset N$ be a hypersurface which is *Cartier*; that is, it is locally defined as zero locus of a single holomorphic function. Assume that $f(M) \not\subset A$. Unless $(f|_{M\setminus I(f)})^{-1}A$ is empty, it is a hypersurface of $M \setminus I(f)$. By Theorem 2.2.5 the closure of $(f|_{M\setminus I(f)})^{-1}A$ is a hypersurface, and is denoted by $f^{-1}A$, which is called the pull-back of A by f. Therefore the pull-back $(f|_{M\setminus I(f)})^{*}A$ as divisor, extends uniquely to a divisor $f^{*}A$ on M.

On singular N we only deal with Cartier divisors defined as follows. Let $N = \bigcup_\alpha U_\alpha$ be an open covering and let ϕ_α be a meromorphic function on each U_α. We assume that ϕ_α/ϕ_β restricted to $U_\alpha \cap U_\beta$ is a holomorphic function without zero for every α and β. Let $N = \bigcup V_\lambda$ and $\{\psi_\lambda\}$ be another such family. We say that $\{\phi_\alpha\}$ and $\{\psi_\lambda\}$ are equivalent if there is a refinement $\{W_\nu\}$ of $\{U_\alpha\}$ and $\{V_\lambda\}$ such that $\phi_{\alpha(\nu)}/\psi_{\lambda(\nu)}$ restricted to W_ν is holomorphic and zero free for every ν. The equivalence class $D = [\{\phi_\alpha\}]$ is called a *Cartier divisor* on N. The restriction $D|_U$ to an open subset $U \subset N$ is naturally defined.

Let M be non-singular. Let $f : M \to N$ be a meromorphic mapping, and let D be a Cartier divisor on N. Assume that D has a representation $\phi_\alpha = \phi_{1\alpha}/\phi_{2\alpha}$ such that $\phi_{2\alpha} \circ f|_{M\setminus I(f)}$ does not vanish identically on any open subset where they are defined. Then the pull-back $(f|_{M\setminus I(f)})^* D$ by $f|_{M\setminus I(f)}$ is defined as a divisor on $M \setminus I(f)$. Then the support $\mathrm{Supp}(f|_{M\setminus I(f)})^* D$ is a hypersurface of $M \setminus I(f)$. Since $\operatorname{codim} I(f) \geqq 2$, it follows from Theorem 2.2.5 that the closure of $\mathrm{Supp}(f|_{M\setminus I(f)})^* D$ in M is a hypersurface of M, and hence the divisor $(f|_{M\setminus I(f)})^* D$ has a unique extension as divisor over M, which is denoted by $f^* D$.

For a holomorphic function ψ on N $(f|_{M\setminus I(f)})^* \psi$ is a holomorphic function on $M \setminus I(f)$. Since $\operatorname{codim} I(f) \geqq 2$, Theorem 2.2.6 (ii) implies that it uniquely extends to a holomorphic function on M denoted by $f^* \psi$. Let ϕ be a meromorphic function on N with local representations $\phi|_{U_\alpha} = \phi_{1\alpha}/\phi_{2\alpha}$, with an open covering $N = \bigcup_\alpha U_\alpha$. Assume that $\phi_{2\alpha} \circ f|_{N\setminus I(f)}$ does not vanish identically on any open subset where they are defined. Then $(f|_{M\setminus I(f)})^* \phi$ is a meromorphic function on $M \setminus I(f)$. Then we have a divisor $f^*(\psi)$ on M as above and so there is a holomorphic function g in a neighborhood U of an arbitrary point $x \in M$ such that $(g) + f^*(\psi)|_U$ is effective. Thus $g \cdot (f|_{U\setminus I(f)})^* \psi$ is holomorphic and by Theorem 2.2.6 (ii) extends to a holomorphic function h on U. Hence the pull-back meromorphic function $f^* \psi$ is defined locally by $\frac{h}{g}$ on U. For a plurisubharmonic function on N its pull-back by f is defined as well by Theorem 2.2.7.

Example 2.3.7 Let $M = \mathbf{C}^m$, $N = \mathbf{P}^n(\mathbf{C})$, and let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. Let $[w_0, \ldots, w_n]$ be a homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$. The hyperplane $\{w_j = 0\}$ is itself an effective divisor. There is an index j with $f(\mathbf{C}^m) \not\subset \{w_j = 0\}$. Changing the indices, we may assume without loss of generality that $f(\mathbf{C}^m) \not\subset \{w_0 = 0\}$. For the pull-back $f^*\{w_0 = 0\}$ as divisor, Theorem 2.2.14 implies the existence of an entire function f_0 on $\mathbf{C}^m$ such that $(f_0) = f^*\{w_0 = 0\}$. As w_j/w_0 is a meromorphic function on $\mathbf{P}^n(\mathbf{C})$, the pull-back $f^*(w_j/w_0)$ is a meromorphic function on $\mathbf{C}^m$, and $f_j = f_0 \cdot f^*(w_j/w_0)$ is holomorphic. By the construction we have

$$\operatorname{codim}\{f_0 = \cdots = f_n = 0\} \geqq 2, \qquad I(f) = \{f_0 = \cdots = f_n = 0\}.$$

We represent $f = [f_0, \ldots, f_n]$, which is called the *reduced representation*.

Let L be a complex space and let $\pi : L \to N$ be a surjective holomorphic mapping. If the following three conditions are satisfied, the triple (L, π, N) or simply L is called a *holomorphic line bundle* (or simply a *line bundle*) over N:

Condition 2.3.8 (i) There is an open covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of N such that the restriction $L|_{V_\lambda} = \pi^{-1}(V_\lambda)$ admits a biholomorphic mapping $\phi_\lambda : L|_{V\lambda} \to V_\lambda \times \mathbf{C}$.

(ii) Whenever $V_\lambda \cap V_\mu \neq \emptyset$, there is a holomorphic function $\phi_{\lambda\mu}$ without zero on $V_\lambda \cap V_\mu$ such that

$$\phi_\lambda \circ \phi_\mu^{-1}|_{(V_\lambda\cap V_\mu)\times\mathbf{C}} : (x,\xi_\mu) \in (V_\lambda \cap V_\mu)\times \mathbf{C}$$
$$\longrightarrow \bigl(x, \phi_{\lambda\mu}(x)\xi_\mu\bigr) \in (V_\mu \cap V_\lambda)\times\mathbf{C}.$$

(iii) If $\lambda = \mu$, $\phi_{\lambda\lambda} = 1$.

In this case $\{V_\lambda\}$ is called a *local trivialization covering* of L and $\{\phi_{\lambda\mu}\}$ is called the *transition function system*.

At every $x \in N$ the inverse image $L_x = \pi^{-1}(x)$ is a one-dimensional complex vector space. For $y_i \in L_x$, $c_i \in \mathbf{C}$, $i = 1, 2$ the natural operation as vector space

$$c_1y_1 + c_2y_2 \in L_x$$

is defined by Condition 2.3.8 (ii).

A mapping $\sigma : W \to L$ from a subset $W \subset N$ into L satisfying $\pi \circ \sigma = \mathrm{id}_W$, is called a *section* of L over W. In particular, when W is open, we denote the set of holomorphic sections of L over W by $H^0(W, L)$, which naturally forms a complex vector space. We denote by $\mathscr{O}(L)$ the sheaf of germs of holomorphic sections of L.

The transition function system $\{\phi_{\lambda\mu}\}$ in Condition 2.3.8 (ii) satisfies the so-called cocycle condition:

$$\begin{aligned} \phi_{\lambda\lambda} &= 1, \\ \phi_{\lambda\mu}\phi_{\mu\lambda} &= 1 \quad (\text{on } V_\lambda \cap V_\mu), \\ \phi_{\lambda\mu}\phi_{\mu\nu}\phi_{\nu\lambda} &= 1 \quad (\text{on } V_\lambda \cap V_\mu \cap V_\nu). \end{aligned} \tag{2.3.9}$$

On the other hand, suppose that we are given a system $\{\phi_{\lambda\mu}\}$ of holomorphic functions satisfying the cocycle condition (2.3.9). Then we may construct as follows a holomorphic line bundle over N whose transition function system is $\{\phi_{\lambda\mu}\}$. First we consider the disjoint union $\sqcup_{\lambda\in\Lambda} V_\lambda \times \mathbf{C}$ of topological spaces. We introduce an equivalence relation $\sim$ for two elements (x_λ, ξ_λ), (x_μ, ξ_μ) by

$$x_\lambda = x_\mu \in N, \qquad \xi_\lambda = \phi_{\lambda\mu}(x_\mu)\xi_\mu.$$

The quotient space $L = (\sqcup_{\lambda\in\Lambda} V_\lambda \times \mathbf{C})/\sim$ constitutes a complex space, as is easily checked, and the mapping $\pi : L \to N$ projecting an equivalence class $[(x_\lambda, \xi_\lambda)]$ to $x_\lambda \in N$ is a holomorphic surjection. It is easily checked that this is what was desired.

Let $W \subset N$ be an open subset and let $S \subset W$ be an analytic subset which is nowhere dense in W. Let $\sigma : W \setminus S \to L$ be a holomorphic section. On an arbitrary $V_\lambda \cap W$ we represent $\phi_\lambda(\sigma(x)) = (x, \sigma_\lambda(x)) \in V_\lambda \times \mathbf{C}$. If $\sigma_\lambda(x)$ is a meromorphic function on $V_\lambda \cap W$, then σ is called a meromorphic section of L on W. Since the transition functions have no zero, a meromorphic section $\sigma : W \to L$ (we write it in this way) defines a Cartier divisor (σ) on W. In particular, when $W = N$, a Cartier divisor (σ) on N is obtained. It is equivalent for σ to be a holomorphic section that $(\sigma) \geqq 0$.

Conversely, let a Cartier divisor D on N be given. We take an open covering $\{V_\lambda\}$ of N such that there exist meromorphic functions σ_λ on V_λ defining $D|_{V_\lambda}$. Then $\sigma_\lambda/\sigma_\mu$ has no zero on $V_\lambda \cap V_\mu$. Put $\phi_{\lambda\mu} = \sigma_\lambda/\sigma_\mu$. Then these are holomorphic functions on $V_\lambda \cap V_\mu$ without zero, which satisfy the cocycle condition (2.3.9). Thus a holomorphic line bundle $L(D)$ is obtained as above. By the construction $L(D)|_{V_\lambda} = V_\lambda \times \mathbf{C}$ and $\{\phi_{\lambda\mu}\}$ is the transition function system. Setting locally $x \in V_\lambda \to (x, \sigma_\lambda(x))$, we obtain a meromorphic section σ on N satisfying $(\sigma) = D$.

In the present book we deal only with holomorphic line bundles, which we call simply line bundles.

Let $\pi_i : L_i \to N$, $i = 1, 2$ be two line bundles over N. If there is a biholomorphic mapping $\psi : L_1 \to L_2$ such that $\pi_1 = \pi_2 \circ \psi$ and $\psi|_{L_{1x}} : L_{1x} \to L_{2x}$ $(x \in N)$ is a linear isomorphism, $\psi : L_1 \to L_2$ is called an *isomorphism*, and L_1 is said to be isomorphic to L_2. We identify isomorphic line bundles with each other. The line bundle $\mathbf{1}_N = N \times \mathbf{C}$ is called a trivial line bundle. A line bundle $L \to N$ is trivial if and only if there exists a holomorphic section on N without zero.

Take a local trivialization covering of both L_1 and L_2. Let $\{\phi_{i\lambda\mu}\}$ be the transition function system of L_i. The product $\{\phi_{1\lambda\mu} \cdot \phi_{2\lambda\mu}\}$ yields a line bundle $L_3 \to N$. This is called the tensor product of L_1 and L_2 and is denoted by $L_1 \otimes L_2$.

The line bundle given by $\{\phi_{1\lambda\mu}^{-1}\}$ is denoted by L_1^{-1}. Then $L_1 \otimes L_1^{-1} = \mathbf{1}_N$. We set

$$L_1^k = L_1 \otimes \cdots \otimes L_1 \quad (k\text{-times}, k \geqq 0),$$
$$L_1^k = L_1^{-1} \otimes \cdots \otimes L_1^{-1} \quad \big(|k|\text{-times}, k < 0\big).$$

For two given divisors D_i, $i = 1, 2$ on N,

$$L(D_1 + D_2) = L(D_1) \otimes L(D_2).$$

Assume for a moment that N is non-singular. Let $\{V_\lambda(x_{\lambda 1}, \ldots, x_{\lambda n})\}_{\lambda \in \Lambda}$ be a covering of holomorphic local coordinate neighborhood system of N. For $V_\lambda \cap V_\mu \neq \emptyset$ we consider the following transition of holomorphic forms:

$$dx_{\lambda 1} \wedge \cdots \wedge dx_{\lambda n} = \kappa_{\mu\lambda}(x) dx_{\mu 1} \wedge \cdots \wedge dx_{\mu n},$$
$$\kappa_{\mu\lambda} = \frac{\partial(x_{\lambda 1}, \ldots, x_{\lambda n})}{\partial(x_{\mu 1}, \ldots, x_{\mu n})} \quad \text{(Jacobian)}.$$

Since $\{\kappa_{\lambda\mu}\}$ satisfies the cocycle condition (2.3.9), a line bundle K_N over N is obtained from it, and K_N is called the *canonical bundle* over N. Meromorphic sections of K_N are identified with meromorphic n-forms.

2.3.2 Differentiable Functions on Complex Spaces

Here we have to be precise on the definition of C^∞ functions on a singular complex space N. There have been more than one such definitions, but here we follow

Fujiki [78a]. Since it is a local notion, we restrict ourselves for a moment to the case where N is an analytic subset of an open set $\Omega \subset \mathbf{C}^n$. Let $\mathscr{I}\langle N\rangle$ be the ideal sheaf of N in the structure sheaf $\mathscr{O}_\Omega$ of holomorphic functions over Ω. The quotient $\mathscr{O}_N = \mathscr{O}_\Omega/\mathscr{I}\langle N\rangle$ is called the structure sheaf (of holomorphic functions) over N. Since $\mathscr{I}\langle N\rangle$ is coherent[3] and we are concerned here only with a local property, we may assume that there are finitely many holomorphic functions,

2.3.10 $\tau_1, \ldots, \tau_l \in \Gamma(\Omega, \mathscr{I}\langle N\rangle)$ over Ω, generating $\mathscr{I}\langle N\rangle_x$ at every point $x \in \Omega$.

Let $\mathscr{E}_\Omega$ denote the sheaf of germs of complex-valued C^∞ functions over Ω. We denote by $\mathscr{E}\mathscr{I}\langle N\rangle$ the sheaf of ideals of $\mathscr{E}_\Omega$ generated by τ_j and $\bar{\tau}_j$, $1 \leqq j \leqq l$, over $\mathscr{E}_\Omega$; that is,

$$\mathscr{E}\mathscr{I}\langle N\rangle = \sum_{j=1}^{l}(\mathscr{E}_\Omega\tau_j + \mathscr{E}_\Omega\bar{\tau}_j) = \sum_{j=1}^{l}(\mathscr{E}_\Omega\Re\tau_j + \mathscr{E}_\Omega\Im\tau_j). \tag{2.3.11}$$

Then we have that $\mathscr{I}\langle N\rangle \subset \mathscr{E}\mathscr{I}\langle N\rangle \subset \mathscr{E}_\Omega$. We define the sheaf of germs of C^∞ functions over N by the quotient

$$\mathscr{E}_N = \mathscr{E}_\Omega/\mathscr{E}\mathscr{I}\langle N\rangle. \tag{2.3.12}$$

Because of the definition, $\mathscr{E}_N$ is well-defined for a complex space N. It is clear that $\mathscr{E}_N$ is a sheaf of rings with unit 1. A section ϕ of $\mathscr{E}_N$ over an open subset $U \subset N$ is called a differentiable or C^∞ function on U, and it uniquely defines a continuous function

$$\phi : U \to \mathbf{C}.$$

If $\phi(x_0) \neq 0$, then in a neighborhood of x_0, ϕ is invertible; this is seen as follows. Since it is a local problem, we let $N \subset \Omega \subset \mathbf{C}^n$ and τ_j be as in 2.3.10. Let ϕ be represented by $\tilde{\phi}, \tilde{\phi}' \in \Gamma(\Omega, \mathscr{E}_\Omega)$ satisfying

$$\tilde{\phi} = \tilde{\phi}' + \sum_{j=1}^{l}(a_j\tau_j + b_j\bar{\tau}_j).$$

Taking $a'_j = a_j/\tilde{\phi}'$ and $b'_j = b_j/\tilde{\phi}'$, one gets

$$\tilde{\phi} = \tilde{\phi}'\left(1 + \sum_{j=1}^{l}\left(a'_j\tau_j + b'_j\bar{\tau}_j\right)\right).$$

[3] K. Oka, in [Iw] VII, [50] and [51], proved three fundamental coherence theorems for (i) $\mathscr{O}_\Omega$, (ii) $\mathscr{I}\langle N\rangle$, and (iii) the normalization of $\mathscr{O}_N$. Cf. H. Cartan [50] for another proof of the coherence of $\mathscr{I}\langle N\rangle$, and Grauert–Remmert [84]. K. Oka called $\mathscr{I}\langle N\rangle$ the *geometric ideal sheaf* (*l'idéal géométrique de domaines indéterminés*). It is interesting to see the comments in Oka [Sp] and Cartan [79].

We may assume that

$$\sum_{j=1}^{l} |a'_j \tau_j| < \frac{1}{2}, \qquad \sum_{j=1}^{l} |b'_j \tau_j| < \frac{1}{2}.$$

Then $|\sum_{j=1}^{l}(a'_j\tau_j + b'_j\bar{\tau}_j)| < 1$, and we have

$$\begin{aligned} \frac{1}{\tilde{\phi}} &= \frac{1}{\tilde{\phi}'} \cdot \frac{1}{1+\sum_{j=1}^{l}(a'_j\tau_j + b'_j\bar{\tau}_j)} \\ &= \frac{1}{\tilde{\phi}'} + \sum_{k=1}^{\infty}(-1)^k \left(\sum_{j=1}^{l} \big(a'_j\tau_j + b'_j\bar{\tau}_j\big) \right)^k \\ &= \frac{1}{\tilde{\phi}'} + \sum_{k=1}^{\infty}(-1)^k \left(\sum_{j=1}^{l} \big(a'_j\tau_j + b'_j\bar{\tau}_j\big) \right)^k . \end{aligned}$$

The second term above is written as an absolute convergent power series in $a'_j\tau_j$, $b'_j\bar{\tau}_j$ without constant term. Therefore there are C^∞ functions a''_j, b''_j in Ω such that

$$\frac{1}{\tilde{\phi}} = \frac{1}{\tilde{\phi}'} + \sum_{j=1}^{l}\big(a''_j\tau_j + b''_j\bar{\tau}_j\big). \tag{2.3.13}$$

Thus $\frac{1}{\tilde{\phi}}$ and $\frac{1}{\tilde{\phi}'}$ define the same section $\frac{1}{\phi} \in \Gamma(N, \mathscr{E}_N)$ such that $\phi \cdot \frac{1}{\phi} = 1$.

We then define the *tangent space* over N. This is again a local object, so that we let $N \subset \Omega \subset \mathbf{C}^n$ and τ_j be as above (cf. 2.3.10). Let $(x_1, \ldots, x_n)$ be the natural coordinate system of $\mathbf{C}^n$. Let $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x_i}$ be a holomorphic tangent vector of $\mathbf{T}(\Omega)$ at $x \in N$. We define

$$\mathbf{T}(N)_x = \left\{ (x, X);\, X(\tau_j) = \sum_{i=1}^{n} X^i \frac{\partial \tau_j}{\partial x_i}(x) = 0,\, 1 \leqq j \leqq l_\lambda \right\}. \tag{2.3.14}$$

This yields an analytic subset

$$\mathbf{T}(N) = \bigcup_{x \in N} \mathbf{T}(N)_x \subset N \times \mathbf{C}^n$$

with the natural projection $\pi : \mathbf{T}(N) \to N$, the fibers of which are vector spaces. We call $\mathbf{T}(N)$ the holomorphic tangent space over N. Because of the coherence of $\mathscr{I}\langle N \rangle$ it is easy to see that the definition of $\mathbf{T}(N)$ is compatible with the change of coordinates (i.e., the embedding into Ω), so that for a general complex space N the holomorphic tangent space $\mathbf{T}(N)$ is defined with the natural projection $\pi : \mathbf{T}(N) \to N$, the fibers of which are vector spaces.

In terms of (2.3.14) we set

$$\bar{\mathbf{T}}(N)=\left\{Y=\sum_{j=1}^{n}Y^j\frac{\partial}{\partial\bar{x}_j};\,Y^j\in\mathbf{C},\,\bar{Y}=\sum_{j=1}^{n}\bar{Y}^j\frac{\partial}{\partial x_j}\in\mathbf{T}(N)\right\},\tag{2.3.15}$$
$$T(N)=\mathbf{T}(N)\oplus\bar{\mathbf{T}}(N).$$

We call $\bar{\mathbf{T}}(N)$ the anti-holomorphic tangent space over N, and $T(N)$ the tangent space over N.

Let $\phi\in\Gamma(U,\mathscr{E}_N)$ be a C^∞ function over an open subset U of a complex space N. We want to define the *differential* $d\phi(X)$ for a tangent vector $X\in\mathbf{T}(N)|_U$. There is locally a C^∞ function $\tilde{\phi}\in\Gamma(\Omega,\mathscr{E}_\Omega)$ which represents ϕ on U. If we take another $\tilde{\phi}'\in\Gamma(\Omega,\mathscr{E}_\Omega)$, then there are C^∞ functions a_j,b_j on Ω such that

$$\tilde{\phi}=\tilde{\phi}'+\sum_{j=1}^{l}(a_j\tau_j+b_j\bar{\tau}_j).\tag{2.3.16}$$

For the differentials we get

$$\partial\tilde{\phi}=\partial\tilde{\phi}'+\sum_{j=1}^{l}(\tau_j\partial a_j+a_j\partial\tau_j+\bar{\tau}_j\partial b_j),\tag{2.3.17}$$
$$\bar{\partial}\tilde{\phi}=\bar{\partial}\tilde{\phi}'+\sum_{j=1}^{l}(\tau_j\bar{\partial}a_j+b_j\bar{\partial}\bar{\tau}_j+\bar{\tau}_j\bar{\partial}b_j).$$

It follows from (2.3.14) that for $X\in\mathbf{T}(N)_x$ $(x\in N)$,

$$\partial\tilde{\phi}(X)=\partial\tilde{\phi}'(X),\qquad\bar{\partial}\tilde{\phi}(\bar{X})=\bar{\partial}\tilde{\phi}'(\bar{X}).$$

Thus $\partial\phi:\mathbf{T}(N)\to\mathbf{C}$ and $\bar{\partial}\phi:\bar{\mathbf{T}}(N)\to\mathbf{C}$ are well-defined, and we have the differential of ϕ:

$$d\phi=\partial\phi\oplus\bar{\partial}\phi:T(N)=\mathbf{T}(N)\oplus\bar{\mathbf{T}}(N)\to\mathbf{C}.$$

From the second equation of (2.3.17) it follows that

$$\partial\bar{\partial}\tilde{\phi}=\partial\bar{\partial}\tilde{\phi}'+\sum_{j=1}^{l}(\partial\tau_j\wedge\bar{\partial}a_j+\tau_j\partial\bar{\partial}a_j+\partial b_j\wedge\bar{\partial}\bar{\tau}_j+\bar{\tau}_j\partial\bar{\partial}b_j).$$

Therefore we can define the *Levi-form* of $\partial\bar{\partial}\phi$ along with the real form $\frac{i}{2}\partial\bar{\partial}\phi$ by

$$\partial\bar{\partial}\phi(X,\bar{Y})=\partial\bar{\partial}\tilde{\phi}(X,\bar{Y})=\partial\bar{\partial}\tilde{\phi}'(X,\bar{Y}),\quad X,Y\in\mathbf{T}(N).\tag{2.3.18}$$

2.3.3 Metrics and Curvature Forms of Line Bundles

Let $L \to N$ be a line bundle over a complex space N. Let $N = \bigcup_\lambda V_\lambda$ be an open covering such that there is a transition function system $\{\phi_{\lambda\mu}\}$ of L. A *hermitian metric* in L is a family $h = \{h_\lambda\}$ of positive real-valued C^∞ functions $h_\lambda \in \Gamma(V_\lambda, \mathscr{E}_N)$ satisfying

$$(2.3.19) \qquad h_\lambda(x) = \left|\phi_{\lambda\mu}(x)\right|^2 h_\mu(x), \quad x \in V_\lambda \cap V_\mu.$$

The line bundle L endowed with h is called a *hermitian line bundle* and is denoted by the pair (L, h); we sometimes write simply L for a hermitian line bundle when there is no confusion. We take a local trivialization

$$L|_{V_\lambda} \cong V_\lambda \times \mathbf{C}.$$

For $v = (x, \xi_\lambda) \in V_\lambda \times \mathbf{C} \subset L$ we set the norm of v with respect to h by

$$\|v\| = \frac{|\xi_\lambda|}{\sqrt{h_\lambda(x)}}.$$

Then this is independent of the choice of V_λ. The norm function $\|v\|$ is also called a hermitian metric in L.

If N is paracompact, we may construct such a hermitian metric in every line bundle L over N. In fact, let $\{V_\lambda\}$ and $\{\phi_{\lambda\mu}\}$ be as above. We may assume that $\{V_\lambda\}$ is locally finite. Taking a partition $\{c_\lambda\}$ of unity subordinated to $\{V_\lambda\}$, we set C^∞ functions on V_λ by $c_\nu(x)\log|\phi_{\lambda\nu}(x)|^2$ extending as 0 on $V_\lambda \setminus V_\nu$, and set

$$h_\lambda(x) = \exp\left\{\sum_\nu c_\nu(x)\log\left|\phi_{\lambda\nu}(x)\right|^2\right\}, \quad x \in V_\lambda.$$

It follows from the cocycle condition (2.3.9) that

$$(2.3.20) \qquad h_\lambda = |\phi_{\lambda\mu}|^2 h_\mu \quad (\text{on } V_\lambda \cap V_\mu \neq \emptyset).$$

Going back to (2.3.19), we want to define the Levi-form $\partial\bar\partial \log h_\lambda$. As in (2.3.16), let h_λ and h'_λ be C^∞ functions such that

$$\tilde h_\lambda = \tilde h'_\lambda + \sum_{j=1}^{l}(a_{\lambda j}\tau_{\lambda j} + b_{\lambda j}\bar\tau_{\lambda j}),$$

where $a_{\lambda j}$ and $b_{\lambda j}$ are C^∞ functions on Ω_λ into which V_λ is embedded, and $\tau_{\lambda j}$ are generators of $\mathscr{I}\langle N\rangle|_{V_\lambda}$. Since h'_λ is positive valued, we have

$$\tilde h_\lambda = \tilde h'_\lambda \cdot \left(1 + \sum_{j=1}^{l}\left(a'_{\lambda j}\tau_{\lambda j} + b'_{\lambda j}\bar\tau_{\lambda j}\right)\right),$$

where $a'_{\lambda j} = a_{\lambda j}/h'_\lambda$ and $b'_{\lambda j} = b_{\lambda j}/h'_\lambda$. Since $\tau_{\lambda j}$ vanish on V_λ, we may assume that $\sum_{j=1}^{l} |a'_{\lambda j}\tau_{\lambda j}| < \frac{1}{2}$ and $\sum_{j=1}^{l} |b'_{\lambda j}\bar{\tau}_{\lambda j}| < \frac{1}{2}$. By making use of the power expansion $\log(1+t) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} t^k$ $(|t| < 1)$, we deduce in the same way as in (2.3.13) that

$$\log \tilde{h}_\lambda = \log \tilde{h}'_\lambda + \sum_{j=1}^{l} \bigl(a''_{\lambda j}\tau_{\lambda j} + b''_{\lambda j}\bar{\tau}_{\lambda j}\bigr),$$

where $a''_{\lambda j}$ and $b''_{\lambda j}$ are C^∞ functions. Hence the Levi-form $\Theta_\lambda = \partial\bar{\partial}\log h_\lambda$ of $\log h_\lambda$ is well-defined, and (2.3.19) implies

$$\Theta_\lambda(x) = \Theta_\mu(x), \quad x \in V_\lambda \cap V_\mu.$$

The *curvature form* $\Theta_{(L,h)}$ of the hermitian line bundle (L,h) is globally defined by

$$\Theta_{(V,h)}|_{V_\lambda}(X, \bar{Y}) = \Theta_\lambda(X, \bar{Y}), \quad X, Y \in \mathbf{T}(N)|_{V_\lambda},$$

and the real form

$$\omega_{(L,h)} = \frac{i}{2\pi}\Theta_{(L,h)} = dd^c \log h_\lambda \text{ in } V_\lambda$$

is called the *Chern form* of (L,h); here we recall

$$d^c = \frac{i}{4\pi}(\bar{\partial} - \partial).$$

When it is not necessary to specify the hermitian metric h, we write ω_L for $\omega_{(L,h)}$. Let $\hat{h}$ be another hermitian metric in L. After taking a refinement $\{V_\lambda\}$ of open coverings of N, we have $h = \{h_\lambda\}$ and $\hat{h} = \{\hat{h}_\lambda\}$, so that $h_\lambda(x)/\hat{h}_\lambda(x)$ is globally defined, independently from the choice $V_\lambda \ni x$. Thus there is a C^∞ function b on N such that

$$\omega_{(L,h)} = \omega_{(L,\hat{h})} + dd^c b. \tag{2.3.21}$$

If N is non-singular, the Chern form $\omega_{(L,h)}$ defines a cohomology class $c_1(L) = [\omega_L] \in H^2(N, \mathbf{R})$, which is independent of the choice of the hermitian metric by (2.3.21) and is called the *Chern class* of L.

If $\omega_{(L,h)}$ is positive (resp. semi-)definite at all points of N, we write $\omega_{(L,h)} > 0$ (resp. $\omega_{(L,h)} \geqq 0$). We write $L > 0$ (resp. $L \geqq 0$) if there exists a hermitian metric h in L with Chern form $\omega_{(L,h)} > 0$ (resp. $\omega_{(L,h)} \geqq 0$); in this case we say that L is positive (resp. semi-positive). If N is non-singular and $L > 0$, ω_L defines a Kähler metric on N.

Let σ_i $(1 \leqq i \leqq p)$ and σ'_j $(1 \leqq j \leqq q)$ be holomorphic sections of L over N (may be singular) and let $\{V_\lambda\}$ be as above. On each V_λ they are given by holomorphic functions $\sigma_{i\lambda}$ $(1 \leqq i \leqq p)$ and $\sigma'_{j\lambda}$ $(1 \leqq j \leqq q)$. Assume that $\sum_j |\sigma'_{j\lambda}|^2 \not\equiv 0$. For $x \in N$ we take $V_\lambda \ni x$ and set

$$\frac{\sum_i |\sigma_i(x)|^2}{\sum_j |\sigma'_j(x)|^2} = \frac{\sum_i |\sigma_{i\lambda}(x)|^2}{\sum_j |\sigma'_{j\lambda}(x)|^2}. \tag{2.3.22}$$

This is independent of the choice of $V_\lambda \ni x$, and defines a function on N outside where the denominator vanishes.

Assume that $\dim H^0(N, L) \geqq 2$ and take linearly independent elements $\sigma_0, \dots, \sigma_n$ of $H^0(N, L)$. Then we get a meromorphic mapping

$$\Phi : x \in N \to \big[\sigma_0(x), \dots, \sigma_n(x)\big] \in \mathbf{P}^n(\mathbf{C}).$$

Note that this is not necessarily reduced representation. If $\dim H^0(N, L) < \infty$ and $\{\sigma_i\}$ is a base of $H^0(N, L)$, we write for the above Φ

$$\Phi_L : N \to \mathbf{P}^n(\mathbf{C}). \tag{2.3.23}$$

Let $S(N)$ denote the set of singular points of N. Then $S(N)$ is a nowhere dense analytic subset of N. Let N be compact. If the meromorphic mapping $\Phi_{L^l} : N \to \mathbf{P}^{n_l}(\mathbf{C})$ $(n_l = \dim H^0(N, K_N^l) - 1)$ for some l has the differential $d\Phi_{K_N^l x}$ of rank equal to $\dim N$ at some point $x \in N \setminus I(\Phi_{K_N^l}) \cup S(N)$, L is said to be *big*. If N is non-singular and K_N is big, N is said to be of *general type*.

If Φ_L gives rise to a holomorphic embedding, L is said to be *very ample*. If there is a number $k \in \mathbf{N}$ with very ample L^k, L is said to be *ample*.

Going back to Example 2.3.7, we denote by $H \to \mathbf{P}^n(\mathbf{C})$ the line bundle determined by the transition function system $\{\phi_{jk} = \frac{w_k}{w_j}\}$ associated to the affine open covering, $\mathbf{P}^n(\mathbf{C}) = \bigcup_{j=0}^n U_j$ with $U_j = \{w_j \neq 0\}$. We call H the *hyperplane bundle* over $\mathbf{P}^n(\mathbf{C})$. Then the functions

$$\rho_j = 1 + \sum_{i \neq j} \frac{|w_i|^2}{|w_j|^2} \tag{2.3.24}$$

on U_i define a hermitian metric in H with the curvature form $\omega_H > 0$. That is, $H > 0$. The Kähler metric h_H is called the *Fubini–Study metric*. The associated Kähler form ω_H is called the Fubini–Study metric form on $\mathbf{P}^n(\mathbf{C})$.

Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping with reduced representation $f = [f_0, \dots, f_n]$. The pull-back of ω_H by f is written as

$$f^*\omega_H = \frac{i}{2\pi} f^*\partial\bar\partial \log \rho_j = \frac{i}{2\pi}\partial\bar\partial \log\left(\sum_{i=0}^n |f_i|^2\right). \tag{2.3.25}$$

The function $\log(\sum_{i=0}^n |f_i|^2)$ is a plurisubharmonic function on $\mathbf{C}^m$.

Let N be a complex projective algebraic variety with a holomorphic embedding $\Psi : N \hookrightarrow \mathbf{P}^n(\mathbf{C})$. The pull-back Ψ^*H is a positive line bundle and a meromorphic mapping into $\mathbf{P}^n(\mathbf{C})$ given by $n+1$ linearly independent sections of $H^0(N, \Psi^*H)$ coincides with Φ. The next theorem gives the converse for it; the non-singular case is due to Kodaira [54], [74] (cf. Nakano [81] for a further generalization).

Theorem 2.3.26 (Grauert [62]) *Let N be a compact complex space and let $L \to N$ be a positive line bundle over N.*

(i) *Let* $\mathscr{I} \to N$ *be a coherent sheaf over a compact complex space* N*. Then there is a number* l_0 *such that* $H^q(N, \mathscr{I} \otimes \mathscr{O}(L^l)) = 0, l \geqq l_0, q \geqq 1$.

(ii) *Let* $E \to N$ *be a line bundle over* N*. Then there is a number* l_0 *such that for every* $l \geqq l_0$ *the meromorphic mapping* $\Phi_{L^l \otimes E} : N \to \mathbf{P}^{n_l - 1}(\mathbf{C})$ *with* $n_l = \dim H^0(N, L^l \otimes E)$ *is a holomorphic embedding, and hence* N *is projective algebraic.*

A divisor D on N defines a line bundle $L(D)$. If $L(D)$ satisfies the above property (ii), D is said to be *ample*; hence, D is ample iff $L(D)$ is positive. If the holomorphic sections of $L(D)$ over N give a holomorphic embedding into a projective space, D is said to be *very ample*.

Let N be a compact complex space. Take a vector subspace $E \subset H^0(N, L)$, $\dim E = l + 1 \geqq 2$. For the divisor (σ) given by $\sigma \in E \setminus \{0\}$ and for $c \in \mathbf{C}^*$ $(c\sigma) = (\sigma)$ holds clearly. Conversely, if $\sigma, \tau \in E \setminus \{0\}$ satisfy $(\sigma) = (\tau)$, then $\sigma = c\tau$ for some $c \in \mathbf{C}^*$. Therefore we have the following isomorphism:

$$\big\{(\sigma); \sigma \in E \setminus \{0\}\big\} \cong \big(E \setminus \{0\}\big)/\mathbf{C}^* = P(E) \cong \mathbf{P}^l(\mathbf{C}).$$

The space $P(E)$ is called a linear system of D, and in particular, when $E = H^0(N, L)$, it is called the *complete linear system* of L denoted by $|L|$. The analytic subset $B(E) = \{x \in N; \sigma(x) = 0, {}^\forall\sigma \in E\}$ is called the *base locus* of E.

In what follows we assume N to be *projective algebraic* unless otherwise mentioned. For any line bundle L over N there are by Theorem 2.3.26 very ample line bundles L_i, $i = 1, 2$, such that $L = L_1 \otimes L_2^{-1}$. By making use of the Chern forms $\omega_{L_i} > 0$ of L_i we obtain

$$\omega_L = \omega_{L_1} - \omega_{L_2}.$$

Noting (2.3.25), we see the following.

Lemma 2.3.27 *Let* N, L *be as above. Let* $f : \mathbf{C}^m \to N$ *be a meromorphic mapping. Then there are plurisubharmonic functions* ξ_i, $i = 1, 2$ *on* $\mathbf{C}^m$ *such that*

$$f^*\omega_L = dd^c\xi_1 - dd^c\xi_2.$$

Let $L \to N$ be a line bundle with a hermitian metric $\|\cdot\|$. For an element $D \in |L|$ we take and fix a holomorphic section $\sigma \in H^0(N, L) \setminus \{0\}$ such that $(\sigma) = D$ and $\|\sigma\| \leqq 1$. The next follows from the Poincaré–Lelong Theorem 2.2.16.

Lemma 2.3.28 *Let* $\|\sigma\|$ *and* ω_L *be as above. Let* $U \subset \mathbf{C}^m$ *be an open subset and let* $f : \mathbf{C}^m \to N$ *be a meromorphic mapping such that the pull-back* f^*D *is defined. Then we have a current equation on* U:

$$dd^c\left[\log \frac{1}{\|\sigma \circ f(z)\|^2}\right] = f^*\omega_L - f^*D.$$

Theorem 2.3.29 (Poincaré duality) *Let N be a compact complex manifold of dimension n. Let $L \to N$ be a hermitian line bundle and let $D \in |L|$. Then*

$$\int_D \eta = \int_N \omega_L \wedge \eta$$

for all d-closed $(n-1, n-1)$ forms η on N.

Proof Let $U \subset N$ be a holomorphic local coordinate neighborhood of N and let $\iota : U \to N$ be the inclusion. Suppose that $\operatorname{Supp} \eta \subset U$. By Lemma 2.3.28 we have

$$\int_U \omega_L \wedge \eta - \int_D \eta = \int_U \left(\log \frac{1}{\|\sigma\|^2} \right) dd^c \iota^* \eta = 0.$$

By making use of the partition of unity, we get the required formula. □

Let $f : \mathbf{C}^m \to N$ be a meromorphic mapping such that the pull-back $f^* D$ is defined. We apply Lemma 2.3.28 for $U = B(r)$ and $\eta = \alpha^{m-1} = (dd^c \|z\|^2)^{m-1}$. Then the equality such as in Theorem 2.3.29 does not hold, and a boundary integral appears:

$$\begin{aligned} &\int_{B(r)} f^* \omega_L \wedge \alpha^{m-1} - \int_{f^* D \cap B(r)} \alpha^{m-1} \\ &\quad = \int_{\partial B(r)} d^c \log \frac{1}{\|\sigma \circ f(z)\|} \wedge \alpha^{m-1}. \end{aligned}$$

The first term of the above equation should be the order function, the second should be the counting function, and the third might be some remainder. Since the boundary integral contains a differentiation, it is inconvenient to handle this formula directly, and we need some more modifications as follows.

The counting function $N(r, f^* D)$ is already defined by (2.2.18). By Lemma 2.3.27 and the result in Sect. 2.1, in particular, by Corollary 2.1.37, we may define the following two quantities:

$$(2.3.30) \qquad \begin{aligned} m_f(r, D) &= \int_{\|z\|=r} \log \frac{1}{\|\sigma \circ f(z)\|} \gamma(z), \\ T_f(r, \omega_L) &= \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} f^* \omega_L \wedge \alpha^{n-1}, \quad r \geqq 1. \end{aligned}$$

We call $m_f(r, D)$ the *proximity function* or the *approximate function* of f for D. We call $T_f(r, \omega_L)$[4] the *order function* of f with respect to ω_L. If $\omega_L \geqq 0$, it follows from

[4] In S. Lang [87] the notation $T_{f,D}$ is used, but this is not proper and mises an essential point: the order function is not dependent on each D, but determined solely by the complete linear system or by its cohomology class, and this is where the First Main Theorem 2.3.31 makes sense.

Lemma 2.1.31 that $T_f(r, \omega_L)$ is a monotone increasing convex function in $\log r$. Let ω'_L be the Chern form of another hermitian metric in L. By (2.3.21) we see that

$$T_f(r, \omega_L) - T_f\big(r, \omega'_L\big) = \frac{1}{2}\int_{\|z\|=r} b \circ f(z)\gamma(z) - \frac{1}{2}\int_{\|z\|=1} b \circ f(z)\gamma(z)$$
$$= O(1) \quad (r \to \infty).$$

Thus the order function of f with respect to L is defined by

$$T_f(r, L) = T_f(r, \omega_L)$$

modulo up to addition with a bounded term in $r \geqq 1$. In the same sense the proximity function $m_f(r, D)$ is determined modulo up to addition with a bounded term in $r \geqq 1$.

We obtain the following important formula from Corollary 2.1.37.

Theorem 2.3.31 (The First Main Theorem) *Let $L \to N$ be a line bundle and let $f : \mathbf{C}^m \to N$ be a meromorphic mapping. For $D \in |L|$ with* Supp $D \not\supset f(\mathbf{C}^m)$,

$$T_f(r, \omega_L) = N\big(r, f^*D\big) + m_f(r, D) - m_f(1, D),$$
$$T_f(r, L) = N\big(r, f^*D\big) + m_f(r, D) + O(1).$$

Corollary 2.3.32 *With the conditions in Theorem 2.3.31 we suppose one of the following*:

(i) $f^*D \neq 0$.
(ii) $f^*\omega_L \geqq 0$ *and* $f^*\omega_L(z_0) > 0$ *at some point* $z_0 \in \mathbf{C}^m$.

Then there is a constant $C > 0$ such that

$$C \log r \leqq T_f(r, \omega_L) + O(1).$$

Proof In the case of (i) there is a $t_0 > 0$ with $n(t_0, f^*D) > 0$. It follows from Theorem 2.2.19 that $n(t, f^*D)$ is a monotone increasing function in t. Therefore

$$N\big(r, f^*D\big) \geqq \int_{t_0}^{r} \frac{n(t_0, f^*D)}{t} dt = n\big(t_0, f^*D\big)(\log r - \log t_0).$$

It follows from Theorem 2.3.31 that

$$T_f(r, \omega_L) \geqq N\big(r, f^*D\big) + O(1).$$

Thus the claim is deduced.

To deal with the case of (ii) we may assume that (i) does not hold; that is, $f^{-1}D = \emptyset$. Then $dd^c \log 1/\|\sigma \circ f\|^2 = f^*\omega_L \geqq 0$, and so $\log 1/\|\sigma \circ f(z)\|^2$ is a plurisubharmonic function. By Lemma 2.1.31

$$\frac{1}{t^{2m-2}} \int_{B(t)} f^*\omega_L \wedge \alpha^{m-1}$$

is a monotone increasing function in t. By assumption we have for $t_0 = \|z_0\| + 1$

$$C_0 = \frac{1}{t_0^{2m-2}} \int_{B(t_0)} f^* \omega_L \wedge \alpha^{m-1} > 0.$$

By definition

$$T_f(r, \omega_L) \geqq \int_{t_0}^{r} \frac{dt}{t^{2m-1}} \int_{B(t)} f^* \omega_L \wedge \alpha^{m-1} \geqq C_0(\log r - \log t_0). \qquad \square$$

Remark 2.3.33 In the above First Main Theorem N is assumed to be projective algebraic, but in fact suffices to be a compact complex manifold. In that case we consider the integral over $\Gamma(f)$ (cf. Theorem 2.2.11). To show Jensen's formula on $\Gamma(f)$ (Lemma 2.1.33) one needs Stokes' Theorem on $\Gamma(f)$ with singularities in general, or a desingularization.

Example 2.3.34 Let $[w_0, \ldots, w_n]$ be a homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$ and let $H \to \mathbf{P}^n(\mathbf{C})$ be the hyperplane bundle. Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping with reduced representation $f = [f_0, \ldots, f_n]$. Then w_j, $0 \leqq j \leqq n$ form a base of $H^0(\mathbf{P}^n(\mathbf{C}), H)$. We take a holomorphic section $\sigma = \sum_{j=0}^{n} c_j w_j$, $(c_j) \in \mathbf{C}^{n+1} \setminus \{0\}$ and the hyperplane $D = (\sigma)$ defined by it. The coefficients (c_j) may be normalized so as

$$\sum_{j=0}^{n} |c_j|^2 = 1.$$

The length of σ with respect to the hermitian metric in H given by (2.3.24) is

$$\|\sigma\| = \frac{|\sum_j c_j w_j|}{\sqrt{(\sum_j |w_j|^2)}} \leqq 1.$$

Then the quantities appearing in the First Main Theorem 2.3.31 are as follows:

$$f^* D = \left(\sum_j c_j f_j\right) = \left(\sum_j c_j f_j\right)_0,$$

$$m_f(r, D) = \int_{\|z\|=1} \frac{\sqrt{\sum_j |f_j(z)|^2}}{|\sum_j c_j f_j(z)|} \gamma(z),$$

$$T_f(r, H) = \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} dd^c \log\left(\sum_j |f_j|^2\right) \wedge \alpha^{m-1}.$$

Here, Jensen's formula (Lemma 2.1.33) applied to the above expression of $T_f(r, H)$ yields

(2.3.35)
$$T_f(r,H)=\int_{\|z\|=r}\log\left(\sum_{j=0}^{n}\left|f_j(z)\right|^2\right)^{1/2}\gamma(z)$$
$$-\int_{\|z\|=1}\log\left(\sum_{j=0}^{n}\left|f_j(z)\right|^2\right)^{1/2}\gamma(z).$$

We take a vector subspace $E\subset H^0(N,L)$ of $\dim E=l+1\geqq 2$ and introduce a homogeneous coordinate system $[u_0,\dots,u_l]$ of $P(E)$. The volume element $\Omega=\omega_0^l$ defined by the Fubini–Study metric form $\omega_0=dd^c\log\sum|u_j|^2$ satisfies

$$\int_{|L|}\Omega=1.$$

The unitary transformation group of the homogeneous coordinate system $u_0,\dots,u_l$ naturally acts on $|L|$ by

$$\big(U,\big[(w_j)\big]\big)\in \mathrm{U}(l+1)\times|L|\to\big[U(w_j)\big]\in|L|.$$

This action is transitive and leaves ω_0 and Ω invariant. For an arbitrary $D\in P(E)$ we take $\sigma=\sum c_ju_j$ so that $(\sigma)=D$, $\sum|c_j|^2=1$. While the vector (c_j) is not uniquely determined, $\frac{(\sum|u_j|^2)^{1/2}}{|\sum c_ju_j|}$ depends only on D. In this sense the following holds:

(2.3.36)
$$\int_{D=(\sigma)\in P(E)}\log\frac{(\sum|u_j|^2)^{1/2}}{|\sum c_ju_j|}\Omega(D)=C(l)>0.$$

Here $C(l)$ is a constant dependent only on l, and a computation yields (H. Weyl–J. Weyl [38])

$$C(l)=\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{l}\right).$$

Theorem 2.3.37 *Let $L\to N$ be a line bundle and let $E\subset H^0(N,L)$ be a vector subspace. Suppose that $B(E)=\emptyset$. Then for an arbitrary meromorphic mapping $f:\mathbf{C}^m\to N$,*

$$T_f(r,L)=\int_{D\in P(E)}N\big(r,f^*D\big)\Omega(D)+O(1).$$

Proof We take bases $\sigma_0,\dots,\sigma_l$ of E. By assumption $l\geqq 1$ and a hermitian metric in L may be assumed to satisfy that for a section $\sigma\in E$

(2.3.38)
$$\left\|\sigma(x)\right\|^2=\frac{|\sigma(x)|^2}{\sum|\sigma_j(x)|^2}.$$

It follows from (2.3.36) and Fubini's theorem that

$$\begin{aligned}\int_{D\in P(E)} m_f(r,D)\Omega(D) &= \int_{\|z\|=r}\int_{D=(\sigma)\in P(E)} \log\frac{1}{\|\sigma(f(z))\|}\Omega(D)\gamma(z)\\ &= C(l).\end{aligned}$$

By the First Main Theorem 2.3.31

$$T_f(r,\omega_L) = \int_{D\in P(E)} N\big(r, f^*D\big)\Omega(D) = T_f(r,L) + O(1). \qquad \square$$

The properties of the order function $T_f(r,L)$ will be summarized in the next section, but here we assume that

$$T_f(r,L)\to\infty \quad (r\to\infty)$$

(cf. Corollary 2.3.32). In this case we define *Nevanlinna's defect*

$$\delta(f,D) = 1 - \varlimsup_{r\to\infty}\frac{N(r,f^*D)}{T_f(r,L)}$$

of $D\in|L|$. This satisfies

$$0\leqq\delta(f,D)\leqq 1.$$

With $k\in\mathbf{N}$ we define the *k-defect* $\delta_k(f,D)$ by

$$\delta_k(f,D) = 1 - \varlimsup_{r\to\infty}\frac{N_k(r,f^*D)}{T_f(r,L)}.$$

In particular, when $f(\mathbf{C}^m)\cap D=\emptyset$, $\delta(f,D)=1$. A divisor D with $\delta(f,D)>0$ is called *Nevanlinna's exceptional divisor*. Theorem 2.3.37 implies the following.

Theorem 2.3.39 (Casorati–Weierstrass) *Let the assumption be the same as in Theorem 2.3.37 and moreover $T_f(r,\omega_L)\to\infty$ $(r\to\infty)$. Then almost all $D\in P(E)$ with respect to the measure Ω satisfy $\delta(f,D)=0$, i.e., they are not Nevanlinna's exceptional divisors.*

2.4 The First Main Theorem for Coherent Ideal Sheaves

2.4.1 Proximity Functions for Coherent Ideal Sheaves

In Theorems 2.3.37 and 2.3.39 the assumption $B(E)=\emptyset$ is essential. To deal with the case of $B(E)\neq\emptyset$ it is necessary to define the proximity function $m_f(r,D)$

not only for divisors but also for cycles of higher codimension. W. Stoll [70] and Bott–Chern [65] dealt with such a case. The First Main Theorem for coherent ideal sheaves was dealt with by Noguchi [03b] and Noguchi–Winkelmann–Yamanoi [08].

Because of a technical reason we assume in what follows that N is a *projective algebraic variety*, possibly singular with reduced structure (cf. Hartshorne [77]). Let $\mathscr{I} \subset \mathscr{O}_N$ be a coherent ideal sheaf of the structure sheaf $\mathscr{O}_N$. Then by definition there are a finite open covering $N = \bigcup U_\lambda$ of N and holomorphic functions $\zeta_{\lambda 1}, \ldots, \zeta_{\lambda l_\lambda}$ on U_λ such that their germs $\underline{\zeta_{\lambda 1}}_x, \ldots, \underline{\zeta_{\lambda l_\lambda}}_x$ generate the stalk $\mathscr{I}_x$ of $\mathscr{I}$ at every point $x \in U_\lambda$.

Lemma 2.4.1 *Let $\mathscr{I}$ be a coherent ideal sheaf of $\mathscr{O}_N$. Then there is a very ample line bundle over N such that $\Gamma(N, \mathscr{I} \otimes \mathscr{O}(L))$ generates the stalk $\mathscr{I}_x \otimes \mathscr{O}_x(L)$ at all $x \in N$; that is, there are bases ϕ_j $(1 \leqq j \leqq l)$ of $\Gamma(N, \mathscr{I} \otimes \mathscr{O}(L))$ such that the germs $\underline{\phi_j}_x$ $(1 \leqq j \leqq l)$ generate $\mathscr{I}_x \otimes \mathscr{O}_x(L)$ as $\mathscr{O}_{N,x}$ module at all $x \in N$.*

Proof In the algebraic sense of coherence the proof is immediate, since U_λ of the aforementioned open covering $\{U_\lambda\}$ are affine varieties and ζ_λ are regular rational functions on U_λ.

If the coherence is taken in the analytic sense, we then use Theorem 2.3.26. For an every fixed point $x \in N$ we denote by the same $\mathscr{I}_x$ the coherent sheaf over N extending $\mathscr{I}_x$ to be a zero sheaf on $N \setminus \{x\}$. Then there is a natural morphism $\kappa_x : \mathscr{I} \to \mathscr{I}_x$ whose kernel is denoted by $\mathscr{K}_x$:

$$0 \to \mathscr{K}_x \to \mathscr{I} \to \mathscr{I}_x \to 0.$$

We fix a positive line bundle L_0 over N. Then we have

$$0 \to \mathscr{K}_x \otimes \mathscr{O}\big(L^\nu\big) \to \mathscr{I} \otimes \mathscr{O}\big(L^\nu\big) \to \mathscr{I}_x \otimes \mathscr{O}\big(L^\nu\big) \to 0.$$

There is a number $\nu_0 \in \mathbf{N}$ such that for $\nu \geqq \nu_0$ $H^1(N, \mathscr{K}_x \otimes \mathscr{O}(L^\nu)) = 0$. Therefore,

$$H^0\big(N, \mathscr{I} \otimes \mathscr{O}\big(L^\nu\big)\big) \to H^0\big(N, \mathscr{I}_x \otimes \mathscr{O}\big(L^\nu\big)\big) \to 0.$$

By the coherence of $\mathscr{I}$, $H^0(N, \mathscr{I} \otimes \mathscr{O}(L^\nu))$ generates $\mathscr{I}_y$ in a neighborhood $U(x)$ of x. Since N is compact we can cover N by a finite number of such $U(x)$'s. If ν is large enough, then L^ν is very ample. □

We extend the bases ϕ_j $(1 \leqq j \leqq l)$ of $\Gamma(N, \mathscr{I} \otimes \mathscr{O}(L))$ in Lemma 2.4.1 to bases ϕ_j $(1 \leqq j \leqq l')$ of $\Gamma(N, \mathscr{O}(L))$. Then in the sense of (2.3.22) we set

$$d_{\mathscr{I}}(x) = \sqrt{\frac{\sum_{j=1}^{l} |\phi_i(x)|^2}{\sum_{j=1}^{l'} |\phi_j(x)|^2}}, \quad x \in N. \tag{2.4.2}$$

A different choice of L yields another $d'_{\mathscr{I}}(x)$, for which there is a constant $C > 0$ satisfying

$$C^{-1} d_{\mathscr{I}}(x) \leqq d'_{\mathscr{I}}(x) \leqq C d_{\mathscr{I}}(x), \quad {}^{\forall} x \in N. \tag{2.4.3}$$

Let $Y = (\operatorname{Supp} \mathscr{O}_N/\mathscr{I}, \mathscr{O}_N/\mathscr{I})$ be the subspace (subscheme), possibly non-reduced of N defined by $\mathscr{I}$ and set $d_Y(x) = d_{\mathscr{I}}(x)$. We call

$$\phi_{\mathscr{I}}(x) = -\log d_{\mathscr{I}}(x), \quad x \in N, \qquad \phi_Y(x) = -\log d_Y(x)$$

the *proximity (approximation) potential* of the coherent ideal sheaf $\mathscr{I}$ (resp. the subspace Y).

Let $f : \mathbf{C}^m \to N$ be a meromorphic mapping such that $f(\mathbf{C}^m) \not\subset \operatorname{Supp} \mathscr{O}_N/\mathscr{I}$. Then as in (2.3.25) and Lemma 2.3.27 we see that $f^*\phi_{\mathscr{I}}$ is written as a difference of two plurisubharmonic functions ξ_1 and ξ_2:

$$f^*\phi_{\mathscr{I}}(z) = \phi_{\mathscr{I}} \circ f(z) = \xi_1(z) - \xi_2(z), \quad z \in \mathbf{C}^m. \tag{2.4.4}$$

The *proximity function* (or approximation function) for $\mathscr{I}$ (or, Y) is defined by

$$m_f(r, \mathscr{I}) = m_f(r, Y) = \int_{\|z\|=r} \phi_{\mathscr{I}} \circ f(z) \gamma(z). \tag{2.4.5}$$

It follows from (2.4.4) that the integral is finite, and then from (2.4.3) that $m_f(r, \mathscr{I})$ is well-defined up to addition of $O(1)$-term.

Moreover, because of (2.4.4) the current $dd^c[\phi_{\mathscr{I}} \circ f(z)]$ is of degree 0; i.e., its coefficients are Radon measures, and the differential form $dd^c\phi_{\mathscr{I}} \circ f(z)$ has coefficients that are locally integrable. We set

$$\begin{aligned} \omega_{\mathscr{I},f} = \omega_{Y,f} &= -2dd^c\phi_{\mathscr{I}} \circ f(z) = -\frac{i}{\pi}\partial\bar{\partial}\phi \circ f(z) \\ &= 2dd^c \log \frac{1}{d_{\mathscr{I}} \circ f(z)}, \quad z \in \mathbf{C}^m. \end{aligned} \tag{2.4.6}$$

The order function of f with respect to $\mathscr{I}$ or Y is defined by

$$T(r, \omega_{\mathscr{I},f}) = T(r, \omega_{Y,f}) = \int_1^r \frac{dt}{t} \int_{B(t)} \omega_{\mathscr{I},f}. \tag{2.4.7}$$

We have to be careful to consider $f^*\mathscr{I}$, or f^*Y. Let $\Gamma(f) \subset \mathbf{C}^m \times N$ be the graph of f, let $p : \Gamma(f) \to \mathbf{C}^m$ and $q : \Gamma(f) \to N$ be respectively the natural projections. Then we have an analytic cycle (a locally finite formal sum of analytic subsets with integral coefficients) $p_*(q^*Y)$, whose supports are of codimension one or more in general. We denote by f^*Y or $f^*\mathscr{I}$ the sum of only those components of $p_*(q^*Y)$ whose supports are of codimension one. Then we have the current equation by Theorem 2.2.11 and Lemma 2.2.12:

$$\begin{aligned} 2dd^c\big[f^*\phi_{\mathscr{I},f}\big] &= \omega_{\mathscr{I},f} - f^*\mathscr{I}, \\ 2dd^c\big[f^*\phi_{Y,f}\big] &= \omega_{Y,f} - f^*Y. \end{aligned} \tag{2.4.8}$$

The counting function for f^*Y or for $f^*\mathscr{I}$ is defined by

$$N\big(r, f^*\mathscr{I}\big) = N\big(r, f^*Y\big) = \int_1^r \frac{dt}{t^{2m-1}} \int_{f^*Y \cap B(t)} \alpha^{m-1}, \quad r \geqq 1.$$

We have also the truncated counting function $N_k(r, f^*\mathscr{I})$ or $N_k(r, f^*Y)$ as in (2.2.18). Then by Jensen's formula (Corollary 2.1.37) and (2.4.8) we have the following.

Theorem 2.4.9 (The First Main Theorem) *Let $f : \mathbf{C}^m \to M$ and let $\mathscr{I}$ be as above. Then we have*

$$T(r, \omega_{\mathscr{I}, f}) = N\big(r, f^*\mathscr{I}\big) + m_f(r, \mathscr{I}) - m_f(1, \mathscr{I}).$$

Let $\mathscr{I}_i$ $(i = 1, 2)$ be two coherent ideal sheaves of $\mathscr{O}_M$ and let Y_i be the subspace defined by $\mathscr{I}_i$, which is possibly non-reduced. We write $Y_1 \supset Y_2$ if $\mathscr{I}_1 \subset \mathscr{I}_2$.

Theorem 2.4.10 *The proximity function for coherent ideal sheaves has the following properties.*

(i) *If $\mathscr{I} \subset \mathscr{J}$, $m_f(r, \mathscr{J}) \leqq m_f(r, \mathscr{I}) + O(1)$; if $Y = \mathscr{O}_N/\mathscr{J} \subset Z = \mathscr{O}_N/\mathscr{I}$, $m_f(r, Y) \leqq m_f(r, Z) + O(1)$.*
(ii) *$m_f(r, \mathscr{I}_1 \otimes \mathscr{I}_2) = m_f(r, \mathscr{I}_1) + m_f(r, \mathscr{I}_2) + O(1)$. In particular, $m_f(r, \mathscr{I}^k) = k m_f(r, \mathscr{I}) + O(1)$, $k \in \mathbf{N}$.*
(iii) *$m_f(r, \mathscr{I}_1 + \mathscr{I}_2) \leqq \min\{m_f(r, \mathscr{I}_1), m_f(r, \mathscr{I}_2)\} + O(1)$.*

Remark 2.4.11 Let D be an effective Cartier divisor on N. Let $\mathscr{I}$ be the ideal sheaf determined by D, and let $L(D)$ be the line bundle determined by D. Then we have

$$T(r, \omega_{\mathscr{I}, f}) = T_f\big(r, L(D)\big) + O(1),$$
$$m_f(r, \mathscr{I}) = m_f(r, D) + O(1),$$
$$N\big(r, f^*\mathscr{I}\big) = N\big(r, f^*D\big).$$

These follow from

$$\big| -\log \|\sigma(x)\| - \phi_{\mathscr{I}}(x) \big| \leqq C, \quad x \in N,$$

where C is a positive constant.

Let $E \subset H^0(N, L)$ be a vector subspace and let $\mathscr{I}_0$ be the coherent ideal sheaf generated by $\{\underline{\sigma}_x; \sigma \in E\}$ at every $x \in N$. Then $\mathscr{I}_0$ decomposes into the common divisor part $\mathscr{I}_1$ and the remaining part $\mathscr{I}_2$, that is,

$$\mathscr{I}_0 = \mathscr{I}_1 \otimes \mathscr{I}_2, \qquad \operatorname{codim} \operatorname{Supp} \mathscr{O}_N/\mathscr{I}_1 = 1, \qquad \operatorname{codim} \operatorname{Supp} \mathscr{O}_N/\mathscr{I}_2 \geqq 2.$$

It may happen that there is no $\mathscr{I}_1$-factor, that is, $\mathscr{I}_0 = \mathscr{I}_2$. Let D_1 be the effective divisor given by $\mathscr{I}_1$, if exists. Note that $D - D_1$ is an effective divisor. The next theorem is due to R. Kobayashi in the case of $D_1 = \emptyset$.

Theorem 2.4.12 *Let the notation be as above. For a meromorphic mapping $f : \mathbf{C}^m \to N$ with $f(\mathbf{C}^m) \not\subset B(E)$ we have the following.*

$$\int_{D \in P(E)} m_f(r, D)\Omega(D) = m_f(r, D_1) + m_f(r, \mathscr{I}_2) + O(1),$$

$$T_f(r, L) = \int_{D \in P(E)} N\big(r, f^*D\big)\Omega(D) + m_f(r, D_1) + m_f(r, \mathscr{I}_2) + O(1).$$

Proof Take $\tau_1 \in H^0(N, L(D_1))$ so that $(\tau_1) = D_1$. Let $\sigma_0, \ldots, \sigma_l$ be the bases of E. Every σ_j is written as

$$\sigma_j = \tau_1 \otimes \tau_{2j}, \quad \tau_{2j} \in H^0\big(N, L(D - D_1)\big), \ 0 \leqq j \leqq l.$$

For an arbitrary $\sigma = \sum c_j \sigma_j = \tau_1 \otimes (\sum c_j \tau_{2j})$,

$$-\log \|\sigma(x)\| = -\log \|\tau_1(x)\| + \phi_{\mathscr{I}_2} + \log \frac{(\sum |\tau_{2j}(x)|^2)^{1/2}}{|\sum c_j \tau_{2j}(x)|} + b(x),$$

where $b(x)$ is a C^∞ function on N. Therefore

$$m_f\big(r, (\sigma)\big) = m_f(r, D_1) + m_f(r, \mathscr{I}_2) + \int_{\|z\|=r} \log \frac{(\sum |\tau_{2j}(f(z))|^2)^{1/2}}{|\sum c_j \tau_{2j}(f(z))|} \gamma(z) + O(1).$$

We integrate this with respect to $\Omega([c_j])$, $[c_j] \in \mathbf{P}^l(\mathbf{C})$; by (2.3.36)

$$\int_{D \in P(E)} m_f(r, D)\Omega(D) = m_f(r, D_1) + m_f(r, \mathscr{I}_2) + O(1).$$

The claimed second formula follows from this and the First Main Theorem 2.3.31. □

Example 2.4.13 In general, even if $D_1 = 0$, $m_f(r, \mathscr{I}_2)$ is not bounded. Consider the following holomorphic map:

$$f : z \in \mathbf{C} \to \big[1, e^z, e^{cz}\big] = [w_0, w_1, w_2] \in \mathbf{P}^2(\mathbf{C}), \quad c > 1.$$

Let $H \to \mathbf{P}^2(\mathbf{C})$ be the hyperplane bundle, and let $E \subset H^0(\mathbf{P}^2(\mathbf{C}), H)$ be the subspace generated by holomorphic sections w_1, w_2. Then $B(E) = \{[1, 0, 0]\}$ and $\mathscr{I}_2$ is the maximal ideal of $\mathscr{O}_{\mathbf{P}^2(\mathbf{C}), [1,0,0]}$. Its proximity potential is

$$\phi_{\mathscr{I}_2} = \frac{1}{2} \log \frac{|w_0|^2 + |w_1|^2 + |w_2|^2}{|w_1|^2 + |w_2|^2}.$$

Following the definition, we calculate $m_f(r, \mathscr{I}_2)$:

$$\begin{aligned}
m_f(r, \mathscr{I}_2) &= \frac{1}{4\pi}\int_{|z|=r} \log \frac{1+|e^z|^2+|e^{cz}|^2}{|e^z|^2+|e^{cz}|^2} d\theta \\
&= \frac{1}{4\pi}\int_{|z|=r} \log\left(1+\frac{1}{e^{2r\cos\theta}+e^{2cr\cos\theta}}\right) d\theta \\
&= \frac{1}{4\pi}\int_{\cos\theta<0} \log\left(1+\frac{1}{e^{2r\cos\theta}+e^{2cr\cos\theta}}\right) d\theta + O(1) \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \log\left(1+\frac{1}{e^{-2r\cos\theta}+e^{-2cr\cos\theta}}\right) d\theta + O(1) \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \log\left(1+\frac{e^{2r\cos\theta}}{1+e^{(1-c)2r\cos\theta}}\right) d\theta + O(1) \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} 2r\cos\theta d\theta + O(1) \\
&= \frac{r}{\pi} + O(1).
\end{aligned}$$

The order function $T_f(r, H)$ is calculated by (2.3.35) as

$$\begin{aligned}
T_f(r, H) &= \frac{1}{4\pi}\int_{|z|=r} \log\bigl(1+|e^z|^2+|e^{cz}|^2\bigr) d\theta + O(1) \\
&= \frac{1}{4\pi}\int_{|z|=r} \log\bigl(1+e^{2r\cos\theta}+e^{2cr\cos\theta}\bigr) d\theta + O(1) \\
&= \frac{1}{4\pi}\int_{\cos\theta>0} \log\bigl(1+e^{2r\cos\theta}+e^{2cr\cos\theta}\bigr) d\theta + O(1) \\
&= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} 2cr\cos\theta d\theta + O(1) \\
&= \frac{cr}{\pi} + O(1).
\end{aligned}$$

2.4.2 The Case of $m = 1$

In the preceding subsection the assumption for N to be projective algebraic was used to deduce (2.4.4), which was used to apply Jensen's formula, Corollary 2.1.37. For Jensen's formula it is sufficient to know that (2.4.4) holds locally in $\mathbf{C}^m$.

In the case of $m = 1$ we give here another simpler way to define the proximity potential function $\phi_{\mathscr{I}}$ without the projective algebraic assumption for N such that (2.4.4) holds in every disk $\Delta(r)$ (cf. Noguchi–Winkelmann–Yamanoi [08]).

Let N be a compact complex space in general, and let $\mathscr{I}$ be a coherent ideal sheaf of $\mathscr{O}_N$. Take an open covering $\{U_j\}$ of N such that

(i) there is a partition of unity $\{c_j\}$ subordinate to $\{U_j\}$,
(ii) there are finitely many sections $\sigma_{jk} \in \Gamma(U_j, \mathscr{I})$, $k = 1, 2, \ldots$, generating every fiber $\mathscr{I}_x$ over $x \in U_j$.

We set

$$\rho_{\mathscr{I}}(x) = C\left(\sum_j c_j(x) \sum_k \left|\sigma_{jk}(x)\right|^2\right)^{1/2}. \tag{2.4.14}$$

Here a positive constant C is chosen so that

$$\rho_{\mathscr{I}}(x) \leqq 1, \quad x \in N.$$

Using the compactness of N, one easily verifies that, up to addition by a bounded function on N, $\log \rho_{\mathscr{I}}$ is independent of the choices of the open covering, the partition of unity, the local generators of the ideal sheaf $\mathscr{I}$, and the constant C.

Let $f : \mathbf{C} \to N$ be a holomorphic mapping, which we will call an *entire curve*, such that $f(\mathbf{C}) \not\subset \operatorname{Supp} Y$. Then the function $\rho_{\mathscr{I}} \circ f(z)$ is smooth over $\mathbf{C} \setminus f^{-1}(\operatorname{Supp} Y)$. For $z_0 \in f^{-1}(\operatorname{Supp} Y)$ there is an open neighborhood U of z_0 and a positive integer ν such that $f^*\mathscr{I} = ((z - z_0)^{\nu_0})$, and then

$$\log \rho_{\mathscr{I}} \circ f(z) = \nu_0 \log |z - z_0| + \psi_0(z), \quad z \in U$$

for some C^∞ function $\psi_0(z)$ defined on U. Setting $\bar{\Delta}(r) \cap \operatorname{Supp} f^*Y = \{z_j\}_{j=1}^h$, we have a finite sum $\sum_{j=1}^h \nu_j \log |z - z_j|$ with $\nu_j \in \mathbf{N}$ such that

$$\psi(z) = \log \rho_{\mathscr{I}} \circ f(z) - \sum_{j=1}^h \nu_j \log |z - z_j|$$

is a C^∞ function in a neighborhood of $\bar{\Delta}(r)$. It is easy to see that a C^∞ function in a neighborhood of $\bar{\Delta}(r)$ is written as a difference of two subharmonic functions in a (possibly smaller) neighborhood of $\bar{\Delta}(r)$. Therefore, at least in a neighborhood of $\bar{\Delta}(r)$, (2.4.4) holds for $\log \rho_{\mathscr{I}} \circ f(z)$.

We then define the proximity function of f for $\mathscr{I}$ or for Y by

$$m_f(r, Y) = m_f(r, \mathscr{I}) = \int_{|z|=r} \log \frac{1}{\rho_{\mathscr{I}}(f(re^{i\theta}))} \frac{d\theta}{2\pi} \quad (\geqq 0). \tag{2.4.15}$$

We define the counting function $N(r, f^*\mathscr{I})$ and $N_l(r, f^*\mathscr{I})$ by using the divisor $\sum \nu_j \{z_j\}$. Moreover we define

$$\begin{aligned} \omega_{\mathscr{I}, f} = \omega_{Y, f} &= -2dd^c \psi(z) = -\frac{i}{\pi} \partial\bar{\partial}\psi(z) \\ &= dd^c \log \frac{1}{\rho_{\mathscr{I}} \circ f(z)} \quad (z \in U), \end{aligned} \tag{2.4.16}$$

which is well-defined on $\mathbf{C}$ as a C^∞ (1,1)-form. The order function of f for $\mathscr{I}$ or Y is defined by

$$T(r, \omega_{\mathscr{I},f}) = T(r, \omega_{Y,f}) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} \omega_{\mathscr{I},f}. \tag{2.4.17}$$

If N is projective algebraic, the difference of the proximity potential functions (cf. (2.4.2), (2.4.14))

$$\log \rho_{\mathscr{I}}(x) - \log d_{\mathscr{I}}(x) \quad (x \in N)$$

is a bounded function.

Therefore we see the following.

Proposition 2.4.18 (i) *The proximity function $m_f(r, \mathscr{I})$ $(=m_f(r, Y))$ in the present subsection differs from that defined in the previous subsection only by a bounded function.*

(ii) *There is no change in the counting function for f^*I $(=f^*Y)$.*

(iii) *Thus the difference of the order functions $T(r, \omega_{\mathscr{I},f})$ $(=T(r, \omega_{Y,f}))$ is bounded, too.*

2.5 Order Functions

Let N be an n-dimensional compact complex space and let $f : \mathbf{C}^m \to N$ be a meromorphic mapping. We will define several order functions of f and will give their comparison. Then we will give a characterization of rational f in terms of the order function, when N is projective algebraic.

2.5.1 Metrics

Let

$$h = \sum_{j,k} h_{j\bar{k}} dx_j \otimes d\bar{x}_k$$

be a hermitian metric on the holomorphic tangent space $\mathbf{T}(N)$ of N; that is, the coefficients of h are C^∞ functions, the quadratic form $h(X, \bar{Y})$ in $X, Y \in \mathbf{T}(N)_x$ $(x \in N)$ is hermitian and positive definite. Let $\omega = \sum_{j,k} \frac{i}{2} h_{j\bar{k}} dx_j \wedge d\bar{x}_k$ be the associated $(1,1)$-form. If $d\omega = 0$, we call ω a Kähler form, h is called a Kähler metric, and N is called a Kähler manifold if N is non-singular. As in (2.3.30), we define the *order function* of f with respect to ω by

$$T_f(r, \omega) = \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} f^*\omega \wedge \alpha^{m-1}, \quad r > 1. \tag{2.5.1}$$

Lemma 2.5.2 *Let ω and ω' be two hermitian metrics on N. Then there is a constant $C > 0$ such that for an arbitrary meromorphic mapping $f : \mathbf{C}^m \to N$*

$$C^{-1}T_f(r,\omega) \leqq T_f(r,\omega) \leqq CT_f\big(r,\omega'\big).$$

Proof Since ω and ω' are both positive definite and N is compact, there is a constant $C > 0$ such that

$$C\omega - \omega' \geqq 0, \qquad C\omega' - \omega \geqq 0.$$

Therefore

$$Cf^*\omega \wedge \alpha^{m-1} - f^*\omega' \wedge \alpha^{m-1} \geqq 0,$$
$$f^*\omega' \wedge \alpha^{m-1} - Cf^*\omega \wedge \alpha^{m-1} \geqq 0.$$

Thus the required inequalities follow. □

We define the symbol $S_f(r,\omega)$ with respect to $T_f(r,\omega)$ as (1.2.4). That is,

(2.5.3) $$S_f(r,\omega) = O\big(\log T_f(r,\omega)\big) + \delta \log r\|_{E(\delta)}.$$

The above definition is independent of the choice of ω by Lemma 2.5.2. When it is not necessary to specify ω, we simply write

$$S_f(r) = S_f(r,\omega).$$

Theorem 2.5.4 *Let ω be a hermitian metric form on N. A meromorphic mapping $f : \mathbf{C}^m \to N$ is constant if and only if $\underline{\lim}_{r\to\infty} T_f(r,\omega)/\log r = 0$. This condition is equivalent to $T_f(r,\omega) = S_f(r,\omega)$, provided that $d\omega = 0$, or $m = 1$.*

Proof The "only if" part is trivial. Assume that ω is a Kähler form on N. Suppose that $d\omega = 0$. Then $f^*\omega$ is a d-closed positive semi-defined $(1,1)$-form. By Poincaré's Lemma for d and for $\bar\partial$ on $\mathbf{C}^m$ there exists a plurisubharmonic function φ on $\mathbf{C}^m$ such that $dd^c\varphi = f^*\omega$ in the sense of currents. (For the present argument, the existence on every ball $B(R)$ $(R > 0)$ is sufficient.) It follows from Lemma 2.1.31 that the function

$$n\big(t, f^*\omega\big) = \frac{1}{t^{2m-2}} \int_{B(t)} f^*\omega \wedge \alpha^{m-1}$$

is monotone increasing in $t > 0$; this monotonicity is trivial without the assumption $d\omega = 0$ if $m = 1$. The constancy of f is equivalent to $f^*\omega \wedge \alpha^{m-1} \equiv 0$ and so to $n(t, f^*\omega) \equiv 0$. Assume that f is not constant. Then there is a $t_0 > 0$ with $n(t_0, f^*\omega) > 0$. For $r > t_0$

$$T_f(r,\omega) = \int_1^r \frac{n(t, f^*\omega)}{t} dt \geqq \int_{t_0}^r \frac{n(t, f^*\omega)}{t} dt$$

(continued)

$$\geqq n\big(t_0, f^*\omega\big)\int_{t_0}^{r}\frac{dt}{t} = n\big(t_0, f^*\omega\big)(\log r - \log t_0).$$

Hence $\underline{\lim}_{r\to\infty} T_f(r,\omega)/\log r \geqq n(t_0, f^*\omega) > 0$.

Assume that $T_f(r,\omega) = S_f(r,\omega)$. Then for every $\delta > 0$ we have

$$\overline{\lim_{r\to\infty}}\,\frac{\log r}{T_f(r,\omega)} \geqq \frac{1}{\delta}.$$

It follows that $\underline{\lim}_{r\to\infty} T_f(r,\omega)/\log r = 0$, and that f is constant. □

Let f be a meromorphic function on $\mathbf{C}^m$. Following after (1.1.10) and (1.1.12), we set

$$\begin{aligned} m(r,f) &= \int_{\|z\|=r}\log^+\big|f(z)\big|\gamma(z),\\ T(r,f) &= m(r,f) + N\big(r,(f)_\infty\big). \end{aligned} \tag{2.5.5}$$

We call $T(r,f)$ *Nevanlinna's order function*. This is convenient in calculating estimates. There are co-prime holomorphic functions f_0 and f_1 on $\mathbf{C}^m$ (i.e., $\operatorname{codim}\{f_0 = f_1 = 0\} \geqq 2$) such that $f = f_1/f_0$. Let $[w_0, w_1]$ be the natural homogeneous coordinate system of $\mathbf{P}^1(\mathbf{C})$, and identify f with a meromorphic mapping

$$f : z \in \mathbf{C}^m \to \big[f_0(z), f_1(z)\big] \in \mathbf{P}^1(\mathbf{C}).$$

By taking the Fubini–Study metric form ω on $\mathbf{P}^1(\mathbf{C})$, we compare $T_f(r,\omega)$ and $T(r,f)$. The following is the Shimizu–Ahlfors theorem and the First Main Theorem.

Theorem 2.5.6 *For a meromorphic function f on $\mathbf{C}^m$*

$$T(r,f) - T_f(r,\omega) = O(1).$$

In particular, for any $a \in \mathbf{C}$

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1).$$

Proof Noting that for every $s \geqq 0$

$$0 \leqq \log(1+s) - \log^+ s \leqq \log 2,$$

we get

$$\begin{aligned} T(r,f) &= \frac{1}{2}\int_{\|z\|=r}\log\left(1 + \left|\frac{f_1(z)}{f_0(z)}\right|^2\right)\gamma(z) + \frac{1}{2}\int_{\|z\|=r}\log|f_0|^2\gamma + O(1)\\ &= \frac{1}{2}\int_{\|z\|=r}\log\big(|f_0|^2 + |f_1|^2\big)\gamma + O(1). \end{aligned}$$

By making use of Lemma 2.1.33 (Jensen's formula), we have

$$\begin{aligned} T(r, f) &= \int_1^r \frac{dt}{t^{2m-1}} \int_{B(t)} f^*\omega \wedge \alpha^{m-1} + O(1) \\ &= T_f(r, \omega) + O(1). \end{aligned}$$

For the latter half, we notice that $T_{1/(f-a)}(r, \omega) = T_{(f-a)}(r, \omega)$. It follows from what was shown that

$$\begin{aligned} T\left(r, \frac{1}{f-a}\right) &= T(r, f-a) + O(1) \\ &= m(r, f-a) + N\big(r, (f-a)_\infty\big) + O(1) \\ &= m(r, f) + N\big(r, (f)_\infty\big) + O(1) \\ &= T(r, f) + O(1). \end{aligned}$$

□

Theorem 2.5.7 *Let $f : \mathbf{C}^m \to N$ be a meromorphic mapping and let $L \to N$ be a hermitian line bundle. Assume that $L \geqq 0$ or that N is projective algebraic. Let $\sigma_0, \sigma_1 \in H^0(N, L)$ be linearly independent sections such that $f(\mathbf{C}^m) \not\subset \{\sigma_0 = 0\}$. Then the meromorphic function $g(z) = \sigma_1 \circ f(z)/\sigma_0 \circ f(z)$ satisfies*

$$T(r, g) \leqq T_f(r, L) + O(1).$$

Proof Let $\|\cdot\|$ be a hermitian metric in L. Then

$$\begin{aligned} \log^+ \left|\frac{\sigma_1}{\sigma_0}\right| &= \log^+ \frac{\|\sigma_1\|}{\|\sigma_0\|} \\ &\leqq \log^+ \frac{1}{\|\sigma_0\|} + \log^+ \|\sigma_1\|. \end{aligned}$$

Since $\|\sigma_1\|$ is bounded on N, there is a constant C such that

$$\log^+ \left|\frac{\sigma_1}{\sigma_0}\right| \leqq \log^+ \frac{1}{\|\sigma_0\|} + C.$$

By definition $(g)_\infty \leqq f^*(\sigma_0)$. Thus

$$N\big(r, (g)_\infty\big) \leqq N\big(r, f^*(\sigma_0)\big).$$

It follows that

$$\begin{aligned} T(r, g) &\leqq m_f\big(r, (\sigma_0)\big) + N\big(r, f^*(\sigma_0)\big) + C \\ &= T_f(r, L) + O(1). \end{aligned}$$

□

Corollary 2.5.8 (Noguchi [76b]) *If σ_0 and σ_1 have no common zero,*

$$T(r, g) = T_f(r, L) + O(1).$$

Proof By the assumption there is a constant $C > 0$ such that

$$\left|\log^+\left|\frac{\sigma_1}{\sigma_0}\right| - \log^+\frac{1}{\|\sigma_0\|}\right| \leqq C.$$

Furthermore, $N(r, (g)_\infty) = N(r, f^*(\sigma_0))$, so that the claim holds. □

If $\dim N = 1$, then the assumption of Corollary 2.5.8 is always satisfied. Let $N = \mathbf{P}^1(\mathbf{C})$ and $L \to \mathbf{P}^1(\mathbf{C})$ be the hyperplane bundle. Then the above corollary is the Shimizu–Ahlfors Theorem 1.1.19. For the composition of a rational function with f we have

Corollary 2.5.9 *Let $Q \circ f$ be the composition of a meromorphic function $f(z)$ on $\mathbf{C}^m$ with a rational function Q in one variable. Assume that f and Q are non-constant. Denoting the degree of Q by d, we have*

$$T(r, Q \circ f) = dT(r, f) + O(1).$$

Proof There are sections $\sigma_0, \sigma_1 \in H^0(\mathbf{P}^1(\mathbf{C}), L^d)$ without common zero such that $Q = \sigma_1/\sigma_0$. Therefore

$$T(r, Q \circ f) = dT_f(r, L) + O(1) = dT(r, f) + O(1).$$ □

2.5.2 Cartan's Order Function

Here we set $N = \mathbf{P}^n(\mathbf{C})$. Let ω be the Fubini–Study metric form on $\mathbf{P}^n(\mathbf{C})$. Let $w = [w_0, \dots, w_n]$ be a homogeneous coordinate system of $\mathbf{P}^n(\mathbf{C})$. Take a meromorphic mapping $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$. Then there are holomorphic functions $f_0, \dots, f_n$ on $\mathbf{C}^m$ such that $\operatorname{codim}\{f_0 = \dots = f_n = 0\} \geqq 2$ and

$$f(z) = \big[f_0(z), \dots, f_n(z)\big].$$

Let $f(z) = [g_0(z), \dots, g_n(z)]$ be another such representation. Then there is a holomorphic function $h(z)$ on $\mathbf{C}^m$ without zero such that

$$f_j(z) = h(z)g_j(z), \quad 0 \leqq j \leqq n,\ z \in \mathbf{C}^m. \tag{2.5.10}$$

H. Cartan [33] defined the order function $T_f(r)$ of $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ by

$$T_f(r) = \int_{\|z\|=r} \log \max_{0\leqq j\leqq n} |f_j(z)|\gamma(z) - \log \max_{0\leqq j\leqq n} |f_j(0)|, \tag{2.5.11}$$

which is called Cartan's order function; in fact, he dealt with the case of $m = 1$. It follows from (2.5.10) that $T_f(r)$ is independent of the representation of f.

Theorem 2.5.12 *Let the notation be as above. Then we have*

$$T_f(r) = \int_{\|z\|=r} \log\Bigl(\sum_{0\leqq j\leqq n} \bigl|f_j(z)\bigr|^2\Bigr)^{1/2} \gamma(z) + O(1)$$
$$= T_f(r,\omega) + O(1).$$

This is immediate by (2.3.35).

A linear form $F_j = \sum c_{jk} w_k$ in the homogeneous coordinates $[w_0, \dots, w_n]$ is identified with a holomorphic section of the hyperplane bundle on $\mathbf{P}^n(\mathbf{C})$. For two linearly independent linear forms F_1 and F_2 we consider the compositions $F_j(f(z)) = \sum c_{jk} f_k(z)$ with $f(z) = [f_0(z), \dots, f_n(z)]$. If $F_1(f(z)) \not\equiv 0$, $g(z) = F_2(f(z))/F_1(f(z))$ is defined independently of the representation of f. The next theorem is due to Toda [70a], Lemma 1 in the case of $m = 1$ and the case of $m \geqq 2$ is similarly proved.

Theorem 2.5.13 *Let $f(z) = [f_0(z), \dots, f_n(z)]$ and $g(z)$ be as above. The following hold*:

(i) $T(r, g) \leqq T_f(r, \omega) + O(1)$.
(ii) *If* $f_k \not\equiv 0$,

$$\frac{1}{n}\sum_{j=0}^{n} T\left(r, \frac{f_j}{f_k}\right) + O(1) \leqq T_f(r,\omega) \leqq \sum_{j=0}^{n} T\left(r, \frac{f_j}{f_k}\right) + O(1).$$

Proof (i) is a special case of Theorem 2.5.7. The first inequality in (ii) is clear by (i). The latter is deduced as follows. We may assume $f_0 \not\equiv 0$ without loss of generality. Then

$$T_f(r,\omega) = \int_{\|z\|=r} \log\left(\max_{1\leqq j\leqq n}\left\{1, \frac{|f_j(z)|}{|f_0(z)|}\right\} \cdot \bigl|f_0(z)\bigr|\right)\gamma(z) + O(1)$$
$$= \int_{\|z\|=r} \log^+ \max_{1\leqq j\leqq n}\left\{\frac{|f_j(z)|}{|f_0(z)|}\right\}\gamma(z)$$
$$+ \int_{\|z\|=r} \log\bigl|f_0(z)\bigr|\gamma(z) + O(1)$$
$$\leqq \sum_{j=1}^{n} \int_{\|z\|=r} \log^+\left|\frac{f_j}{f_0}\right|\gamma + N\bigl(r, (f_0)_0\bigr) + O(1)$$
$$\leqq \sum_{j=1}^{n}\left\{\int_{\|z\|=r} \log^+\left|\frac{f_j}{f_0}\right|\gamma + N\left(r, \left(\frac{f_j}{f_0}\right)_\infty\right)\right\} + O(1)$$
$$= \sum_{j=1}^{n} T\left(r, \frac{f_j}{f_0}\right) + O(1).$$

□

2.5.3 A Family of Rational Functions

First note that w_j/w_k, $0 \leqq j(\neq k) \leqq n$, form a transcendental base of the field extension of the rational function field of $\mathbf{P}^n(\mathbf{C})$ over $\mathbf{C}$. It follows from Theorem 2.5.13 (ii) that up to a positive constant multiple, $T_f(r,\omega)$ and $\sum_{\substack{j=0 \\ j\neq k}}^{n} T(r, f_j/f_k)$ are equivalent. We generalize this to a general projective algebraic manifold.

Lemma 2.5.14 *For an arbitrary point $a \in \mathbf{C}$*

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - a\right| d\theta = \log^+ |a|.$$

The proof is left to the reader.

Lemma 2.5.15 *Let $g(z)$ and $A_1(z), \ldots, A_l(z)$ be meromorphic functions on $\mathbf{C}^m$ satisfying*

$$\big(g(z)\big)^l + A_1(z)\big(g(z)\big)^{l-1} + \cdots + A_l(z) = 0.$$

Then

$$T(r,g) \leqq \sum_{j=1}^{l} T(r, A_j) + \log(l+1).$$

Proof Introducing a variable t, we set

$$B(z,t) = t^l + A_1(z)t^{l-1} + \cdots + A_l(z).$$

For $z \in \mathbf{C}^m \setminus \bigcup_j \mathrm{Supp}(A_j)_\infty$ we let $t_1(z) = g(z), t_2(z), \ldots, t_l(z)$ be roots of $B(z,t) = 0$. Then

$$B(z,t) = \prod_{j=1}^{l}\big(t - t_j(z)\big).$$

Therefore we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|B\big(z, e^{i\theta}\big)\right| d\theta &= \sum_{j=1}^{l} \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{i\theta} - t_j(z)\right| d\theta \\ &\qquad \text{(by Lemma 2.5.14)} \\ &= \log^+\left|g(z)\right| + \sum_{j=2}^{l} \log^+\left|t_j(z)\right| \\ &\geqq \log^+\left|g(z)\right|.\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} \log\left|B\left(z, e^{i\theta}\right)\right| d\theta \\
&\quad = \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{il\theta} + A_1(z)e^{i(l-1)\theta} + \cdots + A_l(z)\right| d\theta \\
&\quad \leqq \sum_{j=1}^{l} \log^+ \left|A_j(z)\right| + \log(l+1).
\end{aligned}$$

It follows that

$$m(r, g) \leqq \sum_{j=1}^{l} m(r, A_j) + \log(l+1). \tag{2.5.16}$$

By making use of co-prime holomorphic functions g_0, g_1 on $\mathbf{C}^m$, we represent $g = g_1/g_0$. Let (A_0) be the minimum common divisor of the polar divisors of $A_1, \ldots, A_l$, where A_0 is a holomorphic function on $\mathbf{C}^m$. Then

$$A_0(g_1)^l = -g_0(z)\left\{A_0A_1(g_1)^{l-1} + \cdots + A_0A_l(g_0)^{l-1}\right\}.$$

Therefore $(g_0) \leqq (A_0)$ as divisors, and

$$N\left(r, (g)_\infty\right) \leqq N\left(r, (A_0)_0\right) \leqq \sum_{j=1}^{l} N\left(r, (A_j)_\infty\right). \tag{2.5.17}$$

It follows from (2.5.16) and (2.5.17) that

$$\begin{aligned}
T(r, g) &\leqq \sum_{j=1}^{l} \left\{m(r, A_j) + N\left(r, (A_j)_\infty\right)\right\} + \log(l+1) \\
&= \sum_{j=1}^{l} T(r, A_j) + \log(l+1).
\end{aligned}$$

□

In general, let N be an n-dimensional complex projective algebraic variety and let $\{\phi_j\}_{j=1}^s$ be a finite subset of the rational function field $\mathbf{C}(N)$ of N. Assume that a meromorphic mapping $f : \mathbf{C}^m \to N$ satisfies $f(\mathbf{C}^m) \not\subset \bigcup_{j=1}^s \mathrm{Supp}(\phi_j)_\infty$. We define the *order function* of f with respect to $\{\phi_j\}$ by

$$T_f\left(r, \{\phi_j\}\right) = \sum_{j=1}^{l} T(r, \phi_j \circ f).$$

In what follows, it is the essential case when $\{\phi_j\}_{j=1}^s$ contains a transcendental base of $\mathbf{C}(N)$ over $\mathbf{C}$.

Let $N \hookrightarrow \mathbf{P}^l(\mathbf{C})$ be an embedding and let L be the restriction of the hyperplane bundle over $\mathbf{P}^l(\mathbf{C})$ to N.

Theorem 2.5.18 (i) *Assume that $\{\phi_j\}_{j=1}^s$ contains a transcendental base of $\mathbf{C}(N)$ over $\mathbf{C}$ and that f is algebraically non-degenerate. Then there is a constant $C > 0$ independent of f such that*

$$C^{-1}T_f(r, L) + O(1) \leqq T_f\big(r, \{\phi_j\}\big) \leqq CT_f(r, L) + O(1).$$

(ii) *Let $f : \mathbf{C}^m \to N$ be an algebraically non-degenerate meromorphic mapping and let $\Phi : N \to M$ be a birational mapping onto another complex projective algebraic variety M. Let $H \to M$ be a positive line bundle. Then there is a constant $C_1 = C_1(L, H) > 0$ such that*

$$C_1^{-1}T_f(r, L) + O(1) \leqq T_{\Phi\circ f}(r, H) \leqq C_1^{-1}T_f(r, L) + O(1).$$

Proof (i) Let $[w_0, \dots, w_l]$ be a homogeneous coordinate system of $\mathbf{P}^l(\mathbf{C})$ and let $f(z) = [f_0, \dots, f_l]$ be a reduced representation of f as a meromorphic mapping into $\mathbf{P}^l(\mathbf{C})$. We may assume $f_0 \not\equiv 0$ without loss of generality. The restrictions ψ_k of w_k/w_0 to N give a transcendental base of $\mathbf{C}(N)$ and then ϕ_j are represented as rational functions in ψ_k,

$$\phi_j = Q_j(\psi_1, \dots, \psi_l).$$

Substituting $f(z)$, we have

$$\phi_j \circ f(z) = Q_j\big(\psi_1 \circ f(z), \dots, \psi_l \circ f(z)\big).$$

Therefore there is a constant $C > 0$ determined by Q_j, $1 \leqq j \leqq n$, such that

$$\begin{aligned} T\big(r, Q_j(\psi_k \circ f)\big) &\leqq C\sum_{k=1}^{l} T(r, \psi_k \circ f) + O(1) \\ &\leqq lCT_f(r, L) + O(1). \end{aligned}$$

We deduce that

$$T\big(r, \{\phi_j\}\big) \leqq lnCT_f(r, L) + O(1).$$

Conversely, the rational functions ψ_k are algebraic over the field generated by $\{\phi_j\}$ over $\mathbf{C}$. Thus there are algebraic relations

$$(\psi_k)^{d_k} + A_{k1}(\phi_j)(\psi_k)^{d_k-1} + \cdots + A_{kd_k}(\phi_j) = 0, \quad 1 \leqq k \leqq l.$$

Substituting f, we obtain algebraic relations of meromorphic functions on $\mathbf{C}^m$:

$$(\psi_k \circ f)^{d_k} + A_{k1}(\phi_j \circ f)(\psi_k \circ f)^{d_k-1} + \cdots + A_{kd_k}(\phi_j \circ f) = 0, \quad 1 \leqq k \leqq l.$$

Lemma 2.5.15 implies that

$$\begin{aligned} T(r, \psi_k \circ f) &\leqq \sum_{h=1}^{d_k} T\bigl(r, A_{kh}(\phi_j \circ f)\bigr) + \log(d_k + 1) \\ &\leqq C' \sum_{j=1}^{n} T(r, \phi_j \circ f) + O(1). \end{aligned}$$

Here $C' > 0$ is a constant depending only on $\{\psi_k\}$ and $\{\phi_j\}$. It follows from Theorem 2.5.13 that

$$T_f(r, L) \leqq \sum_{k=1}^{l} T(r, \psi_k \circ f) \leqq l C' \sum_{j=1}^{n} T(r, \phi_j \circ f) + O(1).$$

(ii) Since $\Phi^* : \mathbf{C}(M) \to \mathbf{C}(N)$ is a field isomorphism over $\mathbf{C}$, the claim is clear by (i). $\square$

In general let N be a compact complex space and let ω be a hermitian metric form on it. For a meromorphic mapping $f : \mathbf{C}^m \to N$ we define the *order* ρ_f by

$$\rho_f = \varlimsup_{r\to\infty} \frac{\log T_f(r, \omega)}{\log r}. \tag{2.5.19}$$

By Lemma 2.5.2 ρ_f is independent of the choice of ω. If N is projective algebraic and f is algebraically non-degenerate, $T_f(r, \omega)$ in (2.5.19) may be replaced with $T_f(r, \{\phi_j\})$ by Theorem 2.5.18 to define the same ρ_f.

The next three propositions are easily deduced from Theorem 2.5.18.

Proposition 2.5.20 *Let $\eta : V \to W$ be a rational mapping between quasi-projective algebraic varieties V and W. Then for an algebraically non-degenerate meromorphic mapping $f : \mathbf{C}^m \to V$*

$$T_{\eta\circ f}(r) = O\bigl(T_f(r)\bigr).$$

Moreover, if η is generically finite, then

$$T_f(r) = O\bigl(T_{\eta\circ f}(r)\bigr).$$

Let V be a quasi-projective algebraic variety, and let $f : \mathbf{C}^m \to V$ be a meromorphic mapping. Taking a projective compactification $\bar{V} \supset V$, and regarding $f : \mathbf{C}^m \to \bar{V}$, we may define the *order* ρ_f of f, which is independent of the choice of the compactification $\bar{V}$.

Corollary 2.5.21 *The above order ρ_f of f is independent of the choice of the compactification $\bar{V}$ of V.*

This is immediate from Proposition 2.5.20.

Proposition 2.5.22 *Let $f : \mathbf{C}^m \to N$ be a meromorphic mapping into a complex projective variety N and let H be a line bundle on N. Assume that H is big, and that f is algebraically non-degenerate. Then*

$$T_f(r, L) = O\big(T_f(r, H)\big)$$

for every line bundle L on M.

If $f : \mathbf{C}^m \to N$ is algebraically degenerate, we may consider the Zariski closure X of $f(\mathbf{C})$ and a desingularization $\tau : \tilde{X} \to X$. Then f lifts to a map to $\tilde{X}$ and $\tau^*(H|_X)$ is big on $\tilde{X}$ for every ample line bundle H on N. As a consequence we obtain

Proposition 2.5.23 *Let $f : \mathbf{C}^m \to N$ be a meromorphic mapping into a complex projective variety N. Let $h(r)$ be a non-negative valued function in $r > 1$. Then $h(r) = S_f(r, H)$ holds for every ample line bundle if and only if it holds for at least one ample line bundle.*

Similarly, the statement "$h(r) \leqq \varepsilon T_f(r, H)\|_\varepsilon$, $^\forall \varepsilon > 0$", respectively "$h(r) = O(T_f(r, H))$" holds for every ample line bundle H if and only if it holds for at least one ample line bundle.

If f is algebraically non-degenerate, the same statements as above hold for big line bundles.

2.5.4 Characterization of Rationality

Let $g \not\equiv 0$ be a holomorphic function on $\mathbf{C}^m$. Then $\log|g(z)|$ is a plurisubharmonic function and hence a subharmonic function on $\mathbf{C}^m \cong \mathbf{R}^{2m}$ (Theorem 2.1.26 (i)). Taking the Poisson integral over the sphere $\{\|z\| = R\}$ of $\mathbf{C}^m$ we have that for $\|z\| < R$

$$\begin{aligned}
\log\big|g(z)\big| &\leqq \int_{\|\zeta\|=R} \log\big|g(\zeta)\big| \frac{(R^2 - \|z\|^2)R^{2m-2}}{\|\zeta - z\|^{2m}} \gamma(\zeta) \\
&\leqq \int_{\|\zeta\|=R} \log^+\big|g(\zeta)\big| \frac{(R^2 - \|z\|^2)R^{2m-2}}{\|\zeta - z\|^{2m}} \gamma(\zeta) \\
&\leqq \frac{(R^2 - \|z\|^2)R^{2m-2}}{(R - \|z\|)^{2m}} \int_{\|\zeta\|=R} \log^+\big|g(\zeta)\big| \gamma(\zeta).
\end{aligned}$$

Therefore we obtain

Lemma 2.5.24[5] *Let g be a holomorphic function on $\mathbf{C}^m$. Then for $0 < r < R$,*

$$T(r, g) \leqq \log^+ \max_{\|z\|=r} |g(z)| \leqq \frac{1-(r/R)^2}{(1-r/R)^{2m}} T(R, g).$$

Lemma 2.5.25 *A holomorphic function g on $\mathbf{C}^m$ is polynomial if and only if $T(r, g) = O(\log r)$.*

Proof If $g(z)$ is a polynomial function, an easy computation yields that $T(r, g) = O(\log r)$. Conversely we assume that $T(r, g) \leqq d \log r + C$. Putting $R = \tau r$, $\tau > 1$, we get by Lemma 2.5.24

$$\log^+ \max_{\|z\|=r} |g(z)| \leqq \frac{(\tau+1)\tau^{2m-2}}{(\tau-1)^{2m-1}} (d \log r + d \log \tau + C).$$

Set $d(\tau) = \frac{(\tau+1)\tau^{2m-2}}{(\tau-1)^{2m-1}}$ and $C(\tau) = d(\tau)(d \log \tau + C)$. We expand $g(z)$ to a Taylor series with multi-indices α,

$$g(z) = \sum_{|\alpha|=0}^{\infty} a_\alpha z^\alpha.$$

From this we obtain

$$\left(\frac{1}{2\pi}\right)^m \int_0^{2\pi} \cdots \int_0^{2\pi} \left|g\left(e^{i\theta_1} z_1, \ldots, e^{i\theta_m} z_m\right)\right|^2 d\theta_1 \cdots d\theta_m$$
$$= \sum_{|\alpha|=0}^{\infty} |a_\alpha|^2 |z_1|^{2\alpha_1} \cdots |z_m|^{2\alpha_m} \leqq \max_{\|z\|=r} |g(z)|^2$$
$$\leqq r^{2dd(\tau)} \cdot e^{2d(\tau)C(\tau)}.$$

Hence $\sum_{|\alpha|>dd(\tau)}^{\infty} |a_\alpha|^2 |z_1|^{2\alpha_1} \cdots |z_m|^{2\alpha_m} = 0$ and so

$$g(z) = \sum_{|\alpha|=0}^{dd(\tau)} a_\alpha z^\alpha.$$

Since $dd(\tau) \to d$ as $\tau \to \infty$, $g(z)$ is a polynomial of degree at most d. □

Theorem 2.5.26 (Stoll [64a], [64b]) *Let E be an effective divisor on $\mathbf{C}^m$. Then E is a divisor determined by a polynomial of degree at most d if and only if*

$$N(r, E) \leqq d \log r + O(1).$$

[5] An estimate of this type in several complex variables is found in Kneser [38] without an explicit formula; cf. Noguchi [75].

This is shown by the Weierstrass–Stoll canonical product which generalizes Weierstrass' canonical product. Cf. Noguchi–Ochiai [90] (Ochiai–Noguchi [84]) for a proof simplified by Lelong. Here we omit the proof.

Theorem 2.5.27 *A meromorphic function $g(z)$ on $\mathbf{C}^m$ is a rational function if and only if*

$$T(r,g) = O(\log r).$$

Proof If g is a rational function, we write $g(z) = \frac{P(z)}{Q(z)}$ with co-prime polynomials $P(z)$, $Q(z)$. By the Shimizu–Ahlfors Theorem 1.1.19 and Theorem 2.5.12 we have

$$\begin{aligned} T(r,g) &= \int_{\|z\|=r} \log \sqrt{|P(z)|^2 + |Q(z)|^2}\gamma + O(1) \\ &= O(\log r). \end{aligned}$$

For the converse, we first note that

$$N\big(r, (g)_\infty\big) \leqq T(r,g) = O(\log r).$$

By Stoll's Theorem 2.5.26 there is a polynomial g_0 such that $(g_0) = (g)_\infty$. If we set $g_1 = gg_0$, g_1 is a holomorphic function and satisfies

$$T(r,g_1) \leqq T(r,g) + T(r,g_0) = O(\log r).$$

By Lemma 2.5.25 g_1 is a polynomial. Thus g is a rational function. □

Theorem 2.5.28 *Let N be a projective algebraic variety and let ω be a hermitian metric form on it. A meromorphic mapping $f : \mathbf{C}^m \to N$ is a rational mapping if and only if*

$$T_f(r,\omega) = O(\log r).$$

Proof Taking an embedding $N \hookrightarrow \mathbf{P}^l(\mathbf{C})$, we may assume $N = \mathbf{P}^l(\mathbf{C})$ with ω the Fubini–Study metric form. The "only if" part is immediate from Theorems 2.5.27 and 2.5.13.

Assume that $T_f(r,\omega) = O(\log r)$. Let $w = [w_0, \ldots, w_l]$ be a homogeneous coordinate system of $\mathbf{P}^l(\mathbf{C})$ such that $f(\mathbf{C}^m) \not\subset \{w_0 = 0\}$. It follows from Theorem 2.5.13 that

$$T\left(r, f^* \frac{w_j}{w_0}\right) = O(\log r).$$

Therefore $f^*(w_j/w_0)$ are rational, and hence f is rational. □

Remark By Lemma 1.1.22 we see that $T_f(r,\omega) = O(\log r)$ if and only if $\underline{\lim}_{r\to\infty} \frac{T_f(r,\omega)}{\log r} < \infty$.

2.6 Nevanlinna's Inequality

We generalize Theorem 1.1.18 to meromorphic functions on $\mathbf{C}^m$ and moreover to the case of meromorphic mappings. This plays an essential role in the proof of the lemma on logarithmic derivatives in the next chapter.

Let $f_1, \ldots, f_n$ be entire functions on $\mathbf{C}^m$ which are linearly independent over $\mathbf{C}$. For a vector $w = (w_j) \in \mathbf{C}^n$ we set

$$I(w) = \int_{\|z\|=1} \log \left| \sum_{j=1}^{n} w_j f_j(z) \right| \gamma(z).$$

Lemma 2.6.1 *The function $I(w)$ is bounded on* $\{\|w\| = 1\}$.

Proof[6] Set $M = \sup\{|\sum_{j=1}^{n} w_j f_j(z)|; \|w\| = 1, \|z\| = 1\}$ $(<\infty)$. Setting $\Gamma = \{w \in \mathbf{C}^n; \|w\| = 1\}$, we see that $I(w) \leqq \log M$ on Γ. Now we show the boundedness of $I(w)$ from below. The function $\log|\sum_{j=1}^{n} w_j f_j(\zeta)|$ in $\zeta \in \mathbf{C}^m \cong \mathbf{R}^{2m}$ is subharmonic by Theorem 2.1.26 (i). By taking the Poisson integral we have

$$\log \left| \sum_{j=1}^{n} w_j f_j(\zeta) \right| \leqq \int_{\|z\|=1} \left(\log \left| \sum_{j=1}^{n} w_j f_j(z) \right| \right) \frac{1 - \|\zeta\|^2}{\|z - \zeta\|^{2m}} \gamma(z), \quad \|\zeta\| < 1.$$

For an arbitrary $a = (a_j) \in \Gamma$ we take $\zeta_0 \in B(1) (\subset \mathbf{C}^m)$ so that

$$\sum_{j=1}^{n} a_j f_j(\zeta_0) \neq 0.$$

There is a neighborhood W of a in Γ such that for every $w \in W$

$$\left| \sum_{j=1}^{n} w_j f_j(\zeta_0) \right| \geqq \frac{1}{2} \left| \sum_{j=1}^{n} a_j f_j(\zeta_0) \right| > 0.$$

Thus for $w \in W$

$$\begin{aligned} \log \frac{1}{2} \frac{|\sum_{j=1}^{n} a_j f_j(\zeta_0)|}{M} &\leqq \int_{\|z\|=1} \left(\log \frac{|\sum_{j=1}^{n} w_j f_j(z)|}{M} \right) \frac{1 - \|\zeta_0\|^2}{\|z - \zeta_0\|^{2m}} \gamma(z) \\ &\leqq \frac{1 - \|\zeta_0\|^2}{(1 + \|\zeta_0\|)^{2m}} \int_{\|z\|=1} \log \frac{|\sum_{j=1}^{n} w_j f_j(z)|}{M} \gamma(z) \\ &\leqq \frac{1 - \|\zeta_0\|^2}{2^{2m}} \big(I(w) - \log M \big). \end{aligned}$$

[6] A discussion on the proof of this lemma with Professors Phong and Demailly at Hayama Symposium on Complex Analysis in Several Variables 2002 was very helpful.

Hence $I(w)$ $(w \in W)$ is bounded from below. Since Γ is compact, $I(w)$ is bounded from below on Γ. □

Remark 2.6.2 In fact, it is shown that $I(w)$ is continuous in $w \in \mathbf{C}^n \setminus \{0\}$. Since the convergence theorem of Lebesgue integrals cannot be used, some more contrivance is necessitated to the proof.

Theorem 2.6.3 (Nevanlinna's inequality) *Let f be a non-constant meromorphic function on $\mathbf{C}^m$. Then there is a constant C such that for every $a \in \mathbf{C} \cup \{\infty\}$*

$$N\big(r, (f-a)_0\big) < T(r, f) + C, \quad r \geqq 1.$$

Proof We write $f = f_2/f_1$ with co-prime entire functions f_1, f_2. We regard f to be a meromorphic mapping $f : z \in \mathbf{C}^m \to [f_1(z), f_2(z)] \in \mathbf{P}^1(\mathbf{C})$. Let $a = [a_2, -a_1] \in \mathbf{P}^1(\mathbf{C}) \cong \mathbf{C} \cup \{\infty\}$. We may assume that $|a_1|^2 + |a_2|^2 = 1$. By Example 2.3.34 we have

$$(f-a)_0 = (a_1 f_1 + a_2 f_2)_0 = f^* a,$$
$$m_f(r, a) = \int_{\|z\|=r} \log \frac{\sqrt{|f_1(z)|^2 + |f_2(z)|^2}}{|a_1 f_1(z) + a_2 f_2(z)|} \gamma(z) \geqq 0.$$

Let ω be the Fubini–Study metric form on $\mathbf{P}^1(\mathbf{C})$. The First Main Theorem 2.3.31 implies that

$$N\big(r, f^* a\big) \leqq T_f(r, \omega) + m_f(1, a),$$
$$\begin{aligned} m_f(1, a) &= \int_{\|z\|=1} \log \frac{\sqrt{|f_1(z)|^2 + |f_2(z)|^2}}{|a_1 f_1(z) + a_2 f_2(z)|} \gamma(z) \\ &= \int_{\|z\|=1} \log \sqrt{\big|f_1(z)\big|^2 + \big|f_2(z)\big|^2} \gamma(z) \\ &\quad - \int_{\|z\|=1} \log \big|a_1 f_1(z) + a_2 f_2(z)\big| \gamma(z). \end{aligned}$$

By Lemma 2.6.1 there is a constant C such that

$$m_f(1, a) < C, \quad {}^\forall a \in \mathbf{P}^1(\mathbf{C}).$$

By Theorem 2.5.6 $T_f(r, \omega) = T(r, f) + O(1)$. Thus the required formula is obtained. □

The above Nevanlinna inequality will suffice for the application in the next chapter, but we extend it to a meromorphic mapping $f : \mathbf{C}^m \to N$ into a projective algebraic variety N. Let $L \to N$ be a hermitian line bundle and take an arbitrary linear subspace $E \subset H^0(N, L)$.

Theorem 2.6.4 *Let the notation be as above. Assume that $f(\mathbf{C}^m) \not\subset \operatorname{Supp}(\sigma)$ for every $\sigma \in E \setminus \{0\}$. Then there is a constant C such that for all $\sigma \in E \setminus \{0\}$*

$$N\bigl(r, f^*(\sigma)\bigr) < T_f(r, L) + C, \quad r \geqq 1.$$

Proof Let $\|\cdot\|$ be the hermitian metric in L. The pull-back f^*L is a line bundle on $\mathbf{C}^m$. On $\mathbf{C}^m$ every line bundle is globally trivial. We fix an isomorphism, $f^*L \cong \mathbf{C}^m \times \mathbf{C}$. Take bases $\sigma_1, \ldots, \sigma_n$ of E. Because of the isomorphism $f^*L \cong \mathbf{C}^m \times \mathbf{C}$, there are entire functions $f_j(z) = (f^*\sigma_j)(z)$, $1 \leqq j \leqq n$, on $\mathbf{C}^m$ and a C^∞ positive-valued function $h(z)$ such that

$$f^*\omega_L = dd^c \log h(z),$$
$$\sum_{j=1}^n \bigl\|\sigma_j\bigl(f(z)\bigr)\bigr\|^2 = \frac{\sum_j |f_j(z)|^2}{h(z)} \leqq 1, \quad 1 \leqq j \leqq n.$$

Write $\sigma = \sum w_j \sigma_j$ with $\|(w_j)\| = 1$. Then by the First Main Theorem 2.3.31 we have

$$N\bigl(r, f^*(\sigma)\bigr) = T_f(r, L) + m_f\bigl(1, (\sigma)\bigr) - m_f\bigl(r, (\sigma)\bigr),$$
$$m_f\bigl(1, (\sigma)\bigr) = \int_{\|z\|=1} \log \frac{\sqrt{h(z)}}{|\sum w_j f_j(z)|} \gamma(z).$$

Notice that $m_f(r, (\sigma)) \geqq 0$. By the choice, the functions f_j, $1 \leqq j \leqq n$, are linearly independent over $\mathbf{C}$. By Lemma 2.6.1 $m_f(1, (\sigma))$ is bounded in $\sigma = \sum w_j \sigma_j$, $\|(w_j)\| = 1$. Therefore there is a constant C such that

$$N\bigl(r, f^*(\sigma)\bigr) < T_f(r, L) + C, \quad r \geqq 1. \qquad \square$$

2.7 Ramified Covers over $\mathbf{C}^m$

Let X be an irreducible normal complex space. We call $X \xrightarrow{\pi} \mathbf{C}^m$ a finite ramified cover over $\mathbf{C}^m$ if π is a proper finite surjective holomorphic mapping. For example, if X is a normal affine algebraic variety, then there exists such a $\pi : X \to \mathbf{C}^m$ due to the "Noether Normalization Lemma".

In this section we summarize known facts on meromorphic mappings $f : X \to N$ from such X into a compact complex space N.

The case where $m = 1$ and $N = \mathbf{P}^1(\mathbf{C})$ is classical and was studied by Rémoundos [27], A. Valiron [29], [31], H.L. Selberg [30], [34], and Ullrich [32], etc. It is an essential case when $X \xrightarrow{\pi} \mathbf{C}^m$ is not algebraic but transcendental.

Let p be the covering number of $\pi : X \to \mathbf{C}^m$. We denote by $S(X)$ the set of all singular points of X. Since X is assumed to be normal, $\operatorname{codim} S(X) \geqq 2$. Let $R(X) = X \setminus S(X)$ be the set of regular (non-singular) points of X. The zero divisor of $\det d\pi|_{R(X)}$ naturally extends to a divisor on X by Theorem 2.2.5. It is called the *ramification divisor* of $\pi : X \to \mathbf{C}^m$ and is denoted by Ξ. Set

$$X(r) = \{x \in X;\ \|\pi(x)\| < r\}, \qquad \partial X(r) = \{x \in X;\ \|\pi(x)\| = r\}.$$

Let ω be a hermitian metric form on N. We define the order function $T_f(r, \omega)$ of f with respect to ω by

$$T_f(r, \omega) = \frac{1}{p} \int_1^r \frac{dt}{t^{2m-1}} \int_{X(t)} f^*\omega \wedge \pi^*\alpha^{m-1}. \tag{2.7.1}$$

For a line bundle L over N we define $T_f(r, L)$ as done previously.

Let E be a Weil divisor on X and let $E = \sum_\lambda k_\lambda E_\lambda$ be the irreducible decomposition. As in (2.2.18) the counting functions are similarly defined:

$$\begin{aligned} n_k(t, E) &= \frac{1}{p} \int_{X(t)\cap(\sum_\lambda \min\{k,k_\lambda\}E_\lambda)} \alpha^{m-1}, \\ N_k(r, E) &= \int_1^r \frac{n_k(t, E)}{t^{2m-1}} dt, \\ n(t, E) &= n_\infty(t, E), \qquad N(r, E) = N_\infty(r, E). \end{aligned} \tag{2.7.2}$$

As in (2.3.30) we define a proximity function for an effective Cartier divisor D on N by

$$m_f(r, D) = \frac{1}{p} \int_{\partial X(r)} \log \frac{1}{\|\sigma \circ f\|} \pi^*\gamma. \tag{2.7.3}$$

For the proofs of the following results, cf. Noguchi [76a], [76b].

Theorem 2.7.4 (The First Main Theorem) *Assume that* $f(X) \not\subset \operatorname{Supp} D$. *Then*

$$T_f\big(r, L(D)\big) = N\big(r, f^*D\big) + m_f(r, D) + O(1).$$

We say that a meromorphic mapping $f : X \to N$ *separates the fiber* of π if there is a point $z \in \mathbf{C}^m$ satisfying that $\pi^{-1}(z) \cap (\Xi \cup I(f)) = \emptyset$ and f takes distinct values on $\pi^{-1}(z)$.

Lemma 2.7.5 (Noguchi [76a]) *For every meromorphic mapping $f : X \to N$ there exist a finite ramified cover $\pi' : X' \to \mathbf{C}^m$, a proper finite holomorphic mapping $\eta : X \to X'$ and a meromorphic mapping $f' : X' \to N$ satisfying the following*:

(i) $\pi = \pi' \circ \eta$, $f = f' \circ \eta$.
(ii) f' *separates the fiber of* π'.
(iii) $T_{f'}(r, \omega) = T_f(r, \omega)$, $N(r, f'^*D) = N(r, f^*D)$, $m_{f'}(r, D) = m_f(r, D)$.

Lemma 2.7.6 (Characterization of algebraicity I; Noguchi [76a]) *The complex space X is affine algebraic and $\pi : X \to \mathbf{C}^m$ is rational if and only if*

$$N(r, \Xi) = O(\log r).$$

In this case we say that $X \overset{\pi}{\to} \mathbf{C}^m$ is algebraic.

Theorem 2.7.7 (Characterization of algebraicity II; Noguchi [76a]) *Let L be the hyperplane bundle on $\mathbf{P}^n(\mathbf{C})$. If a meromorphic mapping $f : X \to \mathbf{P}^n(\mathbf{C})$ separates the fiber of π, then the following holds*:

(i) $N(r, \Xi) \leqq (2p-2)T_f(r, L) + O(1)$.
(ii) *It is necessary and sufficient for X to be algebraic and for $f : X \to \mathbf{P}^n(\mathbf{C})$ to be rational that*

$$T_f(r, L) = O(\log r).$$

H.L. Selberg [30] proved the above (i) in the case of $m = n = 1$.

http://www.springer.com/978-4-431-54570-5

Nevanlinna Theory in Several Complex Variables and Diophantine Approximation
Noguchi, J.; Winkelmann, J.
2014, XIV, 416 p. 6 illus., Hardcover
ISBN: 978-4-431-54570-5