

Chapter 2

Ordinal Utility and Preference Reversal Phenomenon

Chapter 1 described that decision-making phenomena are broadly divisible into those under certainty, those under risk, and others under uncertainty. This chapter first introduces the concept of utility used to explain such decision-making phenomena and briefly presents the idea of traditional ordinal utility theory.

Ordinal utility theory is assumed in many theories in economics. However, it includes some phenomena that constitute counterexamples that cannot be justified from the perspective of behavioral decision theory. Among such cases, this chapter presents specific examination of a case in which the transitivity premised on ordinal utility is not satisfied, and presents a phenomenon known as preference reversal. Transitivity refers to a relation between two alternatives, which is a consistent attribute of preference relation that, for instance, if oranges are preferred to bananas and apples are preferred to oranges, then apples are preferred to bananas. In addition, preference reversal is regarded as a phenomenon that deviates from the procedural invariance that preference cannot be reversed by the preference revelation procedures. In this type of case, for instance, when making a purchase decision, although Brand A is said to be more desirable than Brand B when their values (e.g., the prices considered reasonable) are assessed independently, Brand B is chosen if the two are actually compared.

1 What Is Utility?

Utility is interpreted in daily usage as the subjective value or desirability of a result of selecting an alternative. In decision-making theory, it is often technically considered a real-valued function to express a preference relation; it is therefore sometimes called a utility function. Utility is considered in terms of real numbers because the mathematical analysis of a decision-making phenomenon provides the benefit of facilitating the prediction and explanation of the phenomenon. Expressions using utility are also used in decision-aid based on technologies using computers and other tools that are designed to support decisions with which

decision-makers can be satisfied. Such technologies have yielded practical advances (Kobashi 1988).

The following presents some simple examples of utility. We consider a case of decision-making under certainty, in which a product—either Brand A or Brand B—is to be selected. In this case, utility refers to the relative preference for Brand A to Brand B ($\text{Brand A} \succeq \text{Brand B}$). Only in such a case is the real number that the utility of Brand A ($u(\text{Brand A})$) higher than the utility of Brand B ($u(\text{Brand B})$). In other words, when a relation

$$u(\text{Brand A}) \geq u(\text{Brand B}) \Leftrightarrow \text{Brand A} \succeq \text{Brand B}.$$

holds true, then the preference relation is expressed with a utility function u . Particularly, the type of utility that maintains only the order of preference is called ordinal utility. Ordinal utility does not lose its fundamental meaning even if monotonic increasing conversion of its utility function is performed; it corresponds to the ordinal scale used in psychology and statistics. For instance, if u represents ordinal utility, then the preference relation is maintained even if, for example, $u(\text{Brand A}) = 5$ and $u(\text{Brand B}) = 2$ are changed to $\phi(u(\text{Brand A})) = 8$ and $\phi(u(\text{Brand B})) = 3$ using the function ϕ that increases the values monotonically.

The following expresses ordinal utility slightly more formally. Assuming that Set A of alternatives is finite nonempty and that the preference structure $\langle A, \succeq \rangle$ is a weak order, then the preference structure refers to the set that combines the set of alternatives and a preference relation \succeq of some kind. The weak order in this case represents the relation in which the following two conditions hold:

1. Comparability $\forall x, y \in A, x \succeq y \vee y \succeq x$.

In other words, this is such a relation in which $x, y (\forall x, y \in A), x \succeq y$, or $y \succeq x$ of Set A of alternatives exists. In this case, the symbol \vee is a logical symbol for “or,” which means that at least one of them holds true. Comparability is also called connectedness or completeness. For instance, if the set of brands considered is A and $x \succeq y$ is interpreted as a relation by which y (Brand y) is preferred to x (Brand x) or is interpreted as indifference, then this is a case that can be determined as one in which Brand x is preferred to Brand y or indifference or one in which Brand y is preferred to Brand x or indifference. A situation for which it is unknown “which one is preferred, or whether the chooser is indifferent” does not satisfy comparability.

2. Transitivity $\forall x, y, z \in A, x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$.

In other words, this is a relation in which, if $x \succeq y$ and $y \succeq z, x \succeq z$ holds for the arbitrary elements $x, y, z (\forall x, y, z \in A)$ of A. In this case, the symbol \wedge represents a logical symbol for “and,” which means that both relations hold true. For instance, if A is a set of alternatives of product brands just as in the example presented above and $x \succeq y$ is interpreted, then the transitivity is satisfied if there is a relation by which Brand x is preferred to Brand z or indifference when Brand x is preferred to Brand y or indifference or Brand y is preferred to Brand z or indifference.

If transitivity does not hold, it is a three-cornered deadlock relation. For instance, if the power relation of rock–paper–scissors is \succ , then rock \succ scissors and scissors \succ paper, but not rock \succ paper. Consequently, \succ does not satisfy transitivity.

We know that the following theorem holds for the weak order that satisfies these two characteristics (Krantz et al. 1971).

The theorem related to a weak order on a finite set (Krantz et al. 1971)

If a preference structure of a finite nonempty set A , $\langle A, \succsim \rangle$, is a weak order, then there exists a real-valued function (ordinal utility function) $u : A \rightarrow \mathbb{R}$ on A such that for all $x, y \in A$,

$$x \succsim y \Leftrightarrow u(x) \geq u(y).$$

In other words, this theorem means that if the preference such as a weak order is being made, then it can be expressed with a function that takes real numbers that maintain the preference relation. Therefore, it indicates that the preference relation of a qualitative weak order can be examined by quantifying it using ordinal utility. Although this theorem is based on a finite set in this case, we know that it also applies to a countably infinite set and further to an uncountably infinite set with certain conditions added (Krantz et al. 1971).

In addition, the following theorem holds for ordinal utility (Krantz et al. 1971).

The theorem related to weak order uniqueness on a finite set (Krantz et al. 1971)

If the preference structure $\langle A, \succsim \rangle$ on a finite nonempty set A is a weak order, then $\langle A, \succsim \rangle$ is expressed as $\langle \mathbb{R}, \geq \rangle$ through the real-valued function $u : A \rightarrow \mathbb{R}$ of A indicated in the theorem above; the structure $\langle \langle A, \succsim \rangle, \langle \mathbb{R}, \geq \rangle, u \rangle$ becomes an ordinal scale.

Although this theorem assumes a finite set, we know that it applies also to countably and uncountably infinite sets (Krantz et al. 1971).

Aside from ordinal utility, cardinal utility, which is often used in economics, is a type of utility that does not lose its fundamental meaning even with an interval scale, i.e., positive linear transformation (a linear transformation of multiplication by a constant and adding with a constant), as used in psychology and statistics. Using cardinal utility,

$$\begin{aligned} \forall x, y \in A, x \succsim y &\Leftrightarrow u(x) \geq u(y) \\ &\Leftrightarrow \phi(u(x)) \geq \phi(u(y)) \end{aligned}$$

where $\phi(u(x)) = \alpha u(x) + \beta (\alpha > 0)$ holds true.

2 Does a Weak Order Empirically Hold?

Does the weak order assumed by ordinal utility theory and by cardinal utility theory hold in actual decision-making?

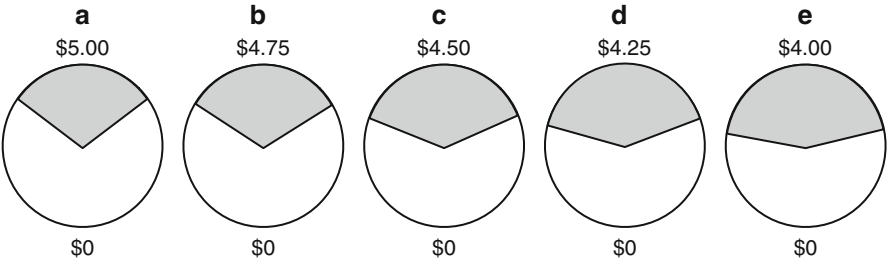


Fig. 2.1 Examples of gambling cards used in the experiment. *Source:* Tversky (1969). Reproduced in part by author

Tversky (1969) used experiments to examine whether the transitivity assumed by weak orders was satisfied in decision-making. He presented two cards with a pie chart test to subjects as presented in Fig. 2.1. He then asked the subjects which *gamble* they would prefer. The respondents were not allowed to express an indifferent preference relation. They were requested to indicate which was preferred. Therefore, this choice indicates a strong preference relation $x \succ y$, i.e., $x \succsim y \wedge \text{not } (y \succsim x)$ (where \succsim is a weak order). The cards had the amount of prize money written above the pie charts. The percentage of the area of the black sector in the circle was presented as the winning percentage.

Although multiple patterns were prepared in the experiment, the typical pattern was that two of five cards as in Table 2.1 were combined and the respondents were asked which one they would prefer. The winning percentage rose and the amount of prize money decreased while moving from *a* to *e*. In the case of comparative judgments such as those between *a* and *b* and between *b* and *c*, a slight difference in the winning percentages was ignored and the graph showing a larger prize tended to be selected. In contrast, for a combination with a significant difference in the winning percentages such as *a* and *e*, then *e*, the choice with a higher winning percentage, tended to be preferred. This tendency illustrates a relation of $a \succ b$, $b \succ c$, $c \succ d$, $d \succ e$, $e \succ a$, which demonstrably does not satisfy transitivity.

Table 2.1 Experimental tasks examining transitivity-1

Gamble	Winning probability	Outcome (in \$)	Expected value (in \$)
a	7/24	5.00	1.46
b	8/24	4.75	1.58
c	9/24	4.50	1.69
d	10/24	4.25	1.77
e	11/24	4.00	1.83

Source: Tversky (1969). Reproduced in part by author

Tversky (1969) also presented the percentile rank points from the assessment of the intelligence, emotional stability, and sociability of five college applicants, as in Table 2.2, to test subjects. He had them answer which applicant should be admitted

to a college based on a paired comparison with the priority on their intelligence. For a comparative judgment such as those between a and b and between b and c , a slight difference in the intelligence assessment was ignored and other factors with a better assessment tended to be selected. In contrast, for a combination with a significant difference in the intelligence assessment such as a and e , then e , with a better intelligence assessment, tended to be preferred. This result also indicates a relation of $a \succ b, b \succ c, c \succ d, d \succ e, e \succ a$, which clearly does not satisfy transitivity.

Table 2.2 Experimental tasks examining transitivity-2

Applicant	Intelligence	Emotional stability	Sociability
a	69	84	75
b	72	78	65
c	75	72	55
d	78	66	45
e	81	60	35

Source: Tversky (1969). Reproduced in part by author

Tversky (1969) proposed a mathematical model called the additive difference model to explain such preferences that do not satisfy the transitivity. This model first posits that a set of alternatives consists of multiple attributes $A = A_1 \times A_2 \times \dots \times A_m$ as shown in Table 2.2. Additionally, each alternative is regarded as comprising the values of multiple attributes such as $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$. The additive difference model is expressed as follows using u_i as a real-valued function and ϕ_i as an increasing function.

$$x \succeq y \Leftrightarrow \sum_{i=1}^n \phi_i[u_i(x_i) - u_i(y_i)] \geq 0$$

where $\phi_i(-\delta_i) = -\phi_i(\delta_i)$, $\delta_i = u_i(x_i) - u_i(y_i)$ for an arbitrary attribute, i .

Assuming that $\phi_i(\delta_i) = t_i(\delta_i)$, $t_i > 0$, then

$$\begin{aligned} & \sum_{i=1}^n \phi_i[u_i(x_i) - u_i(y_i)] \\ &= \sum_{i=1}^n t_i u_i(x_i) - \sum_{i=1}^n t_i u_i(y_i) \end{aligned}$$

can be drawn. Furthermore, assuming that $v_i(x_i) = t_i u_i(x_i)$, Then the result is

$$x \succeq y \Leftrightarrow \sum_{i=1}^n v_i(x_i) \geq \sum_{i=1}^n v_i(y_i)$$

which produces an additive utility model. Although non-transitivity cannot be explained when ϕ_i can be assumed with such linearity, if ϕ_i is a step function with a threshold (e.g., if $\varepsilon \geq \delta$, then $\phi_i(\delta_i) = 0$ where ε is a threshold of ϕ_i), then this additive difference model can explain non-transitivity.

Nakamura (1992) conducted an experimental examination of the conditions that deviated from the transitivity. Results suggest the following: (1) In the case of a preference judgment based on a single attribute, the judgment was clarified, even if the difference in the utility was small. (2) When the utility of two or more attributes was traded off and the difference was slight, the judgment would be ambiguous. (3) If the utility of a certain attribute could be considered equivalent while the utility of other attributes might not be deemed equivalent, then the effect of the attribute that is probably equivalent would be neglected. In an effort to explain the non-transitivity of people's preferences, he proposed a preference model termed the additive fuzzy utility difference structure model, which assumes utility as a set with an ambiguous boundary: a fuzzy set.

The fact that the transitivity in a weak order would not always hold has been described. Based on experience, comparability is unlikely to hold at all times, as well. For example, inadequate knowledge of product brands would make it difficult to present a preference relation that would satisfy comparability in all cases. Furthermore, the study by Tversky (1969) forced test subjects to choose between two alternatives. In reality, however, selecting one from two might be difficult in some cases. Takemura (2007, 2012) expanded the model of Nakamura (1992) and the analytical techniques of Takemura (2000, 2005) and proposed a model that included the weight utility function and attempted to express, approximately, the preference relation that did not satisfy the transitivity.

3 Preference Reversal Phenomenon

Concepts similar to decision-making include the concept of judgment. Although decision-making is the act of selecting one from a group of alternatives, judgment is definable as the act of specifying the subject to a particular position in an assessment continuum. Judgment includes, for instance, assessing the risk of a traffic accident with a probability between zero and one and rating the desirability of a result on a scale of seven or nine levels of "not desirable at all" to "very desirable." Another example of judgment is pricing the value of alternatives in product brands. According to common reasoning, judgment and decision-making differ only in the patterns of reactions. They are expected to reflect preference and assessment in the same direction. As a result of judgment, for example, if the assessed value of an alternative x is found to be higher than that of an alternative y , then a relation by which x is likely to be chosen over y can be expected as a result of decision-making. A situation in which alternative y is likely to be chosen over alternative x in decision-making despite the higher assessed value of alternative x than that of alternative y in judgment is improbable based on common reasoning.

In general, utility theory implicitly assumes that the preference ranking relations of the assessed subjects are maintained even if many different methods are used. This assumption is apparently self-evident considering the measurement of the physical quantity. In other words, when weighing each of two fish that have been

caught, for example, whether comparing their weights on a balance or measuring their weights in grams, there is expected to be little difference in the order relation as to which fish is heavier, even if the scales lack accuracy to some degree.

However, psychological studies conducted in the past suggest that preference order based on judgment and preference order based on decision-making are not necessarily the same and might be reversed in some cases. This phenomenon in which the preference order is reversed because of the difference between the reactions to judgment and decision-making is called the preference reversal phenomenon. This phenomenon was reported first by psychologists such as Lindman (1971) and Lichtenstein and Slovic (1971) as a phenomenon of preference relation inconsistency attributable to the methods of selection and pricing in gambles. The selection problem of these studies necessitated that test subjects choose between Gamble *H*, with a high winning percentage and a small amount of prize money (e.g., the winning percentage is 8/9 and a prize is \$4) and Gamble *L*, with a low winning percentage and a large amount of prize money (e.g., the winning percentage 1/9 and a prize is \$40). The pricing question asked the lowest probable prices at which Gamble *H* and Gamble *L* could be sold if the respondents owned them. In most cases, Gamble *H* was preferred in the selection problem and Gamble *L* was priced higher than the other in the pricing question (Tversky and Thaler 1990). Although many economists were skeptical about this preference reversal phenomenon initially, experimental economists repeatedly discovered effects that led to the recognition that this phenomenon certainly existed. The existence of this phenomenon implies that the type of reaction associated with judgment and decision-making affects the preference order rather than a phenomenon by which each result of judgment and decision-making is simply expressing a certain preference pattern (Tversky et al. 1988).

Tversky et al. (1988) examined other types of preference reversals in decision-making under certainty. In a study, the experimenter gave the following instructions to test subjects under each set of conditions.

Conditions for the selection problem: “In Israel, 600 people die in traffic accidents every year. The Ministry of Transport studied various measures to reduce the victims of traffic accidents. Please consider the following two proposed measures. The annual cost and the number of victims as a result of adopting each of the proposed measures are shown (Table 2.3). Which proposal would you adopt?”

Conditions for the matching problem: In the matching problem, a table resembling Table 2.4 with missing sections was presented to the test subjects, who were asked to deduce the missing information so that Proposed Measure X and Proposed Measure Y would become equivalent.

Table 2.3 Choice task

	Traffic fatalities	Cost
Program x	500 people	\$55 million
Program y	570 people	\$12 million

Source: Tversky et al. (1988). Reproduced in part by author

Table 2.4 Matching task

	Traffic fatalities	Cost
Program x	500 people	?
Program y	570 people	\$12 million

Source: Tversky et al. (1988). Reproduced in part by author

Accepting the assumption that a decrease in the casualties in traffic accidents and a low cost of measures are desirable allows the prediction of the results of the selection problem based on the results of the matching problem. We assume, for example, that a test subject has estimated \$40 million in the matching problem. Therefore, the profile of Proposed Measure X (500 victims and \$40 million) and the profile of Proposed Measure Y (570 victims and \$12 million) are equivalent. Based on the assumption, the profile of Proposed Measure X (500 victims and \$40 million) and the profile of Proposed Measure Y (570 victims and \$12 million) are superior to the profile of Proposed Measure X (500 victims and \$55 million) in the selection problem. Based on such reasoning, Proposed Measure Y is predicted to be selected from the results of the matching problem.

Nonetheless, Tversky et al. (1988) found that most test subjects would select Proposed Measure X in the selection problem and that most would prefer Proposed Measure Y in the matching problem. Such a preference reversal phenomenon has been identified in personal and social decision-making processes and also in risk judgment and other situations (Grether and Plott 1979; Lindman 1971; Lichtenstein and Slovic 1971; Slovic 1995; Slovic et al. 1990; Starmer 2000; Tversky et al. 1988; Takemura 1994, 1996).

The following Chap. 3 will describe the causes of this preference reversal phenomenon and models to explain its occurrence.

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