

Chapter 1

Introduction

Recent dramatic development of theory and experiment for microscopic systems has made it possible to investigate Nature more deeply and to utilize Her often surprising properties for our own purposes. Tests of quantum entanglement is a good early example. In 1935 Einstein, Podolsky, Rosen, and Schrödinger pointed out that quantum mechanics allows the existence of a counterintuitive correlation, now called entanglement, in quantum systems [1, 2]. In 1964 Bell proposed an experiment for testing its existence in practice [3], which motivated the proposal of many similar types of experiments [4]. Such methods are now called tests of Bell-type correlation. In 1981 the first experimental demonstration of such a test was done by Aspect, Grangier, and Roger in a photonic system, clearly establishing the existence of entanglement [5]. Today many experiments for such tests have been done in several types of physical systems, including photons [6], neutrons [7], trapped ions [8, 9], superconducting qubits [10], NV-centers in a diamond [11], and B mesons [12]. This has driven the recent growth of the field of quantum information, with quantum cryptography and quantum computation as good later examples. In 1984 and 1991, Bennett and Brassard, and Ekert independently proposed a cryptographic protocol using quantum systems [13, 14]. Surprisingly, the protocol is secure in principle for the ideal case, and the security of nonideal cases was proved later [15, 16]. In 1994 Shor proposed a computational algorithm for factorizing in prime integers using quantum systems [17]. It is again surprising that the protocol runs in polynomial time, something widely believed to be impossible for classical algorithms. With this cryptographic protocol and computational algorithm which outperform their classical counterparts as motivation, several new protocols and algorithms have been proposed [18, 19], with experimentalists in diverse fields working to implement many of them in their labs [20, 21].

Protocols in quantum cryptography and algorithms in quantum computation are examples of quantum information protocols. When experimentalists try to implement a quantum information protocol in their lab, they need to confirm the successful preparation of specific quantum states and operations. The standard method used for this verification is called quantum tomography [22]. For simplicity, let us consider

quantum state tomography, where, we have many identical copies of an unknown quantum state, and our purpose is to identify the state by performing predetermined measurements on the copies. We then try to estimate the true state from the data obtained. In quantum mechanics, measurement outcomes are obtained probabilistically. Data obtained in quantum state tomography includes two types of errors. One is caused by statistical fluctuation, and the other is due to systematic noise. The former is called statistical error, and the latter is called systematic error. Because of these errors, an estimated state does not coincide with the true state. The difference between the estimated and true state is called an estimation error. In quantum state tomography, it is very important to accurately evaluate the size of the estimation error in order to verify a successful preparation of a quantum state. This importance of accurately evaluating estimation error is also true in the test of Bell-type correlations. In an experimental Bell-type test, we perform measurements on a bipartite system and calculate a correlation function from the outcomes obtained. In this case, the estimation object is the value of the correlation function. If the value is larger than a certain threshold, it indicates that there exists a counterintuitive correlation between the two systems. The estimated value, of course, includes estimation errors. In order to verify the existence of the correlation, we need to evaluate the size of the estimation error and show that it is sufficiently small. In general, to estimate from data fluctuating probabilistically and affected by noise, and to evaluate the size of estimation error are topics in statistical estimation theory. Statistical estimation for quantum systems is called quantum estimation. Usually, the effect of the systematic error is approximated by introducing a model, and is assumed to be known. Therefore, the analysis of the estimation error is usually reduced to that of the statistical error.

In statistical estimation, how to choose the measurements used for the estimation is called an experimental design, and how to calculate the estimation results from the obtained data is called an estimator [23]. It is a key aim of both classical and quantum estimation theory to find a combination of experimental design and estimator which gives us more precise estimation results using fewer measurement trials. Before trying to find better combinations, we need to decide how to evaluate the size of the estimation error for a given combination. In statistical estimation theory, there are two main ways to evaluate the size of estimation errors. One is to use expected loss. We evaluate the difference between the true object and the estimate by what is called a loss function, which we average by taking its statistical expectation. The result is called the expected loss. Mean squared error and variance are the examples of expected loss, and an expected loss is a generalization of error bars. The other way is to use error probability. Like with expected loss, we evaluate the difference between the true object and the estimate by a loss function, but now calculate the probability that we obtain data that gives a difference larger than some threshold. An error probability tells us the probability of obtaining estimation results that deviate from the truth by more than some amount. An expected loss tends to be used for evaluating the rough size of an estimation error, while an error probability is used to more accurately evaluate its size. In quantum estimation theory, there are results for expected losses and error probabilities for infinitely many data. These constitute what

is called the large sample theory or the asymptotic theory of quantum estimation [24]. In real experiments, however, the amount of available data is finite. Compared to the asymptotic analysis, it is more difficult to analyze the size of estimation errors for finite data. Theoretical methods for accurately evaluating estimation errors in tests of Bell-type correlations and in quantum tomography with finite data have not been established. In this thesis we develop the theory of statistical parameter estimation toward a finite sample theory applicable to such real world estimation problems.

In classical statistics, improving the estimation precision is as important a topic as accurately evaluating estimation errors. This is also true in quantum estimation. Suppose that the mathematical model of our estimation object and the way we evaluate estimation error are determined. When we try to improve the estimation precision, we can choose to change the experimental design or the estimator. Let us consider the case in which we try to improve the experimental design. In general the optimal design of experiment depends on the true value of the estimation object. However, we do not know the true value and that is our motivation for estimation. This lack of knowledge of the true value forces us to choose a sub-optimal design of experiment. Adaptive design of experiments is one such sub-optimal design. Suppose that a set of identical copies of an unknown estimation object is given, and we perform a measurement on each copy. The recipe for adaptive design of experiments is as follows. We perform a measurement on the first copy and obtain an outcome. From that outcome, we calculate an estimate. Before performing the next measurement, we tune the measurement apparatus in the optimal way for the estimated value. That is, we treat the estimate as the true value temporarily and tune the apparatus accordingly. Afterwards, we perform the tuned measurement on the second copy, calculate an estimate from the first and second outcomes, and repeat the tuning, measurement, and estimation until we reach the final copy. The class of adaptive designs of experiments clearly includes nonadaptive and independent designs of experiments, and so is expected to outperform them. In this thesis, we propose an adaptive design of experiment for one-qubit state estimation and show that our method gives more accurate estimation results than standard quantum state tomography.

We summarize the contents above. In this thesis we analyze statistical estimation errors in tests of Bell-type correlations and in quantum tomography for finite samples. Chapters 2 and 3 are devoted to preliminary. In Chap. 2, we review the postulates of quantum mechanics used in quantum information and quantum estimation. We also explain the importance of finite sample analysis of estimation errors in the test of Bell-type correlations and quantum tomography. In Chap. 3, we introduce the basic concepts and notation in statistical parameter estimation theory and show the known results necessary for analyzing errors in quantum estimation. Chapters 4–6 treat our results on the accurate evaluation and an improvement of estimation errors. In Chap. 4, we analyze expected losses and error probabilities in the test of Bell-type correlations with finite samples. We derive explicit forms of their upper bounds that are directly applicable for evaluating the validity of the violation of the CHSH inequality in experiments. In Chap. 5, we analyze expected losses and error probabilities in quantum tomography with finite samples. We focus on three estimators, extended linear, extended norm-minimization, and maximum-likelihood, and derive

Table 1.1 Ways to evaluate estimation errors and the sections of this thesis discussing the corresponding results

	Test of bell-type correlations	Quantum tomography	Adaptive design of experiments
Expected loss	Sect. 4.3.2	Sect. 5.2	Sect. 6.2
Error probability	Sect. 4.3.3	Sect. 5.3	–

upper bounds on their estimation errors. In Chap. [6](#), we consider an adaptive design of experiment in one-qubit state estimation. We focus on a measurement update criterion called the A-optimality criterion, and numerically evaluate its performance. Our numerical results indicate that the A-optimal design of experiment gives more precise estimation results than standard quantum state tomography. Table [1.1](#) gives the sections containing our results for each of these topics. In Chap. [7](#), we summarize the results of this thesis.

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