Chapter 2
Supergravities in Four Dimensions

2.1 Superalgebras and Supermultiplets

Supersymmetry is a symmetry between bosons and fermions. In supersymmetric theories bosons and fermions belong to supermultiplets and are related by supertransformations. Supertransformations together with spacetime transformations such as the Poincaré transformations form a superalgebra. There are various kinds of superalgebras depending on the spacetime dimension, the spacetime symmetry and the number of supersymmetries. In this section we discuss supermultiplets of the $D = 4$ super Poincaré algebra with the smallest number of supersymmetries.

The super Poincaré algebra consists of the generators of supertransformations (supercharges) Q_α and those of the Poincaré algebra, i.e., the translation generators P_μ and the Lorentz generators $M_{\mu\nu}(= -M_{\nu\mu})$. (We use $\alpha, \beta, \ldots = 1, 2, 3, 4$ for spinor indices.) When it contains only one Majorana spinor supercharge, it is called the $\mathscr{N} = 1$ super Poincaré algebra, which we consider first. A Majorana spinor ψ is a spinor satisfying the Majorana condition $\psi = \psi^c$, where ψ^c is the charge conjugation of ψ defined by

$$\psi^c = C\bar{\psi}^T. \tag{2.1}$$

Here, the charge conjugation matrix C is a 4×4 matrix satisfying

$$C^{-1}\gamma^\mu C = -\gamma^{\mu T}, \quad C^T = -C, \quad C^\dagger C = 1. \tag{2.2}$$

The (anti)commutation relations of the $\mathscr{N} = 1$ super Poincaré algebra are

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= -\mathrm{i}\eta_{\nu\rho}M_{\mu\sigma} + \mathrm{i}\eta_{\nu\sigma}M_{\mu\rho} + \mathrm{i}\eta_{\mu\rho}M_{\nu\sigma} - \mathrm{i}\eta_{\mu\sigma}M_{\nu\rho}, \\
[M_{\mu\nu}, P_\rho] &= -\mathrm{i}\eta_{\nu\rho}P_\mu + \mathrm{i}\eta_{\mu\rho}P_\nu, \qquad [P_\mu, P_\nu] = 0, \\
[M_{\mu\nu}, Q_\alpha] &= \frac{1}{2}\mathrm{i}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta, \qquad [P_\mu, Q_\alpha] = 0, \\
\{Q_\alpha, \bar{Q}^\beta\} &= -2\mathrm{i}\,(\gamma^\mu)_\alpha{}^\beta P_\mu,
\end{aligned} \tag{2.3}$$

Y. Tanii, *Introduction to Supergravity*, SpringerBriefs in Mathematical Physics,
DOI: 10.1007/978-4-431-54828-7_2, © The Author(s) 2014

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ is the metric of flat Minkowski spacetime and $\gamma^{\mu\nu} = \gamma^{[\mu}\gamma^{\nu]}$ is the antisymmetrized product of gamma matrices defined in Appendix B. The components of the supercharge Q_α are fermionic generators and satisfy anticommutation relations rather than commutation relations.

In supersymmetric theories a certain set of particle states with different spins form a multiplet of the superalgebra called a supermultiplet. We can find possible supermultiplets by studying irreducible representations of the super Poincaré algebra for one particle states. From the fifth commutation relation of (2.3) we see that all the states generated by acting Q_α on a state in a supermultiplet have the same eigenvalue p_μ of the translation generator P_μ. Hence, all such states have the same mass $m = \sqrt{-p_\mu p^\mu}$. To proceed we choose a representation of the gamma matrices

$$\gamma^0 = -\mathrm{i}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = -\mathrm{i}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (i = 1, 2, 3), \tag{2.4}$$

where σ_i are the 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.5}$$

In this representation the chirality matrix $\gamma_5 = \mathrm{i}\,\gamma_0\gamma_1\gamma_2\gamma_3$ is diagonal. The charge conjugation matrix C and the Majorana spinor supercharge Q can be written as

$$C = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix}, \qquad Q = \begin{pmatrix} \mathrm{i}\sigma_2 Q^{\dagger T} \\ Q \end{pmatrix}. \tag{2.6}$$

We can choose the lower two components Q_3, Q_4 as independent components of the supercharge.

Let us first consider supermultiplets of massless one particle states. In this case we can choose a Lorentz frame in which the momentum eigenvalue is $p^\mu = (E, 0, 0, E)$ $(E > 0)$. Then, only non-vanishing anticommutator of Q_α $(\alpha = 3, 4)$ in (2.3) is

$$\{Q_3, (Q_3)^\dagger\} = 4E. \tag{2.7}$$

Since Q_4 anticommutes with all the components of Q including Q_4 itself, we can assume $Q_4 = 0$. If we define $b = (4E)^{-\frac{1}{2}} Q_3$, $b^\dagger = (4E)^{-\frac{1}{2}} Q_3^\dagger$, they satisfy the anticommutation relations of creation and annihilation operators of a fermion

$$\{b, b^\dagger\} = 1, \quad \{b, b\} = 0, \quad \{b^\dagger, b^\dagger\} = 0. \tag{2.8}$$

Therefore, their representation space consists of two states

$$|h_0\rangle\,, \qquad b^\dagger\,|h_0\rangle\,, \tag{2.9}$$

where $|h_0\rangle$ is a state with helicity h_0 satisfying $b\,|h_0\rangle = 0$. From the fourth commutation relation in (2.3) we find $[M_{12}, b^\dagger] = \frac{1}{2}b^\dagger$, which implies that $b^\dagger$ has helicity $\frac{1}{2}$. Therefore, the states (2.9) have helicities $(h_0, h_0 + \frac{1}{2})$. According to the CPT theorem, if a quantum field theory contains a state with helicity h, then it also contains a state with helicity $-h$. Therefore, supermultiplets realized by a quantum field theory are

$$(h_0,\ h_0 + \tfrac{1}{2})\ \oplus\ (-h_0 - \tfrac{1}{2},\ -h_0) \qquad (h_0 = 0,\ \tfrac{1}{2},\ 1, \ldots). \tag{2.10}$$

Massless supermultiplets often used in particle physics are

$$\begin{aligned} \text{chiral multiplet:}\quad & (0,\ \tfrac{1}{2}) \oplus (-\tfrac{1}{2},\ 0), \\ \text{massless vector multiplet:}\quad & (\tfrac{1}{2},\ 1) \oplus (-1,\ -\tfrac{1}{2}), \\ \text{supergravity multiplet:}\quad & (\tfrac{3}{2},\ 2) \oplus (-2,\ -\tfrac{3}{2}). \end{aligned} \tag{2.11}$$

For massive one particle states with a mass m we can choose a Lorentz frame in which the momentum eigenvalue is $p^\mu = (m, 0, 0, 0)$. Then, the anticommutator of the supercharge Q_α $(\alpha = 3,\ 4)$ in (2.3) becomes

$$\{Q_\alpha, (Q_\beta)^\dagger\} = 2m\delta_{\alpha\beta}. \tag{2.12}$$

This implies that $b_\alpha = (2m)^{-\frac{1}{2}} Q_\alpha$ and $b_\alpha^\dagger = (2m)^{-\frac{1}{2}} Q_\alpha^\dagger$ are two sets of creation and annihilation operators of fermions. Therefore, their representation space consists of

$$|s_0\rangle\,, \quad b_\alpha^\dagger\,|s_0\rangle\,, \quad b_3^\dagger b_4^\dagger\,|s_0\rangle\,. \tag{2.13}$$

The state $|s_0\rangle$ satisfies $b_\alpha\,|s_0\rangle = 0$ and collectively represents the $2s_0 + 1$ states of spin s_0. Since $b_\alpha^\dagger$ has spin $\frac{1}{2}$ as can be seen from the fourth commutation relation in (2.3), the states (2.13) have spins

$$(s_0 - \tfrac{1}{2},\ s_0,\ s_0,\ s_0 + \tfrac{1}{2}) \qquad (s_0 = 0,\ \tfrac{1}{2},\ 1, \ldots), \tag{2.14}$$

where the $s_0 - \frac{1}{2}$ state is absent for $s_0 = 0$. Massive supermultiplets often used are

$$\begin{aligned} \text{chiral multiplet:}\quad & (0,\ 0,\ \tfrac{1}{2}), \\ \text{massive vector multiplet:}\quad & (0,\ \tfrac{1}{2},\ \tfrac{1}{2},\ 1). \end{aligned} \tag{2.15}$$

We see that the numbers of bosonic states and fermionic states in a supermultiplet are the same for both of massless and massive supermultiplets.

2.2 Supersymmetric Field Theories

To construct a supersymmetric field theory one introduces supermultiplets of fields. In globally supersymmetric theories one usually uses two kinds of supermultiplets:

$$\begin{aligned} &\text{vector multiplet:} \ \ (A^I_\mu, \ \chi^I, \ D^I) \quad (I = 1, 2, \ldots, \dim G), \\ &\text{chiral multiplet:} \ \ (\phi_i, \ \psi_{-i}, \ F_i) \quad (i = 1, 2, \ldots, n). \end{aligned} \tag{2.16}$$

The vector multiplet consists of vector fields $A^I_\mu(x)$, Majorana spinor fields $\chi^I(x)$ and real scalar fields $D^I(x)$, which all belong to the adjoint representation of a gauge group G. The chiral multiplet consists of complex scalar fields $\phi_i(x)$, $F_i(x)$ and Weyl spinor fields with negative chirality $\psi_{-i}(x)$ ($\gamma_5\psi_{-i} = -\psi_{-i}$), which all belong to a certain n-dimensional representation of G. The fields D^I and F_i are auxiliary fields, which do not have physical degrees of freedom, and can be expressed in terms of other fields by their field equations. To construct a supersymmetric Lagrangian for these fields it is useful to use the superfield method, which we do not discuss here (see, e.g., [16]).

As an example of supersymmetric field theories let us consider a theory consisting of a chiral multiplet $(\phi, \ \psi_-, \ F)$. A supersymmetric Lagrangian is

$$\begin{aligned} \mathscr{L} = &- \partial_\mu\phi^*\partial^\mu\phi - \bar{\psi}_-\gamma^\mu\partial_\mu\psi_- + F^*F + W'(\phi)F + (W'(\phi))^*F^* \\ &- \frac{1}{2}W''(\phi)\bar{\psi}_+\psi_- - \frac{1}{2}(W''(\phi))^*\bar{\psi}_-\psi_+, \end{aligned} \tag{2.17}$$

where the superpotential $W(\phi)$ is a holomorphic function of the complex scalar field ϕ, and $\psi_+ = (\psi_-)^c$ is the charge conjugation of ψ_- and has positive chirality ($\gamma_5\psi_+ = +\psi_+$). This Lagrangian is invariant up to total divergences under the supertransformation

$$\delta_Q\phi = \frac{1}{2}\bar{\varepsilon}_+\psi_-, \quad \delta_Q\psi_- = \frac{1}{2}\gamma^\mu\varepsilon_+\partial_\mu\phi + \frac{1}{2}F\varepsilon_-, \quad \delta_Q F = \frac{1}{2}\bar{\varepsilon}_-\gamma^\mu\partial_\mu\psi_-. \tag{2.18}$$

Therefore, the action obtained by integrating the Lagrangian over spacetime is invariant. The transformation parameter $\varepsilon = \varepsilon_+ + \varepsilon_-$ is a constant Majorana spinor and $\varepsilon_\pm$ are its projections on the chirality eigenstates. Since the transformation parameter is fermionic, the supertransformation exchanges the bosonic fields ϕ, F and the fermionic field ψ. For all the fields the commutator of two supertransformations with parameters ε_1 and ε_2 becomes

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_P(\xi), \qquad \xi^\mu = \frac{1}{4}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1, \tag{2.19}$$

where $\delta_P(\xi)$ is the infinitesimal translation with a parameter ξ^μ. This commutation relation corresponds to the last anticommutation relation in the super Poincaré algebra (2.3).

The field equation of the auxiliary field F derived from the Lagrangian (2.17) is algebraic and can be used to express it in terms of the scalar field ϕ as $F^* = -W'(\phi)$. Substituting this back into (2.17) we obtain a Lagrangian without the auxiliary field. For instance, if we choose the superpotential $W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{6}\lambda\phi^3$, we obtain

$$\begin{aligned}\mathscr{L} = &- \partial_\mu\phi^*\partial^\mu\phi - \bar{\psi}_-\gamma^\mu\partial_\mu\psi_- - \left|m\phi + \frac{1}{2}\lambda\phi^2\right|^2 \\ &- \frac{1}{2}(m+\lambda\phi)\bar{\psi}_+\psi_- - \frac{1}{2}(m+\lambda\phi^*)\bar{\psi}_-\psi_+. \end{aligned} \tag{2.20}$$

The fields ϕ and ψ_- represent two spin 0 bosons and a spin $\frac{1}{2}$ fermion with the same mass m. They form a chiral multiplet $(0,\ 0,\ \frac{1}{2})$ in (2.15). The coupling constants of various interaction terms in (2.20) are related by supersymmetry. For instance, the coupling constant of the $|\phi|^4$ coupling and that of the Yukawa coupling $\phi\bar{\psi}_+\psi_-$ are given by using the same λ. Substituting $F^* = -W'(\phi)$ into the first two equations in (2.18) we obtain the supertransformation of ϕ and ψ_- without the auxiliary field. The Lagrangian with the auxiliary field eliminated is still invariant up to total divergences under this transformation. However, an extra term proportional to the field equation of ψ_- appears on the right-hand side of the commutation relation (2.19) for ψ_-. Thus, when the auxiliary field is eliminated, the commutator algebra closes only on-shell, i.e., only if the field equation is used.

2.3 $\mathscr{N} = 1$ Poincaré Supergravity

Supergravity is a field theory which has local supersymmetry. Since the transformation parameter of supersymmetry is a spinor ε_α, the gauge field of local supersymmetry should be $\psi_{\mu\alpha}(x)$ with a vector index μ and a spinor index α. The transformation law of this gauge field is $\delta_Q\psi_{\mu\alpha} = \partial_\mu\varepsilon_\alpha + \cdots$, where the transformation parameter $\varepsilon_\alpha(x)$ is an arbitrary function of spacetime coordinates x^μ. Such a field $\psi_{\mu\alpha}(x)$ is the Rarita–Schwinger field representing a spin $\frac{3}{2}$ fermion (gravitino). Furthermore, we need another gauge field. In globally supersymmetric theories the commutator of two supertransformations generates a translation as in (2.19). Therefore, we expect that the gauging of supersymmetry leads to the gauging of translation. Since the local translation is the general coordinate transformation, we also need the gravitational field $e_\mu{}^a(x)$ as a gauge field. To summarize, supergravity is a theory which is invariant under the local supersymmetry transformation as well as the general coordinate transformation. It contains the gravitational field $e_\mu{}^a(x)$ and the Rarita–Schwinger field $\psi_{\mu\alpha}(x)$. As we saw in Sect. 2.1, the supergravity multiplet for the $\mathscr{N} = 1$ super Poincaré algebra consists of the states with helicities $(\frac{3}{2}, 2) \oplus (-2, -\frac{3}{2})$, which correspond to a pair of fields $(e_\mu{}^a(x), \psi_{\mu\alpha}(x))$. Hence, we expect that there exists a supergravity theory which contains these two fields. Such a theory was indeed constructed in [2, 9, 10] and is called $\mathscr{N} = 1$ Poincaré supergravity.

The field content of $\mathscr{N} = 1$ Poincaré supergravity is a vierbein $e_\mu{}^a(x)$ and a Majorana Rarita–Schwinger field $\psi_\mu(x)$. As discussed in Chap. 1 the vierbein is related to the metric as $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(-1, +1, +1, +1)$ is the flat Minkowski metric. The Rarita–Schwinger field satisfies the Majorana condition $\psi_\mu^c = \psi_\mu$. The Lagrangian consists of the Einstein term and the Rarita–Schwinger term as

$$\mathscr{L} = e\hat{R} - \frac{1}{2} e\bar{\psi}_\mu \gamma^{\mu\nu\rho} \hat{D}_\nu \psi_\rho, \tag{2.21}$$

where $e = \det e_\mu{}^a$, $\gamma^\mu = \gamma^a e_a{}^\mu$, and $\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$ is the antisymmetrized product of gamma matrices defined in Appendix B. The curvature and the covariant derivative are defined by

$$\begin{aligned}
\hat{R} &= e_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab}, \\
\hat{R}_{\mu\nu}{}^{ab} &= \partial_\mu \hat{\omega}_\nu{}^{ab} - \partial_\nu \hat{\omega}_\mu{}^{ab} + \hat{\omega}_\mu{}^a{}_c \hat{\omega}_\nu{}^{cb} - \hat{\omega}_\nu{}^a{}_c \hat{\omega}_\mu{}^{cb}, \\
\hat{D}_{[\nu} \psi_{\rho]} &= \left(\partial_{[\nu} + \frac{1}{4} \hat{\omega}_{[\nu}{}^{ab} \gamma_{ab} \right) \psi_{\rho]}.
\end{aligned} \tag{2.22}$$

The spin connection $\hat{\omega}_\mu{}^{ab}$ used here is given by

$$\hat{\omega}_{\mu ab} = \omega_{\mu ab} + \frac{1}{8} \left(\bar{\psi}_a \gamma_\mu \psi_b + \bar{\psi}_\mu \gamma_a \psi_b - \bar{\psi}_\mu \gamma_b \psi_a \right), \tag{2.23}$$

where $\omega_{\mu ab}$ is the spin connection without torsion given in (1.21). The spin connection (2.23) has a torsion depending on the Rarita–Schwinger field:

$$\hat{D}_\mu e_\nu{}^a - \hat{D}_\nu e_\mu{}^a = \frac{1}{4} \bar{\psi}_\mu \gamma^a \psi_\nu. \tag{2.24}$$

If one wishes, one can rewrite the Lagrangian using the torsionless spin connection $\omega_{\mu ab}$ as

$$\mathscr{L} = eR - \frac{1}{2} e\bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho + (\text{four-fermi terms}), \tag{2.25}$$

where R and D_ν are defined by using $\omega_{\mu ab}$. Explicit four-fermi terms have appeared.

The Lagrangian (2.21) is invariant up to total divergences under the general coordinate transformation

$$\delta_G(\xi) e_\mu{}^a = \xi^\nu \partial_\nu e_\mu{}^a + \partial_\mu \xi^\nu e_\nu{}^a, \quad \delta_G(\xi) \psi_\mu = \xi^\nu \partial_\nu \psi_\mu + \partial_\mu \xi^\nu \psi_\nu, \tag{2.26}$$

the local Lorentz transformation

$$\delta_L(\lambda) e_\mu{}^a = -\lambda^a{}_b e_\mu{}^b, \quad \delta_L(\lambda) \psi_\mu = -\frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_\mu \tag{2.27}$$

and the $\mathscr{N} = 1$ local supertransformation

$$\delta_Q(\varepsilon)e_\mu{}^a = \frac{1}{4}\bar{\varepsilon}\gamma^a\psi_\mu, \quad \delta_Q(\varepsilon)\psi_\mu = \hat{D}_\mu\varepsilon, \tag{2.28}$$

where the transformation parameters $\xi^\mu(x)$, $\lambda_{ab}(x)$ ($\lambda_{ab} = -\lambda_{ba}$) and $\varepsilon_\alpha(x)$ ($\varepsilon^c = \varepsilon$) are arbitrary infinitesimal functions of spacetime coordinates x^μ. The invariance under the first two bosonic transformations is manifest. The invariance under the local supertransformation is shown in the next section. From (2.28) we can compute the local supertransformation of the spin connection $\hat{\omega}_{\mu ab}$ in (2.23). We see that terms containing derivatives of the transformation parameter $\partial_\mu\varepsilon$ appear from $\delta_Q\omega_{\mu ab}$ but are canceled by the variation of the ψ_μ bilinear terms (see (2.40) below). In general, quantities whose local supertransformation does not contain $\partial_\mu\varepsilon$ are called supercovariant. We often put ˆ on supercovariant quantities.

The above local transformations satisfy the closed commutation relations

$$\begin{aligned}
[\delta_G(\xi_1), \delta_G(\xi_2)] &= \delta_G(\xi_2 \cdot \partial\xi_1 - \xi_1 \cdot \partial\xi_2), \\
[\delta_L(\lambda_1), \delta_L(\lambda_2)] &= \delta_L([\lambda_1, \lambda_2]), \\
[\delta_G(\xi), \delta_L(\lambda)] &= \delta_L(-\xi \cdot \partial\lambda), \\
[\delta_G(\xi), \delta_Q(\varepsilon)] &= \delta_Q(-\xi \cdot \partial\varepsilon), \\
[\delta_L(\lambda), \delta_Q(\varepsilon)] &= \delta_Q(\tfrac{1}{4}\lambda^{ab}\gamma_{ab}\varepsilon), \\
[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] &= \delta_G(\xi) + \delta_L(\lambda) + \delta_Q(\varepsilon),
\end{aligned} \tag{2.29}$$

where the transformation parameters on the right-hand side of the last commutation relation are

$$\xi^\mu = \frac{1}{4}\bar{\varepsilon}_2\gamma^\mu\varepsilon_1, \quad \lambda_{ab} = -\xi^\mu\hat{\omega}_{\mu ab}, \quad \varepsilon = -\xi^\mu\psi_\mu. \tag{2.30}$$

We see that the commutator of two local supertransformations generates a general coordinate transformation as we expected at the beginning of this section. The commutation relations (2.29) except the last one can be easily shown. The last commutation relation is shown in the next section. To obtain the last commutation relation we have to use the Rarita–Schwinger field equation derived from the Lagrangian (2.21). In this sense the commutator algebra closes only on-shell. In the present theory it is possible to close the commutator algebra off-shell by introducing an appropriate set of auxiliary fields, which have no dynamical degrees of freedom [5, 12, 13]. A formulation with an off-shell algebra is more convenient, although not indispensable, when one fixes a gauge of the local symmetries and when one couples matter supermultiplets. For general supergravities (those with highly extended supersymmetry and/or in higher dimensions) such an off-shell formulation is not known.

This theory also has a global symmetry. The Lagrangian (2.21) is invariant under the global chiral $U(1)$ transformation

$$\delta e_\mu{}^a = 0, \qquad \delta\psi_\mu = \mathrm{i}\Lambda\gamma_5\psi_\mu, \tag{2.31}$$

where Λ is a constant infinitesimal transformation parameter. The transformation of ψ_μ is consistent with the Majorana condition on ψ_μ.

The field equations derived from the Lagrangian (2.21) have a Minkowski spacetime solution $e_\mu{}^a = \delta_\mu^a$, $\psi_\mu = 0$. This solution is preserved by the super Poincaré transformations corresponding to the local symmetry transformations with the parameters $\xi^\mu(x) = a^\mu{}_\nu x^\nu + b^\mu$, $\lambda^\mu{}_\nu(x) = a^\mu{}_\nu$, $\varepsilon(x) = \varepsilon$, where $a^\mu{}_\nu$, b^μ and ε are constant. Dynamics of small fluctuations of the fields around this background is subject to the symmetry under these transformations. Substituting this solution and these transformation parameters into (2.29) we find the commutation relations of the super Poincaré algebra (2.3). In general, a supergravity which has a Minkowski spacetime solution is called the Poincaré supergravity.

One can couple matter supermultiplets to the supergravity multiplet $(e_\mu{}^a, \psi_\mu)$. Possible matter multiplets are the chiral multiplet (ϕ_i, ψ_{-i}) and the vector multiplet (A_μ^I, χ^I), which we discussed in Sect. 2.2. For details of matter couplings see [8].

2.4 Local Supersymmetry of $\mathscr{N} = 1$ Poincaré Supergravity

In this section we show the invariance of the action and the commutator algebra of $\mathscr{N} = 1$ Poincaré supergravity discussed in the previous section. We use the identities for spinors and gamma matrices given in Appendix B.

2.4.1 Invariance of the Action

The Lagrangian (2.21) consists of two terms:

$$\begin{aligned}
\mathscr{L} &= \mathscr{L}_\mathrm{E} + \mathscr{L}_\mathrm{RS}, \\
\mathscr{L}_\mathrm{E} &= e\, e_a{}^\mu e_b{}^\nu \hat{R}_{\mu\nu}{}^{ab} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{abcd} e_\rho{}^c e_\sigma{}^d \hat{R}_{\mu\nu}{}^{ab}, \\
\mathscr{L}_\mathrm{RS} &= -\frac{1}{2} e e_a{}^\mu e_b{}^\nu e_c{}^\rho \bar{\psi}_\mu \gamma^{abc}\hat{D}_\nu\psi_\rho = \frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_\nu\gamma_5\hat{D}_\rho\psi_\sigma,
\end{aligned} \tag{2.32}$$

where $\varepsilon^{\mu\nu\rho\sigma}$ and ε_{abcd} are the totally antisymmetric Levi-Civita symbols with components $\varepsilon^{0123} = +1$ and $\varepsilon_{0123} = -1$ (see Appendix A), and $\gamma_5 = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ is the chirality matrix. The Riemann tensor $\hat{R}_{\mu\nu}{}^{ab}$ and the covariant derivative $\hat{D}_\mu$ depend on the fields $e_\mu{}^a$, ψ_μ only through the spin connection $\hat{\omega}_{\mu ab}$. The following observation is useful to show the invariance of the action. If we treat $\hat{\omega}_{\mu ab}$ as an independent field, the variation of the action with respective to it vanishes:

$$\frac{\delta}{\delta\hat{\omega}_{\mu ab}}\int d^4x\, \mathscr{L}(e, \psi, \hat{\omega}) = 0 \tag{2.33}$$

when (2.23) is substituted into (2.33) after the variation. To show this, it is convenient to use the second forms of $\mathscr{L}_{\mathrm{E}}$ and $\mathscr{L}_{\mathrm{RS}}$ in (2.32). Therefore, when we compute the supertransformation of the Lagrangian, we need not consider the variation of the spin connection.

Let us compute the variation of the Lagrangian (2.32) under the local supertransformation (2.28). Using the first form of $\mathscr{L}_{\mathrm{E}}$ in (2.32) we find

$$\begin{aligned}\delta_Q \mathscr{L}_{\mathrm{E}} &= \delta_Q (e\, e_a{}^{\mu} e_b{}^{\nu}) \hat{R}_{\mu\nu}{}^{ab} \\ &= -\frac{1}{2} e \bar{\varepsilon} \gamma^{\mu} \psi_a \left(e_b{}^{\nu} \hat{R}_{\mu\nu}{}^{ab} - \frac{1}{2} e_{\mu}{}^{a} \hat{R} \right),\end{aligned} \tag{2.34}$$

while using the second form of $\mathscr{L}_{\mathrm{RS}}$ in (2.32) we find

$$\begin{aligned}\delta_Q \mathscr{L}_{\mathrm{RS}} = \frac{1}{2} \mathrm{i} \varepsilon^{\mu\nu\rho\sigma} \Big(& \hat{D}_{\mu} \bar{\varepsilon} \gamma_{\nu} \gamma_5 \hat{D}_{\rho} \psi_{\sigma} + \bar{\psi}_{\mu} \gamma_{\nu} \gamma_5 \hat{D}_{\rho} \hat{D}_{\sigma} \varepsilon \\ & + \delta_Q e_{\nu}{}^{a}\, \bar{\psi}_{\mu} \gamma_a \gamma_5 \hat{D}_{\rho} \psi_{\sigma} \Big).\end{aligned} \tag{2.35}$$

By partial integration, the first term of (2.35) becomes

$$-\frac{1}{2} \mathrm{i} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_{\nu} \gamma_5 \hat{D}_{\mu} \hat{D}_{\rho} \psi_{\sigma} - \frac{1}{2} \mathrm{i} \varepsilon^{\mu\nu\rho\sigma} \hat{D}_{\mu} e_{\nu}{}^{a}\, \bar{\varepsilon} \gamma_a \gamma_5 \hat{D}_{\rho} \psi_{\sigma} \tag{2.36}$$

up to total divergences. By using (2.24), the Fierz identity (B.15) and the symmetry properties (B.13) we find that the second term of (2.36) cancels the third term of (2.35). Then, (2.35) becomes

$$\begin{aligned}\delta_Q \mathscr{L}_{\mathrm{RS}} &= -\frac{1}{4} \mathrm{i} \varepsilon^{\mu\nu\rho\sigma} \bar{\varepsilon} \gamma_{\nu} \gamma_5 [\hat{D}_{\mu}, \hat{D}_{\rho}] \psi_{\sigma} + \frac{1}{4} \mathrm{i} \varepsilon^{\mu\nu\rho\sigma} \bar{\psi}_{\mu} \gamma_{\nu} \gamma_5 [\hat{D}_{\rho}, \hat{D}_{\sigma}] \varepsilon \\ &= \frac{1}{2} e \bar{\varepsilon} \gamma^{\mu} \psi_a \left(e_b{}^{\nu} \hat{R}_{\mu\nu}{}^{ab} - \frac{1}{2} e_{\mu}{}^{a} \hat{R} \right)\end{aligned} \tag{2.37}$$

up to total divergences, where in the last equality we have used (1.30) and (B.13). Thus, the variations of $\mathscr{L}_{\mathrm{E}}$ and $\mathscr{L}_{\mathrm{RS}}$ cancel each other and the total Lagrangian (2.21) is invariant under the local supertransformation (2.28) up to total divergences.

2.4.2 Commutator Algebra

Next let us show the commutation relation of two local supertransformations in (2.29). We first consider the commutator acting on the vierbein

$$\begin{aligned}[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]e_\mu{}^a &= \delta_Q(\varepsilon_1)\left(\frac{1}{4}\bar{\varepsilon}_2\gamma^a\psi_\mu\right) - (1 \leftrightarrow 2) \\ &= \frac{1}{4}\bar{\varepsilon}_2\gamma^a\hat{D}_\mu\varepsilon_1 - \frac{1}{4}\bar{\varepsilon}_1\gamma^a\hat{D}_\mu\varepsilon_2 \\ &= \frac{1}{4}\hat{D}_\mu\left(\bar{\varepsilon}_2\gamma^a\varepsilon_1\right), \end{aligned} \tag{2.38}$$

where we have used (B.13). Then, we obtain

$$\begin{aligned}[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]e_\mu{}^a &= \hat{D}_\mu\left(\xi^\nu e_\nu{}^a\right) \\ &= \partial_\mu\xi^\nu e_\nu{}^a + \xi^\nu\hat{D}_\nu e_\mu{}^a + \xi^\nu\left(\hat{D}_\mu e_\nu{}^a - \hat{D}_\nu e_\mu{}^a\right) \\ &= \partial_\mu\xi^\nu e_\nu{}^a + \xi^\nu\partial_\nu e_\mu{}^a + \xi^\nu\hat{\omega}_\nu{}^a{}_b e_\mu{}^b - \frac{1}{4}\xi^\nu\bar{\psi}_\nu\gamma^a\psi_\mu \\ &= \left[\delta_G(\xi) + \delta_L(-\xi\cdot\hat{\omega}) + \delta_Q(-\xi\cdot\psi)\right]e_\mu{}^a, \end{aligned} \tag{2.39}$$

where $\xi^\nu = \frac{1}{4}\bar{\varepsilon}_2\gamma^\nu\varepsilon_1$ and we have used (2.24). This shows the last commutation relation in (2.29) for $e_\mu{}^a$. Similarly, we can compute the commutation relation for ψ_μ. We need the supertransformation of the spin connection $\hat{\omega}_{\mu ab}$, which can be obtained by applying δ_Q on both sides of (2.24) as

$$\delta_Q(\varepsilon)\hat{\omega}_{\mu ab} = -\frac{1}{8}\left(\bar{\varepsilon}\gamma_\mu\psi_{ab} - \bar{\varepsilon}\gamma_a\psi_{b\mu} + \bar{\varepsilon}\gamma_b\psi_{a\mu}\right), \tag{2.40}$$

where $\psi_{\mu\nu} = \hat{D}_\mu\psi_\nu - \hat{D}_\nu\psi_\mu$. By using (2.40), (B.15) and (B.13) we obtain

$$\begin{aligned}[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]\psi_\mu = &\left[\delta_G(\xi) + \delta_L(-\xi\cdot\hat{\omega}) + \delta_Q(-\xi\cdot\psi)\right]\psi_\mu \\ &+ \frac{1}{128}\bar{\varepsilon}_2\gamma^{ab}\varepsilon_1\left(2\gamma_{ab\mu\nu}\mathscr{R}^\nu - \gamma_{ab}\mathscr{R}_\mu - 2e_{\mu a}\mathscr{R}_b\right) \\ &- \frac{1}{16}\xi^\nu\left(\gamma_\nu\mathscr{R}_\mu + 2\gamma_{\mu\nu\lambda}\mathscr{R}^\lambda\right), \end{aligned} \tag{2.41}$$

where $\mathscr{R}^\nu = \gamma^{\nu\rho\sigma}\psi_{\rho\sigma}$. The field equation of the Rarita–Schwinger field is $\mathscr{R}^\nu = 0$. Therefore, the commutator algebra closes on-shell.

2.5 $\mathscr{N} = 1$ Anti de Sitter Supergravity

We can construct a supergravity with a cosmological term [14]. The Lagrangian is

$$\mathscr{L} = e\hat{R} - \frac{1}{2}e\bar{\psi}_\mu\gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho + 6m^2 e + \frac{1}{2}me\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu, \tag{2.42}$$

where m is a real constant parameter. The third term is the cosmological term. Comparing with (1.2) in Chap. 1 we see that the cosmological constant is negative $\Lambda = -3m^2$. The last term is a mass term of the Rarita–Schwinger field and is proportional to the parameter m appearing in the cosmological term. When $m = 0$, this theory reduces to the Poincaré supergravity in Sect. 2.3. A positive cosmological constant corresponds to an imaginary m, for which the Rarita–Schwinger mass term is not real. Therefore, a positive cosmological constant is not allowed.

The Lagrangian (2.42) is invariant up to total divergences under the general coordinate transformation (2.26), the local Lorentz transformation (2.27) and the $\mathcal{N} = 1$ local supertransformation

$$\delta_Q e_\mu{}^a = \frac{1}{4}\bar{\varepsilon}\gamma^a\psi_\mu, \qquad \delta_Q\psi_\mu = \hat{D}_\mu\varepsilon + \frac{1}{2}m\gamma_\mu\varepsilon. \tag{2.43}$$

The term proportional to m has been added to $\delta_Q\psi_\mu$. The commutator algebra of the local transformations has the same form as (2.29) for $m = 0$ except that the parameter of the local Lorentz transformation in (2.30) is replaced by $\lambda_{ab} = -\xi^\mu\hat{\omega}_{\mu ab} - \frac{1}{4}m\bar{\varepsilon}_2\gamma_{ab}\varepsilon_1$. To obtain the closed commutator algebra we have to use the Rarita–Schwinger field equation derived from (2.42). Due to the mass term of the Rarita–Schwinger field the Lagrangian is no longer invariant under the global $U(1)$ transformation (2.31).

Let us consider a classical solution of the field equations derived from the Lagrangian (2.42). When $\psi_\mu = 0$, the Rarita–Schwinger field equation is automatically satisfied and the gravitational field equation becomes

$$R_{\mu\nu} = -3m^2 g_{\mu\nu}. \tag{2.44}$$

Minkowski spacetime $e_\mu{}^a = \delta_\mu^a$ has $R_{\mu\nu} = 0$ and therefore does not satisfy this equation. A solution of this equation is anti de Sitter (AdS) spacetime (see, e.g., [8] for details of AdS spacetime). The Riemann tensor of AdS spacetime can be expressed by using the metric as

$$R_{\mu\nu\rho\sigma} = -m^2\left(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}\right) \tag{2.45}$$

and the Ricci tensor satisfies (2.44). The parameter m is called the inverse radius of AdS spacetime. Supergravities which have an AdS spacetime solution are called anti de Sitter (AdS) supergravities.

AdS spacetime has a large isometry $SO(2,3)$ in the same way as Minkowski spacetime has the isometry of the Poincaré group. The dimensions of $SO(2,3)$ and the Poincaré group are ten. In general, the dimension of the isometry group, i.e., the number of independent Killing vectors, in D-dimensional spacetime is at most $\frac{1}{2}D(D+1)$ (see, e.g., Chap.13 of [15]). Spacetime having an isometry of this dimension is called maximally symmetric. Minkowski spacetime and AdS spacetime are maximally symmetric. Another maximally symmetric spacetime is de Sitter spacetime, which is a solution of the Einstein equation with a positive cosmological constant and has the isometry $SO(1,4)$. It is a general property of maximally symmetric spacetimes that the Riemann tensor can be expressed in terms of the metric as in (2.45). The coefficient on the right-hand side of (2.45) is zero for Minkowski spacetime, positive for de Sitter spacetime and negative for AdS spacetime.

The AdS spacetime solution has a global supersymmetry. For a solution of the field equations to preserve supersymmetry, the supertransformation (2.43) must vanish for that solution. When $\psi_\mu = 0$, the supertransformation of $e_\mu{}^a$ automatically vanishes. Requiring $\delta_Q \psi_\mu = 0$ we obtain the condition

$$\left(D_\mu + \frac{1}{2} m \gamma_\mu \right) \varepsilon = 0. \tag{2.46}$$

This is a partial differential equation on the transformation parameter ε. Spinors ε satisfying (2.46) are called Killing spinors. If Killing spinors exist, the solution is supersymmetric. The consistency of (2.46) requires the integrability condition

$$\begin{aligned} 0 &= \left[D_\mu + \frac{1}{2} m \gamma_\mu ,\ D_\nu + \frac{1}{2} m \gamma_\nu \right] \varepsilon \\ &= \frac{1}{4} \left(R_{\mu\nu}{}^{ab} + 2m^2 e_\mu{}^a e_\nu{}^b \right) \gamma_{ab} \varepsilon. \end{aligned} \tag{2.47}$$

Using (2.45) we see that this condition is satisfied by AdS spacetime. Solutions of (2.46) indeed exist and were explicitly constructed in [1]. Hence, the AdS solution has a global supersymmetry. The supertransformation with Killing spinors as transformation parameters and the $SO(2,3)$ transformation together form a closed superalgebra $OSp(1|4)$. This algebra is an analog of the super Poincaré algebra for Minkowski spacetime and is called a super anti de Sitter (AdS) algebra. We will discuss more general super AdS algebras in Sect. 3.5.

2.6 Extended Supersymmetries

So far we have considered $\mathcal{N} = 1$ supersymmetry, which contains a single Majorana spinor supercharge. More generally, we can consider $\mathcal{N}$-extended supersymmetry [11], which contains $\mathcal{N}$ Majorana spinor supercharges Q^i $(i = 1, 2, \ldots, \mathcal{N})$. The anticommutation relation of the supercharges of the $\mathcal{N}$-extended super Poincaré algebra is

$$\{Q^i_\alpha, \bar{Q}^{j\beta}\} = -2\mathrm{i}\, (\gamma^\mu)_\alpha{}^\beta P_\mu \delta^{ij} + \delta^\beta_\alpha U^{ij} + \mathrm{i} (\gamma_5)_\alpha{}^\beta V^{ij}, \tag{2.48}$$

where $U^{ij} = -U^{ji}$, $V^{ij} = -V^{ji}$ are generators called the central charges and commute with all the generators of the algebra. Other commutation relations have the same form as (2.3).

As in Sect. 2.1, we can find possible supermultiplets of this superalgebra. Let us consider massless supermultiplets with vanishing central charges. In this case we can

Table 2.1 Supergravity multiplets

h	$\mathcal{N}=1$	$\mathcal{N}=2$	$\mathcal{N}=3$	$\mathcal{N}=4$	$\mathcal{N}=5$	$\mathcal{N}=6$	$\mathcal{N}=7$	$\mathcal{N}=8$
$+2$	1	1	1	1	1	1	1	1
$+\frac{3}{2}$	1	2	3	4	5	6	$7+1$	8
$+1$		1	3	6	10	$15+1$	$21+7$	28
$+\frac{1}{2}$			1	4	$10+1$	$20+6$	$35+21$	56
0				$1+1$	$5+5$	$15+15$	$35+35$	70
$-\frac{1}{2}$			1	4	$1+10$	$6+20$	$21+35$	56
-1		1	3	6	10	$1+15$	$7+21$	28
$-\frac{3}{2}$	1	2	3	4	5	6	$1+7$	8
-2	1	1	1	1	1	1	1	1

construct $\mathcal{N}$ pairs of creation and annihilation operators $(b^i, b^{i\dagger})$ $(i = 1, 2, \dots, \mathcal{N})$ satisfying the anticommutation relations of fermions. Therefore, a supermultiplet contains the states

$$|h_0\rangle\,, \quad b^{i\dagger}\,|h_0\rangle\,, \quad b^{i\dagger}b^{j\dagger}\,|h_0\rangle\,, \dots, \quad b^{1\dagger}b^{2\dagger}\cdots b^{\mathcal{N}\dagger}\,|h_0\rangle\,, \tag{2.49}$$

where $|h_0\rangle$ is a state with helicity h_0 and satisfying $b^i\,|h_0\rangle = 0$. Since $b^{i\dagger}$ has helicity $\frac{1}{2}$, helicities of these states are

$$h = h_0,\ h_0 + \tfrac{1}{2},\ h_0 + 1, \dots,\ h_0 + \tfrac{1}{2}\mathcal{N}. \tag{2.50}$$

We see that for $\mathcal{N} > 8$ all the supermultiplets contain states with helicity $|h| > 2$. However, consistent interacting field theories are not known when they contain massless fields with helicity $|h| > 2$. As a consequence, $\mathcal{N} = 8$ is the largest supersymmetry that has been realized by field theories.

Supermultiplets which contain a graviton ($h = \pm 2$) and gravitinos ($h = \pm\frac{3}{2}$) are called supergravity multiplets. The numbers of states in supergravity multiplets are listed in Table 2.1. Since the states with helicity $h_0 + \frac{1}{2}n$ in (2.49) contain n anticommuting $b^{i\dagger}$, the number of such states is ${}_{\mathcal{N}}C_n$. In Table 2.1 we have added helicity flipped states required by the CPT theorem as in (2.10). Note that the $\mathcal{N} = 8$ supergravity multiplet contains all the states required by the CPT theorem without adding helicity flipped states. Note also that the $\mathcal{N} = 7$ and $\mathcal{N} = 8$ supergravity multiplets contain the same states and are expected to give the same theory.

One can construct extended supergravities corresponding to the supergravity multiplets in Table 2.1. The field contents of such theories are summarized in Table 3.3. Extended supergravities generically contain vector fields, spinor fields and scalar fields in addition to the gravitational field and the Rarita–Schwinger fields. As we mentioned above, field theories with $\mathcal{N} > 8$ supersymmetry are not known, and $\mathcal{N} = 8$ supergravity is called the maximal supergravity.

2.7 $\mathscr{N} = 2$ Poincaré Supergravity

The simplest extended supergravity is $\mathscr{N} = 2$ supergravity [4]. The field content is a gravitational field $e_\mu{}^a(x)$, a vector field $B_\mu(x)$ and two Majorana Rarita–Schwinger fields $\psi_\mu^i(x)$ ($i = 1, 2$). The Lagrangian is

$$\begin{aligned}\mathscr{L} &= e\hat{R} - \frac{1}{2}e\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho^i - \frac{1}{4}eF_{\mu\nu}F^{\mu\nu} \\ &\quad + \frac{1}{16}e\varepsilon^{ij}\bar{\psi}_\mu^i\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi_\nu^j\left(F^{\rho\sigma} + \hat{F}^{\rho\sigma}\right),\end{aligned} \tag{2.51}$$

where $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ is the field strength of the $U(1)$ gauge field B_μ and

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \frac{1}{2}\varepsilon^{ij}\bar{\psi}_\mu^i\psi_\nu^j \tag{2.52}$$

is the supercovariant field strength. ε^{ij} is the antisymmetric symbol with a component $\varepsilon^{12} = +1$. The scalar curvature $\hat{R}$ and the covariant derivative $\hat{D}_\mu$ are defined by using the spin connection $\hat{\omega}_{\mu ab}$ given by (2.23) with the replacements $\bar{\psi}_a\gamma_\mu\psi_b \to \bar{\psi}_a^i\gamma_\mu\psi_b^i$, etc. The covariant derivative does not contain a minimal coupling to the $U(1)$ gauge field B_μ. This means that the Rarita–Schwinger fields do not have a non-zero $U(1)$ charge. The coupling of the vector field and the Rarita–Schwinger fields is given by the last term of (2.51), which is called the Pauli term.

The Lagrangian (2.51) is invariant up to total divergences under the general coordinate transformation, the local Lorentz transformation and the $U(1)$ gauge transformation

$$\delta_g e_\mu{}^a = 0, \quad \delta_g B_\mu = \partial_\mu\zeta, \quad \delta_g\psi_\mu^i = 0. \tag{2.53}$$

It is also invariant under the $\mathscr{N} = 2$ local supertransformation

$$\begin{aligned}\delta_Q e_\mu{}^a &= \frac{1}{4}\bar{\varepsilon}^i\gamma^a\psi_\mu^i, \qquad \delta_Q B_\mu = \frac{1}{2}\varepsilon^{ij}\bar{\varepsilon}^i\psi_\mu^j, \\ \delta_Q\psi_\mu^i &= \hat{D}_\mu\varepsilon^i - \frac{1}{8}\varepsilon^{ij}\gamma^{\rho\sigma}\gamma_\mu\varepsilon^j\hat{F}_{\rho\sigma},\end{aligned} \tag{2.54}$$

where the transformation parameters are two Majorana spinors $\varepsilon^i(x)$ ($i = 1, 2$). The commutation relation of two local supertransformations is

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_G(\xi) + \delta_L(\lambda) + \delta_g(\zeta) + \delta_Q(\varepsilon), \tag{2.55}$$

where the transformation parameters on the right-hand side are

$$\xi^\mu = \frac{1}{4}\bar{\varepsilon}_2^i \gamma^\mu \varepsilon_1^i\,, \quad \lambda_{ab} = -\xi^\mu \hat{\omega}_{\mu ab} + \frac{1}{16}\varepsilon^{ij}\bar{\varepsilon}_2^i \gamma_{[a}\gamma^{\mu\nu}\gamma_{b]}\varepsilon_1^j \hat{F}_{\mu\nu},$$
$$\zeta = \frac{1}{2}\varepsilon^{ij}\bar{\varepsilon}_2^i \varepsilon_1^j - \xi^\mu B_\mu, \quad \varepsilon^i = -\xi^\mu \psi_\mu^i. \tag{2.56}$$

To obtain (2.55) we have to use the Rarita–Schwinger field equation.

This theory is called $\mathcal{N} = 2$ Poincaré supergravity since its field equations have a Minkowski spacetime solution $e_\mu{}^a = \delta_\mu^a$, $B_\mu = 0$, $\psi_\mu^i = 0$. For this background the commutator algebra of the local symmetry transformations reduces to the $\mathcal{N} = 2$ super Poincaré algebra. The $U(1)$ transformation $\delta_g(\zeta)$ on the right-hand side of (2.55) corresponds to the central charge in (2.48).

Global $U(2)$ Symmetry

This theory has a global $U(2)$ symmetry in addition to the above local symmetries [3]. This symmetry is an analog of the global $U(1)$ symmetry (2.31) of the $\mathcal{N} = 1$ theory. $SU(2)$ in $U(2) \sim SU(2) \times U(1)$ is a symmetry of the Lagrangian but the remaining $U(1)$ is a symmetry of the field equations.

The Lagrangian (2.51) is invariant under the global transformations with real constant parameters Σ^{ij}, Λ^{ij} satisfying $\Sigma^{ij} = -\Sigma^{ji}$, $\Lambda^{ij} = \Lambda^{ji}$, $\Lambda^{ii} = 0$:

$$\delta e_\mu{}^a = 0, \quad \delta B_\mu = 0, \quad \delta\psi_\mu^i = \left(\Sigma^{ij} + \mathrm{i}\Lambda^{ij}\gamma_5\right)\psi_\mu^j. \tag{2.57}$$

These transformations form the group $SU(2)$. To see this, we decompose ψ_μ into the chirality eigenstates $\psi_{\mu\pm}^i = \frac{1}{2}(1 \pm \gamma_5)\psi_\mu^i$. The transformation of the positive chirality component is $\delta\psi_{\mu+}^i = \left(\Sigma^{ij} + \mathrm{i}\Lambda^{ij}\right)\psi_{\mu+}^j$. The 2×2 matrix $\Sigma + i\Lambda$ is a traceless anti-hermitian matrix and represents an infinitesimal $SU(2)$ transformation. The negative chirality component $\psi_{\mu-}^i$ is the charge conjugation of $\psi_{\mu+}^i$.

This theory also has a global $U(1)$ symmetry. This $U(1)$ is not a symmetry of the Lagrangian or the action but a symmetry of the field equations. The field equations of the vector field can be written as

$$\partial_\mu(e * G^{\mu\nu}) = 0, \qquad \partial_\mu(e * F^{\mu\nu}) = 0, \tag{2.58}$$

where $*$ in $*F^{\mu\nu}$ and $*G^{\mu\nu}$ is the Hodge dual of second rank antisymmetric tensors defined as

$$*F^{\mu\nu} = \frac{1}{2}e^{-1}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{2.59}$$

$*G^{\mu\nu}$ in the first equation of (2.58) is defined by

$$*G^{\mu\nu} = \frac{2}{e}\frac{\partial\mathcal{L}}{\partial F_{\mu\nu}} = -F^{\mu\nu} + \frac{1}{4}\varepsilon^{ij}\bar{\psi}^{\rho i}\gamma_{[\rho}\gamma^{\mu\nu}\gamma_{\sigma]}\psi^{\sigma j}. \tag{2.60}$$

The first equation of (2.58) is the Euler equation derived from the Lagrangian, and the second one is the Bianchi identity, which implies that $F_{\mu\nu}$ can be expressed by the potential B_μ. These two equations correspond to Maxwell's equations of electromagnetism. Since the two equations in (2.58) have the same form, they are invariant under general linear transformations of $(F_{\mu\nu}, G_{\mu\nu})$. However, we should note that $F_{\mu\nu}$ and $G_{\mu\nu}$ are not independent, but are related by (2.60). Taking account of this relation the symmetry of (2.58) is the invariance under the $U(1)$ transformation with a real constant parameter Λ:

$$\delta e_\mu{}^a = 0, \quad \delta \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & \Lambda \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix}, \quad \delta\psi_\mu^i = \frac{1}{2}\mathrm{i}\Lambda\gamma_5\psi_\mu^i. \tag{2.61}$$

The definition of $G_{\mu\nu}$ in (2.60) and the field equations of $e_\mu{}^a$, ψ_μ^i are also invariant under (2.61). The second equation in (2.61) represents an interchange of $F_{\mu\nu}$ and $G_{\mu\nu} = *F_{\mu\nu} + \cdots$, i.e., an interchange of the electric field and the magnetic field. In general, a symmetry which exchanges the field equation and the Bianchi identity as in (2.61) is called the duality symmetry. The duality symmetry also appears in other even-dimensional supergravities and plays a crucial role in applications to string theory. We will discuss the duality symmetry in detail in Sect. 4.2.

Relation to the $\mathscr{N} = 1$ Theory

The Lagrangian of $\mathscr{N} = 1$ supergravity (2.21) can be obtained from that of $\mathscr{N} = 2$ supergravity (2.51) by imposing the conditions $B_\mu = 0$, $\psi_\mu^2 = 0$. Under the supertransformation with the parameter ε^1 in (2.54) these conditions are preserved and the remaining fields $e_\mu{}^a$, ψ_μ^1 transform as in the $\mathscr{N} = 1$ transformation (2.28). Thus, we have obtained the $\mathscr{N} = 1$ theory from the $\mathscr{N} = 2$ theory.

In general, we can obtain the $\mathscr{N}'$-extended theory from the $\mathscr{N}$-extended theory ($\mathscr{N} > \mathscr{N}'$) by a truncation, i.e., by putting a certain set of the fields equal to zero. The truncation must be consistent with the field equations and the supertransformation. The field equations of the fields which we put to zero must be automatically satisfied. Furthermore, the $\mathscr{N}'$-extended supertransformation of the fields which we put to zero must automatically vanish.

2.8 $\mathscr{N} = 2$ Anti de Sitter Supergravity

We can introduce a cosmological term to $\mathscr{N} = 2$ supergravity [6, 7]. The Lagrangian is

$$\begin{aligned} \mathscr{L} &= e\hat{R} + 6m^2 e - \frac{1}{2}e\bar{\psi}_\mu^i\gamma^{\mu\nu\rho}\hat{D}_\nu\psi_\rho^i + \frac{1}{2}me\bar{\psi}_\mu^i\gamma^{\mu\nu}\psi_\nu^i \\ &\quad - \frac{1}{4}eF_{\mu\nu}F^{\mu\nu} + \frac{1}{16}e\varepsilon^{ij}\bar{\psi}_\mu^i\gamma^{[\mu}\gamma_{\rho\sigma}\gamma^{\nu]}\psi_\nu^j\left(F^{\rho\sigma} + \hat{F}^{\rho\sigma}\right). \end{aligned} \tag{2.62}$$

As in the $\mathcal{N} = 1$ case the cosmological term with a negative cosmological constant and the mass term of the Rarita–Schwinger fields are added. In addition to these modifications the covariant derivative on the Rarita–Schwinger fields

$$D_{[\mu}\psi^i_{\nu]} = \left(\partial_{[\mu} + \frac{1}{4}\hat{\omega}_{[\mu}{}^{ab}\gamma_{ab}\right)\psi^i_{\nu]} + \frac{1}{2}m\varepsilon^{ij}B_{[\mu}\psi^j_{\nu]} \tag{2.63}$$

contains a minimal coupling to the $U(1)$ gauge field B_μ, which is not present in the $m = 0$ theory. This corresponds to a gauging of an $SO(2)$ subgroup of the global symmetry $U(2)$ (the Σ^{ij} transformation in (2.57)) of the $m = 0$ theory. The gauge coupling constant is $g = \frac{1}{2}m$ and is proportional to the parameter m appearing in the cosmological term and the Rarita–Schwinger mass term.

This Lagrangian is invariant up to total divergences under the general coordinate transformation, the local Lorentz transformation and the $U(1)$ gauge transformation

$$\delta_g e_\mu{}^a = 0, \quad \delta_g B_\mu = \partial_\mu\zeta, \quad \delta_g\psi^i_\mu = -\frac{1}{2}m\zeta\varepsilon^{ij}\psi^j_\mu. \tag{2.64}$$

It is also invariant under the $\mathcal{N} = 2$ local supertransformation

$$\begin{aligned}\delta_Q e_\mu{}^a &= \frac{1}{4}\bar{\varepsilon}^i\gamma^a\psi^i_\mu, \qquad \delta_Q B_\mu = \frac{1}{2}\varepsilon^{ij}\bar{\varepsilon}^i\psi^j_\mu,\\ \delta_Q\psi^i &= \hat{D}_\mu\varepsilon^i + \frac{1}{2}m\gamma_\mu\varepsilon^i - \frac{1}{8}\varepsilon^{ij}\gamma^{\rho\sigma}\gamma_\mu\varepsilon^j\hat{F}_{\rho\sigma}.\end{aligned} \tag{2.65}$$

The covariant derivative on ε^i contains a minimal coupling to B_μ as in (2.63). The commutation relation of two local supertransformations has the same form as (2.55). The parameters on the right-hand side are the same as (2.56) except that the local Lorentz transformation parameter has the additional term $-\frac{1}{4}m\bar{\varepsilon}_2{}^i\gamma_{ab}\varepsilon^i_1$. The global $U(2)$ symmetry of the $m = 0$ theory is broken by the coupling to the gauge field since only $U(1)$ part of $U(2)$ was gauged.

This theory is called $\mathcal{N} = 2$ anti de Sitter (AdS) supergravity since its field equations have an AdS spacetime solution. It is also called $\mathcal{N} = 2$ gauged supergravity since it has the minimal coupling to the gauge field.

2.9 $\mathcal{N} \geq 3$ Supergravities

Similarly, $\mathcal{N} = 3,\ 4,\ 5,\ 6,\ 8$ extended supergravities can be constructed. Let us have a quick look at these theories.

$\mathcal{N} = 3$ Poincaré supergravity is similar to the $\mathcal{N} = 2$ theory. The field content is a gravitational field, three Majorana Rarita–Schwinger fields, three vector fields and a Majorana spinor field. The action is invariant under the general coordinate transformation, the local Lorentz transformation, the $U(1)^3$ gauge transformation and the $\mathcal{N} = 3$ local supertransformation. The field equations are invariant under the global $U(3)$ transformation, which includes the duality transformation of the vector fields.

One can also construct $\mathcal{N} = 3$ gauged (AdS) supergravity. The Lagrangian has a cosmological term and a mass term of the Rarita–Schwinger fields as in (2.42) of the $\mathcal{N} = 1$ theory. A new feature of the $\mathcal{N} = 3$ theory is that the three vector fields become the non-Abelian $SO(3)$ Yang–Mills field. This corresponds to a gauging of a subgroup $SO(3)$ of the global $U(3)$ in the ungauged theory. The Rarita–Schwinger fields have a minimal coupling to the Yang–Mills field. The gauge coupling constant is $g = \frac{1}{2}m$ as in the $\mathcal{N} = 2$ theory.

$\mathcal{N} \geq 4$ Poincaré supergravities contain scalar fields and are much different from the $\mathcal{N} < 4$ theories. The field content is a gravitational field, $\mathcal{N}$ Majorana Rarita–Schwinger fields, $\frac{1}{2}\mathcal{N}(\mathcal{N}-1)$ vector fields and a certain number of scalar and spinor fields. The scalar fields have non-polynomial interactions. The action is invariant under the general coordinate transformation, the local Lorentz transformation, the $U(1)^{\frac{1}{2}\mathcal{N}(\mathcal{N}-1)}$ gauge transformation and the $\mathcal{N}$ local supertransformation.

A significant feature of the $\mathcal{N} \geq 4$ theories is a global symmetry of a non-compact Lie group G. For instance, the $\mathcal{N} = 4$ theory has a global $G = SU(4) \times SU(1,1)$ symmetry. This symmetry is an analog of $U(\mathcal{N})$ symmetry in the $\mathcal{N} \leq 3$ theories. The $U(1)$ subgroup of $U(4) = SU(4) \times U(1)$ is enlarged to the non-compact group $SU(1,1)$. This G symmetry is a symmetry of the field equations since it contains the duality transformation. In ordinary field theories like the standard theory of particle physics the internal symmetry is a compact Lie group. However, $\mathcal{N} \geq 4$ supergravities can have a non-compact internal symmetry since they contain scalar fields represented by a non-linear sigma model. The non-linear sigma model is a theory of scalar fields which take values in a coset space G/H, where G is a Lie group and H is a subgroup of G. Originally, non-linear sigma models were introduced in order to describe massless Nambu–Goldstone bosons when a symmetry G is spontaneously broken to H. In supergravities G is a non-compact group and H is a maximal compact subgroup of G. We will discuss non-linear sigma models and non-compact symmetries in Chap. 4.

One can also construct gauged supergravities for $\mathcal{N} \geq 4$, which contain minimal couplings to the vector fields. The $\frac{1}{2}\mathcal{N}(\mathcal{N} - 1)$ vector fields in the theory become the Yang–Mills field with the gauge group $SO(\mathcal{N})$, which is a subgroup of the non-compact symmetry G of the ungauged theory. (It is also possible to gauge other (non-compact) subgroup of G.) The Lagrangian contains a potential term of the scalar fields $-g^2 eV(\phi)$ and Yukawa couplings of the fermionic fields and the scalar fields such as $gef(\phi)\bar{\psi}_\mu \gamma^{\mu\nu} \psi_\nu$. Here, g is the gauge coupling constant of the Yang–Mills field. These terms effectively become a cosmological term and fermion mass terms when the scalar fields have vacuum expectation values for which $V(\phi)$ and $f(\phi)$ are non-zero. Then, the field equations have an AdS spacetime solution. We will discuss gauged supergravities in Chap. 7.

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http://www.springer.com/978-4-431-54827-0

Introduction to Supergravity
Tanii, Y.
2014, IX, 130 p., Softcover
ISBN: 978-4-431-54827-0