

# Chapter 2

## The Photon

Theoretical analyses toward building up a semiconductor laser theory are start from this chapter. This book is aimed not require to read other books for understanding basis of the theory. When the readers wish to refer other books to compare our treatments with them, bibliographies are listed in the end of the last chapter.

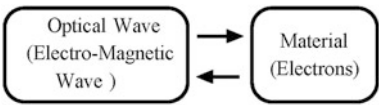
### 2.1 Analytical Approaches

Laser operation is based on the interaction between an optical wave and the semiconducting material, or more specifically between an electromagnetic (EM) wave and electron-hole pairs in the semiconductor as depicted in Fig. 2.1. As already discussed in Sect. 1.1, these interactions are classified according to whether they give rise to optical absorption, stimulated emission, or spontaneous emission. The accompanying noise is understood as a summed effect of fluctuations occurring from these interactions.

Nonetheless, the analytical method of any branch of physics is grounded in *classical mechanics* and *quantum mechanics*. The analytical approaches to these interactions are classified into three types as listed in Table 2.1. The first, called *full quantization*, treats both the optical wave and the material using quantum mechanics. The second, called *semi-classical*, continues to treat the material quantum mechanically but treats the optical wave using classical mechanics. The third, called *fully classical*, treats both optical wave and material using classical mechanics.

If we use classical mechanics to analyze the electron dynamics in the material, we can describe optical absorption by the material, but cannot explain stimulated emission. Spontaneous emission can be theoretically described only using a full quantization approach. The term *photon* is given to the massless particle representing a quantum of light following the *quantization* of the optical wave. However, results using quantized optical fields do not always match those from conventional treatments in electronics originally founded on classical mechanics,

**Fig. 2.1** Interaction between optical wave and material



**Table 2.1** Relationships between the analytical approach and phenomenological treatment

Analytical approach	Mechanics		Phenomena			
	Optical wave	Material	Optical absorption	Stimulated emission	Spontaneous emission	Noise
Full quantization	Quantum	Quantum	Theoretical	Theoretical	Theoretical	Theoretical
Semi-classical	Classical	Quantum	Theoretical	Theoretical	Phenomeno-logical	Phenomeno-logical
Fully classical	Classical	Classical	Theoretical	Invalid	Phenomeno-logical	Phenomeno-logical

such as EM theory, circuit theory, control theory, and signal processing theory. The classical mechanics require us simultaneous representation of the energy and the vibrating phase. From a quantum mechanical viewpoint, there exists an *uncertainty* between the energy and vibrational phase of an optical wave as will be shown in later. The word of *temporal coherency* originates from classical mechanics not from the quantum mechanics. Therefore, in many cases, we apply a semi-classical approach in which the optical wave is analyzed using the classical EM equations of Maxwell. Spontaneous emission and noise are introduced via phenomenological equations aided by results obtained from quantizing the optical wave.

2.2 Principles of Classical Mechanics

Many people find quantum mechanics is challenging to understand and to envisage. This stems from the different logical framework of quantum mechanics from that of Newtonian mechanics. Classical and quantum mechanics are compared in Table 2.2. There are two types of mathematical equations used to describe dynamics. One is a theory based on differential equations corresponding to *causal relationships*. The other is a theory with integral equations corresponding to a *variational principle*. Even in classical mechanics, Newtonian and analytical mechanics are founded on completely different logical frameworks. Let us review these frames of classical mechanics, before moving onto quantum mechanics.

In *Newtonian mechanics*, it is postulated that the rate of change of momentum is given by a force; force is the “cause” resulting in a momentum change, the

**Table 2.2** Logical frameworks and classification of mechanics

	Theory with differential equation (causal relationship)	Theory with integral equation (variational principle)	Reproducibility
Classical	Newtonian mechanics	Analytical mechanics	Assumed
Quantum	?	Quantum mechanics	Uncertain

“effect”. This postulate can be described mathematically using a differential equation

$$\frac{dp}{dt} = F, \quad (2.1)$$

where  $p$  is the momentum of a material object and  $F$  the force applied to that object. In Newtonian mechanics, the relation  $p = m \, dx/dt$  is *presupposed*, where  $x$  and  $m$  are the position and mass of the object, respectively. The Maxwell EM equations are also described in classical theory using differential equations.

In *classical analytical mechanics*, the relation between  $x$  and  $p$  is supposed to be *unknown initially*. The postulate of this mechanics is that the relation between  $x$  and  $\dot{x}$  is determined by the *principle of least action*, i.e., the minimization of the value of the action  $J$  defined as

$$J = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt, \quad (2.2)$$

with the *Lagrangian*

$$L(x, \dot{x}, t) = T(x, \dot{x}, t) - U(x, t), \quad (2.3)$$

where  $T(x, \dot{x}, t)$  is the kinetic energy,  $U(x, t)$  the potential energy, and

$$\dot{x} = \frac{\partial x}{\partial t} \quad (2.4)$$

is the velocity. In classical analytical mechanics, the relation between  $x$  and  $\dot{x}$  is also unknown initially. To find this relation through (2.2) is the fundamental objective of classical analytical mechanics. The momentum  $p$  is defined as

$$p = \frac{\partial L(x, \dot{x}, t)}{\partial \dot{x}}, \quad (2.5)$$

With  $p$  called the *conjugate* of  $x$ . The *classical Hamiltonian*  $H(x, p, t)$  is defined using  $\dot{x}$ ,  $p$ , and  $L(x, \dot{x}, t)$  as

$$H(x, p, t) = \dot{x}p - L(x, \dot{x}, t) = T(x, \dot{x}, t) + U(x, t). \quad (2.6)$$

The Hamiltonian determines the total energy of the mechanical system.

The Lagrangian  $L(x, \dot{x}, t)$  and Hamiltonian  $H(x, p, t)$  are called generating functions. Physical quantities such as  $x$  and  $p$  are obtained by differentiating the classical Hamiltonian:

$$\dot{p} = -\frac{\partial H(x, p, t)}{\partial x}, \quad (2.7)$$

$$\dot{x} = \frac{\partial H(x, p, t)}{\partial p}. \quad (2.8)$$

Note that in classical analytical mechanics the physical quantities  $x$  and  $p$  have been introduced as independent parameters. The relation  $p = m dx/dt$  is obtained as a *derived result of classical analytical mechanics*.

As given in the above discussion, the basic postulate and the logical framework of classical analytical mechanics differ completely from those of Newtonian mechanics.

Here, we find a situation in (2.2) that is strange. The lower limit  $t_1$  of the integration corresponds to the starting time of the dynamics which is in the *past*, whereas the upper limit  $t_2$  corresponds to the finishing time of the dynamics which must be in the *future*. Can we know the future? The answer to this question in classical mechanics is “Yes”. The common assumption in classical mechanics is that any dynamics is *reproducible* for temporal variation. Hence, both Newtonian mechanics and classical analytical mechanics are consistent and compatible with each other. However, we know through personal experiences that nature and human life are never reproducible.

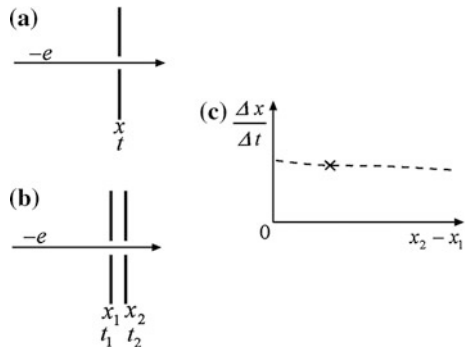
Because the fundamental equation (2.1) in Newtonian mechanics is given for the instantaneous time *now*, everyone can readily accept its outcomes. However, everyone might not accept the outcomes from classical analytical mechanics because (2.2) requires us to suppose that our *future* is reproducible.

Even in Newtonian mechanics, we can predict our future by integrating (2.1) into the future, if that is what we wish to know. The key concept underlying classical mechanics is the assumption of *reproducibility*.

Here, we present an example illustrating the logical difficulty in classical mechanics without the assumption of *reproducibility*. We consider an experiment to measure position and velocity of an electron moving in a vacuum using metal slits, as depicted in Fig. 2.2. When an electron passes a slit set at position  $x$ , the induced electric current in the slit rises to a peak and falls away. We can then fix a time  $t$  when the electron passes this slit, as sketched in Fig. 2.2a. Here, we also suppose that the sensitivity and resolution of the measuring system are sufficient enough to register sharp measurements.

Meanwhile, the instantaneous velocity  $v$  of the moving electron as specified within Newtonian mechanics is

**Fig. 2.2** Example to show the relationship between reproducibility and uncertainty. **a** Position measurement. **b** Velocity measurement. **c** Variation of measured data



$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}. \quad (2.9)$$

Thus, we need to prepare two slits, as shown in Fig. 2.2b, to measure the difference in position and the interval of time

$$\Delta x = x_2 - x_1, \quad (2.10)$$

$$\Delta t = t_2 - t_1. \quad (2.11)$$

We can then plot point  $(x_2 - x_1)/(t_2 - t_1)$  on the graph of  $\Delta x/\Delta t$  versus  $x_2 - x_1$ , as illustrated in Fig. 2.2c. Another requirement in determining the velocity is that we need to repeat the measurement to be able to obtain the extrapolated point  $\lim_{\Delta t \rightarrow 0} \Delta x/\Delta t = \lim_{\Delta x \rightarrow 0} \Delta x/\Delta t$  by reducing the distance  $x_2 - x_1$  between the two slits, as shown in Fig. 2.2c. Here, we pose the following questions:

Is the experiment reproducible? If we repeat the experiment, can we get exactly the same result? Is there any proof that the extrapolated point  $\lim_{\Delta t \rightarrow 0} \Delta x/\Delta t = \lim_{\Delta x \rightarrow 0} \Delta x/\Delta t$  converges to a unique point?

Most people might respond to these questions using a statistical treatment, for example, to repeat the experiment many times and to take an averaged value. However, statistical treatments are only supported on the *assumption of reproducibility*.

If we suppose that we can do the experiment only once and repeating the experiment is not permitted, what sorts of logical differences will be generated. The aim of the experiment is to measure the position and the velocity simultaneously.

We need to use two slits, because the velocity cannot be measured by one slit only. We might set the slit distance  $x_2 - x_1$  as small as possible to obtain  $(x_2 - x_1)/(t_2 - t_1)$ . However, this value is not exactly the value  $v \equiv \lim_{\Delta t \rightarrow 0} \Delta x/\Delta t = \lim_{\Delta x \rightarrow 0} \Delta x/\Delta t$ , because  $t_2 - t_1$  is still nonzero albeit small. The next question is

whether we can ascertain the position  $x$  of the electron exactly when the electron has velocity  $v$ . We can only say that the electron is present in the space between  $x_1$  and  $x_2$  during the time interval  $t_1 - t_2$ . Thus, we cannot measure precisely the position  $x$  and velocity  $v$  simultaneously from only a single measurement. An accurate measurement of the position requires an instantaneous time  $t$ , while in principle the measurement of velocity requires two instantaneous times  $t_1$  and  $t_2$ . Such incompatible requirements are at the root of *uncertainty* in quantum mechanics as we shall explain in the following sections.

The reader should bear in mind that classical mechanics is established under the assumption of reproducibility.

### 2.3 Principles of Quantum Mechanics

The basic philosophy of quantum mechanics is different from that of classical mechanics. Quantum mechanics does not suppose reproducibility for the temporal variation of physical quantities but permits uncertainty in the dynamics. The rules and postulates of quantum mechanics are summarized in four axioms:

- (1) Any *physical quantity* is expressed as an expectation value of an *operator*  $A$  and a *state*  $|\Psi\rangle$  as  $\langle A \rangle = \langle \Psi | A | \Psi \rangle$ , where  $\langle A \rangle$  is said to be the *expectation value* of the physical quantity.
- (2) The temporal variation of any physical quantity is *not* given directly by the dynamics of the *physical quantity* itself, but is given by the dynamics of the quantum mechanical *operator* or *state*.
- (3) The temporal variation of the quantum mechanical operator or state is determined from one of the following equations:

- the *Schrödinger equation* for arbitrary state  $|\Psi\rangle$ ,

$$j\hbar \frac{d|\Psi\rangle}{dt} = H |\Psi\rangle, \quad (2.12)$$

where  $j$  is the imaginary unit and  $H$  is the *quantum mechanical Hamiltonian*.

- the *Heisenberg equation* for arbitrary operator  $A$ ,

$$\frac{dA}{dt} = \frac{1}{j\hbar} [A, H] = \frac{1}{j\hbar} (AH - HA), \quad (2.13)$$

where  $[, ]$  denotes the *commutator* of the two arguments.

- the *commutation relation* for *canonical conjugate operators*,

$$[q, p] = qp - pq = j\hbar. \quad (2.14)$$

The applications of these three equations are different in mathematics. However, their intrinsic characteristics are the same. We are able to develop quantum mechanical properties by applying any one of these equations.

- (4) Any observable physical quantity is represented by a *real number*. Introducing the imaginary unit into quantum mechanics aids the handling of the mathematics more simply. The *imaginary unit*  $j$  (sometimes denoted  $i$ ) merely indicates  $\pi/2$  preceding a *phase variation* for periodically vibrating phenomena.

Let us recall some of the mathematical structure of quantum mechanics:

We introduce  $\langle\Psi|$  and  $|\Psi\rangle$ , called the *bra vector* and *ket vector* respectively, to denote the states of a quantum mechanical system. For each operator  $A$  acting on the ket vector,  $A|\Psi\rangle$ , a *conjugate operation*  $A^+$  is defined that acts on the bra vector,  $\langle\Psi|A^+$ , where the symbol  $^+$  is read as *dagger*. When the relation  $A^+ = A$  holds, this operator is called a *Hermite operator* or *Hermitian*. The expectation value  $\langle A \rangle = \langle A^+ \rangle$  of a Hermite operator is always a *real number*.

## 2.4 Procedure for Quantization

Because any dynamic behavior can be derived from the quantum mechanical Hamiltonian  $H$ , quantum mechanics is categorized as a *theory with an integral equation*, as stated in Table 2.2. Even though the Schrödinger equation and the Heisenberg equation are given in the form of differential equations, these equations do not imply a causal relationship. These equations describe the dynamics of states and operators, and not of physical quantities. Indeed, theories based on causal relations, such as Newton's dynamical equation and Maxwell's equations, are easy to understand. Unfortunately, there is no direct theory to treat the quantum mechanical properties based on causal relationships. As we shall show in Chaps. 5 and 6, the density matrix equation and the rate equations are methods to enable causal relations to be treated by taking into account the quantum mechanical properties in the form of differential equations. However, these are *skillfully modified equations* to cover the weaknesses of quantum mechanics by allowing one to step out of the exact quantum mechanical framework. Therefore, the column of "Theory with differential equation" and "Quantum" in Table 2.2 is noted with " ? " mark.

A way of applying quantum mechanics is to trace and convert categories given in Table 2.2 using the following four steps:

1. Formulate classical differential equations, and analyze them.
2. Calculate the energy associated with the classical Hamiltonian as an integral equation.

3. Regard the dynamic variables as quantum mechanical operators as well as the energy as the quantum mechanical Hamiltonian.
4. Analyze the quantum mechanical features using one of the equations (2.12)–(2.14).

These four steps embody the notion of quantizing a general classical system. Let us now quantize the optical wave by following these steps.

## 2.5 Classical Form of Maxwell's Equations

The first step towards quantization is to formulate classical differential equations and analyze them. Optical waves in the semiconductor laser propagate along the active region and are reflected back by a facet mirror at either end, thereby forming a standing wave in the laser cavity, as sketched in Fig. 1.14. In this chapter, we assume that reflectivities  $R_f$  and  $R_b$  of the two facet mirrors are 100 %, and there is neither gain nor loss in the laser cavity. That is, the entire optical wave is confined within the laser cavity in a steady state.

For EM waves without sources, Maxwell's equations are given as

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \quad (2.15)$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \quad (2.16)$$

$$\nabla \cdot \mathbf{E} = 0, \quad (2.17)$$

$$\nabla \cdot \mathbf{H} = 0, \quad (2.18)$$

where  $\mathbf{E}$  is the *electric field* vector,  $\mathbf{H}$  the *magnetic field* vector,  $\varepsilon$  the dielectric constant, and  $\mu$  the magnetic permeability. Here, we introduce a *vector potential*  $\mathbf{A}$  subject to the *Coulomb gauge condition*

$$\nabla \cdot \mathbf{A} = 0. \quad (2.19)$$

Both  $\mathbf{E}$  and  $\mathbf{H}$  are derived from the vector potential:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad (2.20)$$

$$\mu \mathbf{H} = \nabla \times \mathbf{A}. \quad (2.21)$$

Equations (2.20) and (2.21) automatically satisfy (2.17) and (2.18), respectively. By substituting (2.20) and (2.21) into (2.15) and (2.16) and using the well-known formula from vector calculus

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}, \quad (2.22)$$

we obtain the *wave equation* for the vector potential,

$$\nabla^2 \mathbf{A} = \varepsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2}. \quad (2.23)$$

We propose trial solutions of (2.23) of the form

$$\mathbf{A} = \sqrt{\frac{1}{\varepsilon}} q(t) \boldsymbol{\Phi}(x, y, z), \quad (2.24)$$

where  $\boldsymbol{\Phi}(x, y, z)$  is a *field distribution function* describing the standing wave in the laser cavity. The distribution function actually consists of three spatial components given by the vector

$$\boldsymbol{\Phi}(x, y, z) = \mathbf{e}_x \Phi_x(x, y, z) + \mathbf{e}_y \Phi_y(x, y, z) + \mathbf{e}_z \Phi_z(x, y, z), \quad (2.25)$$

where  $\mathbf{e}_x$ ,  $\mathbf{e}_y$ , and  $\mathbf{e}_z$  are independent unit vectors indicating the spatial directions. The distribution function is normalized to *unity* over the whole volume of the cavity, that is,

$$\iiint_{cav} |\boldsymbol{\Phi}(x, y, z)|^2 dx dy dz = 1. \quad (2.26)$$

The variable  $q(t)$  in (2.24) is the *temporal-varying amplitude* of the vector potential.

By substituting (2.24) into (2.23), the terms involving the spatial distribution and the temporal variation can be separated into two equations:

$$\nabla^2 \boldsymbol{\Phi}(x, y, z) = -\mu \varepsilon \omega^2 \boldsymbol{\Phi}(x, y, z), \quad (2.27)$$

$$\frac{\partial^2 q(t)}{\partial t^2} = -\omega^2 q(t). \quad (2.28)$$

Here, we should keep in mind that the parameter  $\omega$  is defined as an *eigenvalue* that connects these two equations. The mathematical solution to (2.28) is

$$q(t) = c e^{j\omega t} + c^* e^{-j\omega t}, \quad (2.29)$$

where  $c$  is an amplitude given with a complex number. The second term on the right-hand side of (2.29) is the complex conjugate of the first term, because  $q(t)$  itself should be a real number.

Any observable physical quantity should be a real number both in classical and quantum mechanics. Representations with complex numbers such as in (2.29) stem from the mathematical benefits of using imaginary numbers in the treatment of periodically vibrating phenomena. We find that the eigenvalue  $\omega$  can be interpreted as an angular frequency of the temporal variation.

The second step towards quantization is to calculate the energy corresponding to the classical Hamiltonian. The stored energy  $W$  of the optical wave in the laser cavity as obtained from (2.20), (2.21), and (2.24) is

$$W = \frac{1}{2} \iiint_{cav} (\epsilon \mathbf{E}^2 + \mu \mathbf{H}^2) dx dy dz = \frac{1}{2} (\dot{q}^2 + \omega^2 q^2), \quad (2.30)$$

where  $\dot{q}$  is the first derivative of  $q$  with respect to time  $t$ , that is,  $\dot{q} = \partial q / \partial t$  in classical mechanics. Equation (2.30) describes the stored energy of the optical wave and can be expressed in the same form as for a harmonic oscillator. If we suppose  $q(t) = q(0) \cos(\omega t)$ , which gives  $\dot{q}(t) = -\omega q(0) \sin(\omega t)$ , the time-averaged value of the energy  $\bar{W} = \omega^2 q(0)^2 / 2$  is obtained. In deriving (2.30), we used the formula

$$\iiint (\nabla \times \Phi)^2 dx dy dz = \iiint \Phi \{ \nabla \times (\nabla \times \Phi) \} dx dy dz, \quad (2.31)$$

together with (2.19), (2.22), (2.26), and (2.27).

## 2.6 Obtaining Quantum Mechanical Operators from Classical Physical Quantities

The third step towards quantization is to regard the dynamic variables as quantum mechanical operators. In particular, energy becomes the quantum mechanical Hamiltonian. As mentioned previously, the notion of a time derivative for any physical quantity is not *allowed* in quantum mechanics. We suppose that  $\dot{q}$  must correspond to the operator  $p$  which must satisfy (2.14) in relation to its *canonical conjugate* operator  $q$ :

$$q \rightarrow q, \quad (2.32)$$

$$\dot{q} \rightarrow p. \quad (2.33)$$

Operators  $q$  and  $p$  must be Hermitian because the expectation values of these operator should be real numbers. The stored energy  $W$  associated with the optical wave corresponds to the quantum mechanical Hamiltonian  $H$

$$W \rightarrow H. \quad (2.34)$$

Substitution of (2.32) and (2.33) into (2.30) yields the *quantum mechanical Hamiltonian* in the form

$$H = \frac{1}{2}(p^2 + \omega^2 q^2). \quad (2.35)$$

## 2.7 Quantization of the Optical Wave

The final step to quantization is to analyze the quantum mechanical behavior by applying one of (2.12)–(2.14). Here, we introduce two new operators  $a$  and  $a^+$  defined using  $q$  and  $p$ :

$$a = \sqrt{\frac{1}{2\hbar\omega}}(\omega q + jp), \quad (2.36)$$

$$a^+ = \sqrt{\frac{1}{2\hbar\omega}}(\omega q - jp). \quad (2.37)$$

Although  $q$  and  $p$  are Hermitian, the new operators  $a$  and  $a^+$  are not. Inverting the above relations,  $q$  and  $p$  can be expressed in terms of  $a$  and  $a^+$ :

$$q = \sqrt{\frac{2\hbar}{\omega}} \frac{a + a^+}{2}, \quad (2.38)$$

$$p = \sqrt{2\hbar\omega} \frac{a - a^+}{2j}. \quad (2.39)$$

By substituting (2.38) and (2.39) into (2.14), the commutation relation for  $a$  and  $a^+$  is found to be

$$[a, a^+] = aa^+ - a^+a = 1. \quad (2.40)$$

By substituting (2.38), (2.39), and (2.40) into (2.35), the Hamiltonian is rewritten with  $a$  and  $a^+$  in the form

$$H = \hbar\omega \left( a^+a + \frac{1}{2} \right). \quad (2.41)$$

Based on a more detailed analysis of the operators  $a$  and  $a^+$ , which is given in Appendix 1, it is known that the operator  $a$  acts on the states  $|s\rangle$  to reduce the number  $s$  by one and  $a^+$  acts to increase this number by one:

$$a|s\rangle = \sqrt{s}|s-1\rangle, \quad (2.42)$$

$$a^+|s\rangle = \sqrt{s+1}|s+1\rangle, \quad (2.43)$$

where  $|s\rangle$  is an eigenstate of operator  $a^+a$  corresponding to eigenvalue  $s$ , a positive integer. For this reason  $a$  is called an *annihilation operator* and  $a^+$  a *creation operator*. Derivations of (2.42) and (2.43) are rather tedious and are given in Appendix 1. From (2.42) and (2.43), we verify the relation

$$a^+a|s\rangle = s|s\rangle. \quad (2.44)$$

Hence, the *eigenequation* of the Hamiltonian is written as

$$H|s\rangle = \hbar\omega\left(s + \frac{1}{2}\right)|s\rangle. \quad (2.45)$$

This states that the eigenvalue  $W$  of the *stored energy* of the optical wave in the laser cavity is given by

$$W = \left(s + \frac{1}{2}\right)\hbar\omega. \quad (2.46)$$

Because the energy is characterized by a nonnegative integer  $s$ ,  $s$  is called the *photon number*. The constant  $\hbar\omega/2$  remains even if  $s = 0$ , and is called the *zero-point energy*. Spontaneous emission is induced by this zero-point energy as already explained in Chap. 1 and is proved in Appendix 5.

## 2.8 Remark on the Photon

It is worthwhile to recapitulate the manner in which the photon number was obtained. The vector potential  $\mathbf{A}$  of the optical wave is represented by the amplitude  $q(t)$  and spatial distribution  $\Phi(x, y, z)$ . Quantum mechanics is applied only to the *temporal-varying terms*. In particular, the stored energy or Hamiltonian of the optical wave in a laser cavity is represented by such terms. The photon number is derived via the eigenvalue for the stored energy in the whole cavity. Meanwhile, the spatial distribution of the optical wave  $\Phi(x, y, z)$  is defined over the entire volume of the laser cavity and is characterized by the eigenfunctions determined by (2.27). That means (2.27) is common to both classical mechanics and quantum mechanics, because the spatial distribution is time-independent.

Because the photon refers to a particle-like object, most people imagine that it is a spatially localized particle even in free space, and the output light emitted from a laser is like a volley from a machine gun. *This image is wrong.* When we obtain *discrete eigenvalues* for any physical quantity, we can associate this basic quantum unit to a physical characteristic of a *particle* for the respective phenomenon. The spatial distribution of the photon needs to be determined subject to the spatial boundary conditions as performed for any classical wave.

## 2.9 The Uncertainty Principle

One important feature of quantum mechanics is the uncertainty principle, which is given in the following theorem.

### Theorem

Given the commutation relation  $[A, B] = jC$ , the relation  $\Delta A \cdot \Delta B \geq \frac{1}{2} |\langle \Psi | C | \Psi \rangle|$  then holds.

Here,  $A$  and  $B$  are Hermite operators and  $C$  can be a real number or another Hermite operator.  $\Delta A$  and  $\Delta B$  are the uncertainty values corresponding to the respective operators and are defined as the “root mean square” of the expectation value:

$$\Delta A = \sqrt{\langle \Psi | (A - \bar{A})^2 | \Psi \rangle}, \quad (2.47)$$

$$\text{with } \bar{A} = \langle \Psi | A | \Psi \rangle, \quad (2.48)$$

$$\Delta B = \sqrt{\langle \Psi | (B - \bar{B})^2 | \Psi \rangle}, \quad (2.49)$$

$$\text{with } \bar{B} = \langle \Psi | B | \Psi \rangle. \quad (2.50)$$

Proof of this theorem is presented in Appendix 2. Here, we apply this theorem to several examples.

The first example concerns the *relation between the electric and magnetic fields*. The electric field  $E$  is given in classical mechanics by the time derivative of the vector potential as stated in (2.20) and (2.24). The time derivative of the amplitude  $q$  should be replaced with another operator  $p$  as supposed in (2.33). Then, the electric field  $E$  in quantum mechanics is written as

$$E = -\sqrt{\frac{1}{\epsilon}} p \Phi(x, y, z). \quad (2.51)$$

The magnetic field  $\mathbf{H}$  is given by the rotation (or curl) of the vector potential as formulated in (2.21). This operation is not a temporal variation but a spatial distribution. Hence, the magnetic field  $\mathbf{H}$  in quantum mechanics is written as

$$\mathbf{H} = \frac{1}{\mu\sqrt{\epsilon}} q \nabla \times \Phi(x, y, z). \quad (2.52)$$

Therefore, the commutation relation between the electric and magnetic fields is

$$[\mathbf{E}, \mathbf{H}] = -\frac{1}{\mu\epsilon} \Phi \cdot \nabla \times \Phi (pq - qp) = \frac{j\hbar}{\mu\epsilon} \Phi \cdot \nabla \times \Phi. \quad (2.53)$$

By applying the above theorem to (2.53), we obtain the uncertainty relation

$$\Delta \mathbf{E} \cdot \Delta \mathbf{H} \geq \frac{\hbar}{2\mu\epsilon} \Phi(x, y, z) \cdot \nabla \times \Phi(x, y, z). \quad (2.54)$$

Because the right-hand side is nonzero, this uncertainty relation implies that a simultaneous representation for both electric and magnetic fields is not allowed in quantum mechanics, although a simultaneous representation is essential in the classical mechanics. Therefore, the quantum mechanical treatment of the optical wave does not match with its conventional treatment in electronics which is based on classical mechanics.

The second example concerns the *relation between energy and time*. The Heisenberg equation has been given in (2.13). If we regard the operator  $A$  in (2.13) as time  $t$ , the commutation relation on the right-hand side is

$$\frac{1}{j\hbar} [A, H] = \frac{1}{j\hbar} [t, H]. \quad (2.55)$$

The left-hand side of (2.13) is the time derivative of  $t$ , giving the simple relation

$$\frac{dt}{dt} = 1. \quad (2.56)$$

With the expectation value of the Hamiltonian denoted by  $W$ , we obtain the uncertainty relation

$$\Delta W \cdot \Delta t \geq \frac{\hbar}{2}, \quad (2.57)$$

which means that the expectation value of an energy should be evaluated over a sufficiently broad time interval  $\Delta t$ .

A third example results by substituting (2.46) into (2.57) yielding

$$\Delta W \cdot \Delta t = \hbar \left\{ \Delta \omega \left( s + \frac{1}{2} \right) + \omega \Delta s \right\} \Delta t \geq \frac{\hbar}{2} \quad (2.58)$$

which means that, for *very short time intervals*  $\Delta t$ , *neither frequency  $\omega$  nor photon number  $s$  can be evaluated exactly.*

## 2.10 Coherent State

In (2.46), the photon number  $s$  is introduced as an eigenvalue associated with the eigenstate of the number operator  $a^+a$  to characterize the optical energy. Meanwhile, the operators corresponding to the electric field  $\mathbf{E}$  and magnetic field  $\mathbf{H}$  can be expressed in terms of  $a^+$  and  $a$  as derived from (2.38), (2.39), (2.51), and (2.52):

$$\mathbf{E} = j \sqrt{\frac{\hbar \omega}{2\epsilon}} (a - a^+) \mathbf{\Phi}(x, y, z), \quad (2.59)$$

$$\mathbf{H} = \frac{1}{\mu} \sqrt{\frac{\hbar}{2\epsilon \omega}} (a + a^+) \nabla \times \mathbf{\Phi}(x, y, z). \quad (2.60)$$

We need now to find the eigenstates of  $a^+$  and  $a$  to obtain the expectation values of the electric and magnetic fields. As also found in classical mechanics, energy is a time-independent quantity, whereas the electric and magnetic fields are time dependent quantities. Therefore, the representation of both these fields with finite photon number is in principle not simple.

For the eigenstates of  $a^+$  and  $a$ , the following equation for  $|\phi\rangle$  has been proposed and is called the *coherent state* or *Glauber state*,

$$|\phi\rangle = \sum_s |s\rangle \left( \frac{N^s e^{-N}}{s!} \right)^{1/2} \exp \left[ -j \left( s + \frac{1}{2} \right) (\omega t + \theta) \right] \quad (2.61)$$

Here,  $N$  is the average photon number for this state given by

$$\langle \phi | a^+ a | \phi \rangle = \sum_s \frac{N^s e^{-N}}{s!} = N e^{-N} \sum_s \frac{N^{s-1}}{(s-1)!} = N. \quad (2.62)$$

The expectation values of the operators  $a^+$  and  $a$  are

$$\langle \phi | a^+ | \phi \rangle = \sqrt{N} \exp[j(\omega t + \theta)], \quad (2.63)$$

$$\langle \phi | a | \phi \rangle = \sqrt{N} \exp[-j(\omega t + \theta)]. \quad (2.64)$$

The electric and magnetic fields are then represented in this coherent state as

$$\langle \phi | \mathbf{E} | \phi \rangle = \left( \frac{2\hbar\omega N}{\varepsilon} \right)^{1/2} \sin(\omega t + \theta) \boldsymbol{\Phi}(x, y, z), \quad (2.65)$$

$$\langle \phi | \mathbf{H} | \phi \rangle = \frac{1}{\mu} \left( \frac{2\hbar N}{\varepsilon\omega} \right)^{1/2} \cos(\omega t + \theta) \nabla \times \boldsymbol{\Phi}(x, y, z). \quad (2.66)$$

These two expressions agree with the *classical EM fields*.

The energy of the optical wave calculated from these two expressions is

$$W = \frac{1}{2} \iiint_{cav} \left( \varepsilon \langle \phi | \mathbf{E} | \phi \rangle^2 + \mu \langle \phi | \mathbf{H} | \phi \rangle^2 \right) dx dy dz = \hbar\omega N. \quad (2.67)$$

By comparing this result with (2.46), we find that the energy derived from the field expectation values of (2.65) and (2.66) gives the energy component  $N\hbar\omega$  with the average photon number  $N$ . However, calculations from these field expectation values never produce the zero-point energy  $(1/2)\hbar\omega$ , which is derived by the direct quantization of the optical wave resulting in (2.46).

The distribution of the photon number in the coherent state follows the Poisson distribution

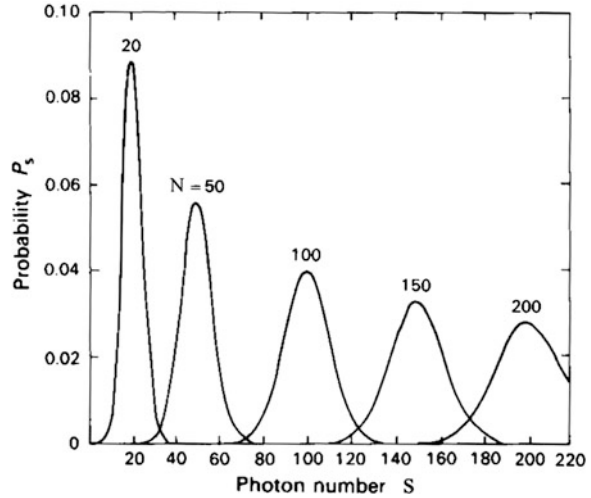
$$P_s \equiv |\langle s | \phi \rangle|^2 = \frac{N^s e^{-N}}{s!}, \quad (2.68)$$

which are illustrated in Fig. 2.3. This equation is interpreted as meaning that  $P_s$  is the probability to measure the photon number of the existing optical wave to be  $s$ . After repeating the measurement many times, the average value of the measured photon number approaches  $N$ .

## 2.11 Interaction Between the Optical Wave and a Charged Particle

The Hamiltonian of the optical wave has been derived from Maxwell's classical EM equations presented in (2.41). However, when a charged particle is present along with the optical wave, the dynamics of the particle will be affected by the optical wave. The motion of the charged particle is characterized by another Hamiltonian. The Hamiltonian for the charged particle and wave is of the form

**Fig. 2.3** Poisson distribution.  $S$  is the measured photon number and  $N$  is the averaged photon number after repeating the measurement many times



$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A})^2 + eU(\mathbf{r}), \quad (2.69)$$

where  $m$  and  $\mathbf{p}$  are the respective mass and momentum of the charged particle,  $e$  its charge, and  $\mathbf{A}$  and  $U$  are the respective vector and scalar potentials describing the optical wave. Equation (2.69) is valid for both classical and quantum mechanics. In quantum mechanics,  $\mathbf{p}$  and  $\mathbf{A}$  are treated as operators.

The validity of (2.69) is confirmed by applying Hamilton's principle in classical analytical mechanics to (2.69). We can derive a Newtonian dynamic equation for the charged particle experiencing a *Lorentz force* as

$$m\ddot{\mathbf{r}} = e[\mathbf{E} + \mu(\dot{\mathbf{r}} \times \mathbf{H})]. \quad (2.70)$$

The derivation of this equation is given in Appendix 3.

In quantum mechanics, the observable physical quantities such as  $\dot{\mathbf{r}}$  and  $\ddot{\mathbf{r}}$  should be obtained as expectation values based on the appropriate quantum mechanical expressions. Using (2.69) and the Schrödinger equation, we can also derive the dynamic equation for the expectation value of the charged particle corresponding to (2.70) as

$$m \frac{d^2 \langle \mathbf{r} \rangle}{dt^2} = e[\langle \mathbf{E} \rangle + \mu(\langle \mathbf{v} \rangle \times \langle \mathbf{H} \rangle)]. \quad (2.71)$$

The derivation of this equation is given in Appendix 4. Although we have derived a classical Newtonian-like dynamic equation from the Schrödinger equation, we should be careful in its application because it is given after the determination of the expectation values of  $\langle \mathbf{r} \rangle$ ,  $\langle \mathbf{v} \rangle$ ,  $\langle \mathbf{E} \rangle$  and  $\langle \mathbf{H} \rangle$ , under several assumptions and approximations, as explained in Appendix 4.

## 2.12 Analysis of Electron Transition Using a Fully Quantized Treatment

As mentioned in [Sect. 2.1](#), the analysis of the interaction between an optical wave and a material (electron) is classified according to three different approaches: fully classical, semi-classical and full quantized. For the fully classical and semi-classical approaches, the dynamics of the electron is represented via polarization, dielectric constant or conductivity, and is introduced into Maxwell's classical EM equations. An analysis of the laser is then performed by applying theories from material science and electronics.

In contrast, for the fully quantized approach, the interaction is analyzed via transition probabilities between energy states of the optical wave and the material. Although an energy conservation rule for the interaction can be derived, the representations of other physical quantities such as the vibrational phase or coherency of the optical wave become complicated. It might be helpful to emphasize that the concept of *coherent light* is founded in *classical mechanics*.

Therefore, the *modus operandi* of this book is that the development will be based mainly on a semi-classical approach. The required modification resulting from full quantization will be added when necessary.

An analysis of electron transition probability based on a fully quantized treatment is presented in Appendix 5. Readers who are more interested in these quantum mechanical aspects are encouraged to consult Appendix 5 for further details.

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