Chapter 2
Causal Dynamical Triangulation

2.1 Matter-Coupled CDT

In physics, when one encounters a recondite issue, it is often very effective to introduce a simple toy model. This ought to capture its nature and substance by getting rid of extraneous data. In the case of profound quantum gravity, one promising toy model is the 2-dimensional theory. A method called dynamical triangulation (DT) nicely expresses physics of the 2-dimensional Euclidean quantum gravity. Especially multicritical models of DT have been designed to describe the 2-dimensional Euclidean quantum gravity coupled to matters by Kazakov [1]. On this line of thought, Staudacher has succeeded in identifying the first multicritical point with a rational minimal conformal field theory characterized by the negative central charge, $c = -22/5$, coupled to the 2-dimensional Euclidean quantum gravity [2]. A possible way to mount the next step is to investigate a Lorentzian model for the 2-dimensional quantum gravity. One of the candidates is causal dynamical triangulation (CDT) [3]. In [4], we have proposed new multicritical models which naturally capture physics of the matter-coupled CDT in the continuum limit. In the following, we indict its essence.

2.1.1 Causal Random Geometries Coupled to Dimers

In Sect. 1.2.3, we have introduced the unrestricted triangulations as geometries discretized by any kinds of polygons. Following this thought, we introduce the potential:

$$V(z) = \frac{1}{2}z^2 - gz - \frac{g}{3}z^3 - \frac{g^3\xi}{2}z^4. \tag{2.1}$$

This potential generates geometries discretized by 1-gons, triangles and squares. Viewing each square as two triangles, one can think of the squares as part of the triangulation, but with a dimer placed on the diagonal, with a fugacity $\tilde{\xi} = g\xi$. In

Y. Sato, *Space-Time Foliation in Quantum Gravity*, Springer Theses,
DOI: 10.1007/978-4-431-54947-5_2,

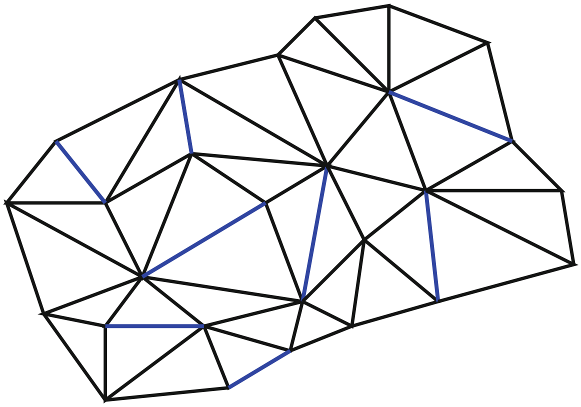

Fig. 2.1 Dimers (*blue edges*) on a triangulation

this way the model describes dimers put on random triangulations in a special way, such that there is at most one dimer per triangle. On the graph dual to the triangulation the dimers are precisely hard dimers: one dimer is allowed to be attached to each vertex at most (see Fig. 2.1). We will call them hard dimers also on the triangulation, even if the rule of putting down the dimers is slightly different from the standard hard dimer rule. Similarly we will denote ξ the fugacity of the dimers, although it is strictly speaking $\tilde{\xi}$ which serves as the fugacity. When considering the sphere topology and imposing the single-cut structure, the solution of the loop equation becomes

$$w(g,z) = \frac{1}{2g_s}\left(V'(z) - \sum_{k=1}^{3} M_k (z-a)^{k-1}\sqrt{(z-a)(z-b)}\right), \tag{2.2}$$

where $w(g,z)$ is the generating function for the boundary loop (sometimes called resolvent or disk function); g_s is the string coupling constant; (a,b) are end points of the cut. From now on we call $w(g,z)$ *resolvent*. From the asymptotic behavior of $w(g,z)$ in $|z| \gg |a-b|$, one obtains a set of equations:

$$M_3 = -2g^3\xi, \tag{2.3}$$

$$M_2 = -g + \frac{M_3}{2}(5a+b), \tag{2.4}$$

$$M_1 = 1 + \frac{M_2}{2}(3a+b) + \frac{M_3}{8}(b^2 - 10ab - 15a^2), \tag{2.5}$$

$$g_s = \frac{1}{16}\left[M_1(b-a)^2 + \frac{1}{2}M_2(b-a)^3 + \frac{5}{16}M_3(b-a)^4\right], \tag{2.6}$$

$$g = \frac{M_1}{2}(a+b) + \frac{M_2}{8}(b^2 - 6ab - 3a^2) + \frac{M_3}{16}(b^3 - 5ab^2 + 15a^2b + 5a^3). \quad (2.7)$$

Remember the moments (1.44). The moments can be written as follows:

$$\begin{aligned} M_k &= \oint_C \frac{d\omega}{2\pi i} \frac{V'(\omega)}{(\omega - a)^{k+1/2}(\omega - b)^{1/2}} \\ &= \oint_{\bar{C}} \frac{d\bar{\omega}}{2\pi i} \left[\frac{\bar{\omega}^{k-1}}{(1-\bar{\omega}a)^{k+1/2}(1-\bar{\omega}b)^{1/2}} \right] V'(1/\bar{\omega}), \end{aligned} \quad (2.8)$$

where $\bar{\omega} = 1/\omega$ and the path $\bar{C}$ enclosing 0 in the $\bar{\omega}$-patch. If $\xi > 0$, one can only set M_1 as zero. However, if $\xi < 0$, one can set not only M_1 but also M_2 as zero. The critical point characterized by $M_1 = M_2 = 0$ is an example of the multicritical points. Let us investigate the multicritical behavior of this model. To begin, we impose the condition for the multicritical point:

$$M_1 = M_2 = 0. \quad (2.9)$$

Plugging (2.9) into the Eqs. (2.3)–(2.7), we find

$$a = \frac{1}{6}\left[\left(-\frac{128 g_s}{5 g^3 \xi} \right)^{1/4} - \frac{1}{g^2 \xi} \right], \quad b = \frac{1}{6}\left[-5\left(-\frac{128 g_s}{5 g^3 \xi} \right)^{1/4} - \frac{1}{g^2 \xi} \right], \quad (2.10)$$

$$b^3 - 5ab^2 + 15a^2b + 5a^3 - 8(5a+b) = 0, \quad 4 = g^3 \xi (b^2 - 10ab - 15a^2). \quad (2.11)$$

For every value of g_s except for 0, approaching the critical point defined by (2.9) yields the Liouville field theory with the central charge $c = -22/5$. What we are interested in is the "CDT" point specified with $g_s = 0$. If one can find a non-trivial scaling relation around that point outside of the universality class of the plain CDT or the Liouville theory with the negative central charge, it turns out that one has succeeded in formulating CDT coupled to dimers. It has been achieved in [4]. Imposing the condition,

$$M_1 = M_2 = g_s = 0, \quad (2.12)$$

the critical values can be obtained (see Fig. 2.2):

$$a_* = b_* = \sqrt{3}, \quad g_* = \frac{1}{a_*}, \quad \xi_* = -\frac{a_*}{6}. \quad (2.13)$$

Then we renormalize the string coupling constant as follows:

$$g_s = G_s \varepsilon^4, \quad (2.14)$$

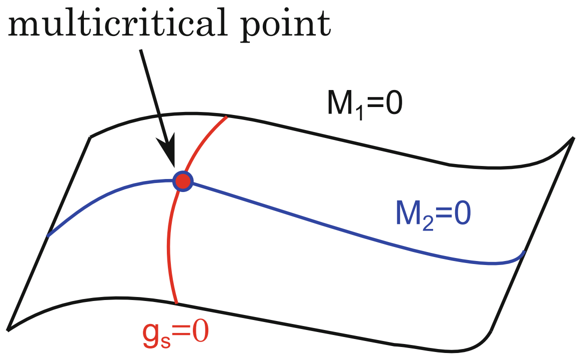

Fig. 2.2 Multicritical point in 3-dimensional coupling constant space (g, ξ, g_s)

where G_s is the renormalized string coupling constant and ε is the lattice spacing. Sitting on the critical line $M_1 = M_2 = 0$, one can expand g, ξ and a:

$$g_c(g_s) = g_* \left(1 - \frac{\sqrt{5}}{9} G_s^{1/2} \varepsilon^2 - \frac{16\sqrt{5}}{27} G_s^{3/4} \varepsilon^3 \right) + \mathscr{O}(\varepsilon^4), \tag{2.15}$$

$$\xi_c(g_s) = \xi_* \left(1 - \frac{\sqrt{5}}{9} G_s^{1/2} \varepsilon^2 + \frac{16\sqrt{5}}{27} G_s^{3/4} \varepsilon^3 \right) + \mathscr{O}(\varepsilon^4), \tag{2.16}$$

$$a_c(g_s) = a_* \left(1 + \frac{2}{3 \cdot 5^{1/4}} G_s^{1/4} \varepsilon \right) + \mathscr{O}(\varepsilon^2). \tag{2.17}$$

The perturbation away from $g_c(g_s)$, $\xi_c(g_s)$ which leads to the potential $V'(a, g, \xi)$ of order ε^3, assuming the boundary cosmological constant is perturbed as $z = a_c(g_s) + \varepsilon Z$, can be parametrized as

$$g = g_* + \tilde{\Lambda}\varepsilon^2 - \Lambda\,\varepsilon^3, \quad \xi = \xi_* - \frac{1}{2}\tilde{\Lambda}\varepsilon^2. \tag{2.18}$$

As in the ordinary multicritical model situation one finds a two-parameter set of solutions depending on Λ, $\tilde{\Lambda}$. Let us choose a convenient "background", using the notation from ordinary matrix models [5], which we call *CDT-background*, namely $\tilde{\Lambda} = 0$. By this choice of the background, one finds

$$\begin{aligned} V'(z; g, \xi) = &\left(\Lambda + \frac{1}{9} Z^3 + \frac{1}{3}\alpha Z^2 G_s^{1/4} + \frac{1}{3}\alpha^2 Z G_s^{1/2} \right) \varepsilon^3 \\ &+ G_s^{3/4} \left(\gamma + \frac{1}{9}\alpha^3 \right) \varepsilon^3 + \mathscr{O}(\varepsilon^4), \end{aligned} \tag{2.19}$$

where

$$\alpha = \frac{2}{5^{1/4}\sqrt{3}}, \quad \gamma = \frac{32\sqrt{5}}{27\sqrt{3}}. \tag{2.20}$$

When applying the CDT variables, $(Z_{\mathrm{cdt}}, \Lambda_{\mathrm{cdt}}, \tilde{\Lambda}_{\mathrm{cdt}})$, defined by

$$Z = Z_{\mathrm{cdt}} - \alpha G_s^{1/4}, \quad \Lambda = \Lambda_{\mathrm{cdt}} - \gamma G_s^{3/4}, \tag{2.21}$$

(2.19) becomes drastically simpler:

$$V'(z; g, \xi) = \left(\Lambda_{\mathrm{cdt}} + \frac{1}{9} Z_{\mathrm{cdt}}^3\right) \varepsilon^3 + \mathscr{O}(\varepsilon^4). \tag{2.22}$$

One may now calculate $w(g, z)$ in the CDT limit $G_s \to 0$ where any creation of baby universes is suppressed:

$$w(g, z) = \varepsilon^{-1} W_{\mathrm{CDT}}^{(3)}(Z_{\mathrm{cdt}}) + \mathscr{O}(\varepsilon^0), \tag{2.23}$$

where

$$W_{\mathrm{CDT}}^{(3)}(Z_{\mathrm{cdt}}) = \frac{1}{Z_{\mathrm{cdt}} + \Lambda_{\mathrm{cdt}}^{1/3}}. \tag{2.24}$$

Next, let us take a look at observables. A leading singularity of the free energy F at the critical point provides the so-called *string susceptibility*, γ_{str}:

$$F = N^2 f_0 (g - g_*)^{2-\gamma_{\mathrm{str}}} + \mathscr{O}(N), \tag{2.25}$$

where f_0 is some constant. The disk function (2.24) in the limit $Z_{\mathrm{cdt}} \to \infty$ is nothing but the genus-zero free energy with a marked point: because of the limit, the boundary loop shrinks to zero yielding the sphere with an infinitesimal boundary, a marked point [see (1.34)]; to mark a point on the sphere, one needs to act the derivative with respect to g. Picking up the first non-analytic structure of the disk function in the limit $Z_{\mathrm{cdt}} \to \infty$, one finds

$$W_{\mathrm{CDT}}^{(3)} \sim (g - g_*)^{1-\gamma_{\mathrm{str}}}, \tag{2.26}$$

where

$$\gamma_{\mathrm{str}} = \frac{2}{3}. \tag{2.27}$$

In the plain CDT, $\gamma_{\mathrm{str}} = 1/2$ [3]; one can find the same value in the so-called *branched polymer phase* of DT. On the other hand, (2.27) coincides with γ_{str} of the branched polymer coupled to dimers [4, 6]. One can compute another exponent called *edge singularity*, σ, defined as

$$\frac{d\log g_*}{d\xi} \sim (\xi - \xi_*)^\sigma. \tag{2.28}$$

From (2.18), one finds

$$\sigma = \frac{1}{2}. \tag{2.29}$$

This is the same as that of DT in the first multicritical point.[1]

2.1.2 Field Theory Qrising from CDT Scaling

We will show that the field-theoretic description can be obtained via the loop equation [7] (and implicitly shown in [4]). The loop equation with the potential (2.1) can be written as

$$\partial_z^2 \left(g_s w(g,z)^2 - V'(z) w(g,z) \right) = 4g^3\xi. \tag{2.30}$$

Plugging the scaling relations, (2.18) and (2.21), into the loop Eq. (2.30) and taking the limit, $\varepsilon \to 0$, one obtains

$$\partial_{Z_{\mathrm{cdt}}}^2 \left[G_s W_{\mathrm{GCDT}}^{(3)}(Z_{\mathrm{cdt}})^2 - \partial_{Z_{\mathrm{cdt}}} \left(\lim_{\varepsilon\to 0} \frac{V(z)}{\varepsilon^4} \right) W_{\mathrm{GCDT}}^{(3)}(Z_{\mathrm{cdt}}) \right] = -\frac{2}{9}, \tag{2.31}$$

where

$$\partial_{Z_{\mathrm{cdt}}} \left(\lim_{\varepsilon\to 0} \frac{V(z)}{\varepsilon^4} \right) = \Lambda_{\mathrm{cdt}} + \frac{1}{9} Z_{\mathrm{cdt}}^3 \equiv \partial_{Z_{\mathrm{cdt}}} \mathscr{V}(Z_{\mathrm{cdt}}); \tag{2.32}$$

$$w(g,z) = \varepsilon^{-1} W_{\mathrm{GCDT}}^{(3)}(Z_{\mathrm{cdt}}) + \mathscr{O}(\varepsilon^0). \tag{2.33}$$

This implies the existence of field theory defined by the potential $\mathscr{V}(Z_{\mathrm{cdt}})$. Thus, the loop equation in the continuum limit becomes

$$G_s W_{\mathrm{GCDT}}^{(3)}(Z_{\mathrm{cdt}})^2 = \partial_{Z_{\mathrm{cdt}}} \mathscr{V}(Z_{\mathrm{cdt}}) W_{\mathrm{GCDT}}^{(3)}(Z_{\mathrm{cdt}}) - Q(Z_{\mathrm{cdt}}), \tag{2.34}$$

[1] This value is considered to be the gravity-dressed edge singularity of the dimer model [2]. In the dimer model, $\sigma = 1/6$.

where $Q(Z_{\mathrm{cdt}})$ is a polynomial of degree 2. The solution of the loop equation (2.34) is the continuous disk function of the generalized CDT coupled to dimers. When prohibiting spatial topology change, i.e., $G_s \to 0$, one can recover the continuous disk function of CDT coupled to dimers (2.24). From this matrix model in the continuum with the potential $\mathscr{V}$, the string field theory of the generalized CDT can be constructed [7] (see Sect. 2.2.1.1 for the string field theory of the generalized CDT).

2.1.3 Discussion

The multicritical model of CDT discussed in this section is the first analytic example of CDT coupled to matter.[2] Overviewing some unsolved problems in DT and CDT, we will mention the status of our model.

Only a few riddles are left in 2d Euclidean quantum gravity coupled to matter. One of them is the behavior of the Hausdorff dimension d_h as a function of the central charge c of the conformal theory coupled to gravity. A formula was derived by Watabiki some years ago [8]

$$d_h = 2\frac{\sqrt{49-c}+\sqrt{25-c}}{\sqrt{25-c}+\sqrt{1-c}}. \tag{2.35}$$

Most likely this formula is correct for $c \le 0$. For $c = 0$ agrees with what is known to be the correct answer [10–12]. For $c = -2$ there are very reliable computer simulations which show agreement with the formula [13, 14]. Finally for $c \to -\infty$ it gives 2, something one would indeed expect from semiclassical Liouville theory. However, for $0 < c \le 1$ the numerical agreement is less conclusive [15, 16], and the possibility that $d_h = 4$ in this range was pointed out, and the idea has recently been resurrected [17]. For $c > 1$ the Liouville formulas become complex and expressions like (2.35) are not valid, but it is believed that there is a universal phase where the world sheet degenerates to branched polymers (BP).

Surprisingly we have a somewhat similar situation in CDT: from numerical simulations d_h seems unchanged (and equal 2) when matter with central charge $0 \le c \le 1$ is coupled to the CDT ensemble [18–20] and recently it was shown that there might be a kind of universal phase for $c > 1$ [21]. However, to the extent we can really view the hard dimer models as corresponding to conformal field theories, it seems that for $c < 0$ the matter systems can change fractal structure of the CDT ensemble. Qualitatively the changes are like in the full Euclidean models, d_h decreases with decreasing c. In the $c = 0$ case the CDT model can be understood as an effective Euclidean model, where baby universes have been integrated out. Whether such an interpretation is possible also when matter is coupled to CDT is presently unknown,

[2] While completing the article [4] we were informed by Zohren that he and Atkin [9] have obtained results which are identical to some of our results. We thank Stefan for informing us of these results prior to publication.

but since the multicritical model captures the critical behavior of both CDT and ordinary 2d Euclidean quantum gravity coupled to certain matter systems, depending on how we scale g_s, we have a chance to answer this question in the context of analytic models like the one discussed here.

2.2 Extended Interactions in CDT

In this section, we quest for possibilities to extend the generalized CDT without changing scaling dimensions of space and time in 2 dimensions [22]: we extend the generalized CDT applying the method in the non-critical string field theory (SFT) techniques in [23, 24]; we solve the Schwinger-Dyson equation (SDE) for the disk amplitude by a perturbation of the string coupling constant. We also show dual matrix models in the continuum limit. The aim of our work [22] is to propose CDT coupled to the Ising model.

2.2.1 Generalized CDT

2.2.1.1 From String Field Theory

We review the non-critical string field theory (SFT) of the generalized CDT formulated in [25]. This model reproduces the disk amplitude derived in the continuum limit of CDT in the case that the string coupling constant is zero. The model requires closed strings with length L are created and annihilated from the vacuum, $|0\rangle$ ($\langle 0|$) by operators, $\psi^\dagger(L)$ and $\psi(L)$, respectively:

$$\langle 0|\psi^\dagger(L) \;= \psi(L)|0\rangle = 0. \tag{2.36}$$

These creation and annihilation operators obey the following commutation relation:

$$[\psi(L), \psi^\dagger(L')] = \delta(L - L'), \tag{2.37}$$

and others are zero. The string world-sheet can be seen as the Universe in 2 dimensions. Corresponding Hamiltonian can be written as follows:

$$H_0 = \int_0^\infty dL \psi^\dagger(L) \mathscr{H}_0(L, \Lambda_{\text{cdt}}) \psi(L)$$
$$+ G_s \int_0^\infty dL_1 \int_0^\infty dL_2 \psi^\dagger(L_1)\psi^\dagger(L_2)\psi(L_1 + L_2)(L_1 + L_2)$$

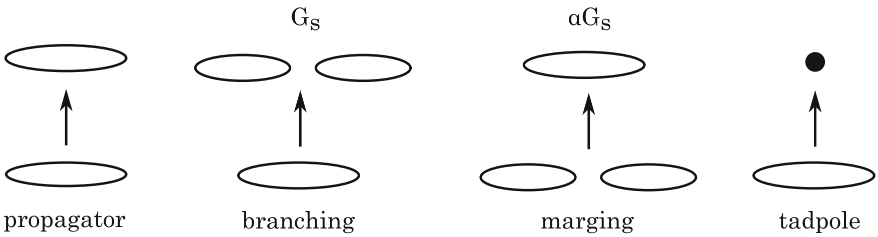

Fig. 2.3 Hamiltonian of generalized CDT

$$+\alpha G_s \int_0^\infty dL_1 \int_0^\infty dL_2 \psi^\dagger(L_1+L_2)\psi(L_2)\psi(L_1)L_2L_1 - \int_0^\infty dL\delta(L)\psi(L), \tag{2.38}$$

where

$$\mathscr{H}_0(L,\Lambda_{\rm cdt}) = -L\partial_L^2 + \Lambda_{\rm cdt}L. \tag{2.39}$$

G_s and $\Lambda_{\rm cdt}$ are the renormalized string coupling constant and the bulk cosmological constant, respectively (see Fig. 2.3); ∂_L is the derivative with respect to L. The parameter α in (2.38) was introduced to count the number of genus. In the following discussion we will take $\alpha = 0$, which suppresses the creation of genus. The Hamiltonian above can be determined under the following scaling dimensions:

$$[S] = \varepsilon, \quad [\psi^\dagger(L)] = \varepsilon^0, \quad [\psi(L)] = \varepsilon^{-1}, \quad [G_s] = \varepsilon^{-3}, \tag{2.40}$$

where ε is the scaling dimension of L, and $[S]$ is the scaling dimension of time. A crucial difference between the Hamiltonian of the non-critical SFT constructed by Ishibashi and Kawai [23] and that of generalized CDT is the existence of propagator term, $\int dL\psi^\dagger(L)\mathscr{H}_0\psi(L)$: it exists in the generalized CDT, but in IK's theory there is no such a term. This difference comes from the fact that both theories have quite different definition of time.

The authors in [25] derived the Schwiner-Dyson equation (SDE) for the disk amplitude Laplace-transformed, $W_{\rm GCDT}(Z_{\rm cdt}) = \int_0^\infty dLe^{-LZ_{\rm cdt}}\langle 0|e^{-SH_0}\ \psi^\dagger(L)|0\rangle|_{S\to\infty}$, in the generalized CDT as[3]:

$$\partial_{Z_{\rm cdt}}\left[(\Lambda_{\rm cdt} - Z_{\rm cdt}^2)W_{\rm GCDT}(Z) + G_s W_{\rm GCDT}(Z_{\rm cdt})^2\right] + 1 = 0. \tag{2.41}$$

The solution was derived by a perturbative expansion w.r.t. the string coupling constant as well [25]:

[3] The authors derived the more general result with arbitrary α, but here we restricted our situation to that with $\alpha = 0$.

$$W_{\mathrm{GCDT}}(Z_{\mathrm{cdt}}) = \frac{1}{Z_{\mathrm{cdt}} + \sqrt{\Lambda_{\mathrm{cdt}}}} - G_s \frac{Z_{\mathrm{cdt}} + 3\sqrt{\Lambda_{\mathrm{cdt}}}}{4\Lambda_{\mathrm{cdt}}(Z + \sqrt{\Lambda_{\mathrm{cdt}}})^3} + \mathcal{O}(G_s^2). \tag{2.42}$$

The first term is equivalent to the CDT solution [3]. In this formalism, the contributions from baby universes are weighted by the string coupling constant G_s.

2.2.1.2 From Matrix Model

A new scaling limit of the hermitian $N \times N$ matrix model was introduced [26]. We start with the matrix integral,

$$\int d\phi\ \exp\left[-\frac{N}{g_s}\mathrm{tr}\left(\frac{1}{2}\phi^2 - g\phi - \frac{g}{3}\phi^3\right)\right] = \int d\phi\ e^{-\frac{N}{g_s}\mathrm{tr}V(\phi)}, \tag{2.43}$$

where ϕ, g and g_s are the $N \times N$ hermitian matrix, 'tHooft coupling constant and string coupling constant, respectively. One can expand the coupling constants around the critical point found in [26], using the lattice spacing ε:

$$g_s = \frac{1}{2}\varepsilon^3 G_s, \quad \phi = \hat{I} - \varepsilon\Phi + \mathcal{O}(\varepsilon^2), \quad g = \frac{1}{2}\left(1 - \frac{1}{2}\varepsilon^2\Lambda_{\mathrm{cdt}} + \mathcal{O}(\varepsilon^4)\right), \tag{2.44}$$

where $\hat{I}$ is the unit $N \times N$ matrix; G_s, Φ and Λ_{cdt} are corresponding renormalized values. Substituting the fine-tuned values above into the potential $\frac{N}{g_s}V(\phi)$, one finds

$$\frac{N}{g_s}\mathrm{tr}V(\phi) = \frac{N}{G_s}\mathrm{tr}\left(\frac{1}{3}\Phi^3 - \Lambda_{\mathrm{cdt}}\Phi\right) + (\text{terms independent of } \Phi) + \mathcal{O}(\varepsilon). \tag{2.45}$$

Since the potential term scales at the new critical point as well as the "singular term" with fractional power, a field theory description can be anticipated. One can define it as the matrix model in the continuum limit [26, 27]. It has the following partition function:

$$Z = \int d\Phi\ \exp\left[-\frac{N}{G_s}\mathrm{tr}\left(\frac{1}{3}\Phi^3 - \Lambda_{\mathrm{cdt}}\Phi\right)\right]. \tag{2.46}$$

In the large-N limit, the saddle-point equation coincides with the SDE of the generalized CDT (2.41).[4]

[4] In [27], the authors derived the general saddle-point equation beyond the large-N limit. The general saddle-point equation indeed coincides with the SDE with arbitrary α by the treatment, $\alpha = 1/N^2$.

2.2.2 *Generalized CDT with Extended Interactions*

2.2.2.1 From String Field Theory

Applying the method in [24], we will construct the non-critical SFT Hamiltonian of the generalized CDT with extended interactions. The propagator term in (2.38), $\int dL \psi^\dagger(L)\mathscr{H}_0(L, \Lambda_{\rm cdt})\psi(L)$, induces causal geometries. Letting this propagator unchanged, one should take the scaling dimension of space and time as

$$[L] = \varepsilon, \quad [S] = \varepsilon. \tag{2.47}$$

We extend the non-critical SFT of the generalized CDT such that the scaling above is unchanged. Since we think that the causality is an identity of CDT, this sort of extension is meaningful to get some deep understanding of what CDT is. First, we consider strings with different charges: the (+)-type and (−)-type. The creation and annihilation operators for the (+)-type string, $\Psi_+^\dagger(L)$ and $\Psi_+(L)$, and for the (−)-type string, $\Psi_-^\dagger(L)$ and $\Psi_-(L)$, are defined as the following vacuum conditions, respectively:

$$\langle 0|\psi_+^\dagger(L) \ = \psi_+(L)|0\rangle = \langle 0|\psi_-^\dagger(L) \ = \psi_-(L)|0\rangle = 0. \tag{2.48}$$

We assume these operators obey the following commutation relations:

$$[\psi_+(L), \psi_+^\dagger(L')] = [\psi_-(L), \psi_-^\dagger(L')] = \delta(L - L'), \tag{2.49}$$

and the others are zero. Additionally, we assume the same scaling dimensions with those of the generalized CDT:

$$[\psi_\pm^\dagger(L)] = \varepsilon^0, \quad [\psi_\pm(L)] = \varepsilon^{-1}, \quad [G_s] = \varepsilon^{-3}, \tag{2.50}$$

where G_s is the string coupling constant. Under the conditions above, one can extend the Hamiltonian of the generalized CDT applying the interaction for spin clusters introduced by Ishibashi and Kawai [24]. Here we call such an interaction *IK-type interaction*. It is based on the so-called *peeling procedure* in a discrete random surface. For example, considering a randomly triangulated surface coupled to Ising spins with one boundary and then assuming that triangles attached to the boundary have homogeneous spins (all spins are up-type or down-type), one peels triangles along with the boundary as if one peels an apple. If one continues to peel off triangles over the boundary triangles and one encounters the triangle carrying a different type of spin, then one surrounds them by the triangles carrying the same spin as the boundary triangles. In short, the randomly triangulated surface is separated by domain walls. The SDE in their approach coincides with the loop equation for the chain-type two-matrix model describing random geometries coupled to Ising spins. We emphasize here that the above closed strings are not seen as the spin boundary as in the case

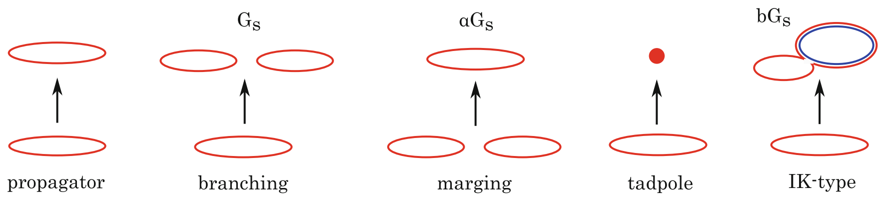

Fig. 2.4 Hamiltonian of generalized CDT with extended interactions

of IK but seen as the equal-time hypersurfaces with different charges. Applying the IK-type interaction, one can write the extended Hamiltonian of the generalized CDT:

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\begin{aligned}
H_m = & \int_0^\infty dL \psi_+^\dagger(L) \mathscr{H}_0(L, \Lambda_{\mathrm{cdt}}) \psi_+(L) \\
& + G_s \int_0^\infty dL_1 \int_0^\infty dL_2 \psi_+^\dagger(L_1) \psi_+^\dagger(L_2) \psi_+(L_1 + L_2)(L_1 + L_2) \\
& + bG_s \int_0^\infty dL_1 \int_0^\infty dL_2 \psi_+^\dagger(L_1 + L_2) \psi_-^\dagger(L_2) \psi_+(L_1) L_1 \\
& + \alpha G_s \int_0^\infty dL_1 \int_0^\infty dL_2 \psi_+^\dagger(L_1 + L_2) \psi_+(L_2) \psi_+(L_1) L_2 L_1 \\
& - \int_0^\infty dL \delta(L) \psi_+(L) + \left[\psi_+(\,\psi_+^\dagger) \leftrightarrow \psi_-(\,\psi_-^\dagger) \right],
\end{aligned}
\tag{2.51}
$$

where α and b are dimension-less constants (see Fig. 2.4).[5]

For simplicity, we restrict topology to the disk. This can be realized by the following Hamiltonian:

$$H_m^D \equiv \lim_{\alpha \to 0} H_m. \tag{2.52}$$

We will then derive the SDE in our model. The SDE corresponds to the Wheeler-DeWitt equation for the wave function of the Universe. We define the partition function and disk amplitudes:

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[5] In fact, it is possible to include the interactions, $\int dL \psi_-^\dagger(L) \mathscr{H}_0(L, \Lambda_{\mathrm{cdt}}) \psi_+(L)$ and its spin-flipped term. However, because of the $\mathbb{Z}_2$-symmetry as to the spin reflection, such terms merely cause a constant shift of the string coupling constant, so that we have not included these terms in the Hamiltonian.

$$Z = \lim_{S\to\infty} \langle 0|e^{-SH_m^D}|0\rangle \equiv 1, \tag{2.53}$$

and

$$W_\pm(L) \equiv \lim_{S\to\infty} \langle 0|e^{-SH_m^D}\psi_\pm^\dagger(L)|0\rangle. \tag{2.54}$$

The SDE for $W_\pm(L)$ is

$$\lim_{S\to\infty} \frac{\partial}{\partial S}\langle 0|e^{-SH_m^D}\psi_\pm^\dagger(L)|0\rangle = 0. \tag{2.55}$$

Using the equation, $H_m^D|0\rangle = 0$, and the commutation relations (2.49), one can rewrite the SDE as follows:

$$\begin{aligned} 0 =& - L\partial_L^2 W_\pm(L) + \Lambda L W_\pm(L) - \delta(L) \\ &+ G_s L \int_0^\infty dL_1 \lim_{S\to\infty} \langle 0|e^{-SH_m^D}\psi_\pm^\dagger(L_1)\psi_\pm^\dagger(L-L_1)|0\rangle \\ &+ bG_s L \int_0^\infty dL_1 \lim_{S\to\infty} \langle 0|e^{-SH_m^D}\psi_\pm^\dagger(L+L_1)\psi_\mp^\dagger(L+L_1)|0\rangle. \end{aligned} \tag{2.56}$$

Here we introduce the factorization theorem:

$$\begin{aligned} \lim_{S\to\infty} \langle 0|e^{-SH_m^D}\psi_\pm^\dagger(L_1)\psi_\pm^\dagger(L_2)|0\rangle = \lim_{S\to\infty} \langle 0|e^{-SH_m^D}\psi_\pm^\dagger(L_1)|0\rangle \lim_{S\to\infty} \langle 0|e^{-SH_m^D} \\ \psi_\pm^\dagger(L_2)|0\rangle. \end{aligned} \tag{2.57}$$

Applying the above factorization theorem, the SDE (2.57) becomes

$$\begin{aligned} 0 = -L\partial_L^2 W_\pm(L) + \Lambda_{\text{cdt}} L W_\pm(L) - \delta(L) + G_s L \int_0^\infty dL_1 W_\pm(L_1) W_\pm(L-L_1) \\ + bG_s L \int_0^\infty dL_1 W_\pm(L+L_1) W_\mp(L_1). \end{aligned} \tag{2.58}$$

In fact, our system has $\mathbb{Z}_2$-symmetry w.r.t. the spin-reflection. Thus, we focus on a $\mathbb{Z}_2$-symmetric solution of the SDE:

$$W_+(L) = W_-(L) \equiv W_{\text{IK}}(L). \tag{2.59}$$

Implementing the Laplace transformation, $\mathscr{L}[W_{\mathrm{IK}}(L)] \equiv \int_0^\infty dL e^{-LZ_{\mathrm{cdt}}} W_{\mathrm{IK}}(L) \equiv W_{\mathrm{IK}}(Z_{\mathrm{cdt}})$, (2.58) becomes

$$0 = \partial_{Z_{\mathrm{cdt}}} \left[(Z_{\mathrm{cdt}}^2 - \Lambda_{\mathrm{cdt}}) W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) - G_s W_{\mathrm{IK}}(Z_{\mathrm{cdt}})^2 \right] - 1 + bG_s \mathscr{L} \left[L \int dL_1 W_{\mathrm{IK}}(L + L_1) W_{\mathrm{IK}}(L_1) \right]. \tag{2.60}$$

One notices that the last term diverges in $Z \to \infty$. To regularize this divergence, we symmetrize the term w.r.t. the reflection, $Z_{\mathrm{cdt}} \leftrightarrow -Z_{\mathrm{cdt}}$ [24, 28–30]:

$$\int_0^\infty dL \int_0^\infty dL_1 e^{-Z_{\mathrm{cdt}}(L+L_1)} W_{\mathrm{IK}}(L + L_1) e^{+Z_{\mathrm{cdt}} L_1} W_{\mathrm{IK}}(L_1) + (Z_{\mathrm{cdt}} \leftrightarrow -Z_{\mathrm{cdt}}) = W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}). \tag{2.61}$$

Subtracting the SDE with the reflection ($Z \to -Z$) from the original SDE (2.60), one obtains the finite SDE:

$$\partial_{Z_{\mathrm{cdt}}} \Big[(Z_{\mathrm{cdt}}^2 - \Lambda_{\mathrm{cdt}}) \left(W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) + W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}) \right) - G_s \left(W_{\mathrm{IK}}(Z_{\mathrm{cdt}})^2 + W_{\mathrm{IK}}(-Z_{\mathrm{cdt}})^2 + b W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}) \right) \Big] = 0. \tag{2.62}$$

Integrating the SDE above over Z_{cdt} yields

$$(Z_{\mathrm{cdt}}^2 - \Lambda_{\mathrm{cdt}}) \left(W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) + W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}) \right) - G_s \left(W_{\mathrm{IK}}(Z_{\mathrm{cdt}})^2 + W_{\mathrm{IK}}(-Z_{\mathrm{cdt}})^2 + b W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}) \right) = c. \tag{2.63}$$

where c is a constant. We calculate a perturbative solution for the SDE above around the weak coupling region, $G_s < 1$, expanding the loop amplitude and c like:

$$W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) = \sum_{n=0}^{\infty} G_s^n W_n(Z_{\mathrm{cdt}}), \quad c = \sum_{n=0}^{\infty} G_s^n c_n. \tag{2.64}$$

As for $W_0(Z_{\mathrm{cdt}})$, one finds

$$W_0(Z_{\mathrm{cdt}}) = \frac{1}{Z_{\mathrm{cdt}} + \sqrt{\Lambda_{\mathrm{cdt}}}}, \tag{2.65}$$

where we have chosen an overall constant for $W_0(Z)$ so as to coincide with that of the plain CDT. Assuming that the disk amplitude behaves as $1/Z_{\mathrm{cdt}}$ in the large value of $|Z_{\rm cdt}|$, we can determine that $c_1 = -(b+1)/2\Lambda_{\rm cdt}$. Furthermore, we can extract $W_1(Z_{\rm cdt})$ considering that $W_1(Z_{\rm cdt})$ is analytic in the region, $|Z_{\rm cdt}| > 0$. Thus, the perturbative solution is

$$W_{\rm IK}(Z_{\rm cdt}) = \frac{1}{Z_{\rm cdt}+\sqrt{\Lambda_{\rm cdt}}} - G_s\frac{1}{4\Lambda_{\rm cdt}}\left[\frac{Z_{\rm cdt}+3\sqrt{\Lambda_{\rm cdt}}}{(Z_{\rm cdt}+\sqrt{\Lambda_{\rm cdt}})^3} + \frac{b}{(Z_{\rm cdt}+\sqrt{\Lambda_{\rm cdt}})^2}\right] + \mathscr{O}(G_s^2). \tag{2.66}$$

The solution with $b = 0$ is equivalent to that of the plain generalized CDT (2.42).

2.2.2.2 From Matrix Model

We start with the following matrix integral:

$$\int d\phi_+ d\phi_- e^{-\frac{N}{g_s}\mathrm{tr}V(\phi_+,\phi_-)}, \tag{2.67}$$

where

$$V(\phi_+,\phi_-) = \frac{1}{2}(\phi_+^2+\phi_-^2) - g(\phi_+ + \phi_-) - \frac{g}{3}(\phi_+^3+\phi_-^3) + x\phi_+\phi_-. \tag{2.68}$$

In the integral above, $\phi_\pm$, g, g_s and x are $N \times N$ hermitian matrices, the 'tHooft coupling constant, string coupling constant and coupling constant characterizing the interaction, respectively. We then expand the fields and coupling constants w.r.t. the lattice spacing ε as follows:

$$\phi_+ = \hat{I} - \varepsilon(A+B) + \mathscr{O}(\varepsilon^2), \quad \phi_- = \hat{I} - \varepsilon(A-B) + \mathscr{O}(\varepsilon^2), \tag{2.69}$$

and

$$g_s = \varepsilon^3 G_s, \quad g = \frac{1}{2}\left(1 - \frac{1}{2}\varepsilon^2(\Lambda_{\rm cdt} - 2X) + \mathscr{O}(\varepsilon^4)\right), \quad x = X\varepsilon^2, \tag{2.70}$$

where A and B are $N \times N$ hermitian matrices; $\hat{I}$ is the unit matrix; G_s, Φ, Λ and X are corresponding renormalized values. This construction implies that the cut-length shrinks to zero ($g_s \to 0$), and the strength of the interaction falls off ($x \to 0$). The causality induces the scaling, $g_s \to 0$ and taking the limit, $x \to 0$, the model can be seen as the weakly interacting model. Substituting the fine-tuned values, one can write the partition function of the matrix model in the continuum limit:

$$Z = \int dAdB \exp\left[-\frac{N}{G_s}\mathrm{tr}\left(\frac{1}{3}A^3 + AB^2 - \Lambda_{\rm cdt}A\right)\right]. \tag{2.71}$$

An interesting thing is that in the matrix model having this type of potential, the Gaussian integral over B can be performed by introducing the eigenvalues λ_i's for the matrix A [31]:

$$Z \propto \int \prod_i d\lambda_i \prod_{i<j} (\lambda_j - \lambda_i)^2 \prod_{i,j} (\lambda_i + \lambda_j)^{-1/2} e^{-\frac{N}{G_s} V}, \tag{2.72}$$

where

$$V = \sum_{i=1}^{N} V(\lambda_i) = \sum_{i=1}^{N} \left(\frac{1}{3}\lambda_i^3 - \Lambda_{\mathrm{cdt}} \lambda_i \right). \tag{2.73}$$

In the large-N limit, the saddle point equation becomes

$$\frac{2}{N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = \frac{1}{N} \sum_j \frac{1}{\lambda_i + \lambda_j} + \frac{1}{G_s} V'(\lambda_i), \tag{2.74}$$

where $V'(\lambda_i) = \lambda_i^2 - \Lambda_{\mathrm{cdt}}$. Here we define the resolvent for A as $W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) \equiv \frac{1}{N}\mathrm{tr}(Z_{\mathrm{cdt}} - A)^{-1}$, and the distribution of eigenvalues as $\rho(\lambda) \equiv \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$. Multiplying (2.74) by $1/(Z - \lambda_i)$ and summing over i, one obtains the loop equation in the large-N limit:

$$\begin{aligned} &(Z_{\mathrm{cdt}}^2 - \Lambda_{\mathrm{cdt}}) \left(W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) + W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}) \right) \\ &\quad - G_s \left(W_{\Lambda_{\mathrm{cdt}}}(Z_{\mathrm{cdt}})^2 + W_{\mathrm{IK}}(Z_{\mathrm{cdt}}) W_{\mathrm{IK}}(-Z_{\mathrm{cdt}}) + W_{\mathrm{IK}}(-Z_{\mathrm{cdt}})^2 \right) = 2 \int d\lambda \rho(\lambda)\lambda. \end{aligned} \tag{2.75}$$

Remember the SDE derived in the non-critical SFT approach (2.66). One can find a great similarity between the two. Namely, when setting $b = 1$ in the SDE, the two equations are exactly same. Thus, this matrix model in the continuum limit can reproduce the generalized CDT with extended interactions in $b = 1$.

One can extend the matrix model in the continuum limit above to the general $O(n)$ vector model [31] such that:

$$Z = \int dA dB_1 \ldots dB_n e^{-\frac{N}{G_s}\mathrm{tr} U(A, B_1, \ldots, B_n)}, \tag{2.76}$$

where

$$U(A, B_1, \ldots, B_n) = A(B_1^2 + \cdots + B_n^2) + \frac{1}{3}A^3 - \Lambda_{\mathrm{cdt}} A; \tag{2.77}$$

$A, B_1, \ldots, B_n$ are $N \times N$ hermitian matrices. Notice that the previous matrix model in the continuum limit is $O(1)$ vector model. Integrating out all B_i's, the loop equation in the large-N limit is

$$
\begin{aligned}
&(Z_{\text{cdt}}^2 - \Lambda_{\text{cdt}})\,(W_{\text{IK}}(Z_{\text{cdt}}) + W_{\text{IK}}(-Z_{\text{cdt}})) \\
&\quad - G_s\left(W_{\Lambda_{\text{cdt}}}(Z_{\text{cdt}})^2 + nW_{\text{IK}}(Z_{\text{cdt}})W_{\text{IK}}(-Z_{\text{cdt}}) + W_{\text{IK}}(-Z_{\text{cdt}})^2\right) = 2\int d\lambda\rho(\lambda)\lambda.
\end{aligned} \tag{2.78}
$$

The loop equation of the $O(n)$ vector model coincides with the SDE labeled by the free parameter b (2.63) only if one identifies n with b.

2.2.3 Discussion

We have shown the equivalence between the two different field theories at the level of differential equations, the Schwinger-Dyson equation in the non-critical SFT and the loop equation of the matrix model in the continuum limit. In the following, we will examine the extended models from different point of view.

To begin with, we discuss our model in terms of the SFT approach. Although we have used the IK-type interaction to construct the extended SFT of the generalized CDT, we do not understand if our model is on the critical point of the Ising model characterized by the Curie temperature. In the following, we try to explain two complications around this issue. First, at the critical point of Ising spins the spin configuration must be random: spins are supposed to fluctuate all length scales between the lattice spacing and the correlation length. Contrary to that, in our model the homogeneous spin (charge) configurations survive as the propagators. Second, the definition of time associated with our Hamiltonian (2.51) is different from the would-be generalized CDT coupled to Ising spins. Namely, we consider the closed strings in our model as not spin-cluster boundaries but spatial boundaries, so that we pursue the time flow of spatial boundaries. Thus, our time is nothing but the proper time. This proper time is crucially different from the time defined via the spin-cluster boundary [32, 33]. Considering our time as the one defined via the spin-cluster boundary is equivalent to treating our model as the generalized CDT coupled to Ising spins; the scaling dimension of time may be different from the lattice spacing ε according to [32]. This contradicts our first setup (2.47). The free parameter b, in one way or another, might be the key to know what our model is.

In addition, it is possible to extend our non-critical SFT to the multi-"colored" system:

$$
\begin{aligned}
H_m^{(n)} &= \sum_{i=1}^{n}\int_0^\infty dL\,\psi_i^\dagger(L)\mathscr{H}_0(L,\Lambda_{\text{cdt}})\psi_i(L) \\
&\quad + G_s\sum_{i=1}^{n}\int_0^\infty dL_1\int_0^\infty dL_2\,\psi_i^\dagger(L_1)\psi_i^\dagger(L_2)\psi_i(L_1+L_2)(L_1+L_2)
\end{aligned}
$$

$$+ G_s \sum_{i=1}^{n} \sum_{j \neq i}^{n} b_{ij} \int_0^\infty dL_1 \int_0^\infty dL_2 \psi_i^\dagger(L_1 + L_2) \psi_j^\dagger(L_2) \psi_i(L_1) L_1$$
$$+ \alpha G_s \sum_{i=1}^{n} \int_0^\infty dL_1 \int_0^\infty dL_2 \psi_i^\dagger(L_1 + L_2) \psi_i(L_2) \psi_i(L_1) L_2 L_1$$
$$- \sum_{i=1}^{n} \int_0^\infty dL \delta(L) \psi_i(L). \tag{2.79}$$

One can derive the free parameter b in our model from the multi-“colored” system under the treatment, $W_1(L) = \cdots = W_n(L) \equiv W_\Lambda(L)$; $b_{ij} = 0$ for $j = i$; $b_{ij} = 1$ for $j \neq i$.

Next, we closely look at our matrix model. We consider the direct product of the two copies of the potential. Each of them yields the plain generalized CDT. Introducing the linear combinations of matrices such that $\Phi_+ = A + B$ and $\Phi_- = A - B$, one finds

$$\frac{1}{\tilde{G_s}} \left(\frac{1}{3} \Phi_+^3 - \Lambda_{\mathrm{cdt}} \Phi_+ + \frac{1}{3} \Phi_-^3 - \Lambda_{\mathrm{cdt}} \Phi_- \right) = \frac{1}{G_s} \left(\frac{1}{3} A^3 + AB^2 - \Lambda_{\mathrm{cdt}} A \right), \tag{2.80}$$

where $\tilde{G_s} = 2G_s$. This is nothing but the potential of our $O(1)$ vector model in the continuum limit. Diagonalizing the matrix A as $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ and integrating out the matrix B, one gets the effective theory for the eigenvalues of A with the potential,

$$\underbrace{\left[\frac{1}{G_s} \sum_i \left(\frac{1}{3} \lambda_i^3 - \Lambda_{\mathrm{cdt}} \lambda_i \right) - \frac{1}{N} \log \prod_{i<j} (\lambda_j - \lambda_i)^2 \right]}_{\text{terms appeared in the plain generalized CDT}}$$
$$+ \text{ terms induced by the integration over } B. \tag{2.81}$$

An important point here is that our model is slightly different from the plain generalized CDT matrix model because integrating out the matrix B the extra correction is added to terms appeared in the plain generalized CDT. From the matrix A’s point of view, the matrix B can be seen like some external field. The strength of such an external field can be bigger by inserting the integrated-out matrices. It leads to the $O(n)$ vector model in the continuum limit.

References

1. Kazakov, V. A. (1986). Ising model on a dynamical planar random lattice: Exact solution. *Physics Letter A*, *119*, 140.
2. Staudacher, M. (1990). The Yang-lee edge singularity on a dynamical planar random surface. *Nuclear Physics B*, *336*, 349.

3. Ambjorn, J., & Loll, R. (1998). Nonperturbative Lorentzian quantum gravity, causality and topology change. *Nuclear Physics B*, *536*, 407. [hep-th/9805108].
4. Ambjorn, J., Glaser, L., Gorlich, A., & Sato, Y. (2012). New multicritical matrix models and multicritical 2d CDT. *Physics Letter B*, *712*, 109. arXiv:1202.4435 [hep-th].
5. Moore, G. W., Seiberg, N., & Staudacher, M. (1991). *Nuclear Physics B*, *362*, 665.
6. Glaser, L. (2012). Coupling dimers to CDT. arXiv:1210.4063 [hep-th].
7. Atkin, M. R., & Zohren, S. (2012). On the quantum geometry of multi-critical CDT. *JHEP*, *1211*, 037. arXiv:1203.5034 [hep-th].
8. Watabiki, Y. (1993). Analytic study of fractal structure of quantized surface in two-dimensional quantum gravity. *Progress of Theoretical Physics Supplement*, *114*, 1.
9. Atkin, M. R., & Zohren, S. (2012). An analytical analysis of CDT coupled to dimer-like matter. *Physics Letter B*, *712*, 445. arXiv:1202.4322 [hep-th].
10. Kawai, H., Kawamoto, N., Mogami, T., & Watabiki, Y. (1993). Transfer matrix formalism for two-dimensional quantum gravity and fractal structures of space-time. *Physics Letter B*, *306*, 19. [hep-th/9302133].
11. Ambjorn, J., & Watabiki, Y. (1995). Scaling in quantum gravity. *Nuclear Physics B*, *445*, 129. [hep-th/9501049].
12. Aoki, H., Kawai, H., Nishimura, J., & Tsuchiya, A. (1996). Operator product expansion in two-dimensional quantum gravity. *Nuclear Physics B*, *474*, 512. [hep-th/9511117].
13. Ambjorn, J., Anagnostopoulos, K. N., Ichihara, T., Jensen, L., Kawamoto, N., Watabiki, Y., et al. (1997). Quantum geometry of topological gravity. *Physics Letter B*, *397*, 177. [hep-lat/9611032].
14. Ambjorn, J., Anagnostopoulos, K., Ichihara, T., Jensen, L., Kawamoto, N., Watabiki, Y., et al. (1998). The quantum space-time of c $= -2$ gravity. *Nuclear Physics B*, *511*, 673. [hep-lat/9706009].
15. Ambjorn, J., & Anagnostopoulos, K. N. (1997). Quantum geometry of 2-D gravity coupled to unitary matter. *Nuclear Physics B*, *497*, 445. [hep-lat/9701006].
16. Ambjorn, J., Anagnostopoulos, K. N., Magnea, U., & Thorleifsson, G. (1996). Geometrical interpretation of the KPZ exponents. *Physics Letter B*, *388*, 713. [hep-lat/9606012].
17. Duplantier, B. (2011) The hausdorff dimension of two-dimensional quantum gravity. arXiv:1108.3327 [math-ph].
18. Ambjorn, J., Anagnostopoulos, K. N., Loll, R., & Pushkina, I. (2009). Shaken, but not stirred: Potts model coupled to quantum gravity. *Nuclear Physics B*, *807*, 251. arXiv:0806.3506 [hep-lat].
19. Ambjorn, J., Anagnostopoulos, K. N., & Loll, R. (1999). A new perspective on matter coupling in 2-D quantum gravity. *Physics Review D*, *60*, 104035. [hep-th/9904012].
20. Ambjorn, J., Anagnostopoulos, K. N., & Loll, R. (2000). Crossing the c $= 1$ barrier in 2-D Lorentzian quantum gravity. *Physics Review D*, *61*, 044010. [hep-lat/9909129].
21. Ambjorn, J., Goerlich, A. T., Jurkiewicz, J., & Zhang, H.-G. (2012). Pseudo-topological transitions in 2D gravity models coupled to massless scalar fields. *Nuclear Physics B*, *863*, 421. arXiv:1201.1590 [gr-qc].
22. Fuji, H., Sato, Y., & Watabiki, Y. (2011). Causal dynamical triangulation with extended interactions in $1 + 1$ dimensions. *Physics Letter B*, *704*, 582. arXiv:1108.0552 [hep-th].
23. Ishibashi, N., & Kawai, H. (1993). String field theory of noncritical strings. *Physics Letter B*, *314*, 190. [hep-th/9307045].
24. Ishibashi, N., & Kawai, H. (1994). String field theory of c ¡= 1 noncritical strings. *Physics Letter B*, *322*, 67. [hep-th/9312047].
25. Ambjorn, J., Loll, R., Watabiki, Y., Westra, W., & Zohren, S. (2008). A string field theory based on causal dynamical triangulations. *JHEP*, *0805*, 032. arXiv:0802.0719 [hep-th].
26. Ambjorn, J., Loll, R., Watabiki, Y., Westra, W., & Zohren, S. (2008). A new continuum limit of matrix models. *Physics Letter B*, *670*, 224. arXiv:0810.2408 [hep-th].
27. Ambjorn, J., Loll, R., Watabiki, Y., Westra, W., & Zohren, S. (2008). A matrix model for 2D quantum gravity defined by causal dynamical triangulations. *Physics Letter B*, *665*, 252. arXiv:0804.0252 [hep-th].

28. Kostov, I. K. (1989). *Nuclear Physics B*, *326*, 583.
29. Kostov, I. K. (1991). *Physics Letter B*, *266*, 42.
30. Kostov, I. K. (1992). *Nuclear Physics B*, *376*, 539.
31. Kostov, I. K. (1989). O (n) vector model on a planar random lattice: Spectrum of anomalous dimensions. *Modern Physics Letters A*, *4*, 217.
32. Ambjorn, J., Anagnostopoulos, K. N., Jurkiewicz, J., & Kristjansen, C. F. (1998). The concept of time in 2-D gravity. *JHEP*, *9804*, 016. [hep-th/9802020].
33. Ambjorn, J., Kristjansen, C., & Watabiki, Y. (1997). The two point function of $c = -2$ matter coupled to 2-D quantum gravity. *Nuclear Physics B*, *504*, 555. [hep-th/9705202].

http://www.springer.com/978-4-431-54946-8

Space-Time Foliation in Quantum Gravity
Sato, Y.
2014, XIII, 100 p. 16 illus., 12 illus. in color., Hardcover
ISBN: 978-4-431-54946-8