# Chapter 2
# Almost Convergence of Double Sequences

The notion of almost convergence for ordinary (single) sequences was given by Lorentz [63], and for double sequences by Moricz and Rhoades [83]. In this chapter, we discuss the notion of almost convergence and almost Cauchy for double sequences. Some more related spaces for double sequences, associated sublinear functionals, and various inclusion relations are also studied.

## 2.1 Introduction

A double sequence $x = (x_{jk})$ of real or complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{j,k} |x_{jk}| < \infty$. The space of all bounded double sequences is denoted by $\mathcal{M}_u$.

A double sequence $x = (x_{jk})$ is said to *converge to the limit $L$ in Pringsheim's sense* (shortly, *$P$-convergent to $L$*) [108] if for every $\varepsilon > 0$, there exists an integer $N$ such that $|x_{jk} - L| < \varepsilon$ whenever $j, k > N$. In this case $L$ is called the $P$-limit of $x$. If, in addition, $x \in \mathcal{M}_u$, then $x$ is said to be *boundedly convergent to L in Pringsheim's sense* (shortly, *BP-convergent to L*). The sets of $P$-convergent and $BP$-convergent double sequences $x = (x_{jk})$ will be denoted by $\mathcal{C}_p$ and $\mathcal{C}_{bp}$, respectively.

A double sequence $x = (x_{jk})$ is said to *converge regularly to $L$* (shortly, *$R$-convergent to $L$*) if $x \in \mathcal{C}_p$ and the limits $x^j := \lim_k x_{jk}$ $(j \in \mathbb{N})$ and $x^k := \lim_j x_{jk}$ $(k \in \mathbb{N})$ exist. Note that, in this case, the limits $\lim_j \lim_k x_{jk}$ and $\lim_k \lim_j x_{jk}$ exist and are equal to the $P$-limit of $x$.

In general, for any notion of convergence $\nu$, the space of all $\nu$-convergent double sequences will be denoted by $\mathcal{C}_\nu$, the space of all $\nu$-convergent to 0 double sequences by $\mathcal{C}_{\nu 0}$, and the limit of a $\nu$-convergent double sequence $x$ by $\nu\text{-}\lim_{j,k} x_{jk}$, where $\nu \in \{P, BP, R\}$.

The idea of almost convergence for single sequences was introduced by Lorentz [63], and for double sequences by Moricz and Rhoades [83].

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A double sequence $x = (x_{jk})$ of real numbers is said to be *almost convergent* to a limit $L$ if

$$\lim_{p,q\to\infty} \sup_{m,n>0} \left| \frac{1}{pq} \sum_{j=m}^{m+p-1} \sum_{k=n}^{n+q-1} x_{jk} - L \right| = 0. \tag{2.1}$$

In this case, $L$ is called the $\mathcal{F}$-limit of $x$, and we shall denote by $\mathcal{F}$ the space of all almost convergent double sequences, i.e.,

$$\mathcal{F} = \left\{ x = (x_{jk}) : \lim_{p,q\to\infty} \left|\tau_{pqst}(x) - L\right| = 0, \text{ uniformly in } s, t \right\},$$

where

$$\tau_{pqst}(x) = \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t}.$$

Note that throughout the book, the notation lim for double sequences will represent $P$-lim.

If $m = n = 1$ in (2.1), then we get the $(C, 1, 1)$-convergence, and in this case, we write $x_{jk} \to \ell(C, 1, 1)$, where $\ell = (C, 1, 1)\text{-}\lim x$.

Recently, the concept of Banach limits for double sequences was defined in [101] as follows.

A linear functional $\mathcal{L}$ on $\mathcal{M}_u$ is said to be a *Banach limit* if it has the following properties:

(i) $\mathcal{L}(x) \geq 0$ if $x \geq 0$ (i.e., $x_{jk} \geq 0$ for all $j, k$),
(ii) $\mathcal{L}(\mathbf{e}) = 1$, where $\mathbf{e} = (e_{jk})$ with $e_{jk} = 1$ for all $j, k$, and
(iii) $\mathcal{L}(S_{11}x) = \mathcal{L}(x) = \mathcal{L}(S_{10}x) = \mathcal{L}(S_{01}x)$,

where the shift operators $S_{01}$, $S_{10}$, and $S_{11}$ are defined by

$$S_{01}x = (x_{j,k+1}), \qquad S_{10}x = (x_{j+1,k}), \qquad S_{11}x = (x_{j+1,k+1}).$$

Let $\mathcal{B}_2$ be the set of all Banach limits on $\mathcal{M}_u$. A double sequence $x = (x_{jk})$ is said to be *almost convergent* to a number $L$ if $\mathcal{L}(x) = L$ for all $\mathcal{L} \in \mathcal{B}_2$.

As in case of single sequences [63], Theorem 2.5 gives the equivalence of these two definitions.

The idea of strong almost convergence for single sequences is due to Maddox [64], and for double sequences, to Başarir [12].

A double sequence $x = (x_{jk})$ is said to be *strongly almost convergent* to a number $L$ if

$$P\text{-}\lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} |x_{j+s,k+t} - L| = 0,$$

uniformly in $s$, $t$. By $[\mathcal{F}]$ we denote the space of all strongly almost convergent double sequences. Note that $[\mathcal{F}] \subset \mathcal{F} \subset \mathcal{M}_u$.

In [26], Čunjalo introduced the idea of almost Cauchy for double sequences. A double sequence $x = (x_{jk})$ is said to be *almost Cauchy* if for every $\epsilon > 0$, there exits $k \in \mathbb{N}$ such that

$$\left| \frac{1}{p_1 q_1} \sum_{j=0}^{p_1-1} \sum_{k=0}^{q_1-1} x_{j+s_1,k+t_1} - \frac{1}{p_2 q_2} \sum_{j=0}^{p_2-1} \sum_{k=0}^{q_2-1} x_{j+s_2,k+t_2} \right| < \epsilon$$

for all $p_1, p_2, q_1, q_2 > k$ and $(s_1, t_1), (s_2, t_2) \in \mathbb{N} \times \mathbb{N}$.

## 2.2 Some Auxiliary Results

Note that, in contrast to the single sequences, a $P$-convergent double sequence need not be almost convergent. However, every bounded convergent double sequence is almost convergent, and every almost convergent double sequence is also bounded, i.e., $\mathcal{C}_{BP} \subset \mathcal{F} \subset \mathcal{M}_u$, and each inclusion is proper.

We start with the following basic result.

**Theorem 2.1** *Let a double sequence $x = (x_{jk})$ be BP-convergent to $L$. Then it is almost convergent to $L$, but the converse is not true in general.*

*Proof* For a given $\varepsilon > 0$, we choose $N, M \in \mathbb{N}$ with

$$|x_{jk} - L| < \frac{\varepsilon}{2} \quad (j, k \geq N), \tag{2.2}$$

and for $N$, there are $p_0, q_0 \in \mathbb{N}$ such that $p_0 > N,\ q_0 > M$, and

$$\frac{2(N+1)(M+1)\|x\|_\infty}{p_0 q_0} < \frac{\varepsilon}{2}. \tag{2.3}$$

Then, for $s, t \in \mathbb{N}$ and $p \geq p_0$, $q \geq q_0$, we obtain

$$\left| \frac{1}{pq} \sum_{j=s}^{s+p-1} \sum_{k=t}^{t+q-1} x_{jk} - L \right| \leq \frac{1}{pq} \sum_{j=s}^{s+p-1} \sum_{k=t}^{t+q-1} |x_{jk} - L|$$

$$\leq \begin{cases} \frac{1}{pq} pq \frac{\varepsilon}{2} & \text{if } s \geq N,\ t \geq M, \\ \frac{1}{pq} \sum_{j=s}^{N} \sum_{k=t}^{M} |x_{jk} - L| & \\ \quad + \frac{1}{pq} \sum_{j=N+1}^{s+p-1} \sum_{k=M+1}^{t+q-1} |x_{jk} - L| & \text{if } s < N,\ t < M, \end{cases}$$

$$\leq \begin{cases} \frac{\varepsilon}{2} & \text{if } s \geq N,\ t \geq M, \\ \frac{1}{pq}(N - s + 1)(M - t + 1)2\|x\|_\infty & \\ \quad + \frac{1}{pq}(s + p - 1 - N)(t + q - 1 - M)\frac{\varepsilon}{2} & \text{if } s < N,\ t < M, \end{cases}$$

$$< \varepsilon \quad \big(\text{by (2.2), (2.3) and the choice of } p_0, q_0 \text{ and } N, M\big).$$

Thus, $x = (x_{jk})$ is almost convergent to $L$.

For converse, consider the double sequence $x = (x_{jk})$ defined by

$$x_{jk} = \begin{cases} 1 & \text{if } j = k \text{ odd,} \\ -1 & \text{if } j = k \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $x = (x_{jk})$ is almost convergent to zero but not $P$-convergent, that is, $\mathcal{C}_{BP} \subsetneq \mathcal{F}$. □

**Theorem 2.2** *Every almost convergent double sequence $x = (x_{jk})$ is bounded, but the converse is not true in general.*

*Proof* Let $x = (x_{jk}) \in \mathcal{F}$ and $r_0, q_0 \in \mathbb{N}$ with

$$\frac{1}{rq}\left|\sum_{j=s}^{s+r-1}\sum_{k=t}^{t+q-1} x_{jk}\right| \leq 1 \quad (r \geq r_0,\ q \geq q_0,\ \text{and } s, t \in \mathbb{N}) \tag{2.4}$$

be given. Then, for $s, t \in \mathbb{N}$, we have

$$\begin{aligned} |x_{st}| &= \left|\sum_{j=s}^{s+r_0-1}\sum_{k=t}^{t+q_0-1} x_{jk} - \sum_{j=s+1}^{s+r_0-1}\sum_{k=t+1}^{t+q_0-1} x_{jk}\right| \\ &= \left|\sum_{j=s}^{s+r_0-1}\sum_{k=t}^{t+q_0-1} (x_{jk} - L) - \sum_{j=s+1}^{s+r_0-1}\sum_{k=t+1}^{t+q_0-1} (x_{jk} - L) + L\right| \\ &\leq (r_0 + 1)(q_0 + 1) + r_0 q_0 + |L| \quad \text{by (2.4).} \end{aligned}$$

That is, $x = (x_{jk}) \in \mathcal{M}_u$.

For converse, let $x = (x_{jk})$ be defined as

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

i.e., in each row, there is one 1, then two 0s, then four 1s, then eight 0s, then sixteen 1s, etc.

Now, for $n, m$ odd, the sum of the first $2^j 2^k$ elements will be at least $(2^{k-1} + 2^{k-3})2^j$, and so

$$\frac{1}{2^n 2^m} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} \geq \frac{2^n(2^{m-1} + 2^{m-3})}{2^n 2^m} = \frac{5}{8}.$$

Similarly, for $n, m$ even,

$$\frac{1}{2^n 2^m} \sum_{j=1}^{n} \sum_{k=1}^{m} x_{jk} \leq \frac{2^n(2^{m-2} + 2^{m-3})}{2^n 2^m} = \frac{3}{8},$$

i.e., $x \notin \mathcal{F}$, but $x \in \mathcal{M}_u$. □

**Theorem 2.3** *Every double sequence $x = (x_{jk})$ is almost convergent if and only if it is almost Cauchy.*

*Proof* Let $x = (x_{jk})$ be almost convergent double sequence. Then, for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\left| \frac{1}{pq} \sum_{j=0}^{p-1} \sum_{k=0}^{q-1} x_{j+s,k+t} - L \right| < \epsilon$$

for all $p, q > k$ and $(n, m) \in \mathbb{N} \times \mathbb{N}$. Therefore,

$$\begin{aligned}
&\left| \frac{1}{p_1 q_1} \sum_{j=0}^{p_1-1} \sum_{k=0}^{q_1-1} x_{j+s_1,k+t_1} - \frac{1}{p_2 q_2} \sum_{j=0}^{p_2-1} \sum_{k=0}^{q_2-1} x_{j+s_2,k+t_2} \right| \\
&\quad \leq \left| \frac{1}{p_1 q_1} \sum_{j=0}^{p_1-1} \sum_{k=0}^{q_1-1} x_{j+s_1,k+t_1} - L \right| + \left| \frac{1}{p_2 q_2} \sum_{j=0}^{p_2-1} \sum_{k=0}^{q_2-1} x_{j+s_2,k+t_2} - L \right| \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

for all $p_1, p_2, q_1, q_2 > k$ and $(s_1, t_1), (s_2, t_2) \in \mathbb{N} \times \mathbb{N}$. Hence, $x = (x_{jk})$ is almost Cauchy.

Conversely, let $x = (x_{jk})$ be almost Cauchy. Then, for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that

$$\left| \frac{1}{p_1 q_1} \sum_{j=0}^{p_1-1} \sum_{k=0}^{q_1-1} x_{j+s_1,k+t_1} - \frac{1}{p_2 q_2} \sum_{j=0}^{p_2-1} \sum_{k=0}^{q_2-1} x_{j+s_2,k+t_2} \right| < \frac{\epsilon}{2}$$

for all $p_1, p_2, q_1, q_2 > k$ and $(s_1, t_1), (s_2, t_2) \in \mathbb{N} \times \mathbb{N}$. Taking $s_1 = s_2 = s_0$ and $t_1 = t_2 = t_0$ in the above equation, we obtain that $(\frac{1}{pq} \sum_{j=0}^{p-1} \sum_{k=0}^{q-1} x_{j+s_0,k+t_0})_{p,q=1}^{\infty}$ is a Cauchy sequence and hence convergent. Let

$$P\text{-}\lim_{p,q\to\infty}\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}x_{j+s_0,k+t_0}=L.$$

Then, for every $\epsilon>0$, there exists $k_1\in\mathbb{N}$ such that

$$\left|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}x_{j+s_0,k+t_0}-L\right|<\frac{\epsilon}{2}$$

for all $p,q>k_1$. It follows that

$$\begin{aligned}&\left|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}x_{j+s,k+t}-L\right|\\ &\quad\le\left|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}x_{j+s,k+t}-\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}x_{j+s_0,k+t_0}\right|+\left|\frac{1}{pq}\sum_{j=0}^{p-1}\sum_{k=0}^{q-1}x_{j+s_0,k+t_0}-L\right|\\ &\quad<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon\end{aligned}$$

for all $p,q>\max(k,k_1)$ and $(n,m)\in\mathbb{N}\times\mathbb{N}$. Hence, $x$ is almost convergent to $L$. $\square$

## 2.3 Some Related Spaces of Double Sequences

In this section we introduce the following spaces involving the idea of Banach limit and almost convergence for double sequences. Such type of spaces for single sequences were studied by Das and Sahoo [31], and for double sequences by Mursaleen and Mohiuddine [100, 101].

Let $\mathbf{e}=(e_{jk})$ with $e_{jk}=1$ for all $j,k$. Then

$$w_2=\left\{x=(x_{jk}):\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}(x-L\mathbf{e})\longrightarrow 0\text{ as }m,n\longrightarrow\infty,\right.$$
$$\left.\text{uniformly in } s,t,\ \text{for some } L\right\},$$

$$[w_2]=\left\{x=(x_{jk}):\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\tau_{pqst}(x-L\mathbf{e})\right|\longrightarrow 0\text{ as }m,n\longrightarrow\infty,\right.$$
$$\left.\text{uniformly in } s,t,\ \text{for some } L\right\},$$

$$[w]_2=\Bigg\{x=(x_{jk}):\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}\big(|x-L\mathbf{e}|\big)\longrightarrow 0\text{ as }m,n\longrightarrow\infty,$$

$$\text{uniformly in } s,t,\ \text{for some } L\Bigg\},$$

$$\hat{w}_2=\Bigg\{x=(x_{jk}):\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|d_{pqst}-d_{p-1,q,s,t}-d_{p,q-1,s,t}+d_{p-1,q-1,s,t}|$$

$$\text{converges uniformly in } s,t\Bigg\},$$

$$\hat{\hat{w}}_2=\Bigg\{x=(x_{jk}):$$

$$\sup_{s,t}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|d_{pqst}-d_{p-1,q,s,t}-d_{p,q-1,s,t}+d_{p-1,q-1,s,t}|<\infty\Bigg\},$$

where

$$d_{m,n,s,t}=d_{m,n,s,t}(x)=\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}(x)$$

and

$$d_{0,0,s,t}(x)=\tau_{0,0,s,t}=x_{s,t},\qquad d_{-1,0,s,t}(x)=\tau_{-1,0,s,t}(x)=x_{s-1,t},$$
$$d_{0,-1,s,t}(x)=\tau_{0,-1,s,t}(x)=x_{s,t-1},\qquad d_{-1,-1,s,t}(x)=\tau_{-1,-1,s,t}(x)=x_{s-1,t-1}.$$

By $(C_2,2)$, we denote the space of *Cesàro summable* double sequences of order 2, which is defined by

$$(C_2,2)=\Bigg\{x=(x_{jk}):\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{p,q,0,0}(x)\longrightarrow L\text{ as }m,n\longrightarrow\infty\Bigg\},$$

and by $[C_2,2]$, we denote the space of *strongly Cesàro summable* double sequences of order 2, which is defined by

$$[C_2,2]=\Bigg\{x=(x_{jk}):\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\big|\tau_{p,q,0,0}(x)-L\big|\longrightarrow 0$$

$$\text{as }m,n\longrightarrow\infty\Bigg\}.$$

*Remark 2.4* If $[w_2]$-$\lim x = L$, that is,

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} |\tau_{pqst} - L| \longrightarrow 0$$

as $m, n \longrightarrow \infty$, uniformly in $s$, $t$, then

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \frac{1}{p+1} \sum_{j=0}^{p} \tau_{jqst} - L \right| \longrightarrow 0$$

and

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left| \frac{1}{q+1} \sum_{k=0}^{q} \tau_{pkst} - L \right| \longrightarrow 0.$$

## 2.4 Associated Sublinear Functionals

Let $G$ be any sublinear functional on $\mathcal{M}_u$. We write $\{\mathcal{M}_u, G\}$ to denote the set of all linear functionals $F$ on $\mathcal{M}_u$ such that $F \le G$, i.e., $F(x) \le G(x)$ for all $x = (x_{jk}) \in \mathcal{M}_u$.

Now we define the following functionals on the space $\mathcal{M}_u$ of real bounded double sequences:

$$\phi(x) = \limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x),$$

$$\psi(x) = \limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \big|\tau_{pqst}(x)\big|,$$

$$\theta(x) = \limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}\big(|x|\big),$$

$$\xi(x) = \limsup_{p,q} \sup_{s,t} \tau_{pqst}(x),$$

$$\eta(x) = \limsup_{p,q} \sup_{s,t} \tau_{pqst}\big(|x|\big),$$

where $|x| = (|x_{jk}|)_{j,k=1}^{\infty}$.

It can be easily verified that each of the above functionals is finite, well defined, and sublinear on $\mathcal{M}_u$.

A sublinear functional $G$ is said to *generate* Banach limits if $F \in \{\mathcal{M}_u, G\}$ is a Banach limit, and it is said to *dominate* Banach limits if $F \in \mathcal{B}_2$ implies $F \in \{\mathcal{M}_u, G\}$.

In the following theorem, we characterize the space $\mathcal{M}_u \cap w_2$ in terms of the sublinear functional $\phi$.

**Theorem 2.5** *We have the following*:

(i) *The sublinear functional $\phi$ both dominates and generates Banach limits*, *i.e.*, $\phi(x) = \xi(x)$ *for all* $x = (x_{jk}) \in \mathcal{M}_u$.
(ii) $\mathcal{M}_u \cap w_2 = \{x = (x_{jk}) \in \mathcal{M}_u : \phi(x) = -\phi(-x)\} = \mathcal{F}$.

*Proof* (i) By the definition of $\xi$, for given $\epsilon > 0$, there exist $p_0, q_0$ such that

$$\tau_{pqst}(x) < \xi(x) + \epsilon$$

for $p > p_0$, $q > q_0$ and all $s, t$. This implies that

$$\phi(x) \leq \xi(x) + \epsilon$$

for all $x = (x_{jk}) \in \mathcal{M}_u$. Since $\epsilon$ is arbitrary, we have $\phi(x) \leq \xi(x)$ for all $x = (x_{jk}) \in \mathcal{M}_u$, and hence,

$$\{\mathcal{M}_u, \phi\} \subset \{\mathcal{M}_u, \xi\} = \mathcal{B}_2 \tag{2.5}$$

i.e., $\phi$ generates Banach limits.

Conversely, suppose that $\mathcal{L} \in \mathcal{B}_2$. Since $\mathcal{L}$ is shift invariant, i.e., $\mathcal{L}(S_{11}x) = \mathcal{L}(x) = \mathcal{L}(S_{10}x) = \mathcal{L}(S_{01}x)$, we have

$$\begin{aligned}
\mathcal{L}(x) &= \mathcal{L}\left(\frac{1}{(p+1)(q+1)} \sum_{j=0}^{p} \sum_{k=0}^{q} x_{j+s,k+t}\right) \\
&= \mathcal{L}\big(\tau_{pqst}(x)\big) \\
&= \mathcal{L}\left(\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x)\right).
\end{aligned} \tag{2.6}$$

But it follows from the definition of $\phi$ that for given $\epsilon > 0$, there exist $m_0, n_0$ such that

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x) < \phi(x) + \epsilon \tag{2.7}$$

for $m > m_0$, $n > n_0$ and all $s, t$. Hence, by (2.7) and properties (i) and (ii) of Banach limits, we have

$$\mathcal{L}\left(\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x)\right) < \mathcal{L}\big((\phi(x)+\epsilon)\mathbf{e}\big) = \phi(x) + \epsilon \tag{2.8}$$

for $m > m_0$, $n > n_0$ and all $s$, $t$, where $\mathbf{e}$ is defined at the beginning of Sect. 2.3. Since $\epsilon$ was arbitrary, it follows from (2.6) and (2.8) that

$$\mathcal{L}(x) \leq \phi(x) \quad \text{for all } x = (x_{jk}) \in \mathcal{M}_u.$$

Hence,

$$\mathcal{B}_2 \subset \{\mathcal{M}_u, \phi\}, \tag{2.9}$$

that is, $\phi$ dominates Banach limits.

Combining (2.5) and (2.9), we get

$$\{\mathcal{M}_u, \xi\} = \{\mathcal{M}_u, \phi\},$$

which implies that $\phi$ dominates and generates Banach limits and $\phi(x) = \xi(x)$ for all $x \in \mathcal{M}_u$.

(ii) As a consequence of the Hahn–Banach theorem, $\{\mathcal{M}_u, \phi\}$ is nonempty, and a linear functional $F \in \{\mathcal{M}_u, \phi\}$ is not necessarily uniquely defined at any particular value of $x$. This is evident in the manner the linear functionals are constructed. But in order that all the functionals $\{\mathcal{M}_u, \phi\}$ coincide at $x = (x_{jk})$, it is necessary and sufficient that

$$\phi(x) = -\phi(-x) \tag{2.10}$$

and

$$\begin{aligned}\limsup_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x) \\ = \liminf_{m,n} \sup_{s,t} \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x).\end{aligned} \tag{2.11}$$

Equality (2.11) holds if and only if

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x) \longrightarrow L(\text{say}) \quad \text{as } m, n \longrightarrow \infty,$$

uniformly in $s$, $t$, and hence, $x = (x_{jk}) \in w_2 \cap \mathcal{M}_u$ for all $x = (x_{jk}) \in \mathcal{M}_u$, while Eq. (2.10) is equivalent to

$$\xi(x) = -\xi(-x),$$

which holds if and only if $x = (x_{jk}) \in \mathcal{F}$. □

In the following theorem, we characterize the spaces $[w_2] \cap \mathcal{M}_u$ and $[w]_2 \cap \mathcal{M}_u$ in terms of the sublinear functionals.

**Theorem 2.6** *We have the following*:

$$[w_2] \cap \mathcal{M}_u = \left\{x = (x_{jk}) : \psi(x - L\mathbf{e}) = 0,\ \textit{for some } L\right\}$$
$$= \left\{x = (x_{jk}) : F(x - L\mathbf{e}) = 0\, \textit{for all } F \in \{\mathcal{M}_u, \psi\}\, \textit{for some } L\right\}.$$

*Proof* Without loss of generality, we assume that $L = 0$. Now, as in Theorem 2.5(ii), as a consequence of the Hahn–Banach theorem, $\{\mathcal{M}_u, \psi\}$ is nonempty, and a linear functional $F \in \{\mathcal{M}_u, \psi\}$ is not necessarily uniquely defined at any particular value of $x$. In order that all the functionals $\{\mathcal{M}_u, \psi\}$ coincide at $x = (x_{jk})$, it is necessary and sufficient that

$$\psi(x) = -\psi(-x).$$

Hence, in general, the construction of the sublinear functional $\psi$ and the definition of $[w_2]$ together suggest that $x = (x_{jk}) \in [w_2] \cap \mathcal{M}_u$ if and only if

$$\psi(x - Le) = -\psi(L\mathbf{e} - x). \tag{2.12}$$

Since $\psi(x) = \psi(-x)$, (2.12) reduces to

$$\psi(x - L\mathbf{e}) = 0. \tag{2.13}$$

Now, if $F \in \{\mathcal{M}_u, \psi\}$, then from (2.13) and from the linearity of $F$ we have

$$F(x - L\mathbf{e}) = 0.$$

Conversely, suppose that $F(x - L\mathbf{e}) = 0$ for all $F \in \{\mathcal{M}_u, \psi\}$ and hence, by the Hahn–Banach theorem, there exists $F_0 \in \{\mathcal{M}_u, \psi\}$ such that $F_0(x) = \psi(x)$. Hence,

$$0 = F_0(x - L\mathbf{e}) = \psi(x - L\mathbf{e}).$$

This completes the proof of the theorem. □

## 2.5 Some Basic Lemmas

In this section, we give some important lemmas, which will be used in the proofs of our main results.

**Lemma 2.7** (Abel's transformation for double series)

$$\sum_{j=1}^{p}\sum_{k=1}^{q} v_{jk}(u_{jk} - u_{j+1,k} - u_{j,k+1} + u_{j+1,k+1})$$
$$= \sum_{j=1}^{p}\sum_{k=1}^{q} u_{jk}(\triangle_{11} v_{jk}) - \sum_{j=1}^{p} u_{j,q+1}(\triangle_{10} v_{jq}) - \sum_{k=1}^{q} u_{p+1,k}(\triangle_{01} v_{pk}) + u_{p+1,q+1} v_{pq},$$

*where*

$$\triangle_{10} v_{jq} = v_{jq} - v_{j-1,q}, \qquad \triangle_{01} v_{pk} = v_{pk} - v_{p,k-1} \quad \textit{and}$$
$$\triangle_{11} v_{jk} = v_{jk} - v_{j-1,k} - v_{j,k-1} + v_{j-1,k-1}.$$

*Proof* Abel's transformation for single series is

$$\sum_{i=1}^{m} v_i(u_i \mp u_{i+1}) = \sum_{i=1}^{m} u_i(v_i \mp v_{i-1}) \mp u_{m+1} v_m. \tag{2.14}$$

Now we prove Abel's transformation for double series:

$$\begin{aligned}
&\sum_{j=1}^{p}\sum_{k=1}^{q} v_{jk}(u_{jk} - u_{j+1,k} - u_{j,k+1} + u_{j+1,k+1}) \\
&\quad = \sum_{k=1}^{q}\left[\sum_{j=1}^{p} v_{jk}(u_{jk} - u_{j+1,k}) - \sum_{j=1}^{p} v_{jk}(u_{j,k+1} - u_{j+1,k+1})\right]
\end{aligned}$$

by using (2.14) we have

$$\begin{aligned}
&= \sum_{k=1}^{q}\left[\sum_{j=1}^{p} u_{jk}(v_{jk} - v_{j-1,k}) - u_{p+1,k} v_{pk}\right. \\
&\quad \left. - \sum_{j=1}^{p} u_{j,k+1}(v_{jk} - v_{j-1,k}) + u_{p+1,k+1} v_{pk}\right] \\
&= \sum_{j=1}^{p}\left[\sum_{k=1}^{q} v_{jk}(u_{jk} - u_{j,k+1}) - \sum_{k=1}^{q} v_{j-1,k}(u_{jk} - u_{j,k+1})\right] - \sum_{k=1}^{q} u_{p+1,k} v_{pk} \\
&\quad + \sum_{k=1}^{q} u_{p+1,k+1} v_{pk}
\end{aligned}$$

and now again using (2.14), we get

$$\begin{aligned}
&= \sum_{j=1}^{p}\left[\sum_{k=1}^{q} u_{jk}(v_{jk} - v_{j,k-1}) - u_{j,q+1} v_{jq} - \sum_{k=1}^{q} u_{jk}(v_{j-1,k} - v_{j-1,k-1})\right. \\
&\quad \left. + u_{j,q+1} v_{j-1,q}\right] - \sum_{k=1}^{q} u_{p+1,k} v_{pk} + \sum_{k=1}^{q} u_{p+1,k} v_{p,k-1} + u_{p+1,q+1} v_{pq} \\
&= \sum_{j=1}^{p}\sum_{k=1}^{q} u_{jk}(v_{jk} - v_{j,k-1} - v_{j-1,k} + v_{j-1,k-1}) - \sum_{j=1}^{p} u_{j,q+1} v_{jq}
\end{aligned}$$

$$+\sum_{j=1}^{p} u_{j,q+1}v_{j-1,q} - \sum_{k=1}^{q} u_{p+1,k}v_{pk} + \sum_{k=1}^{q} u_{p+1,k}v_{p,k-1} + u_{p+1,q+1}v_{pq}$$

$$= \sum_{j=1}^{p}\sum_{k=1}^{q} u_{jk}(\triangle_{11}v_{jk}) - \sum_{j=1}^{p} u_{j,q+1}(\triangle_{10}v_{jq}) - \sum_{k=1}^{q} u_{p+1,k}(\triangle_{01}v_{pk})$$

$$+ u_{p+1,q+1}v_{pq}. \qquad \square$$

Another form of Abel's transformation for double series is given by Altay and Başar [4].

**Lemma 2.8** $[w_2]$-$\lim x = L$ *if and only if*

(i) $w_2$-$\lim x = L$;
(ii) $\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v} |T_1(m,n,s,t) - L| \longrightarrow 0 \ (u, v \longrightarrow \infty)$ *uniformly in* $s, t$;
(iii) $\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v} |T_2(m,n,s,t) - L| \longrightarrow 0 \ (u, v \longrightarrow \infty)$ *uniformly in* $s, t$;
(iv) $\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v} |\tau_{mnst} + d_{mnst} - T_1(m,n,s,t) - T_2(m,n,s,t)| \longrightarrow 0 \ (u, v \longrightarrow \infty)$ *uniformly in* $s, t$, *where*

$$T_1(m,n,s,t) = \frac{1}{(m+1)}\sum_{p=0}^{m} \tau_{pnst} \quad \textit{and} \quad T_2(m,n,s,t) = \frac{1}{(n+1)}\sum_{q=0}^{n} \tau_{mqst}.$$

*Proof* Let $[w_2]$-$\lim x = L$. Then, obviously, $w_2$-$\lim x = L$. (ii) and (iii) follow immediately from Remark 2.4. Now we have

$$\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\tau_{mnst} + d_{mnst} - T_1(m,n,s,t) - T_2(m,n,s,t)\right|$$

$$= \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\left|\tau_{mnst} - L + d_{mnst} - L - T_1(m,n,s,t) + L - T_2(m,n,s,t) + L\right|$$

$$\le \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\Big(|\tau_{mnst} - L| + |d_{mnst} - L| + \left|T_1(m,n,s,t) - L\right|$$

$$+ \left|T_2(m,n,s,t) - L\right|\Big)$$

$\longrightarrow 0$ as $u, v \longrightarrow \infty$, uniformly in $s, t$, since

(a) $[w_2]$-$\lim x = L$ implies that the first sum tends to zero;
(b) (ii) and (iii) imply that the third and fourth sums tend to zero;
(c) (i) implies that $d_{mnst} \longrightarrow L \ (m, n \longrightarrow \infty)$ uniformly in $s, t$, and so the second sum tends to zero.

Conversely, suppose that the conditions hold. We have

$$\begin{aligned}
&\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}|\tau_{mnst}-L| \\
&\quad\leq \frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\Big|\tau_{mnst}+d_{mnst}-T_1(m,n,s,t)-T_2(m,n,s,t)\Big| \\
&\qquad+\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}|d_{mnst}-L|+\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\Big|T_1(m,n,s,t)-L\Big| \\
&\qquad+\frac{1}{uv}\sum_{m=1}^{u}\sum_{n=1}^{v}\Big|T_2(m,n,s,t)-L\Big|
\end{aligned}$$

$\longrightarrow 0$ as $u, v \longrightarrow \infty$, uniformly in $s, t$. □

**Lemma 2.9** *We have*

$$\begin{aligned}
&\tau_{mnst}+d_{mnst}-T_1(m,n,s,t)-T_2(m,n,s,t) \\
&\quad= mn[d_{mnst}-d_{m-1,n,s,t}-d_{m,n-1,s,t}+d_{m-1,n-1,s,t}].
\end{aligned}$$

*Proof* We shall use the equality

$$\begin{aligned}
&d_{mnst}-d_{m-1,n,s,t}-d_{m,n-1,s,t}+d_{m-1,n-1,s,t} \\
&\quad=\left[\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}-\frac{1}{m(n+1)}\sum_{p=0}^{m-1}\sum_{q=0}^{n}\tau_{pqst}\right] \\
&\qquad-\left[\frac{1}{(m+1)n}\sum_{p=0}^{m}\sum_{q=0}^{n-1}\tau_{pqst}-\frac{1}{mn}\sum_{p=0}^{m-1}\sum_{q=0}^{n-1}\tau_{pqst}\right]. \qquad (2.15)
\end{aligned}$$

First, we solve the expression in the first bracket:

$$\begin{aligned}
&\left[\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}-\frac{1}{m(n+1)}\sum_{p=0}^{m-1}\sum_{q=0}^{n}\tau_{pqst}\right] \\
&\quad=\frac{1}{m(m+1)(n+1)}\left[\sum_{q=0}^{n}\left(m\sum_{p=0}^{m}\tau_{pqst}-(m+1)\sum_{p=0}^{m-1}\tau_{pqst}\right)\right] \\
&\quad=\frac{1}{m(m+1)(n+1)}\sum_{q=0}^{n}\left[m\tau_{mqst}-\sum_{p=0}^{m-1}\tau_{pqst}\right]
\end{aligned}$$

$$= \frac{1}{m(m+1)(n+1)} \sum_{q=0}^{n} \left[ (m+1)\tau_{mqst} - \sum_{p=0}^{m} \tau_{pqst} \right]$$

$$= \frac{1}{m(n+1)} \sum_{q=0}^{n} \left[ \tau_{mqst} - \frac{1}{(m+1)} \sum_{p=0}^{m} \tau_{pqst} \right]$$

$$= \frac{1}{m(n+1)} \sum_{q=0}^{n} \tau_{mqst} - \frac{1}{m(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}$$

$$= \frac{1}{m(n+1)} \sum_{q=0}^{n} \tau_{mqst} - \frac{1}{m} d_{mnst}. \tag{2.16}$$

For the expression in the second bracket, we have

$$\frac{1}{(m+1)n} \sum_{p=0}^{m} \sum_{q=0}^{n-1} \tau_{pqst} - \frac{1}{mn} \sum_{p=0}^{m-1} \sum_{q=0}^{n-1} \tau_{pqst}$$

$$= \frac{1}{n} \sum_{q=0}^{n-1} \left[ \frac{1}{m(m+1)} \left\{ m \sum_{p=0}^{m} \tau_{pqst} - (m+1) \sum_{p=0}^{m-1} \tau_{pqst} \right\} \right]$$

$$= \frac{1}{n} \sum_{q=0}^{n-1} \left[ \frac{1}{m(m+1)} \left\{ m\tau_{mqst} - \sum_{p=0}^{m-1} \tau_{pqst} \right\} \right]$$

$$= \frac{1}{n} \sum_{q=0}^{n-1} \left[ \frac{1}{m(m+1)} \left\{ (m+1)\tau_{mqst} - \sum_{p=0}^{m} \tau_{pqst} \right\} \right]$$

$$= \frac{1}{mn} \sum_{q=0}^{n-1} \tau_{mqst} - \frac{1}{m(m+1)n} \sum_{p=0}^{m} \sum_{q=0}^{n-1} \tau_{pqst}$$

$$= \frac{1}{mn} \sum_{q=0}^{n-1} \tau_{mqst} - \frac{1}{m} d_{m,n-1,s,t}. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we get

$$d_{mnst} - d_{m-1,n,s,t} - d_{m,n-1,s,t} + d_{m-1,n-1,s,t}$$

$$= \frac{1}{m(n+1)} \sum_{q=0}^{n} \tau_{mqst} - \frac{1}{mn} \sum_{q=0}^{n-1} \tau_{mqst} - \frac{1}{m} d_{mnst} + \frac{1}{m} d_{m,n-1,s,t}$$

$$= \frac{1}{mn(n+1)} \left[ n \sum_{q=0}^{n} \tau_{mqst} - (n+1) \sum_{q=0}^{n-1} \tau_{mqst} \right] - \frac{1}{m} (d_{mnst} - d_{m,n-1,s,t})$$

$$= \frac{1}{mn(n+1)}\left[n\tau_{mnst} - \sum_{q=0}^{n-1}\tau_{mqst}\right] - \frac{1}{m}(d_{mnst} - d_{m,n-1,s,t})$$

$$= \frac{1}{mn(n+1)}\left[(n+1)\tau_{mnst} - \sum_{q=0}^{n}\tau_{mqst}\right] - \frac{1}{m}(d_{mnst} - d_{m,n-1,s,t})$$

$$= \frac{1}{mn}\left[\tau_{mnst} - \frac{1}{(n+1)}\sum_{q=0}^{n}\tau_{mqst}\right] - \frac{1}{m}(d_{mnst} - d_{m,n-1,s,t}). \tag{2.18}$$

We know that

$$d_{mnst} = \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}$$

$$= \frac{1}{(m+1)(n+1)}\left[\sum_{p=0}^{m-1}\sum_{q=0}^{n}\tau_{pqst} + \sum_{q=0}^{n}\tau_{mqst}\right] \tag{2.19}$$

and

$$d_{m-1,n,s,t} = \frac{1}{m(n+1)}\sum_{p=0}^{m-1}\sum_{q=0}^{n}\tau_{pqst}. \tag{2.20}$$

From (2.19) and (2.20) we have

$$(m+1)d_{mnst} - md_{m-1,n,s,t} = \frac{1}{(n+1)}\sum_{q=0}^{n}\tau_{mqst}. \tag{2.21}$$

Thus, (2.18) becomes

$$= \frac{1}{mn}\left[\tau_{mnst} - (m+1)d_{mnst} + md_{m-1,n,s,t}\right] - \frac{1}{m}(d_{mnst} - d_{m,n-1,s,t})$$

$$= \frac{1}{mn}\left[\tau_{mnst} - d_{mnst} - m(d_{mnst} - d_{m-1,n,s,t}) - n(d_{mnst} - d_{m,n-1,s,t})\right]. \tag{2.22}$$

Also, (2.21) can be written as

$$m(d_{mnst} - d_{m-1,n,s,t}) = \frac{1}{(n+1)}\sum_{q=0}^{n}\tau_{mqst} - d_{mnst}. \tag{2.23}$$

Similarly, we can write

$$n(d_{mnst} - d_{m,n-1,s,t}) = \frac{1}{(m+1)}\sum_{q=0}^{m}\tau_{pnst} - d_{mnst}. \tag{2.24}$$

Using (2.23) and (2.24) in (2.22), we get

$$= \frac{1}{mn}\left[\tau_{mnst} + d_{mnst} - \frac{1}{(m+1)}\sum_{p=0}^{m}\tau_{pnst} - \frac{1}{(n+1)}\sum_{q=0}^{n}\tau_{mqst}\right],$$

which implies that

$$\tau_{mnst}+d_{mnst}-T_m(n)-T_n(m)=mn[d_{mnst}-d_{m-1,n,s,t}-d_{m,n-1,s,t}+d_{m-1,n-1,s,t}].$$

□

**Lemma 2.10** *Let* $V_{mnst} = \sum_{p=m}^{\infty}\sum_{q=n}^{\infty}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}|$. *Then*

$$\begin{aligned} &V_{mnst} - V_{m,n+1,s,t} - V_{m+1,n,s,t} + V_{m+1,n+1,s,t} \\ &\quad = |d_{mnst} - d_{m-1,n,s,t} - d_{m,n-1,s,t} + d_{m-1,n-1,s,t}|. \end{aligned}$$

*Proof* We have

$$\begin{aligned} &V_{mnst} - V_{m,n+1,s,t} - V_{m+1,n,s,t} + V_{m+1,n+1,s,t} \\ &= \sum_{p=m}^{\infty}\sum_{q=n}^{\infty}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &\quad - \sum_{p=m}^{\infty}\sum_{q=n+1}^{\infty}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &\quad - \sum_{p=m+1}^{\infty}\sum_{q=n}^{\infty}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &\quad + \sum_{p=m+1}^{\infty}\sum_{q=n+1}^{\infty}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &= \sum_{p=m}^{\infty}\left[\sum_{q=n}^{\infty} - \sum_{q=n+1}^{\infty}\right]|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &\quad - \sum_{p=m+1}^{\infty}\left[\sum_{q=n}^{\infty} - \sum_{q=n+1}^{\infty}\right]|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &= \sum_{p=m}^{\infty}|d_{pnst} - d_{p-1,n,s,t} - d_{p,n-1,s,t} + d_{p-1,n-1,s,t}| \\ &\quad - \sum_{p=m+1}^{\infty}|d_{pnst} - d_{p-1,n,s,t} - d_{p,n-1,s,t} + d_{p-1,n-1,s,t}| \end{aligned}$$

$$= \left[ \sum_{p=m}^{\infty} - \sum_{p=m+1}^{\infty} \right] |d_{pnst} - d_{p-1,n,s,t} - d_{p,n-1,s,t} + d_{p-1,n-1,s,t}|$$

$$= |d_{mnst} - d_{m-1,n,s,t} - d_{m,n-1,s,t} + d_{m-1,n-1,s,t}|. \qquad \square$$

## 2.6 Inclusion Relations

We establish here some inclusion relations between the spaces defined in Sect. 2.3.

**Theorem 2.11** *We have the following inclusions with the limit preserved in each case, while the reverse inclusions do not hold in general.*

(i) $\hat{w}_2 \subset [w_2]$ *if conditions* (ii) *and* (iii) *of Lemma* 2.8 *hold.*
(ii) $\hat{w}_2 \subset \hat{\hat{w}}_2$.

*Proof* (i) We have to show that $\hat{w}_2 \subset [w_2]$. If $x \in \hat{w}_2$, then we have

$$V_{mnst} = \sum_{p=m}^{\infty} \sum_{q=n}^{\infty} |d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \tag{2.25}$$

$\longrightarrow 0$ as $m, n \longrightarrow \infty$, uniformly in $s, t$, and

$$d_{pqst} \longrightarrow L \quad \text{(say) as } p, q \longrightarrow \infty \text{ uniformly in } s, t,$$

that is, $w_2\text{-}\lim x = L$.

In order to prove that $x \in [w_2]$, it is enough to show that condition (iv) of Lemma 2.8 holds. By Lemmas 2.9 and 2.10 we have

$$\begin{aligned} &\tau_{pqst} + d_{pqst} - T_1(p, q, s, t) - T_2(p, q, s, t) \\ &\quad = pq[d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}] \end{aligned}$$

and

$$\begin{aligned} &|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \\ &\quad = V_{pqst} - V_{p,q+1,s,t} - V_{p+1,q,s,t} + V_{p+1,q+1,s,t}, \end{aligned}$$

so that

$$\begin{aligned} &\frac{1}{mn} \sum_{p=1}^{m} \sum_{q=1}^{n} \left| \tau_{pqst} + d_{pqst} - T_1(p, q, s, t) - T_2(p, q, s, t) \right| \\ &\quad = \frac{1}{mn} \sum_{p=1}^{m} \sum_{q=1}^{n} pq[V_{pqst} - V_{p,q+1,s,t} - V_{p+1,q,s,t} + V_{p+1,q+1,s,t}] \end{aligned}$$

using Lemma 2.7 for Abel's transformation, we have

$$= \frac{1}{mn}\left[\sum_{p=1}^{m}\sum_{q=1}^{n} V_{pqst} - m\sum_{q=1}^{n} V_{m+1,q,s,t} - n\sum_{p=1}^{m} V_{p,n+1,s,t} + mnV_{m+1,n+1,s,t}\right]$$

$\longrightarrow 0$ as $m, n \longrightarrow \infty$, uniformly in $s, t$ (by 2.25). Hence, by Lemma 2.8, $x \in [w_2]$.

(ii) Let $x \in \hat{w}_2$. We have to show that

$$\sup_{s,t}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \leq K,$$

where $K$ is an absolute constant. As $x \in \hat{w}_2$, there exist integers $p_0, q_0$ such that

$$\sum_{p>p_0}\sum_{q>q_0}|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| < 1, \quad \text{for all } s, t. \tag{2.26}$$

Hence, it is left to show that, for fixed $p, q$,

$$|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \leq K \quad \text{for all } s, t.$$

From (2.26) we have that

$$|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| < 1 \tag{2.27}$$

for every fixed $p > p_0$, $q > q_0$ and all $s, t$. Since

$$\begin{aligned} &m(m+1)n(n+1)(d_{mnst} - d_{m-1,n,s,t} - d_{m,n-1,s,t} + d_{m-1,n-1,s,t}) \\ &\quad = \sum_{p=1}^{m}\sum_{q=1}^{n} pq(\tau_{pqst} - \tau_{p-1,q,s,t} - \tau_{p,q-1,s,t} + \tau_{p-1,q-1,s,t}), \end{aligned} \tag{2.28}$$

we have

$$\begin{aligned} &mn\big[(m+1)(n+1)(d_{mnst} - d_{m-1,n,s,t} - d_{m,n-1,s,t} + d_{m-1,n-1,s,t}) \\ &\quad - (m-1)(n+1)(d_{m-1,n,s,t} - d_{m-2,n,s,t} - d_{m-1,n-1,s,t} + d_{m-2,n-1,s,t}) \\ &\quad - (m+1)(n-1)(d_{m,n-1,s,t} - d_{m-1,n-1,s,t} - d_{m,n-2,s,t} + d_{m-1,n-2,s,t}) \\ &\quad + (m-1)(n-1)(d_{m-1,n-1,s,t} - d_{m-2,n-1,s,t} - d_{m-1,n-2,s,t} + d_{m-2,n-2,s,t})\big] \\ &= \sum_{q=1}^{n} q\left[\sum_{p=1}^{m} p(\tau_{pqst} - \tau_{p-1,q,s,t} - \tau_{p,q-1,s,t} + \tau_{p-1,q-1,s,t})\right. \\ &\quad \left. - \sum_{p=1}^{m-1} p(\tau_{pqst} - \tau_{p-1,q,s,t} - \tau_{p,q-1,s,t} + \tau_{p-1,q-1,s,t})\right] \end{aligned}$$

$$
\begin{aligned}
&-\sum_{q=1}^{n-1} q\left[\sum_{p=1}^{m} p(\tau_{pqst}-\tau_{p-1,q,s,t}-\tau_{p,q-1,s,t}+\tau_{p-1,q-1,s,t})\right.\\
&\left.-\sum_{p=1}^{m-1} p(\tau_{pqst}-\tau_{p-1,q,s,t}-\tau_{p,q-1,s,t}+\tau_{p-1,q-1,s,t})\right]\\
&=\sum_{q=1}^{n} q\big[m(\tau_{mqst}-\tau_{m-1,q,s,t}-\tau_{m,q-1,s,t}+\tau_{m-1,q-1,s,t})\big]\\
&-\sum_{q=1}^{n-1} q\big[m(\tau_{mqst}-\tau_{m-1,q,s,t}-\tau_{m,q-1,s,t}+\tau_{m-1,q-1,s,t})\big].
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&(m+1)(n+1)(d_{mnst}-d_{m-1,n,s,t}-d_{m,n-1,s,t}+d_{m-1,n-1,s,t})\\
&\quad-(m-1)(n+1)(d_{m-1,n,s,t}-d_{m-2,n,s,t}-d_{m-1,n-1,s,t}+d_{m-2,n-1,s,t})\\
&\quad-(m+1)(n-1)(d_{m,n-1,s,t}-d_{m-1,n-1,s,t}-d_{m,n-2,s,t}+d_{m-1,n-2,s,t})\\
&\quad+(m-1)(n-1)(d_{m-1,n-1,s,t}-d_{m-2,n-1,s,t}-d_{m-1,n-2,s,t}+d_{m-2,n-2,s,t})\\
&=[\tau_{mnst}-\tau_{m-1,n,s,t}-\tau_{m,n-1,s,t}+\tau_{m-1,n-1,s,t}].
\end{aligned}
\tag{2.29}
$$

Using (2.27) and (2.29), we have that

$$
|\tau_{mnst}-\tau_{m-1,n,s,t}-\tau_{m,n-1,s,t}+\tau_{m-1,n-1,s,t}|\le K(m,n) \tag{2.30}
$$

for every fixed $m>p_0$, $n>q_0$ and all $s$, $t$, where $K(m,n)$ is a constant depending on $m$, $n$. Again, from the definition of $\tau_{mnst}$ we obtain, similarly to (2.25),

$$
\begin{aligned}
&\tau_{mnst}-\tau_{m-1,n,s,t}-\tau_{m,n-1,s,t}+\tau_{m-1,n-1,s,t}\\
&\quad=\frac{1}{m(m+1)n(n+1)}\sum_{u=1}^{m}\sum_{v=1}^{n} uv a_{u+s,v+t},
\end{aligned}
\tag{2.31}
$$

so that

$$
\begin{aligned}
&a_{m+s,n+t}\\
&=(m+1)(n+1)(\tau_{mnst}-\tau_{m-1,n,s,t}-\tau_{m,n-1,s,t}+\tau_{m-1,n-1,s,t})\\
&\quad-(m-1)(n+1)(\tau_{m-1,n,s,t}-\tau_{m-2,n,s,t}-\tau_{m-1,n-1,s,t}+\tau_{m-2,n-1,s,t})\\
&\quad-(m+1)(n-1)(\tau_{m,n-1,s,t}-\tau_{m-1,n-1,s,t}-\tau_{m,n-2,s,t}+\tau_{m-1,n-2,s,t})\\
&\quad+(m-1)(n-1)(\tau_{m-1,n-1,s,t}-\tau_{m-2,n-1,s,t}-\tau_{m-1,n-2,s,t}+\tau_{m-2,n-2,s,t}).
\end{aligned}
$$

Hence, it follows from (2.30) that, for any fixed $m > p_0$ and $n > q_0$,

$$|a_{m+s,n+t}| \leq K(m,n) \quad \text{for all } s,t. \tag{2.32}$$

Now choose $m = m_0 + 1$, $n = n_0 + 1$. Let

$$K = \max\{K(m_0+1, n_0+1), |a_{1,n_0+1}|, |a_{m_0+1,1}|, |a_{2,n_0+1}|, |a_{m_0+1,2}|, \ldots, \\ |a_{n_0+1,n_0+1}|\}.$$

It follows from (2.32) that

$$|a_{u,v}| \leq K \quad \text{for all } u, v$$

where $K$ is independent of $u$, $v$. By (2.31) we have

$$|\tau_{mnst} - \tau_{m-1,n,s,t} - \tau_{m,n-1,s,t} + \tau_{m-1,n-1,s,t}| \leq K \quad \text{for all } m,n,s,t. \tag{2.33}$$

Also, from (2.28) and (2.33) we have

$$|d_{pqst} - d_{p-1,q,s,t} - d_{p,q-1,s,t} + d_{p-1,q-1,s,t}| \leq K \quad \text{for all } p,q,s,t. \qquad \square$$

**Theorem 2.12** *We have the following proper inclusions with the limit preserved in each case*:

$$[\mathcal{F}] \subset [w]_2 \subset [w_2] \subset w_2 \subset (C_2, 2).$$

*Proof* Let $x \in [\mathcal{F}]$ with $[\mathcal{F}]$-$\lim x = L$, say. Then

$$\tau_{pqst}\big(|x - L\mathbf{e}|\big) \longrightarrow 0 \quad \text{as } p,q \longrightarrow \infty, \text{ uniformly in } s,t.$$

This implies that

$$\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}\big(|x - L\mathbf{e}|\big) \longrightarrow 0 \quad \text{as } p,q \longrightarrow \infty, \text{ uniformly in } s,t.$$

This proves that $x \in [w]_2$ and $[\mathcal{F}]$-$\lim x = [w]_2$-$\lim x = L$.

Since

$$\begin{aligned} \frac{1}{(m+1)(n+1)} \left| \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}(x - L\mathbf{e}) \right| &\leq \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \left|\tau_{pqst}(x - L\mathbf{e})\right| \\ &\leq \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} \tau_{pqst}\big(|x - L\mathbf{e}|\big), \end{aligned}$$

this implies that $[w]_2 \subset [w_2] \subset w_2$ and

$$[w]_2\text{-}\lim x = [w_2]\text{-}\lim x = w_2\text{-}\lim x = L.$$

Since

$$\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\tau_{pqst}(x-\ell\mathbf{e})$$

converges uniformly in $s, t$ as $m, n \longrightarrow \infty$, we have the convergence for $s = 0 = t$. It follows that $w_2 \subset (C_2, 2)$ and $w_2$-$\lim x = (C_2, 2)$-$\lim x = L$. □

*Example 2.13* $[w]_2 \cap \mathcal{M}_u \subsetneq [w_2] \cap \mathcal{M}_u$.

Let $x = (x_{jk})$ be defined by

$$x_{jk} = (-1)^k \quad \text{for all } j,$$

that is,

$$\begin{pmatrix} -1 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ -1 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ -1 & 1 & -1 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Then

$$\begin{aligned}\left|\tau_{pqst}(x-0)\right| &= \left|\frac{1}{(p+1)(q+1)}\sum_{j=0}^{p}\sum_{k=0}^{q}x_{j+s,k+t}\right| \\ &\le \frac{q+1}{(p+1)(q+1)} = \frac{1}{p+1} \quad \text{uniformly for } s, t.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\left|\tau_{pqst}(x-0)\right| &\le \frac{1}{(m+1)(n+1)}\sum_{p=0}^{m}\sum_{q=0}^{n}\frac{1}{p+1} \\ &= \frac{1}{(n+1)(m+1)}\sum_{p=0}^{m}\frac{n+1}{p+1} \\ &= \frac{1}{(m+1)}\sum_{p=0}^{m}\frac{1}{p+1} \to 0 \quad \text{as } m \to \infty,\end{aligned}$$

i.e., $x = (x_{jk}) \in [w_2] \cap \mathcal{M}_u$, and hence, by Theorem 2.5, $x \in \mathcal{F}$. But $x \notin [w]_2 \cap \mathcal{M}_u$, and hence $x \notin [\mathcal{F}]$.

## 2.7 Exercises

**1** Check whether the double sequence $x = (x_{nk})$ defined by

$$x_{nk} = \begin{cases} 1 & \text{if } n \text{ and } k \text{ are squares,} \\ 0 & \text{otherwise} \end{cases}$$

is almost convergent or not. If yes, then what is its almost limit?

**2** Show that the inclusion

$$[\mathcal{F}] \subset \mathcal{F}$$

is proper.

**3** Prove that

$$\begin{aligned} [w]_2 \cap \mathcal{M}_u &= \left\{x = (x_{jk}) : \theta(x - L\mathbf{e}) = 0 \text{ for some } L\right\} \\ &= \left\{x = (x_{jk}) : F(x - L\mathbf{e}) = 0 \text{ for all } F \in \{\mathcal{M}_u, \theta\} \text{ and some } L\right\}. \end{aligned}$$

**4** Prove that

$$[\mathcal{F}] \subset [w_2] \subset w_2$$

and

$$[w_2]\text{-}\lim x = w_2\text{-}\lim x = L.$$

**5** Show that the inclusion

$$[w]_2 \subset [w_2]$$

is proper.

**6** Prove the following proper inclusions:

$$[\mathcal{F}] \subset [w]_2 \cap \mathcal{M}_u \subset [w_2] \cap \mathcal{M}_u \subset \mathcal{F}.$$

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