Chapter 2
Behaviour of Solutions of Linear Homogeneous Differential Equations of Third Order

This chapter is concerned with the oscillation and nonoscillation of solutions of a class of third-order linear homogeneous differential equations of the form

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \tag{2.1}$$

where $a \in C^2([\sigma, \infty), R)$, $b \in C^1([\sigma, \infty), R)$, $c \in C([\sigma, \infty), R)$ and $\sigma \in R$.

As dealt with in Sect. 1.2, according to the signs of $a(t)$, $b(t)$ and $c(t)$, one may consider the following eight different cases: (i) $a(t) \geq 0$, $b(t) \leq 0$, $c(t) > 0$, (ii) $a(t) \leq 0$, $b(t) \leq 0$, $c(t) > 0$, (iii) $a(t) \leq 0$, $b(t) \leq 0$, $c(t) < 0$, (iv) $a(t) \geq 0$, $b(t) \leq 0$, $c(t) < 0$, (v) $a(t) \geq 0$, $b(t) \geq 0$, $c(t) > 0$, (vi) $a(t) \leq 0$, $b(t) \geq 0$, $c(t) > 0$, (vii) $a(t) \geq 0$, $b(t) \geq 0$, $c(t) < 0$ and (viii) $a(t) \leq 0$, $b(t) \geq 0$, $c(t) < 0$.

Equation (2.1) may be written as

$$\big(r(t)x''\big)' + q(t)x' + p(t)x = 0, \tag{2.2}$$

where $r(t) = e^{\int_0^t a(s)\,ds}$, $q(t) = r(t)b(t)$ and $p(t) = r(t)c(t)$. Since Eqs. (2.1) and (2.2) are equivalent, we refer to either of them according to our convenience. The adjoint of (2.1) is given by

$$\Big(\big(x' - a(t)x\big)' + b(t)x\Big)' - c(t)x = 0,$$

that is,

$$x''' - a(t)x'' + \big(b(t) - 2a'(t)\big)x' - \big(c(t) - b'(t) + a''(t)\big)x = 0. \tag{2.3}$$

The transformation

$$x(t) = z(t)e^{-\frac{1}{3}\int_\sigma^t a(s)\,ds}$$

transforms (2.1) into

$$z''' + Q(t)z' + P(t)z = 0, \tag{2.4}$$

where $P(t) = \frac{1}{27}(2a^3(t) - 9a(t)b(t) + 27c(t) - 9a''(t))$ and $Q(t) = \frac{1}{3}(3b(t) - 3a'(t) - a^2(t))$. Clearly (2.1) is oscillatory, if and only if (2.4) is oscillatory.

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This chapter has been divided into nine sections. First-six sections deal with the oscillation, nonoscillation of solutions of (2.1) and their asymptotic behaviour for the six among the above eight different cases. Sufficient conditions have been obtained for oscillation of solutions of Eq. (2.1) of which the Euler equation

$$x'''(t) + \frac{a_0}{t}x''(t) + \frac{b_0}{t^2}x'(t) + \frac{c_0}{t^3}x(t) = 0, \tag{2.5}$$

where a_0, b_0 and c_0 are reals, is a particular case. Section 2.7 deals with the oscillation, nonoscillation, property A and B of a third-order disconjugate equation. Some remarks and open problems are given in Sect. 2.8. Section 2.9 contains a note for the readers for easy reference.

The results presented in this chapter on the oscillation and nonoscillation of (2.1) and (2.2) are new and were not presented in the monograph [14].

2.1 Behaviour of Solutions of $x''' + a(t)x'' + b(t)x' + c(t)x = 0$ with $a(t) \geq 0$, $b(t) \leq 0$ and $c(t) > 0$

This section deals with the oscillation and asymptotic behaviour of solutions of (2.1) with $a(t) \geq 0$, $b(t) \leq 0$ and $c(t) > 0$. The objective here is to present results generalising the observations in Proposition 1.2.1 to Eq. (2.1). Further, sufficient conditions are given for the oscillation of (2.1), which, in particular, is applicable to the Euler equation (2.5). Necessary and sufficient condition is given in Theorems 2.1.1 and 2.1.7 for (2.1) to be oscillatory. Theorems 2.1.9 and 2.1.10 give a sufficient condition for the nonoscillation of (2.1) when $a(t) \equiv 0$. Theorems 2.1.11 and 2.1.12 show that (2.1) admits at most one nonoscillatory solution $x(t)$ (neglecting linear dependence) with the property $x(t)x'(t) < 0$, $x(t)x''(t) > 0$ for $t \in [\sigma, \infty)$ and $\lim_{t\to\infty} x(t) = 0$, provided that (2.1) has an oscillatory solution.

In the particular case, when $a(t) \equiv 0$, $b(t) \equiv 0$ and $c(t) > 0$, $t \geq \sigma$, (2.1) reduces to

$$x''' + c(t)x = 0. \tag{2.6}$$

There is a well-known oscillation criterion for (2.6) of the form

$$\int^{\infty} t^{2-\epsilon} c(t)\,dt = \infty, \quad \text{for some } \epsilon > 0. \tag{2.7}$$

This condition has been improved several times. Chanturiya [7] proved that if

$$\liminf_{t\to\infty} t^2 \int_t^{\infty} c(s)\,ds > \frac{2}{3\sqrt{3}}, \tag{2.8}$$

then (2.6) is oscillatory. Lazer [23] proved that if

$$\int^{\infty} \left[c(t) - \frac{2}{3\sqrt{3}}\big(-b(t)\big)^{3/2}\right] dt = \infty, \tag{2.9}$$

then

$$x''' + b(t)x' + c(t)x = 0 \tag{2.10}$$

is oscillatory. We note that (2.10) can be obtained from (2.1) with $a(t) \equiv 0$. We start this section with the following lemma.

Lemma 2.1.1 *Equation* (2.1) *is of type* C_I.

Proof Suppose that $x(t)$ is a solution of (2.1) with $x(t_0) = x'(t_0) = 0$ and $x''(t_0) > 0$ for some $t_0 > \sigma$. We show that $x(t) > 0$ for $t \in [\sigma, t_0)$. If possible, suppose that $x(t_1) = 0$ for some $t_1 \in [\sigma, t_0)$. There exists a $t_2 \in (t_1, t_0)$ such that $x'(t_2) = 0$. Consequently, there exists a $t_3 \in (t_2, t_0)$ such that $x''(t_3) = 0$ and $x(t) > 0$, $x'(t) < 0$ and $x''(t) > 0$ for $t \in (t_3, t_0)$. Since $x(t)$ is a solution of (2.1), it satisfies (2.2). Integrating (2.2) from t_3 to t_0, we obtain

$$0 < r(t_0)x''(t_0) = \int_{t_3}^{t_0} \big[-q(t)x'(t) - p(t)x(t)\big]\,dt < 0,$$

a contradiction. Thus $x(t) > 0$ for $t \in [\sigma, t_0)$. The lemma is proved. □

Remark 2.1.1 If $x(t)$ is a solution of (2.1) with $x(t_0) \geq 0$, $x'(t_0) \leq 0$ and $x''(t_0) > 0$, $t_0 \in (\sigma, \infty)$, then $x(t) > 0$, $x'(t) < 0$, $x''(t) > 0$ and $x'''(t) < 0$ for $t \in [\sigma, t_0)$.

Lemma 2.1.2 *If* $2c(t) - a(t)b(t) - b'(t) \geq 0$ *but* $\not\equiv 0$ *on any subinterval of* $[\sigma, \infty)$, *then* (2.1) *is of type* C_I.

Proof Suppose that $x(t)$ is a solution of (2.1) with $x(t_0) = x'(t_0) = 0$ and $x''(t_0) > 0$. If possible, suppose that $x(t_1) = 0$ for some $t_1 \in [\sigma, t_0)$. Since $x(t)$ is a solution of (2.1), it satisfies (2.2). Multiplying (2.2) through by $x(t)$ and integrating the resulting identity from t_1 to t_0, we obtain

$$-r(t_1)\big(x'(t_1)\big)^2 \geq \int_{t_1}^{t_0} \big(2p(t) - q'(t)\big)x^2(t)\,dt,$$

that is,

$$0 \geq -r(t_1)\big(x'(t_1)\big)^2 \geq \int_{t_1}^{t_0} \big(2c(t) - a(t)b(t) - b'(t)\big)r(t)x^2(t)\,dt > 0,$$

a contradiction. Hence $x(t) > 0$ for $t \in [\sigma, t_0)$. This completes the proof of the lemma. □

One may observe that the condition $b(t) \leq 0$ and $c(t) > 0$ has not been assumed explicitly in Lemma 2.1.2.

Theorem 2.1.1 *Equation (2.1) is oscillatory, if and only if all nonoscillatory solutions of the second-order differential equation*

$$z'' + 3zz' + a(t)z' + z^3 + a(t)z^2 + b(t)z + c(t) = 0 \tag{2.11}$$

are eventually negative.

Proof Suppose that all nonoscillatory solutions of (2.11) are eventually negative. It is required to show that (2.1) admits an oscillatory solution. If possible, let all solution of (2.1) be nonoscillatory. By Lemma 1.5.4, there exists at least one nonoscillatory solution $u(t)$ of (2.1) which does not satisfy the condition $u(t)u'(t) < 0$. Without any loss of generality, we may take $u(t) > 0$ for $t \geq t_0 \geq \sigma$. From Lemma 1.5.3, it follows that $u'(t) > 0$ for $t \geq t_1 > t_0$. Setting $z(t) = \frac{u'(t)}{u(t)}$, $t > t_1$, it is easy to verify that $z(t)$ is a nonnegative nonoscillatory solution of (2.11), a contradiction. Hence (2.1) admits an oscillatory solution.

Conversely, suppose that (2.1) has an oscillatory solution. If possible, let $z(t)$ be a positive nonoscillatory solution of (2.11). It may be verified that $v(t) = \exp(\int_{t_0}^{t} z(s)\,ds)$ is a positive increasing solution of (2.1), which contradicts Lemma 1.5.4. Hence the proof of the theorem is completed. □

Theorem 2.1.2 *Suppose that $a'(t) \leq 0$. If*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right] dt = \infty, \tag{2.12}$$

then (2.1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of (2.1). From Lemma 1.5.3, it follows that there exists a $t_0 \in [\sigma, \infty)$ such that $x'(t) \leq 0$ or ≥ 0 for $t \in [t_0, \infty)$. In view of Lemma 1.5.4 and the second part of Lemma 1.5.3, it is sufficient to prove that $x(t)x'(t) \geq 0$ for $t > t_0$ does not hold.

Suppose that $x(t)x'(t) \geq 0$ for $t > t_0$. Setting $u(t) = \frac{x'(t)}{x(t)}$, $t \geq t_0$, we see that $u(t)$ is a solution of the second-order Riccati equation

$$z'' + 3zz' + a(t)z' = -F\big(u(t), t\big), \tag{2.13}$$

where

$$F\big(u(t), t\big) = u^3(t) + a(t)u^2(t) + b(t)u(t) + c(t).$$

Clearly, $F(u(t), t)$ attains its minimum value for $u(t) \geq 0$ at

$$u(t) = \frac{1}{3}\left[-a(t) + \big(a^2(t) - 3b(t)\big)^{1/2}\right]$$

and the minimum of $F(u(t), t)$ is given by

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}.$$

So,

$$\begin{aligned}
&u''(t) + 3u(t)u'(t) + a(t)u'(t) \\
&\quad \le -\left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t)\right)^{3/2}\right].
\end{aligned}$$

Integrating the above inequality from t_0 to t, we obtain

$$\begin{aligned}
u'(t) &\le u'(t_0) + \frac{3}{2}u^2(t_0) + a(t_0)u(t_0) - \frac{3}{2}u^2(t) - a(t)u(t) + \int_{t_0}^{t} a'(s)u(s)\,ds \\
&\quad - \int_{t_0}^{t}\left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s)\right)^{3/2}\right]ds \\
&\le u'(t_0) + \frac{3}{2}u^2(t_0) + a(t_0)u(t_0) \\
&\quad - \int_{t_0}^{t}\left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s)\right)^{3/2}\right]ds.
\end{aligned}$$

From (2.12), it follows that $\lim_{t\to\infty} u'(t) = -\infty$. Hence $u(t) < 0$ for large t, a contradiction. This completes the proof of the theorem. □

Theorem 2.1.3 *Let $b(t) - a'(t) \le 0$ and*

$$\begin{aligned}
&\int_{\sigma}^{\infty}\left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \frac{a(t)a'(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right]dt \\
&\quad = \infty,
\end{aligned} \tag{2.14}$$

then (2.1) *is oscillatory.*

Remark 2.1.2 Theorem 2.1.2 generalises the observation (i) in Proposition 1.2.1. However, it fails to hold for Eq. (2.5). Instead of setting $u(t) = \frac{x'(t)}{x(t)}$ in proof of Theorem 2.1.2, if we set $u(t) = \frac{t^2 x'(t)}{x(t)}$ for $t \ge t_0$, we see that $u(t)$ is a positive solution of the second-order Riccati equation

$$\left[z' - \frac{4}{t}z + \frac{3}{2t^2}z^2 + a(t)z\right]' = -G\big(u(t), t\big), \tag{2.15}$$

where

$$\begin{aligned}
G\big(u(t), t\big) &= \frac{u^3(t)}{t^4} - \frac{3u^2(t)}{t^3} + \frac{a(t)u^2(t)}{t^2} + \frac{2u(t)}{t^2} - a'(t)u(t) \\
&\quad - \frac{2a(t)u(t)}{t} + b(t)u(t) + t^2 c(t).
\end{aligned} \tag{2.16}$$

If $ta(t) \le 3$ and $1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t) \ge 0$, then a simple calculation shows that $G(u(t),t)$ attains the minimum

$$\min G\big(u(t),t\big) = \frac{2t^2a^3(t)}{27} - \frac{t^2a(t)b(t)}{3} + t^2c(t) - \frac{2}{3}a(t) + tb(t) + \frac{t^2a(t)a'(t)}{3}$$
$$- ta'(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{3/2}$$

for $u(t) > 0$ at

$$u(t) = t\left[\left(1 - \frac{ta(t)}{3}\right) + \frac{1}{\sqrt{3}}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{1/2}\right].$$

Hence (2.15) yields

$$\left[u'(t) - \frac{4u(t)}{t} + \frac{3}{2}\frac{u^2(t)}{t^2} + a(t)u(t)\right]' \le -\min G\big(u(t),t\big).$$

Integrating the above inequality from t_0 to t, we obtain

$$u'(t) - \frac{4u(t)}{t} + \frac{3}{2}\frac{u^2(t)}{t^2} \le K - \int_{t_0}^{t} \min G\big(u(s),s\big)\,ds.$$

Since $\frac{3}{2}\frac{u^2(t)}{t^2} - \frac{4u(t)}{t}$ attains the minimum $-\frac{8}{3}$ for $u(t) > 0$ at $u(t) = \frac{4t}{3}$, the above integral inequality yields

$$u'(t) \le K + \frac{8}{3} - \int_{t_0}^{t} \min G\big(u(s),s\big)\,ds.$$

The above inequality leads us to the following theorem.

Theorem 2.1.4 *Let $ta(t) \le 3$ and $1 - t^2b(t) + \frac{t^2a^2(t)}{3} + t^2a'(t) \ge 0$. If*

$$\int_{\sigma}^{\infty}\left[\frac{2t^2a^3(t)}{27} - \frac{t^2a(t)b(t)}{3} + t^2c(t) - \frac{2}{3}a(t) + tb(t) + \frac{t^2a(t)a'(t)}{3}\right.$$
$$\left. - ta'(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{3/2}\right]dt = \infty, \qquad (2.17)$$

then (2.1) is oscillatory.

On the other hand, if $ta(t) \ge 3$ and $2ta(t) - t^2b(t) + t^2a'(t) > 2$, then

$$\left(1 - \frac{ta(t)}{3}\right) + \frac{1}{\sqrt{3}}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{1/2} > 0.$$

In this case, one can prove the following theorem:

Theorem 2.1.5 *Suppose that $ta(t) \geq 3$ and $2ta(t) - t^2b(t) + t^2a'(t) > 2$ hold. If (2.17) is satisfied, then (2.1) is oscillatory.*

Remark 2.1.3 Theorem 2.1.4 can be applied to find sufficient conditions for the oscillation of the Euler equation (2.5). If $0 \leq a_0 \leq 3$, $b_0 \leq 0$, $c_0 > 0$, $1 - b_0 + \frac{a_0^2}{3} - a_0 > 0$ and

$$\frac{2a_0^3}{27} + \frac{a_0}{3} + b_0 + c_0 - \frac{a_0^2}{3} - \frac{a_0b_0}{3} - \frac{2}{3\sqrt{3}}\left(1 + \frac{1}{3}a_0^2 - a_0 - b_0\right)^{3/2} > 0, \quad (2.18)$$

then (2.5) is oscillatory.

Applying Theorem 2.1.5 to the Euler equation (2.5), we see that (2.5) is oscillatory if $a_0 \geq 3$, $b_0 \leq 0$, $c_0 > 0$, $a_0 - b_0 > 2$ and (2.18) hold.

Example 2.1.1 Consider

$$x''' + \frac{1}{t}x'' - \frac{1}{t^2}x' + \frac{3}{t^3}x = 0, \quad t \geq 1.$$

Since all the conditions of Theorem 2.1.4 are satisfied, this equation has an oscillatory solution. In particular, $x_1(t) = t^{3/2}\cos(\frac{\sqrt{3}}{2}\log t)$ and $x_2(t) = t^{3/2}\sin(\frac{\sqrt{3}}{2}\log t)$ are the oscillatory solutions of this equation.

Example 2.1.2 The equation

$$x''' + \frac{6}{t}x'' - \frac{1}{t^2}x' + \frac{25}{t^3}x = 0, \quad t \geq 1$$

satisfies the hypothesis of Theorem 2.1.5. Hence this equation has an oscillatory solution. In particular, $x_1(t) = t\cos(2\log t)$ and $x_2(t) = t\sin(2\log t)$ are the oscillatory solutions of this equation.

In the particular case, when $a(t) \equiv 0$ and $b(t) \equiv 0$, there is a well-known Kneser-type condition for oscillation of Eq. (2.6). Hanan [16] proved that (2.6) is oscillatory, if

$$\liminf_{t\to\infty} t^3 c(t) > \frac{2}{3\sqrt{3}}. \quad (2.19)$$

From (2.19), it follows that there exist $\epsilon > 0$ and $T \geq \sigma$ such that

$$c(t) - \frac{2}{3\sqrt{3}t^3} \geq \epsilon t^{-3}, \quad (2.20)$$

and

$$t^2\left[c(t) - \frac{2}{3\sqrt{3}t^3}\right] \geq \frac{\epsilon}{t} \quad (2.21)$$

for all $t \geq T$.

For Eq. (2.6), the conditions (2.8) and (2.17) may be rewritten as

$$\liminf_{t\to\infty} t^2 \int_\sigma^\infty \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds > 0 \tag{2.22}$$

and

$$\int^\infty s^2 \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds = \infty \tag{2.23}$$

respectively. Observe that (2.19) implies (2.22) and (2.21) implies (2.23). Now, we compare (2.22) and (2.23). For this, we suppose that

$$c(t) - \frac{2}{3\sqrt{3}t^3} \geq 0 \tag{2.24}$$

for sufficiently large t.

Remark 2.1.4 Assume that (2.24) is satisfied and (2.22) holds. So there exist $\delta > 0$ and $T_1 \geq \sigma$ such that

$$t^2 \int_t^\infty \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds \geq \delta \quad \text{for all } t \geq T_1.$$

If $\int^\infty s^2 (c(s) - \frac{2}{3\sqrt{3}s^3})\, ds < \infty$, then there exists $T_2 \geq T_1$ such that

$$\int_{T_2}^\infty s^2 \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds \leq \frac{\delta}{2}.$$

So, we have

$$\begin{aligned}
\frac{\delta}{2} &\geq \int_{T_2}^\infty s^2 \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds \\
&= \liminf_{t\to\infty} \left[\int_{T_2}^t s^2 \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds + \int_t^\infty s^2 \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds \right] \\
&\geq \liminf_{t\to\infty} \int_t^\infty s^2 \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds \\
&\geq \liminf_{t\to\infty} t^2 \int_t^\infty \left(c(s) - \frac{2}{3\sqrt{3}s^3} \right) ds \\
&\geq \delta,
\end{aligned}$$

a contradiction. Hence (2.23) is satisfied.

Now, we consider the equation

$$x''' + b_0 t^\beta x' + c_0 t^\delta x = 0, \tag{2.25}$$

where $b_0 < 0$ and $c_0 > 0$ are some constants, and $\delta \geq -3, 2\delta \geq 3\beta$.

For $\beta = -2$ and $\delta = -3$, Eq. (2.25) becomes the Euler equation. Lazer's condition (2.9) is not applicable to the Euler equation. The necessary and sufficient condition for oscillation of Euler's equation (2.25) is

$$b_0 + c_0 - \frac{2}{3\sqrt{3}}(1 - b_0)^{3/2} > 0, \tag{2.26}$$

which follows from (2.18) with $a_0 \equiv 0$.

Remark 2.1.5 Let $\delta = -3, \beta < -2$ and $c_0 > \frac{2}{3\sqrt{3}}$. Then (2.25) is oscillatory. Indeed, since

$$(1+x)^{3/2} = 1 + \frac{3}{2}x + \frac{3}{8}x^2 + \frac{3x^2}{4}\sum_{k=1}^{\infty}\frac{(-1)^k(2k-1)!!}{2^k(k+2)!}x^k, \quad |x| < 1, \tag{2.27}$$

where $(2k-1)!! = (1)(3)(5)\cdots(2k-1)$, substituting the coefficient of Eq. (2.25) to the left-hand side of (2.17) for $t > \sigma_0 \geq (-b_0)^{-1/(\beta+2)}$ with $a(t) \equiv 0$, we obtain

$$\begin{aligned}
&\int_{\sigma_0}^{\infty}\frac{1}{t}\left[c_0 + b_0 t^{\beta+2} - \frac{2}{3\sqrt{3}}\left(1 - b_0 t^{\beta+2}\right)^{3/2}\right]dt \\
&\quad = \int_{\sigma_0}^{\infty}\frac{1}{t}\left[c_0 + b_0 t^{\beta+2} - \frac{2}{3\sqrt{3}}\left(1 - \frac{3}{2}b_0 t^{\beta+2} + \frac{3}{8}b_0^2 t^{2\beta+4} + \cdots\right)\right]dt \\
&\quad = \int_{\sigma_0}^{\infty}\frac{1}{t}\left[c_0 - \frac{2}{3\sqrt{3}} + b_0 t^{\beta+2} - \frac{2}{3\sqrt{3}}\left(-\frac{3}{2}b_0 t^{\beta+2} + \frac{3}{8}b_0^2 t^{2\beta+4} + \cdots\right)\right]dt.
\end{aligned}$$

Since $c_0 > \frac{2}{3\sqrt{3}}$ and $\beta + 2 < 0$, it is easy to see that condition (2.17) is satisfied. Thus, the proof follows immediately from Theorem 2.1.4.

In case $\delta > -3, 2\delta = 3\beta$, Lazer's condition (2.9) may be applied only when $\delta \geq -1$ and $c_0 > \frac{2(-b_0)^{3/2}}{3\sqrt{3}}$.

Remark 2.1.6 Let $\delta > -3, 2\delta = 3\beta$ and $c_0 > \frac{2(-b_0)^{3/2}}{3\sqrt{3}}$. Then (2.25) is oscillatory. Indeed, since $\delta > -3$ and $2\delta = 3\beta$, there exists $\epsilon > 0$ such that $\delta = -3 + \epsilon$ and $\beta = -2 + \frac{2}{3}\epsilon$. Let $t > \sigma_0 \geq (-b_0)^{\frac{-3}{2\epsilon}}$. Substituting the coefficients of Eq. (2.25) in the left-hand side of (2.17) with $a(t) \equiv 0$ and using (2.27), we get

$$\begin{aligned}
&\int_{\sigma_0}^{\infty}\left[c_0 t^{-1+\epsilon} + b_0 t^{-1+2\epsilon/3} - \frac{2}{3\sqrt{3}t}\left(-b_0 t^{\frac{2\epsilon}{3}}\right)\left(1 - \frac{t^{\frac{-2\epsilon}{3}}}{b_0}\right)^{3/2}\right]dt \\
&\quad = \int_{\sigma_0}^{\infty} t^{-1+\epsilon}\left[c_0 + b_0 t^{\frac{-\epsilon}{3}} - \frac{2}{3\sqrt{3}}(-b_0)^{3/2}\left(1 - \frac{3t^{-\frac{2\epsilon}{3}}}{2b_0} + \frac{3t^{-\frac{4\epsilon}{3}}}{8b_0^2} + \cdots\right)\right]dt \\
&\quad = \int_{\sigma_0}^{\infty} t^{-1+\epsilon}\left[c_0 - \frac{2}{3\sqrt{3}}(-b_0)^{3/2} + b_0 t^{\frac{-\epsilon}{3}} - \frac{2}{3\sqrt{3}}(-b_0)^{3/2}\right.
\end{aligned}$$

$$\times \left(-\frac{3t^{-\frac{2\epsilon}{3}}}{2b_0} + \frac{3t^{-\frac{4\epsilon}{3}}}{8b_0^2} + \cdots \right) \Bigg] dt$$

$$= \infty$$

since $c_0 - \frac{2(-b_0)^{3/2}}{3\sqrt{3}} > 0$. By Theorem 2.1.4, Eq. (2.25) is oscillatory.

Remark 2.1.7 Let $\delta > -3$ and $2\delta > 3\beta$. Then Eq. (2.25) is oscillatory. In fact, similar to the above, after substituting the coefficients of Eq. (2.25) with $a(t) = 0$ and using (2.27) for sufficiently large t, we obtain

$$\int_{\sigma_0}^{\infty} \left[c_0 t^{\delta+2} + b_0 t^{\beta+1} - \frac{2}{3\sqrt{3}t} (-b_0)^{3/2} t^{3(\beta+2)/2} \left(1 - \frac{3t^{-\beta-2}}{2b_0} - \frac{3t^{-2\beta-4}}{8b_0^2} - \cdots \right) \right] dt$$

for $\beta + 2 > 0$;

$$\int_{\sigma_0}^{\infty} \left[c_0 t^{\delta+2} + b_0 t^{-1} - \frac{2}{3\sqrt{3}t} (1 - b_0)^{3/2} \right] dt,$$

for $\beta + 2 = 0$; and

$$\int_{\sigma_0}^{\infty} \left[c_0 t^{\delta+2} + b_0 t^{\beta+1} - \frac{2}{3\sqrt{3}t} \left(1 - \frac{3}{2} b_0 t^{\beta+2} + \frac{3}{8} b_0^2 t^{2\beta+4} + \cdots \right) \right] dt$$

for $\beta + 2 < 0$. It is easy to check that $\delta + 2 > \beta - 1$, and hence $\delta + 2 > -1$, and all the integrals above satisfy (2.17) with $a(t) \equiv 0$. Then Theorem 2.1.4 implies that (2.25) is oscillatory.

Theorem 2.1.6 *Suppose that $a(t) + b(t) + 1 \le 0$. If*

$$\frac{2(a(t)+3)^3}{27} - \frac{(a(t)+3)(a(t)+b(t)+1)}{3} + c(t)$$
$$- \frac{2}{3\sqrt{3}} \left(\frac{(a(t)+3)^2}{3} - \big(a(t)+b(t)+1\big) \right)^{3/2} > 0 \tag{2.28}$$

for $t \ge t_0 > \sigma$, then (2.1) is oscillatory.

Proof For the sake of contradiction, suppose that all solutions of (2.1) are nonoscillatory. Then by Lemma 1.5.4, there exists a solution $x(t)$ of (2.1) satisfying the property $x(t)x'(t) > 0$ for $t \ge t_0 \ge \sigma$. Let $x(t) > 0$ and $x'(t) > 0$ for $t \ge t_0$. Set

$$e^z = \frac{x'(t)}{x(t)}. \tag{2.29}$$

Differentiating successively, we obtain

$$\frac{x''(t)}{x(t)} = e^z + e^{2z} \tag{2.30}$$

and

$$\frac{x'''(t)}{x(t)} = e^{3z} + 3e^{2z} + e^{z}. \tag{2.31}$$

Dividing (2.1) throughout by $x(t)$ and using (2.29), (2.30) and (2.31) in the resulting equation, we obtain

$$F\big(e^{z}, t\big) = e^{3z} + \big(3 + a(t)\big)e^{2z} + \big(a(t) + b(t) + 1\big)e^{z} + c(t) = 0. \tag{2.32}$$

The minimum of $F(e^{z}, t)$ attains at

$$e^{z} = \frac{1}{3}\Big[-a(t) - 3 + \sqrt{\big(a(t)+3\big)^2 - 3\big(a(t) + b(t) + 1\big)}\Big]$$

and the minimum value of $F(e^{z}, t)$ is given by

$$\min_{z} F\big(e^{z}, t\big) = \frac{2A^3(t)}{27} - \frac{A(t)B(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{A^2(t)}{3} - B(t)\right)^{3/2}, \tag{2.33}$$

where $A(t) = a(t) + 3$ and $B(t) = a(t) + b(t) + 1$. Combining (2.32) and (2.33), we have the inequality

$$\frac{2A^3(t)}{27} - \frac{A(t)B(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{A^2(t)}{3} - B(t)\right)^{3/2} \le 0,$$

which contradicts (2.28). This completes the proof. □

Theorem 2.1.7 *Let $M(t)$ be a solution of (2.1) with $M(t) > 0$ and $M'(t) < 0$, $t \ge \sigma$. Equation (2.1) is oscillatory if and only if the second-order differential equation*

$$z'' + \left(\frac{3M'(t) + a(t)M(t)}{M(t)}\right)z' + \left(\frac{3M''(t) + 2a(t)M'(t) + b(t)M(t)}{M(t)}\right)z = 0 \tag{2.34}$$

is oscillatory.

Proof Let (2.1) be oscillatory. Let $x(t)$ be an oscillatory solution of the equation. Since $M(t) > 0$ for $t \ge \sigma$, it is easy to verify that $(\frac{x(t)}{M(t)})'$ is an oscillatory solution of (2.34), and hence (2.34) is oscillatory.

Conversely, suppose that (2.34) is oscillatory. If possible, let (2.1) be nonoscillatory. By Lemma 1.5.4, there exists at least one nonoscillatory solution $u(t)$ of (2.1) not satisfying the condition $u(t)u'(t) < 0$ for large t. Without any loss of generality we may assume that $u(t) > 0$ for $t \ge \sigma$. From Lemma 1.5.3, it follows that $u'(t) \ge 0$, $t \ge t_0 \ge \sigma$. Clearly, $x'(t)$ is an oscillatory solution of (2.34), where $x(t) = \frac{u(t)}{M(t)}$, $t > t_0 \ge \sigma$. But $x'(t) = (M(t)u'(t) - M'(t)u(t))M^{-2}(t) > 0$, $t \ge t_0$, a contradiction. Thus (2.1) is oscillatory. Hence the theorem is proved. □

Theorem 2.1.8 *Equation* (2.1) *has a solution* $x(t)$ *with the following properties*:

$$x'''(t)x''(t)x'(t)x(t) \neq 0, \quad t \in [\sigma, \infty),$$
$$\operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t) = \operatorname{sgn} x'''(t),$$
$$\lim_{t \to \infty} x''(t) = \lim_{t \to \infty} x'(t) = 0,$$

and $x(t)$ *is asymptotic to a finite constant.*

Proof For every positive integer $n > \sigma$, let $x_n(t)$ be a solution of (2.1) satisfying the initial conditions

$$x_n(n) = 0, \qquad x'_n(n) = 0, \qquad x''_n(n) > 0.$$

By Remark 2.1.1, we have

$$x_n(t) > 0, \qquad x'_n(t) < 0, \qquad x''_n(t) > 0 \tag{2.35}$$

for $t \in [\sigma, n)$. Let $\{u_1(t), u_2(t), u_3(t)\}$ be a set of linearly independent solutions of (2.1). It is possible to write

$$x_n(t) = c_{1n}u_1(t) + c_{2n}u_2(t) + c_{3n}u_3(t), \tag{2.36}$$

where

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1. \tag{2.37}$$

Since the sequence $\langle c_{in} \rangle$, $i = 1, 2, 3$, is bounded, there exists a subsequence $\langle c_{in_j} \rangle$ which converges to c_i, $i = 1, 2, 3$. From (2.37), we obtain

$$c_1^2 + c_2^2 + c_3^2 = 1. \tag{2.38}$$

Now, consider the equation

$$x(t) = c_1u_1(t) + c_2u_2(t) + c_3u_3(t) \tag{2.39}$$

of (2.1). Since the sequences $\langle x_{nj}(t) \rangle$, $\langle x'_{nj}(t) \rangle$ and $\langle x''_{nj}(t) \rangle$ converge uniformly to $x(t)$, $x'(t)$ and $x''(t)$, respectively, on any finite subinterval of $[\sigma, \infty)$, it follows from (2.35) that

$$x(t) \geq 0, \qquad x'(t) \leq 0, \qquad x''(t) \geq 0$$

and

$$x'''(t) = -a(t)x''(t) - b(t)x'(t) - c(t)x(t) \leq 0 \tag{2.40}$$

for $t \in [\sigma, \infty)$. If $x(t_0) = 0$ for some $t_0 \geq \sigma$, then $x(t) \equiv 0$, $t \in [t_0, \infty)$, which contradicts (2.38) and (2.39). Thus, $x(t) > 0$ for $t \in [\sigma, \infty)$. If possible, suppose that $x'(t_0) = 0$ for some $t_0 \geq \sigma$. Since $x''(t) \geq 0$, it follows that $x'(t_0) \equiv 0$. Consequently, $x(t)$ reduces to a nonzero constant. From (2.1), we obtain $c(t)x = 0$, a contradiction.

Hence $x'(t) < 0$ for $t \in [\sigma, \infty)$. $x''(t_1) = 0$ for some $t_1 \geq \sigma$ implies that $x''(t) \equiv 0$, and hence a contradiction from (2.1). Hence $x''(t) > 0$ for $t \in [\sigma, \infty)$. From (2.40), it follows that $x'''(t) < 0$ for $t \in [\sigma, \infty)$. Thus

$$\lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0,$$

and $x(t)$ is asymptotic to a finite constant. This completes the proof of theorem. □

Lemma 2.1.3 *Consider* (2.1) *with* $a(t) \equiv 0$. *If* $u(t)$ *is a nonoscillatory solution of* (2.1) *and* $v(t)$ *is any solution of* (2.1), *then* $(u(t)v'(t) - u'(t)v(t))$ *is a solution of*

$$\left(\frac{x'}{u(t)}\right)' + \left(\frac{u''(t) + b(t)u(t)}{u^2(t)}\right)x = 0.$$

Theorem 2.1.9 *Consider* (2.1) *with* $a(t) \equiv 0$. *If* $\int_0^\infty c(t)\,dt < \infty$, $2c(t) - b'(t) \geq 0$, $\lim_{t\to\infty} b(t) = 0$ *and the second-order differential equation*

$$z'' + \left[b(t) + \frac{3}{2}\int_t^\infty c(s)\,ds\right]z = 0 \tag{2.41}$$

is nonoscillatory, then (2.1) *is nonoscillatory.*

Proof If possible, let $v(t)$ be an oscillatory solution of (2.1). From Theorem 1.1 of Lazer [23] and Theorem 6 due to Jones [19], it follows that (2.1) admits a nonoscillatory solution $u(t)$ such that $u(t) > 0$, $u'(t) < 0$, $u''(t) > 0$, $u'''(t) < 0$ for $t \in [\sigma, \infty)$ and

$$\lim_{t\to\infty} u(t) = \lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u''(t) = 0.$$

If

$$F\big[u(t)\big] = \big(u'(t)\big)^2 - 2u(t)u''(t) - b(t)u^2(t), \tag{2.42}$$

then $\lim_{t\to\infty} F(u(t)) = 0$. Clearly, for $T > t$, we get

$$F\big[u(T)\big] = F\big[u(t)\big] + \int_t^T \big(2c(s) - b'(s)\big)u^2(s)\,ds.$$

Hence, taking limit as $T \to \infty$, we obtain

$$2u(t)u''(t) - \big(u'(t)\big)^2 = -b(t)u^2(t) + \int_t^\infty \big(2c(s) - b'(s)\big)u^2(s)\,ds,$$

that is,

$$\frac{2u''(t)}{u(t)} - \left(\frac{u'(t)}{u(t)}\right)^2 = -b(t) + \int_t^\infty \big(2c(s) - b'(s)\big)\frac{u^2(s)}{u^2(t)}\,ds.$$

Since $u(t)$ is positive and decreasing, we have

$$\frac{2u''(t)}{u(t)} - \left(\frac{u'(t)}{u(t)}\right)^2 \le -b(t) + \int_t^\infty \big(2c(s) - b'(s)\big)\,ds. \tag{2.43}$$

From Lemma 2.1.3, it follows that

$$x(t) = u(t)v'(t) - u'(t)v(t) = u^2(t)\left(\frac{v(t)}{u(t)}\right)'$$

is an oscillatory solution of

$$\left(\frac{x'(t)}{u(t)}\right)' + \left(\frac{u''(t) + b(t)u(t)}{u^2(t)}\right)x = 0. \tag{2.44}$$

The substitution $x = w(t)u^{1/2}(t)$ transforms (2.44) into

$$w'' + \left[b(t) + \frac{3}{4}\left(\frac{2u''(t)}{u(t)} - \frac{(u'(t))^2}{u^2(t)}\right)\right]w = 0. \tag{2.45}$$

We may note that (2.44) is oscillatory, if and only if (2.45) is oscillatory. Now using (2.43), we get

$$\begin{aligned} b(t) + \frac{3}{4}\left(\frac{2u''(t)}{u(t)} - \frac{(u'(t))^2}{u^2(t)}\right) &\le \frac{1}{4}b(t) + \frac{3}{4}\int_t^\infty \big(2c(s) - b'(s)\big)\,ds \\ &\le b(t) + \frac{3}{2}\int_t^\infty c(s)\,ds. \end{aligned} \tag{2.46}$$

From Sturm's comparison theorem, it follows that (2.41) is oscillatory, a contradiction. Hence (2.1) is nonoscillatory. This completes the proof of the theorem. □

Remark 2.1.8 If $\lim_{t\to\infty} b(t) = 0$ is not assumed in Theorem 2.1.9, one may proceed as follows: since

$$F\big[u(t)\big] \ge \big(u'(t)\big)^2 - 2u(t)u''(t),$$

we have $\lim_{t\to\infty} F(u(t)) \ge 0$. Consequently,

$$2u(t)u''(t) - \big(u'(t)\big)^2 \le -b(t)u^2(t) + \int_t^\infty \big(2c(s) - b'(s)\big)u^2(s)\,ds$$

and hence we obtain (2.43). Proceeding as in Theorem 2.1.9, we get

$$b(t) + \frac{3}{4}\left(\frac{2u''(t)}{u(t)} - \frac{(u'(t))^2}{u^2(t)}\right) \le \frac{1}{4}b(t) + \frac{3}{4}\int_t^\infty \big(2c(s) - b'(s)\big)\,ds.$$

Thus, one may restate Theorem 2.1.9 as follows:

Theorem 2.1.10 *Consider* (2.1) *with* $a(t) \equiv 0$. *If* $2c(t) - b'(t) \geq 0$, $\int_\sigma^\infty (2c(t) - b'(t))\,dt < \infty$ *and the second-order differential equation*

$$z'' + \left[\frac{1}{4}b(t) + \frac{3}{4}\int_t^\infty \big(2c(s) - b'(s)\big)\,ds\right]z = 0$$

is nonoscillatory, then (2.1) *is nonoscillatory.*

Example 2.1.3 Consider

$$x''' - e^{-t}x' + e^{-t}x = 0, \quad t \geq 0.$$

In this case, Eq. (2.41) takes the form $z'' + \frac{1}{2}e^{-t}z = 0$. From Hille's theorem (p. 45, [38]) it follows that this equation is nonoscillatory. Hence from Theorem 2.1.9, it follows that all solutions of the given third-order equation are nonoscillatory.

Example 2.1.4 Consider

$$x''' - \big(1 + e^{-t}\big)x' + e^{-t}x = 0, \quad t \geq 0.$$

In this equation, $2c(t) - b'(t) = e^{-t}$ and

$$\frac{1}{4}b(t) + \frac{3}{4}\int_t^\infty \big(2c(s) - b'(s)\big)\,ds = -\frac{1}{4} + \frac{1}{2}e^{-t} < 0$$

for large t. Thus the second-order differential equation associated with Theorem 2.1.10 is nonoscillatory. By Theorem 2.1.10, all solutions of the equation are nonoscillatory. In particular, $x(t) = e^t$ is a nonoscillatory solution of the equation. We may note that $\lim_{t\to\infty} b(t) \neq 0$.

Now, we provide some results concerning the asymptotic behaviour of nonoscillatory solutions of (2.1) in the presence of an oscillatory solution.

Theorem 2.1.11 *Suppose that* (2.1) *has an oscillatory solution and* $\lim_{t\to\infty} t^2 b(t) \neq 0$. *If* $u(t)$ *is a nonoscillatory solutions of* (2.1), *then* $\lim_{t\to\infty} u(t) = 0$.

Proof From Lemma 1.5.4, we obtain

$$u(t)u'(t)u''(t) \neq 0, \quad \operatorname{sgn} u(t) = \operatorname{sgn} u''(t) \neq \operatorname{sgn} u'(t) = \operatorname{sgn} u'''(t)$$

for $t \geq \sigma$ and

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u''(t) = 0, \qquad \lim_{t\to\infty} u(t) = \lambda \neq \pm\infty.$$

Without any loss of generality, we may assume that $u(t) > 0$ for $t \geq \sigma$. So $u'(t) < 0$, $u''(t) > 0$, $u'''(t) < 0$ for $t \geq \sigma$ and $\lim_{t\to\infty} u(t) = \lambda$, where $0 \leq \lambda < \infty$. If possible, let $\lambda \neq 0$. Further, without loss of generality, we may assume that $\lambda = \frac{1}{2}$. Since $u'(t) < 0$, $t \geq \sigma$, it is possible to find $t_0 \geq \sigma$ such that $\frac{1}{2} < u(t) < 1$ for $t \geq t_0$.

Suppose that $v(t)$ is an oscillatory solution of (2.1). Then $v(t)$ is also a solution of (2.2). It may be verified that $W(u, v)(t) = u(t)v'(t) - v(t)u'(t)$ is an oscillatory solution of

$$\left(\frac{r(t)z'}{u(t)}\right)' + \left(\frac{r(t)u''(t) + q(t)u(t)}{u^2(t)}\right)z = 0. \tag{2.47}$$

Proceeding as in Jones [19], one may get $\lim_{t\to\infty} t^2 u''(t) = 0$. Since $r(t)$ is monotonic increasing, we have

$$\frac{r(t)}{u(t)} > r(t) > r(\sigma) = 1 \quad \text{for } t \geq t_0.$$

Further,

$$\lim_{t\to\infty} t^2\big[u''(t) + b(t)u(t)\big] < 0$$

implies that

$$\lim_{t\to\infty} t^2\big[r(t)u''(t) + q(t)u(t)\big] = \lim_{t\to\infty} t^2 r(t)\big[u''(t) + b(t)u(t)\big] < 0.$$

Hence the equation

$$z'' + \left[\frac{r(t)u''(t) + q(t)u(t)}{u^2(t)}\right]z = 0 \tag{2.48}$$

is nonoscillatory (see p. 45, [38]). From Sturm's comparison theorem, it follows that (2.47) is nonoscillatory, a contradiction. Hence $\lambda = 0$. The theorem is proved. □

Remark 2.1.9 The condition $\lim_{t\to\infty} t^2 b(t) \neq 0$ in Theorem 2.1.11 may be replaced by the assumption "$r(t)$ is bounded". Indeed, if $r(t)$ is bounded, then there exists a $k > 0$ such that $r(t) < k$ for $t \geq \sigma$. Now

$$\lim_{t\to\infty} t^2 r(t)\big[u''(t) + b(t)u(t)\big] < k \lim_{t\to\infty} t^2 u''(t) = 0.$$

So (2.48) is nonoscillatory.

Theorem 2.1.12 *Suppose that* (2.1) *has an oscillatory solution and* $\int_\sigma^\infty c(t)\,dt = \infty$. *If* $u(t)$ *is a nonoscillatory solution of* (2.1), *then* $\lim_{t\to\infty} u(t) = 0$.

Proof From Lemma 1.5.4, it follows that

$$u(t)u'(t)u''(t) \neq 0, \quad \operatorname{sgn} u(t) = \operatorname{sgn} u''(t) \neq \operatorname{sgn} u'(t) = \operatorname{sgn} u'''(t)$$

for $t \geq \sigma$ and

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u''(t) = 0, \qquad \lim_{t\to\infty} u(t) = \lambda \neq \pm\infty.$$

Without any loss of generality, we may assume that $u(t) > 0$ for $t \ge \sigma$. So $u'(t) < 0$, $u''(t) > 0$, and $u'''(t) < 0$ for $t \ge \sigma$ and $\lim_{t\to\infty} u(t) = \lambda$, $0 \le \lambda < \infty$. We claim that $\lambda = 0$. If not, then $\lambda > 0$. Since $u(t)$ satisfies (2.2), integrating (2.2) from σ to t, we obtain

$$\begin{aligned} r(t)u''(t) &= r(\sigma)u''(\sigma) - \int_\sigma^t q(s)u'(s)\,ds - \int_\sigma^t p(s)u(s)\,ds \\ &\le r(\sigma)u''(\sigma) - u(t)\int_\sigma^t p(s)\,ds \\ &\le r(\sigma)u''(\sigma) - r(\sigma)u(t)\int_\sigma^t c(s)\,ds \\ &\le r(\sigma)u''(\sigma) - \lambda\int_\sigma^t c(s)\,ds. \end{aligned}$$

This, in turn, implies that $u''(t) < 0$ for large t, a contradiction. Hence $\lambda = 0$. This completes the proof of the theorem. □

Theorem 2.1.13 *Suppose that* (2.1) *admits an oscillatory solution. If $x(t)$ is a nonoscillatory solution of* (2.1) *with* $\lim_{t\to\infty} x(t) = 0$, *then every nonoscillatory solution of* (2.1) *is a constant multiple of $x(t)$.*

Proof Let $u_1(t)$ and $u_2(t)$ be two solutions of (2.2) on $[\sigma, \infty)$ with initial conditions

$$u_1(\sigma) = u_1'(\sigma) = 0, \qquad r(\sigma)u_1''(\sigma) = 1$$

and

$$u_2(\sigma) = u_2''(\sigma) = 0, \qquad u_2'(\sigma) = -1.$$

From Lemma 2.1.1 and Lemma 1.5.10, it follows that both $u_1(t)$ and $u_2(t)$ are oscillatory.

Clearly, $W_1(t) = W_1(u_1, u_2)(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t)$ is a solution of the adjoint equation

$$\big(\big(r(t)z'\big)' + q(t)z\big)' - p(t)z = 0, \tag{2.49}$$

which is the adjoint of Eq. (2.2). Observe that

$$W_1(\sigma) = W_1'(\sigma) = 0 \quad \text{and} \quad \big(rW_1'\big)'(\sigma) > 0.$$

It is easy to see that $W_1(t) > 0$ for $t > \sigma$. Indeed, $(rW_1')'(\sigma) > 0$ and the continuity of $(rW_1')'(t)$ imply that $(rW_1')'(t) > 0$ for $t \in [\sigma, \sigma + \delta)$, for some $\delta > 0$. This, in turn, implies that $W_1'(t) > 0$ for $t \in (\sigma, \sigma + \delta)$ and hence $W_1(t) > 0$ for $t \in (\sigma, \sigma + \delta)$. We claim that $W_1(t) > 0$ for $t > \sigma$. If not, then there is a $t_1 > \sigma$ such that $W_1(t_1) = 0$ and $W_1(t) > 0$ for $t \in (\sigma, t_1)$. Since $W_1(t)$ is a solution of (2.49), we have

$$\big(\big(r(t)W_1'\big)' + q(t)W_1(t)\big)' = p(t)W_1(t) > 0, \quad t \in (\sigma, t_1)$$

implies that $(rW_1')'(t)+q(t)W_1(t)$ is nondecreasing on $[\sigma,t_1)$. Hence $(rW_1')'(t)>0$ for $t\in[\sigma,t_1]$. Consequently, $W_1'(t)>0$ for $t\in(\sigma,t_1]$, a contradiction. Hence our claim holds. Further, as $u_1(t)$ and $u_2(t)$ are linearly independent oscillatory solutions of the second-order differential equation, we have

$$\begin{vmatrix} u_1(t) & u_2(t) & y \\ u_1'(t) & u_2'(t) & y' \\ u_1''(t) & u_2''(t) & y'' \end{vmatrix}=0,$$

that is,

$$\left(\frac{y'}{W_1(t)}\right)'+\left(\frac{(rW_1')'(t)+q(t)W_1(t)}{r(t)W_1^2(t)}\right)y=0. \tag{2.50}$$

So any nontrivial linear combination of $u_1(t)$ and $u_2(t)$ is oscillatory. Clearly, $\{u_1(t),u_2(t),x(t)\}$ forms a basis of solution space of (2.2).

Without any loss of generality, we may assume that $x(t)>0$ for $t\ge t_0^*\ge\sigma$.

If possible, let $w(t)$ be a nonoscillatory solution of (2.1) on $[\sigma,\infty)$ such that $x(t)$ and $w(t)$ are linearly independent. Then

$$w(t)=\lambda_1u_1(t)+\lambda_2u_2(t)+\lambda_3x(t),$$

where λ_1, λ_2 and λ_3 are constants. $\lambda_3=0$ implies that $w(t)$ is oscillatory. Hence $\lambda_3\ne0$. Dividing $w(t)$ by λ_3, we get $z(t)=x(t)+c_1u_1(t)+c_2u_2(t)$, where $z(t)=\frac{w(t)}{\lambda_3}$, $c_1=\frac{\lambda_1}{\lambda_3}$ and $c_2=\frac{\lambda_2}{\lambda_3}$. Clearly, c_1 and c_2 cannot be equal to zero simultaneously. Since $c_1u_1(t)+c_2u_2(t)$ is oscillatory, there exists a $t_0\ge t_0^*$ such that $x(t)$ and $z(t)$ are of the same sign for $t\ge t_0$.

Setting $x_1(t)=-c_1u_1(t)-c_2u_2(t)$, we obtain $z(t)=x(t)-x_1(t)$. Clearly, $x_1(t)$ is an oscillatory solution of (2.50) and (2.2). Let $t_1>t_0$ be a zero of $x_1(t)$ such that $x_1'(t_1)>0$. Let $x_2(t)$ be a solution of (2.2) on $[t_1,\infty)$ such that $x_2(t_1)=x_2'(t_1)=0$ and $x_2''(t_1)=1$. From Lemma 2.1.1 and Lemma 1.5.10, it follows that $x_2(t)$ is oscillatory. Clearly, $W(t)=W(x_1,x_2)(t)=x_1(t)x_2'(t)-x_1'(t)x_2(t)$ is a solution of (2.49) with $W(t_1)=0=W'(t_1)$ and $(rW')'(t_1)>0$. Hence $W(t)>0$ for $t>t_1$. Consequently, it follows from (2.49) that $(rW')'+qW$ is increasing in $[t_1,\infty)$. So, for $t>t_1$,

$$\begin{aligned}\big(rW'\big)'(t)&\ge\big(rW'\big)'(t)+q(t)W(t)\\&\ge\big(rW'\big)'(t_1)+q(t_1)W(t_1)\\&=\big(rW'\big)'(t_1)>0.\end{aligned} \tag{2.51}$$

This, in turn, implies that $W'(t)>0$ for $t>t_1$. Clearly, $\{x_1(t),x_2(t),x(t)\}$ forms a basis of solution space of (2.2), because $x_1(t)$ and $x_2(t)$ are linearly independent oscillatory solutions of the second-order equation

$$\left(\frac{x'}{W(t)}\right)'+\left(\frac{(rW')'(t)+q(t)W(t)}{r(t)W^2(t)}\right)x=0. \tag{2.52}$$

So

$$\begin{vmatrix} x_1(t) & x_2(t) & x(t) \\ x_1'(t) & x_2'(t) & x'(t) \\ r(t)x_1''(t) & r(t)x_2''(t) & r(t)x''(t) \end{vmatrix} = K,$$

a nonzero constant, that is,

$$K = W(t)r(t)x''(t) - W'(t)r(t)x'(t) + \big((rW')'(t) + q(t)W(t)\big)x(t).$$

From Lemma 1.5.4, it follows that $x'(t) < 0$ and $x''(t) > 0$ for $t > t_1$. So (2.51) yields $K > 0$ and, for $t > t_1$,

$$0 < \big((rW')'(t) + q(t)W(t)\big)x(t) < K.$$

Let $\langle \sigma_n \rangle$ be an increasing sequence of maximum points of $x_1(t)$ such that $\sigma_n > t_1$ and $\sigma_n \to \infty$ as $n \to \infty$. So $x_1(\sigma_n) > 0$ and $x_1'(\sigma_n) = 0$. Since $z(t) > 0$ for $t \geq t_0$, $x_1(t) < x(t)$. From (2.51), we obtain $(rW')'(t) + q(t)W(t) > 0$ for $t \geq t_1$. Hence

$$\begin{aligned} 0 &< \big[(rW')'(\sigma_n) + q(\sigma_n)W(\sigma_n)\big]x_1(\sigma_n) \\ &< \big[(rW')'(\sigma_n) + q(\sigma_n)W(\sigma_n)\big]x(\sigma_n) \\ &< K. \end{aligned}$$

Further, since $\lim_{n\to\infty} x(\sigma_n) = 0$, we have $\lim_{n\to\infty} x_1(\sigma_n) = 0$. Consequently,

$$\lim_{n\to\infty} \big[(rW')'(\sigma_n) + q(\sigma_n)W(\sigma_n)\big]x_1^2(\sigma_n) = 0. \tag{2.53}$$

On the other hand, if

$$H(t) = r(t)W(t)\big(x_1'(t)\big)^2 + \big((rW')'(t) + q(t)W(t)\big)x_1^2(t),$$

then

$$\begin{aligned} H'(t) = {} & 2r(t)W(t)x_1'(t)x_1''(t) + (rW)'(t)\big(x_1'(t)\big)^2 \\ & + p(t)W(t)x_1^2(t) + 2\big((rW')'(t) + q(t)W(t)\big)x_1(t)x_1'(t). \end{aligned}$$

Since $x_1(t)$ is a solution of (2.52), we have

$$r(t)W(t)x_1''(t) = r(t)W'(t)x_1'(t) - \big[(rW')'(t) + q(t)W(t)\big]x_1(t)$$

and hence, for $t > t_1$,

$$H'(t) = 2r(t)W(t)\big(x_1'(t)\big)^2 + (rW)'(t)\big(x_1'(t)\big)^2 + p(t)W(t)x_1^2(t) > 0,$$

because $(rW)'(t) = r'(t)W(t) + r(t)W'(t) > 0$. So $H(t)$ is a positive increasing function. But from (2.53), we obtain $\lim_{n\to\infty} H(\sigma_n) = 0$, a contradiction. So $x(t)$ and $z(t)$ are linearly dependent. Consequently, $x(t)$ and $w(t)$ are linearly dependent. Hence the theorem is proved. □

Corollary 2.1.1 *Suppose that (2.1) has an oscillatory solution and* $\lim_{t\to\infty} t^2 b(t) \neq 0$ *or* $\int_\sigma^\infty c(t)\,dt = \infty$. *Then all nonoscillatory solutions of (2.1) tend to zero as* $t \to \infty$.

Theorem 2.1.14 *Let* $(2c(t) - a(t)b(t) - b'(t))r(t) \geq d > 0$. *If (2.1) is oscillatory, then every nonoscillatory solution* $u(t)$ *of (2.1) satisfies the property* $u(t) \to 0$, $u'(t) \to 0$ *and* $u''(t) \to 0$ *as* $t \to \infty$.

Proof Lemma 2.1.1, or Lemma 2.1.2, guarantees the existence of a nonoscillatory solution of (2.1). If $u(t)$ has a zero in $[\sigma, \infty)$, then Lemma 1.5.10 implies that $u(t)$ is oscillatory. Hence $u(t) \neq 0$ for $t \geq \sigma$. Since (2.1) is oscillatory, by Lemma 1.5.4, $u(t)$ satisfies the property

$$\begin{aligned} &u(t)u'(t)u''(t) \neq 0, \\ &\operatorname{sgn} u(t) = \operatorname{sgn} u''(t) \neq \operatorname{sgn} u'(t) = \operatorname{sgn} u'''(t), \quad t \geq \sigma \end{aligned}$$

and

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u''(t) = 0, \qquad \lim_{t\to\infty} u(t) = \lambda \neq \pm\infty.$$

Let $u(t) > 0$ for $t \geq \sigma$. Suppose that $\lambda \neq 0$. Hence for $0 < \epsilon < \lambda$, there exists $T > \sigma$ such that $u(t) > \lambda - \epsilon$ for $t \geq T$. Since (2.1) is oscillatory, from Lemma 1.5.6, it follows that $F[u(t)] < 0$ for $t \geq \sigma$, where

$$\begin{aligned} F\big[u(t)\big] &= r(t)\big(u'(t)\big)^2 - 2r(t)u(t)u''(t) - q(t)u^2(t) \\ &= F\big[u(\sigma)\big] + \int_\sigma^t r'(s)\big(u'(s)\big)^2\,ds + \int_\sigma^t \big(2p(s) - q'(s)\big)u^2(s)\,ds. \end{aligned} \tag{2.54}$$

Hence

$$\begin{aligned} d\int_\sigma^\infty u^2(t)\,dt &\leq \int_\sigma^\infty \big(2c(t) - a(t)b(t) - b'(t)\big)r(t)u^2(t)\,dt \\ &\leq \int_\sigma^\infty \big(2p(t) - q'(t)\big)u^2(t)\,dt \leq -F\big[u(\sigma)\big] < \infty \end{aligned}$$

implies that

$$\int_\sigma^\infty u^2(t)\,dt < \infty.$$

On the other hand,

$$\int_\sigma^\infty u^2(t)\,dt > \int_T^\infty u^2(t)\,dt = \infty,$$

a contradiction. Hence $\lambda = 0$. This completes the proof of the theorem. □

Theorem 2.1.15 *Assume that $c_2(t)$ does not vanish identically on any subinterval of $[\sigma, \infty)$, and*

$$b_1(t) \le b_2(t) \le 0 \quad \text{and} \quad 0 \le c_1(t) \le c_2(t)$$

for all $t \ge \sigma$. If the differential equation

$$x''' + a(t)x'' + b_1(t)x' + c_1(t)x = 0$$

is oscillatory on $[\sigma, \infty)$, then the equation

$$x''' + a(t)x'' + b_2(t)x' + c_2(t)x = 0$$

is oscillatory on $[\sigma, \infty)$.

2.2 Behaviour of Solutions of $x''' + a(t)x'' + b(t)x' + c(t)x = 0$ with $a(t) \le 0$, $b(t) \le 0$ and $c(t) > 0$

This section deals with Eq. (2.1) with $a(t) \le 0$, $b(t) \le 0$ and $c(t) > 0$. We state a sufficient condition for the oscillation of (2.1). Further, the asymptotic behaviour of nonoscillatory solutions of (2.1) is given in the presence of an oscillatory solution of (2.1).

Lemma 2.2.1 *Equation (2.1) is of type C_I. Hence (2.49) is of type C_{II}.*

Theorem 2.2.1 *Suppose that $b(t) - a'(t) \le 0$. If*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right] dt = \infty, \tag{2.55}$$

then (2.1) is oscillatory.

Proof Suppose that $x(t)$ is a nonoscillatory solution of (2.1). From Lemma 1.5.3, it follows that there exists a $t_0 \in [\sigma, \infty)$ such that $x'(t) \le 0$ or ≥ 0 for $t \in [t_0, \infty)$. In view of Lemma 1.5.4 and the second part of Lemma 1.5.3, it is enough to prove that $x(t)x'(t) \ge 0$ for $t \ge t_0$ does not hold. Setting $u(t) = \frac{x'(t)}{x(t)}$, $t \ge t_0$, one may verify that $u(t) \ge 0$ is a solution of the Riccati equation (2.13). Integrating (2.13) from t_0 to t, we obtain

$$\begin{aligned} u'(t) &= u'(t_0) - \frac{3}{2}u^2(t) + \frac{3}{2}u^2(t_0) - a(t)u(t) + a(t_0)u(t_0) \\ &\quad - \int_{t_0}^t \left[u^3(s) + a(s)u^2(s) + \big(b(s) - a'(s)\big)u(s) + c(s)\right] ds. \end{aligned} \tag{2.56}$$

The minimum of $u^3(s) + a(s)u^2(s) + (b(s) - a'(s))u(s) + c(s)$, for $u(s) \geq 0$, is attained at

$$u(s) = \frac{1}{3}\big[-a(s) + \big(a^2(s) - 3\big(b(s) - a'(s)\big)\big)^{1/2}\big]$$

and its minimum value is given by

$$\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + \frac{a(s)a'(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{3/2}.$$

If $H(u(t), t) = -\frac{3}{2}u^2(t) - a(t)u(t)$, then the maximum of $H(u(t), t)$, for $u(t) \geq 0$, is attained at $u(t) = \frac{-a(t)}{3}$ and the maximum value of $H(u(t), t)$ is given by $\frac{a^2(t)}{6}$. Hence from (2.56), we have

$$\begin{aligned}
u'(t) &\leq u'(t_0) + \frac{3}{2}u^2(t_0) + a(t_0)u(t_0) + \frac{a^2(t)}{6} \\
&\quad - \int_{t_0}^{t}\left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + \frac{a(s)a'(s)}{3} + c(s)\right. \\
&\quad \left. - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{3/2}\right] ds \\
&\leq u'(t_0) + \frac{3}{2}u^2(t_0) + a(t_0)u(t_0) + \frac{a^2(t_0)}{6} \\
&\quad - \int_{t_0}^{t}\left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{3/2}\right] ds.
\end{aligned}$$

Hence (2.55) implies that $\lim_{t\to\infty} u'(t) = -\infty$. Consequently, $u(t) < 0$ for large t, a contradiction. Then (2.1) is oscillatory. This completes the proof of the theorem. □

Theorem 2.2.2 *Let* $-\infty < \lim_{t\to\infty} ta(t) \leq 0$ *and* $1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t) \geq 0$. *If* (2.17) *holds, then* (2.1) *is oscillatory.*

Proof If possible, let (2.1) be nonoscillatory. From Lemma 1.5.3 and Lemma 1.5.4, it follows that Eq. (2.1) admits a nonoscillatory solution $x(t)$ such that $x(t)x'(t) \geq 0$ for $t \geq t_0 \geq \sigma$. Setting $u(t) = \frac{t^2x'(t)}{x(t)}$ for $t \geq t_0$, we observe that $u(t) \geq 0$ is a solution of (2.15), where $G(u(t), t)$ is given by (2.16). A simple calculation shows that $G(u(t), t)$ attains the minimum value

$$\begin{aligned}
&\frac{2t^2a^3(t)}{27} - \frac{t^2a(t)b(t)}{3} + t^2c(t) - \frac{2}{3}a(t) + tb(t) + \frac{t^2a(t)a'(t)}{3} - ta'(t) \\
&\quad - \frac{2}{3\sqrt{3}t}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{3/2}
\end{aligned}$$

for $u(t) \geq 0$ at

$$u(t) = t\left[\left(1 - \frac{ta(t)}{3}\right) + \frac{1}{\sqrt{3}}\left(1 - t^2 b(t) + \frac{1}{3}t^2 a^2(t) + t^2 a'(t)\right)^{1/2}\right].$$

Integrating (2.15) from t_0 to t and by using the minimum of $\frac{3}{2}\frac{u^2(t)}{t^2} - \frac{4u(t)}{t} + a(t)u(t)$ as $-\frac{1}{6}(ta(t) - 4)^2$ for $u(t) > 0$ at $U(t) = \frac{t^2}{3}(\frac{4}{t} - a(t))$, we obtain

$$u'(t) \leq k + \frac{1}{6}\big(ta(t) - 4\big)^2 - \int_{t_0}^{t} \min G\big(u(s), s\big)\, ds.$$

From the given hypothesis, it follows that $u'(t) \to -\infty$ as $t \to \infty$ and hence $u(t) < 0$ for large t, a contradiction. Thus, the theorem is proved. □

Remark 2.2.1 Theorem 2.2.2 holds for the Euler equation (2.5) with $a_0 \leq 0$, $b(t) \leq 0$ and $c_0 > 0$. Indeed, if (2.18) holds, then (2.5) is oscillatory.

Theorem 2.2.3 *Suppose that* (2.1) *has an oscillatory solution and a nonoscillatory solution which tends to zero as $t \to \infty$. Then all nonoscillatory solutions of* (2.1) *tend to zero as $t \to \infty$.*

Proof Suppose that $x_0(t)$ is a nonoscillatory solution of (2.1) such that $x_0(t) \to 0$ as $t \to \infty$. Without any loss of generality, we may assume that $x_0(t) > 0$ for $t \geq t_0 \geq \sigma$. Let $x(t)$ be any nonoscillatory solution of (2.1). We claim that $\lim_{t\to\infty} x(t) = 0$. If not, assume that $\lim_{t\to\infty} x(t) = \mu_1 \neq 0$. Clearly, $x_0(t)$ and $x(t)$ are linearly independent.

Define the solutions $u_1(t)$ and $u_2(t)$ of (2.2) with the initial conditions

$$u_1(\sigma) = u_1'(\sigma) = 0, \qquad r(\sigma)u_1''(\sigma) = 1$$

and

$$u_2(\sigma) = u_2''(\sigma) = 0, \qquad u_2'(\sigma) = -1.$$

From Lemma 1.5.10, it follows that both $u_1(t)$ and $u_2(t)$ are oscillatory. Let $W_1(t) = W(u_1, u_2)(t) = u_1(t)u_2'(t) - u_1'(t)u_2(t)$. Proceeding as in Theorem 2.1.13, we may get $W_1(t) > 0$ for $t > \sigma$.

Clearly, $u_1(t)$ and $u_2(t)$ are solutions of the second-order differential equation (2.50). Consequently, $\{u_1(t), u_2(t), x_0(t)\}$ forms a basis for the solution space of (2.1), and hence for (2.2). Let $x(t) = c_1x_0(t) + c_2u_1(t) + c_3u_2(t)$. Clearly, $c_1 \neq 0$. Setting $z(t) = \frac{x(t)}{c_1}$ and $x_1(t) = -[\frac{c_2}{c_1}u_1(t) + \frac{c_3}{c_1}u_2(t)]$, we have $z(t) = x_0(t) - x_1(t)$. Again $x_0(t) > 0$ for large t and $x_1(t)$ oscillatory imply that $z(t)$ cannot eventually be negative. Hence $z(t) > 0$ for large t. Consequently, $\mu > 0$ where $\mu = \frac{\mu_1}{c_1} = \lim_{t\to\infty} z(t) > 0$.

Let $\langle \sigma_n \rangle$ be an increasing sequence of maxima of $x_1(t)$ such that $x_1(\sigma_n) \geq 0$, $x_1'(\sigma_n) = 0$ for $\sigma_n > \sigma$. From Lemma 1.5.14, it follows that $z'(t) < 0$. Clearly,

$\lim_{n\to\infty} z(\sigma_n) = \mu$. Again $z(t) > 0$ implies that $x_0(\sigma_n) > x_1(\sigma_n)$. Now

$$0 < \mu = \lim_{n\to\infty} z(\sigma_n) = \lim_{n\to\infty} \big(x_0(\sigma_n) - x_1(\sigma_n)\big) = 0,$$

a contradiction. Hence $\lim_{t\to\infty} z(t) = 0$. Consequently, $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Lemma 2.2.2 *If $z_1(t)$ is an oscillatory solution of* (2.49), *then there exists an oscillatory solution $z_2(t)$ of* (2.49) *such that $W(t) = W(z_1, z_2)(t) = z_1(t)z_2'(t) - z_1'(t)z_2(t) > 0$ for large t.*

Proof Let $\beta \ge \sigma$ be such that $z_1(\beta) \ne 0$. Suppose that $\{t_n\}_{n=1}^{\infty}$ is a sequence of zeros of $z_1(t)$ in (β, ∞) such that $t_n \to \infty$ as $n \to \infty$. Define a sequence $\langle x_n(t)\rangle_{n=1}^{\infty}$ of solutions of (2.49) on $[\beta, \infty)$ with the boundary conditions

$$x_n(\beta) = x_n(t_n) = 0, \qquad x_n'(t_n) > 0.$$

Suppose that $\{u_1(t), u_2(t), u_3(t)\}$ forms a basis for the solution space of (2.49). Then there exist real constants c_{1n}, c_{2n} and c_{3n} such that

$$x_n(t) = c_{1n}u_1(t) + c_{2n}u_2(t) + c_{3n}u_3(t)$$

with $c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$.

We claim that the zeros of $z_1(t)$ and $x_n(t)$ separate in (β, t_n). Let α_1 and α_2 ($\alpha_1 < \alpha_2$) be two consecutive zeros of $z_1(t)$ in (β, t_n). If possible, let $x_n(t) > 0$ or < 0 for $t \in [\alpha_1, \alpha_2]$. From Lemma 1.5.13, it follows that there exists a constant λ such that $z_1(t) - \lambda x_n(t)$ has a double zero in (α_1, α_2). This is a contradiction, because (2.49) is of type C_{II} and $z_1(t) - \lambda x_n(t)$ has a zero at t_n. Hence $x_n(t)$ has a zero in $[\alpha_1, \alpha_2]$. Again $x_n(t_n) = z_1(t_n) = 0$ implies that $x_n(\alpha_1) \ne 0$ and $x_n(\alpha_2) \ne 0$ (see Theorem 2.10, [16]). Similarly, if α_1 and α_2 are consecutive zeros of $x_n(t)$ in (β, t_n), then $z_1(t)$ has a zero in (α_1, α_2). Thus the claim holds, that is, the zeros of $z_1(t)$ and $z_2(t)$ separate each other in (β, t_n).

The bounded sequences $\langle c_{i_n}\rangle_{n=1}^{\infty}$, $i = 1, 2, 3$ admit a convergent subsequence, say $\langle c_{i_{n_j}}\rangle$, $i = 1, 2, 3$, respectively. Let $c_i = \lim_{n_j\to\infty} c_{i_{n_j}}$, $i = 1, 2, 3$. So $\langle x_{n_k}(t)\rangle$ converges uniformly to the solution $z_2(t) = c_1u_1(t) + c_2u_2(t) + c_3u_3(t)$ of (2.49). Thus $z_2(\beta) = 0$ and the zeros of $z_1(t)$ and $z_2(t)$ separate in (β, ∞). Since $z_1(\beta) \ne 0$, it follows that $z_1(t)$ and $z_2(t)$ are linearly independent oscillatory solutions of (2.49).

Next, we show that every linear combination of $z_1(t)$ and $z_2(t)$ is oscillatory. If possible, suppose that $\mu_1 z_1(t) + \mu_2 z_2(t)$ is nonoscillatory, where μ_1 and μ_2 are some constants. Without any loss of generality, we suppose that $\mu_1 z_1(t) + \mu_2 z_2(t) > 0$ for $t \ge t_0 > \beta$. Let t_1, t_2 and t_3 be successive zeros of $z_1(t)$ in $[t_0, \infty)$. So $\mu_2 z_2(t_i) > 0$, $i = 1, 2, 3$. This contradicts the fact that the zeros of $z_1(t)$ and $z_2(t)$ separate in (β, ∞). Hence every linear combination of $z_1(t)$ and $z_2(t)$ is oscillatory. Next, we prove that $W[z_1, z_2](t) \ne 0$ for $t \in (\beta, \infty)$. If not, then there exists a γ, $\gamma > \beta$ such that $W[z_1, z_2](\gamma) = 0$, that is, $z_1(\gamma)z_2'(\gamma) - z_1'(\gamma)z_2(\gamma) = 0$. Since $z_1(t)$ and $z_2(t)$ separate their zeros in (β, ∞), both of $z_1(\gamma)$ and $z_2(\gamma)$ are not equal to zero. Hence $v(t) = z_1(\gamma)z_2(t) - z_1(t)z_2(\gamma)$ is a nontrivial solution of (2.49) with $v(\gamma) = v'(\gamma) = 0$ and $v''(t) \neq 0$. Consequently, $v(t) > 0$ or < 0 for $t > \gamma$, a contradiction to the fact that every linear combination of $z_1(t)$ and $z_2(t)$ is oscillatory. Hence $W[z_1, z_2](t) > 0$ or < 0 for $t > \beta$. If $W[z_1, z_2](t) < 0$ for $t > \beta$, then one may take $-z_2(t)$ in place of $z_2(t)$ to obtain the required result. This completes the proof of the lemma. □

Lemma 2.2.3 *$z_1(t)$ and $z_2(t)$ are linearly independent oscillatory solutions of the second-order differential equation*

$$\left(\frac{r(t)z'}{u(t)}\right)' + \left(\frac{r(t)u''(t) + q(t)u(t)}{u^2(t)}\right)z = 0, \tag{2.57}$$

where $u(t) = r(t)W(t)$.

Proof Clearly, $z_1(t)$ and $z_2(t)$ are solutions of the second-order differential equation

$$\begin{vmatrix} z_1(t) & z_2(t) & y(t) \\ z_1'(t) & z_2'(t) & y'(t) \\ (rz_1')'(t) & (rz_2')'(t) & (ry')'(t) \end{vmatrix} = 0.$$

Expanding the above determinant, we obtain (2.57). □

Remark 2.2.2 $u(t)$ is a positive solution of (2.2).

Lemma 2.2.4 *Suppose that $a(t) \leq \alpha < 0$ for $t \geq \beta \geq \sigma$. If $z(t)$ is a solution of* (2.49) *with the initial conditions $z(\beta) = z'(\beta) = 0$, $z''(\beta) > 0$, then $z(t)$ has the following properties*:

$$z(t) > 0, \qquad z'(t) > 0, \qquad z''(t) > 0 \quad \textit{and} \quad \big(rz'\big)'(t) > 0$$

for $t \geq \beta$. Moreover, for every real number $M > 0$, there exists a real $N > 0$ such that

$$z(t) > -\frac{\alpha M}{2} r(t)\left(\int_\beta^t \frac{ds}{r(s)}\right)^2$$

for $t > N$.

Proof Since (2.49) is of type C_{II}, we have $z(t) > 0$ for $t > \beta$. From (2.49), we have $(rz')'(t) + q(t)z(t) > (rz')'(\beta) + q(\beta)z(\beta) > 0$ for $t > \beta$. Integrating this inequality from β to t, we get $r(t)z'(t) > r(\beta)z''(\beta)(t - \beta)$. Then, $z'(t) > 0$ for $t > \beta$. Moreover, for every real number $M > 0$, there exists an integer $N_1 > 0$ such that $z'(t) > \frac{M}{r(t)}$ for $t > N_1$. Again $(rz')'(t) > r(\beta)z''(\beta)$ implies that $r(t)z''(t) > -r'(t)z'(t) + r(\beta)z''(\beta) > 0$. Thus, $z''(t) > 0$ for $t > \beta$ and

$$z''(t) > -\frac{r'(t)}{r(t)}z'(t) + \frac{r(\beta)z''(\beta)}{r(t)} > -a(t)\frac{M}{r(t)} + z''(\beta) > -\frac{\alpha M}{r(t)} + z''(\beta).$$

Integrating the above inequality from N_1 to t, we have

$$z'(t) > z'(N_1) - \alpha M \int_{N_1}^{t} \frac{ds}{r(s)} + z''(\beta)(t - N_1).$$

But there exists $N_2 > N_1$ such that for $t > N_2$, we have

$$-\alpha M \int_{\beta}^{N_1} \frac{ds}{r(s)} < z''(\beta)(t - N_1).$$

Hence, for $t > N_2$,

$$z'(t) > z'(N_1) - \alpha M \int_{\beta}^{t} \frac{ds}{r(s)}.$$

Dividing both sides by $r(t)$ and integrating the resulting inequality from N_2 to t, we obtain

$$\frac{z(t)}{r(t)} > \frac{z(N_2)}{r(N_2)} + z'(N_1)(t - N_2) - \alpha M \int_{N_2}^{t} \left(\frac{1}{r(s)} \int_{s}^{\beta} \frac{d\theta}{r(\theta)} \right) ds. \tag{2.58}$$

It is possible to choose $N > 0$ such that for $t > N > N_2$,

$$-\alpha M \int_{\beta}^{N_2} \left(\frac{1}{r(s)} \int_{\beta}^{s} \frac{d\theta}{r(\theta)} \right) ds < z'(N_1)(t - N_2).$$

Hence (2.58) yields

$$z(t) > -\alpha M r(t) \int_{\beta}^{t} \left(\frac{1}{r(s)} \int_{s}^{\beta} \frac{d\theta}{r(\theta)} \right) ds = -\frac{\alpha M}{2} r(t) \left(\int_{\beta}^{t} \frac{ds}{r(s)} \right)^2.$$

The proof is complete. □

Theorem 2.2.4 *If $a(t) \le \alpha < 0$ for $t \ge \sigma$ and (2.1) is nonoscillatory, then (2.1) admits a nonoscillatory solution which tends to zero as $t \to \infty$.*

Proof Clearly (2.1) is oscillatory, if and only if (2.2) is oscillatory. From Lemma 1.5.11, it follows that (2.49) is oscillatory. Suppose that $z_1(t)$ is an oscillatory solution of (2.49). By Lemma 2.2.2, there exists an oscillatory solution $z_2(t)$ of (2.49) such that $W[z_1, z_2](t) > 0$ for $t > \beta$. From Remark 2.2.2, it follows that $u(t) = r(t)W(t)$ is a positive solution of (2.2) and hence of (2.1). So $u'(t) < 0$, $u''(t) > 0$ for $t \ge \beta$, $\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} u''(t) = 0$ and $\lim_{t\to\infty} u(t) = \lambda \ne \infty$. If possible, suppose that $\lambda > 0$.

Let $z(t)$ be a solution of (2.49) with $z(\beta) = z'(\beta) = 0$ and $z''(\beta) > 0$. So, $z(t) > 0$ for $t > \beta$. Clearly, the set $\{z_1(t), z_2(t), z(t)\}$ forms a basis for the solution space of (2.49) and

$$\frac{d}{dt}\left\{ r(t) \begin{vmatrix} z_1(t) & z_2(t) & z(t) \\ z_1'(t) & z_2'(t) & z'(t) \\ (rz_1')'(t) & (rz_2')'(t) & (rz')'(t) \end{vmatrix} \right\} = 0. \tag{2.59}$$

Integrating (2.59) from β to t, we have

$$\begin{vmatrix} z_1(t) & z_2(t) & z(t) \\ z_1'(t) & z_2'(t) & z'(t) \\ (rz_1')'(t) & (rz_2')'(t) & (rz')'(t) \end{vmatrix} = \frac{W(\beta)z''(\beta)}{r(t)}. \tag{2.60}$$

Expanding (2.60), we get

$$\left(\frac{r(t)z'(t)}{u(t)}\right)' + \left(\frac{r(t)u''(t) + q(t)u(t)}{u^2(t)}\right)z(t) = \frac{W(\beta)z''(\beta)}{u^2(t)}. \tag{2.61}$$

Since $u'(t) < 0$ for $t \geq \beta$, from Lemma 2.2.4, it follows that $(\frac{r(t)z'(t)}{u(t)})' > 0$ for $t \geq \beta$. Hence, for $t \geq \beta$, (2.61) yields

$$\big(r(t)u''(t) + q(t)u(t)\big)z(t) < W(\beta)z''(\beta). \tag{2.62}$$

Since $z_1(t)$ and $z_2(t)$ are oscillatory solutions of (2.57), from Theorem 2.3 due to Ohriska [25], we conclude that

$$\limsup_{t\to\infty}\left\{\left(\int_\beta^t \frac{u(s)}{r(s)}\,ds\right)^2 \frac{r(t)}{u(t)}\left(\frac{r(t)u''(t) + q(t)u(t)}{u^2(t)}\right)\right\} = \mu \geq \frac{1}{4}. \tag{2.63}$$

So there exists a sequence $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\left(\int_\beta^{t_n} \frac{u(s)}{r(s)}\,ds\right)^2 \frac{r(t_n)}{u(t_n)}\left(\frac{ru'' + qu}{u^2}\right)(t_n) > 0, \tag{2.64}$$

for every n and

$$\lim_{n\to\infty}\left\{\left(\int_\beta^{t_n} \frac{u(s)}{r(s)}\,ds\right)^2 \frac{r(t_n)}{u(t_n)}\left(\frac{ru'' + qu}{u^2}\right)(t_n)\right\} = \mu.$$

For $M = -\frac{16u^2(\beta)W(\beta)z''(\beta)}{\alpha\lambda^3} > 0$, there exists a real number $N > \beta$ (by Lemma 2.2.4) such that

$$\left(\int_\beta^{t_n} \frac{u(s)}{r(s)}\,ds\right)^2 \leq u^2(\beta)\left(\int_\beta^{t_n} \frac{ds}{r(s)}\right)^2 \leq -\frac{2u^2(\beta)z(t_n)}{\alpha M r(t_n)}, \quad t_n \geq N.$$

This, in turn, because of (2.64) and (2.62), implies that

$$\left(\int_\beta^{t_n} \frac{u(s)}{r(s)}\,ds\right)^2 \frac{r(t_n)}{u(t_n)}\left(\frac{ru'' + qu}{u^2}\right)(t_n) \leq -\frac{2u^2(\beta)W(\beta)z''(\beta)}{\alpha M\lambda^3} = \frac{1}{8}.$$

Then $\frac{1}{4} \leq \mu \leq \frac{1}{8}$, a contradiction. Consequently, $\lim_{t\to\infty} u(t) = 0$. This completes the proof of the theorem. □

Theorem 2.2.5 *Suppose that $a(t) \leq \alpha < 0$ for $t \geq \sigma$. Then (2.1) is oscillatory, if and only if all nonoscillatory solutions of (2.1) tend to zero as $t \to \infty$.*

Proof Suppose that (2.1) is oscillatory. From Theorems 2.2.3 and 2.2.4, it follows that all nonoscillatory solutions of (2.1) tend to zero as $t \to \infty$.

Conversely, suppose that all nonoscillatory solutions of (2.1) tend to zero as $t \to \infty$. Let $u(t)$ be a nonoscillatory solution of (2.1). From Lemma 1.5.2, it follows that $u(t)u'(t) \geq 0$ or ≤ 0 for $t \geq t_0 \geq \sigma$. Since $\lim_{t\to\infty} u(t) = 0$, it follows that $u(t)u'(t) < 0$ for $t \geq t_0$. Then (2.1) is oscillatory, by Lemma 1.5.4. The proof is complete. □

Remark 2.2.3 Theorem 2.2.5 holds good if we replace the condition "$a(t) \leq \alpha < 0$" by "$\int_\sigma^\infty a(s)\,ds \neq -\infty$", which we give here as a theorem:

Theorem 2.2.6 *Suppose that $\int_\sigma^\infty a(s)\,ds \neq -\infty$. Then (2.1) is oscillatory, if and only if all nonoscillatory solutions of (2.1) tend to zero as $t \to \infty$.*

Proof In view of Theorem 2.2.3 and the second part of Theorem 2.2.5, it is enough to show that (2.2) admits a nonoscillatory solution which tends to zero as $t \to \infty$.

From Remark 2.2.2, it follows that $u(t) = r(t)W(t)$ is a positive solution of (2.2) and hence of (2.1). From Lemma 1.5.4, we have $u'(t) < 0$ for $t \geq \beta$. If possible, let $\lim_{t\to\infty} u(t) = \lambda > 0$. Without any loss of generality, we may take $\lambda = \frac{1}{2}$. Let $K = \exp(\int_\sigma^\infty a(s)\,ds)$. Let $M > 0$ be such that $u(t) < 1$ for $t > M$. Hence, for $t > M$, $\frac{r(t)}{u(t)} > K$. From Lemmas 2.2.3 and 1.5.23, it follows that the second-order differential equation

$$z'' + \frac{1}{K}\left(\frac{r(t)u''(t) + q(t)u(t)}{u^2(t)}\right)z = 0 \tag{2.65}$$

is oscillatory. Proceeding as in Jones [19], we may show that (2.65) is nonoscillatory, which is a contradiction. Hence $\lim_{t\to\infty} u(t) = 0$. The proof is complete. □

Example 2.2.1 By Theorem 2.2.1, the equation

$$x''' - \left(1 - \frac{1}{t^2}\right)x'' - \frac{2}{t^3}x' + \left(2 - \frac{1}{t^2} - \frac{2}{t^3}\right)x = 0$$

is oscillatory. Theorem 2.2.5 is applicable to this equation, whereas Theorem 2.2.6 is not applicable to this example. Clearly, $x(t) = e^{-t}$ is a nonoscillatory solution of this equation which tends to zero as $t \to \infty$.

Example 2.2.2 By Theorem 2.2.1, the equation

$$x''' - \frac{1}{t^2}x'' - \frac{2}{t^3}x' + \left(8 + \frac{4}{t^2} - \frac{4}{t^3}\right)x = 0, \quad t \geq 5$$

is oscillatory. Theorem 2.2.6 is applicable to this example, whereas Theorem 2.2.5 cannot be applied to this example. This equation admits a nonoscillatory solution $x(t) = e^{-2t}$, which tends to zero as $t \to \infty$.

2.3 Behaviour of Solutions of $x''' + a(t)x'' + b(t)x' + c(t)x = 0$ with $a(t) \leq 0$, $b(t) \leq 0$ and $c(t) < 0$

In this section, we study the oscillation, nonoscillation and their asymptotic behaviour of Eq. (2.1), when $a(t) \leq 0$, $b(t) \leq 0$ and $c(t) < 0$.

Lemma 2.3.1 *Equation* (2.1) *is of type* C_{II}.

Proof Let $x(t)$ be a solution of (2.1) with $x(t_0) = 0 = x'(t_0)$ and $x''(t_0) > 0$ for $t_0 > \sigma$. From the continuity of $x''(t)$, it follows that $x''(t) > 0$, $t \in (t_0, t_0 + \delta)$ for some $\delta > 0$. We claim that $x''(t) > 0$ for $t > t_0$. If not, there exists a $t_1 > t_0$ such that $x''(t_1) = 0$ and $x(t) > 0$, $x'(t) > 0$ and $x''(t) > 0$ for $t \in (t_0, t_1)$. Since $x(t)$ satisfies (2.2), integrating (2.2) from t_0 to t_1, we obtain

$$0 > -r(t_0)x''(t_0) = -\int_{t_0}^{t_1} \left[q(t)x'(t) + p(t)x(t)\right] dt > 0,$$

a contradiction. Hence our claim holds. This in turn implies that $x(t) > 0$ for $t > t_0$. The proof of the lemma is complete. □

Remark 2.3.1 From Lemma 2.3.1, it follows that a solution $x(t)$ of (2.1) with the property $x(t_0) = x'(t_0) = 0$, $x''(t_0) > 0$ satisfies $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ for $t > t_0$.

In Lemma 2.3.2, we do not assume that $b(t) \leq 0$ and $c(t) < 0$.

Lemma 2.3.2 *If* $2c(t) - b'(t) - a(t)b(t) \leq 0$ *but* $\not\equiv 0$ *on any subinterval of* $[\sigma, \infty)$, *then* (2.1) *is of type* C_{II}.

Proof Suppose that $x(t)$ is a solution of (2.1) with $x(t_0) = x'(t_0) = 0$ and $x''(t_0) > 0$, $t_0 \geq \sigma$. We claim that $x(t) > 0$ for $t > t_0$. If not, then $x(t_1) = 0$ for some $t_1 > t_0$. Since $x(t)$ is a solution of (2.1), it satisfies (2.2). Multiplying (2.2) through by $x(t)$ and integrating the resulting identity from t_0 to t_1, we obtain

$$r(t_1)\big(x'(t_1)\big)^2 \leq \int_{t_0}^{t_1} \big(2p(t) - q'(t)\big)x^2(t)\, dt,$$

that is,

$$0 < r(t_1)\big(x'(t_1)\big)^2 \le \int_{t_0}^{t_1} \big(2c(t) - b'(t) - a(t)b(t)\big) r(t) x^2(t)\,dt < 0,$$

a contradiction. Hence $x(t) > 0$ for $t > t_0$. This completes the proof of the lemma. □

Lemma 2.3.3 *Equation* (2.49) *is of type* C_I.

Lemma 2.3.4 *Equation* (2.2) *is oscillatory if and only if* (2.49) *is oscillatory.*

Theorem 2.3.1 *Suppose that* $2c(t) - b'(t) - a(t)b(t) \le 0$ *but* $\not\equiv 0$ *on any subinterval of* $[\sigma, \infty)$ *such that*

$$\int_\sigma^\infty \big(2c(t) - b'(t) - a(t)b(t)\big) \exp\left(\int_\sigma^t a(s)\,ds\right) dt = -\infty. \tag{2.66}$$

Then Eq. (2.1) *is oscillatory, if and only if all nonoscillatory solutions of the second-order differential equation* (2.11) *are eventually positive.*

Proof Suppose that all nonoscillatory solutions of (2.11) are eventually positive. We have to show that (2.1) has an oscillatory solution. If possible, let all solutions of (2.1) be nonoscillatory. From Lemma 1.5.19, it follows that there exists a nonoscillatory solution $u(t)$ of (2.1) which does not satisfy the conditions

$$\left.\begin{aligned} &u(t)u'(t)u''(t) \ne 0 \quad \text{for } t \ge t_0 \ge \sigma,\\ &\operatorname{sgn} u(t) = \operatorname{sgn} u'(t) = \operatorname{sgn} u''(t), \quad t \ge t_0 \ge \sigma. \end{aligned}\right\} \tag{2.67}$$

Without any loss of generality, we may assume that $u(t) > 0$ for $t \ge t_0 \ge \sigma$. From Lemma 1.5.14, it follows that all solutions of the second-order nonhomogeneous differential equation

$$\big(r(t)x'(t)\big)' + q(t)x(t) = p(t)u(t), \quad t > t_0,$$

are nonoscillatory. Since $-u'(t)$ satisfies the above equation, $-u'(t)$ is nonoscillatory. Let $u'(t) > 0$ or < 0 for $t \ge t_1 \ge t_0$. If $\operatorname{sgn} u(t) = \operatorname{sgn} u'(t)$, then $u'(t) > 0$ for $t \ge t_1$. Since $u(t)$ satisfies (2.2), multiplying (2.2) through by $u(t)$ and integrating the resulting identity from t_1 to t, we have

$$F\big[u(t)\big] = F\big[u(t_1)\big] + \int_{t_1}^t \big(2p(s) - q'(s)\big)u^2(s)\,ds + \int_{t_1}^t r'(s)\big(u'(s)\big)^2\,ds, \tag{2.68}$$

where $F[u(t)]$ is given in (2.54). Since $r'(t) \le 0$ and $u(t)$ is increasing, we have

$$\begin{aligned} F\big(u(t)\big) &\le F\big(u(t_1)\big) + \int_{t_1}^t \big(2p(s) - q'(s)\big)u^2(s)\,ds\\ &\le F\big(u(t_1)\big) + u^2(t_1)\int_{t_1}^t \big(2c(s) - b'(s) - a(s)b(s)\big)r(s)\,ds. \end{aligned}$$

Hence $F(u(t)) \to -\infty$ as $t \to \infty$. Let $t_2 > t_1$ such that $F(u(t)) < 0$ for $t \geq t_2$. Hence

$$-2r(t)u(t)u''(t) < -r(t)\big(u'(t)\big)^2 + q(t)u^2(t) < 0, \quad t > t_2.$$

Consequently, $u(t)u''(t) > 0$ for $t > t_2$. This shows that $u(t)$ satisfies (2.67), a contradiction. Hence $\operatorname{sgn} u(t) \neq \operatorname{sgn} u'(t)$. Now taking $z(t) = \frac{u'(t)}{u(t)}$, it may be easily verified that $z(t)$ is a negative nonoscillatory solution of (2.11), a contradiction again. Hence (2.1) admits an oscillatory solution.

Conversely, suppose that (2.1) has an oscillatory solution. We have to prove that all nonoscillatory solutions of (2.11) are eventually positive. If possible, suppose that $z(t)$ is a negative nonoscillatory solution of (2.11) in $[t_3, \infty)$, $t_3 \geq \sigma$. It may easily be shown that $u(t) = \exp(\int_{t_3}^t z(s)\,ds)$ is a positive nonoscillatory solution of (2.1) with $u'(t) < 0$. It contradicts Lemma 1.5.19. Hence all nonoscillatory solutions of (2.11) are eventually positive. This completes the proof of the theorem. □

Theorem 2.3.2 *Suppose that $a'(t) \geq 0$, $b(t) - 2a'(t) \leq 0$ and $c(t) - b'(t) + a''(t) < 0$. If*

$$\int_\sigma^\infty \Big[-\frac{2}{27}a^3(t) + \frac{1}{3}a(t)b(t) - c(t) - \frac{2}{3}a(t)a'(t) + b'(t) - a''(t) - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - b(t) + 2a'(t)\Big)^{3/2}\Big]\,dt = \infty, \tag{2.69}$$

then (2.1) *is oscillatory.*

Proof Since (2.1) is of type C_{II}, from Lemma 1.5.11 it follows that (2.1) is oscillatory if and only if its adjoint (2.3) is oscillatory, that is, if and only if

$$x''' - a(t)x'' + \big(b(t) - 2a'(t)\big)x' + \big(c(t) - b'(t) + a''(t)\big)x = 0$$

is oscillatory. Clearly, Eq. (2.3) satisfies hypotheses of Theorem 2.1.2. So it is oscillatory, and hence (2.1) is oscillatory. The theorem is proved. □

Note that Theorem 2.3.2 does not contain any sign restrictions on $b(t)$ and $c(t)$.

Theorem 2.3.3 *Suppose that $b(t) - 2a'(t) \leq 0$, $c(t) - b'(t) + a''(t) < 0$, $ta(t) \geq -3$ and $1 - t^2b(t) + \frac{t^2a^2(t)}{3} + t^2a'(t) \geq 0$. If*

$$\int_\sigma^\infty \Big[-\frac{2t^2a^3(t)}{27} + \frac{t^2a(t)b(t)}{3} - t^2c(t) + \frac{2}{3}a(t) + tb(t) - \frac{t^2a(t)a'(t)}{3} - ta'(t) - t^2a''(t) + t^2b'(t) - \frac{2}{3\sqrt{3}t}\Big(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\Big)^{3/2}\Big]\,dt = \infty, \tag{2.70}$$

then (2.1) *is oscillatory.*

Theorem 2.3.4 *Suppose that* $b(t) - 2a'(t) \le 0$, $c(t) - b'(t) + a''(t) < 0$, $ta(t) \le -3$ *and* $2ta(t) + t^2 b(t) - t^2 a'(t) < -2$. *If* (2.70) *holds*, *then* (2.1) *is oscillatory.*

Remark 2.3.2 Applying Theorem 2.3.3 to the Euler equation (2.5), it follows that (2.5) is oscillatory if $-3 \le a_0 < 0$, $b_0 < 0$, $c_0 < 0$ and

$$-\frac{2a_0^3}{27} + \frac{a_0^2}{3} - \frac{a_0}{3} + \frac{a_0 b_0}{3} - b_0 - c_0 - \frac{2}{3\sqrt{3}}\left(1 - b_0 + \frac{a_0^2}{3} - a_0\right)^{3/2} > 0 \quad (2.71)$$

holds. Similarly, applying Theorem 2.3.4 to the Euler equation (2.5), we observe that (2.5) is oscillatory if $a_0 \le -3$, $b_0 < 0$, $c_0 < 0$, $3a_0 + b_0 < -2$ and (2.71) holds.

One may apply Theorem 2.3.2 to Eq. (1.5). We find that (1.5) is oscillatory if (1.9) holds. Thus, one may treat Theorem 2.3.2 as a generalisation of Proposition 1.2.3(i).

Example 2.3.1 By Theorem 2.3.3, the equation

$$x''' - \frac{1}{t}x'' - \frac{1}{t^2}x' - \frac{8}{t^3}x = 0, \quad t \ge 1$$

is oscillatory. In particular, $x_1(t) = \cos(\sqrt{2}\log t)$ and $x_2(t) = \sin(\sqrt{2}\log t)$ are the two oscillatory solutions of this equation.

Example 2.3.2 Consider

$$x''' - \frac{6}{t}x'' - \frac{1}{t^2}x' - \frac{63}{t^3}x = 0, \quad t \ge 1.$$

Since all conditions of Theorem 2.3.4 are satisfied, this equation has an oscillatory solution. One may observe that $x_1(t) = \cos(\sqrt{7}\log t)$ and $x_2(t) = \sin(\sqrt{7}\log t)$ are the oscillatory solutions of the equation.

Theorem 2.3.5 *Suppose that* $2c(t) - b'(t) - a(t)b(t) < 0$ *but* $\not\equiv 0$ *on any subinterval of* $[\sigma, \infty)$ *such that* (2.66) *holds and* (2.1) *has a nonoscillatory solution* $M(t)$ *such that* $M(t) > 0$ *and* $M'(t) > 0$ *for* $t > \sigma$. *Then* (2.1) *is oscillatory if and only if the second-order differential equation* (2.34) *is oscillatory.*

Proof If (2.1) has an oscillatory solution $x(t)$, then it is easy to verify that $(\frac{x(t)}{M(t)})'$ is an oscillatory solution of (2.34). Hence (2.1) being oscillatory implies that (2.34) is oscillatory.

Conversely, suppose that (2.34) is oscillatory. Further, if possible, suppose that all solutions of (2.1) are nonoscillatory. By Lemma 1.5.19, there exists at least one nonoscillatory solution $u(t)$ of (2.1) which does not satisfy the condition

$$u(t)u'(t)u''(t) \ne 0 \quad \text{for } t \ge t_0 \ge \sigma.$$

$$\operatorname{sgn} u(t) = \operatorname{sgn} u'(t) = \operatorname{sgn} u''(t) \quad \text{for } t \ge t_0 \ge \sigma.$$

Without any loss of generality, let us assume that $u(t) > 0$ for $t \geq t_0 \geq \sigma$. Clearly, $u(t)$ is a solution of (2.2) implies that $u'(t)$ is a solution of the second-order differential equation

$$\big(r(t)z'\big)' + q(t)z = -p(t)u, \quad t > t_0. \tag{2.72}$$

From Lemma 1.5.14, it follows that all solutions of (2.72) are nonoscillatory. So $u'(t)$ is nonoscillatory. Proceeding as in Theorem 2.3.1, we obtain $u'(t) < 0$ for $t \geq t_1 \geq t_0$. It is easy to verify that $(\frac{u(t)}{M(t)})'$, $t \geq t_1$, is a solution of (2.34). Since all solutions of (2.34) are oscillatory, $(\frac{u(t)}{M(t)})'$ is oscillatory. But

$$\left(\frac{u(t)}{M(t)}\right)' = \big[M(t)u'(t) - M'(t)u(t)\big]M^{-2}(t) < 0, \quad t \geq t_1,$$

a contradiction. Hence (2.1) is oscillatory. This completes the proof of the theorem. □

Remark 2.3.3 Proceeding as in Lemma 2.3.1, one may show that (2.1) admits a solution $M(t)$ with the initial conditions $M(\sigma) = 0 = M'(\sigma)$ and $M''(\sigma) > 0$ which satisfies conditions $M(t) > 0$ and $M'(t) > 0$ for $t > \sigma$.

The following theorem shows that, if (2.1) has an oscillatory solution, then the set of all oscillatory solutions of (2.1) forms a two-dimensional subspace of its solution space.

Theorem 2.3.6 *Suppose that $b(t) \not\equiv 0$ on any subinterval of $[\sigma, \infty)$. If (2.1) has an oscillatory solution, then there exist two linearly independent oscillatory solutions $x_1(t)$ and $x_2(t)$ of (2.1) whose zeros separate and such that any oscillatory solution of (2.1) can be expressed as a linear combination of $x_1(t)$ and $x_2(t)$.*

In the following, we provide some results which are interesting in themselves and helpful in establishing Theorem 2.3.6.

Theorem 2.3.7 *Equation (2.49) admits a nonoscillatory solution $N(t)$ satisfying $N(t) > 0$, $N'(t) < 0$ and $(rN')'(t) + q(t)N(t) > 0$ for $t \in [\sigma, \infty)$.*

Proof Let $\{u_1(t), u_2(t), u_3(t)\}$ be a basis for the solution space of (2.49). Let $z_n(t)$ be a solution of (2.49) with $z_n(n) = z_n'(n) = 0$, $z_n''(n) > 0$, $n \geq \sigma$. So there exist constants c_{1n}, c_{2n}, c_{3n} such that

$$z_n(t) = c_{1n}u_1(t) + c_{2n}u_2(t) + c_{3n}u_3(t),$$

with $c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$. Since $\langle c_{in}\rangle_{n=1}^{\infty}$, $i = 1, 2, 3$ is a bounded sequence of real numbers, it admits a convergent subsequence. Without any loss of generality, we denote this convergent subsequence by $\langle c_{in_j}\rangle_{n=1}^{\infty}$, $i = 1, 2, 3$. Let $\lim_{n_j \to \infty} c_{in_j} = c_i$ for $i = 1, 2, 3$. Let

$$N(t) = c_1u_1(t) + c_2u_2(t) + c_3u_3(t).$$

Since (2.49) is of type C_I, we have $z_n(t) > 0$ for $t \in [\sigma, n)$. Now from (2.49) it follows that

$$\left[\left(r(t)z_n'(t)\right)' + q(t)z_n(t)\right]' = p(t)z_n(t) < 0, \quad t \in [\sigma, n).$$

Thus, for $\sigma \le t < n$,

$$\begin{aligned}\left(rz_n'\right)'(t) + q(t)z_n(t) &> \left(rz_n'\right)'(n) + q(n)z_n(n)\\ &= r(n)z_n''(n) + r'(n)z_n'(n) + q(n)z_n(n)\\ &= r(n)z_n''(n) > 0,\end{aligned}$$

that is,

$$\left(rz_n'\right)'(t) > -q(t)z_n(t) > 0.$$

This, in turn, implies that $r(t)z_n'(t)$ is increasing in $[\sigma, n)$. Hence, for $\sigma \le t < n$, $r(t)z_n'(t) < r(n)z_n'(n) = 0$ implies that $z_n'(t) < 0$.

Since the sequences $\langle z_n\rangle$, $\langle z_n'\rangle$ and $\langle (rz_n')' + qz_n\rangle$ converge uniformly to N, N' and $(rN')' + qN$, respectively, on any compact interval of $[\sigma, \infty)$, it follows that $N(t) \ge 0$, $N'(t) \le 0$ and $(rN')'(t) + q(t)N(t) \ge 0$ for $t \in [\sigma, \infty)$. If there is a point $t_1 \ge \sigma$ such that $N(t_1) = 0$, then $N(t) \equiv 0$ for $t \ge t_1$, because $N'(t) \le 0$ for all $t \in [\sigma, \infty)$. But $c_1^2 + c_2^2 + c_3^2 = 1$ implies that $N(t)$ is a nontrivial solution of (2.49), a contradiction. Thus $N(t) > 0$ for $t \in [\sigma, \infty)$. Further, $(rN')'(t) + q(t)N(t) \ge 0$, $t \ge \sigma$ implies that $(rN')'(t) \ge -q(t)N(t) \ge 0$. Consequently, $r(t)N'(t)$ is nondecreasing on $[\sigma, \infty)$. If $(rN')'(t_2) = 0$ for some $t_2 \ge \sigma$, it follows that $r(t)N'(t) = 0$ for $t \ge t_2$. Hence $N'(t) = 0$, $t \ge t_2$. So $N(t)$ is a positive constant. Now $(rN')'(t) + q(t)N(t) \ge 0$ for $t \ge \sigma$ implies that $q(t)N(t) \ge 0$ for $t \ge \sigma$, a contradiction. Hence $N'(t) < 0$ for $t \ge \sigma$. Further, if $[(rN')' + qN](t_3) = 0$ for some $t_3 \ge \sigma$, then $[(rN')' + qN](t) \equiv 0$ for $t \ge t_3$. This in turn implies that $[(rN')'(t) + q(t)N(t)]' = p(t)N(t) \equiv 0$, a contradiction. So $(rN')'(t) + q(t)N(t) > 0$ for all $t \ge \sigma$.

Hence the proof of the theorem is complete. □

Lemma 2.3.5 *The following statements hold.*

(i) $\lim_{t\to\infty} r(t)N'(t) = 0$
(ii) $\lim_{t\to\infty} tr(t)N'(t) = 0$
(iii) $\lim_{t\to\infty} t^2[(rN')'(t) + q(t)N(t)] = 0.$

Theorem 2.3.8 *If* (2.1) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions*, $u_1(t)$ *and* $u_2(t)$, *of*

$$\left(\frac{x'(t)}{N(t)}\right)' + \left(\frac{(rN')'(t) + q(t)N(t)}{r(t)N^2(t)}\right)x(t) = 0, \tag{2.73}$$

which satisfy (2.1).

Proof Since (2.1) has an oscillatory solution, (2.2) has an oscillatory solution. From Lemma 2.3.4 it follows that (2.49) is oscillatory. It is clear from Lemmas 2.3.3 and 1.5.10 that a solution of (2.49) which has at least one zero is oscillatory.

Let $z_1(t)$ and $z_2(t)$ be two linearly independent solutions of (2.49) with

$$z_1(\sigma) = 0, \qquad z_1'(\sigma) = 0, \qquad z_1''(\sigma) = 1,$$
$$z_2(\sigma) = 0, \qquad z_2'(\sigma) = 1, \qquad \left(rz_2'\right)'(\sigma) = 0.$$

So $z_1(t)$ and $z_2(t)$ are oscillatory. It is easy to verify that

$$w_1(t) = N(t)z_1'(t) - N'(t)z_1(t) = N^2(t)\left(\frac{z_1}{N}\right)'(t)$$

and

$$w_2(t) = N(t)z_2'(t) - N'(t)z_2(t) = N^2(t)\left(\frac{z_2}{N}\right)'(t)$$

are oscillatory solutions of

$$\left(\frac{r(t)x'(t)}{N(t)}\right)' + \left[\frac{(rN')'(t) + q(t)N(t)}{N^2(t)}\right]x = 0. \tag{2.74}$$

Consequently, $u_1(t) = r(t)w_1(t)$ and $u_2(t) = r(t)w_2(t)$ are oscillatory solutions of (2.73). It may easily be verified that $u_1(t)$ and $u_2(t)$ satisfy (2.2), and hence (2.1).

To complete the proof of the theorem, it remains to show that $u_1(t)$ and $u_2(t)$ are linearly independent. If possible, let $u_1(t)$ and $u_2(t)$ are linearly dependent. So there exist constants c_1 and c_2 not both zero, such that $c_1u_1(t) + c_2u_2(t) = 0$ for $t \in [\sigma, \infty)$, that is, $c_1w_1(t) + c_2w_2(t) = 0$ for $t \in [\sigma, \infty)$. Since $w_1(t)$ and $w_2(t)$ are nontrivial solutions of (2.74), $c_1 = 0$ implies that $c_2 = 0$ and $c_2 = 0$ implies that $c_1 = 0$. Hence $c_1 \neq 0$ and $c_2 \neq 0$. Now $w_1(t) + \lambda w_2(t) = 0$ for $t \in [\sigma, \infty)$, where $\lambda = \frac{c_2}{c_1}$, implies that

$$N(t)\big(z_1'(t) + \lambda z_2'(t)\big) - N'(t)\big(z_1(t) + \lambda z_2(t)\big) = 0,$$

that is,

$$\frac{z_1'(t) + \lambda z_2'(t)}{z_1(t) + \lambda z_2(t)} = \frac{N'(t)}{N(t)},$$

which gives upon integration $N(t) = c(z_1(t) + \lambda z_2(t))$, $c \neq 0$. Consequently, $z_1(t) + \lambda z_2(t)$ is nonoscillatory. Hence there exists a $t_1 \geq \sigma$ such that $z_1(t) + \lambda z_2(t)$ has one sign for $t \geq t_1$. Let t_2 and t_3 $(t_1 < t_2 < t_3)$ be consecutive zeros of $z_1(t)$. From Lemma 1.5.13, it follows that there exists a constant $\mu \neq 0$ such that $(z_1(t) + \lambda z_2(t)) + \mu z_1(t)$ has a double zero in (t_2, t_3) with a zero at $t = \sigma$. This contradicts the fact that (2.49) is of type C_I. Thus $u_1(t)$ and $u_2(t)$ are linearly independent. This completes the proof of the theorem. □

Remark 2.3.4 (i) Any solution of (2.73) is a solution of (2.1).

(ii) It is possible to choose two linearly independent solutions $x_1(t)$ and $x_2(t)$ of (2.73) such that $W(t) > 0$, where $W(t) = x_1(t)x_2'(t) - x_1'(t)x_2(t)$. Since (2.73) is oscillatory, $x_1(t)$ and $x_2(t)$ are oscillatory. Moreover, $x_1(t)$ and $x_2(t)$ are solutions of (2.1). By the Sturm separation theorem, the zeros of $x_1(t)$ and $x_2(t)$ separate each other in $[\sigma, \infty)$. This observation leads us to obtain the following corollary:

Corollary 2.3.1 *If* (2.1) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions u and v of* (2.1) *such that their zeros separate, and any linear combination of u and v is also oscillatory.*

Theorem 2.3.9 *N and W are linearly dependent. In fact, $W(t) = \lambda N(t)$, where $\lambda > 0$ is a constant.*

Proof We have chosen linearly independent solutions $x_1(t)$ and $x_2(t)$ of (2.73) such that $W(t) > 0$ for $t \geq \sigma$. Clearly, $x_1(t)$ and $x_2(t)$ are solutions of

$$\begin{vmatrix} x & x_1(t) & x_2(t) \\ x' & x_1'(t) & x_2'(t) \\ r(t)x'' & r(t)x_1''(t) & r(t)x_2''(t) \end{vmatrix} = 0,$$

$t \in [\sigma, \infty)$, that is, of

$$x'' - \frac{W'(t)}{W(t)}x' + \left(\frac{(rW')'(t) + q(t)W(t)}{r(t)W(t)}\right)x = 0. \tag{2.75}$$

Equation (2.73) may be written as

$$x'' - \frac{N'(t)}{N(t)}x' + \left(\frac{(rN')'(t) + q(t)N(t)}{r(t)N(t)}\right)x = 0. \tag{2.76}$$

Since $x_1(t)$ and $x_2(t)$ are linearly independent solutions of both (2.75) and (2.76), so it is clear that (2.75) and (2.76) have the same solution space. If $u(t)$ is a solution of (2.75), then it is a solution of (2.76) and hence $u(t)$ is a solution of the first-order equation

$$a_1(t)x' + b_1(t)x = 0,$$

where

$$a_1(t) = \frac{N'(t)}{N(t)} - \frac{W'(t)}{W(t)}$$

and

$$b_1(t) = \frac{(rW')'(t) + q(t)W(t)}{r(t)W(t)} - \frac{(rN')'(t) + q(t)N(t)}{r(t)N(t)}.$$

Hence, in particular,

$$a_1(t)x_1'(t) + b_1(t)x_1(t) = 0$$

and

$$a_1(t)x_2'(t) + b_1(t)x_2(t) = 0.$$

Since $W(t) \neq 0$ for $t \geq \sigma$, we have $a_1(t) = 0$ and $b_1(t) = 0$ for $t \geq \sigma$. But $a_1(t) = 0$, $t \geq \sigma$, implies that

$$\frac{N'(t)}{N(t)} = \frac{W'(t)}{W(t)}, \quad t \geq \sigma.$$

Integrating the above equality, we have $W(t) = \lambda N(t)$, where λ is a constant. Further $W(t) > 0$ and $N(t) > 0$ for $t \geq \sigma$ imply that $\lambda > 0$. Thus, $W = \lambda N(t)$ for $t \geq \sigma$. The theorem is proved. □

Remark 2.3.5 Theorems 2.3.7 and 2.3.8 and Lemma 2.3.5 hold good when $N(t)$ is replaced by $W(t)$ in view of Theorem 2.3.9.

Theorem 2.3.10 *For any solution $x(t)$ of (2.52), the function $G(x(t))$ is a decreasing function of t, where*

$$G\big(x(t)\big) = r(t)W(t)\big(x'(t)\big)^2 + \big(\big(rW'\big)'(t) + q(t)W(t)\big)x^2(t).$$

Proof of Theorem 2.3.6 From the Remark 2.3.4, it follows that there exist two linearly independent oscillatory solutions $x_1(t)$ and $x_2(t)$ of (2.1) whose zeros separate. To complete the proof of theorem, it is enough to show that any oscillatory solution of (2.1) can be expressed as a linear combination of $x_1(t)$ and $x_2(t)$.

Let $x_3(t)$ be a solution of (2.1) with $x_3(\sigma) = 0 = x_3'(\sigma)$, $x_3''(\sigma) > 0$. From Lemma 2.3.1, it follows that $x_3(t) > 0$ for $t > \sigma$. Consequently, Remark 2.3.1 implies that $x_3'(t) > 0$ and $x_3''(t) > 0$ for $t > \sigma$. Clearly, $\{x_1(t), x_2(t), x_3(t)\}$ is a linearly independent set of solutions of (2.1). Hence

$$\begin{vmatrix} x_1(t) & x_2(t) & x_3(t) \\ x_1'(t) & x_2'(t) & x_3'(t) \\ r(t)x_1''(t) & r(t)x_2''(t) & r(t)x_3''(t) \end{vmatrix} = k,$$

where $k \neq 0$ is a constant. Thus

$$\begin{vmatrix} x_1(t) & x_2(t) & u(t) \\ x_1'(t) & x_2'(t) & u'(t) \\ r(t)x_1''(t) & r(t)x_2''(t) & r(t)u''(t) \end{vmatrix} = 1,$$

where $u(t) = \frac{x_3(t)}{k}$. Expanding the above determinant,

$$r(t)W(t)u''(t) - r(t)W'(t)u'(t) + \big(\big(rW'\big)'(t) + q(t)W(t)\big)u(t) = 1. \tag{2.77}$$

We may note that $k < 0$ implies that $u(t) < 0$, $u'(t) < 0$ and $u''(t) < 0$. This, in turn, leads to a contradiction in (2.77), where the left-hand side becomes negative and the right-hand side is positive. Thus $k > 0$.

Let $z(t)$ be an oscillatory solution of (2.1). We claim that $z(t)$ can be expressed as a linear combination of $x_1(t)$ and $x_2(t)$. If not, there exist c_1, c_2 and c_3, $c_3 \neq 0$, such that $z(t) = c_1x_1(t) + c_2x_2(t) + c_3u(t)$. We may note that c_1 and c_2 cannot be zero simultaneously. If

$$z_1(t) = \frac{z(t)}{c_3} \quad \text{and} \quad x(t) = -\frac{c_1x_1(t) + c_2x_2(t)}{c_3},$$

then $z_1(t) = u(t) - x(t)$. Clearly $x(t)$ is a nontrivial oscillatory solution of (2.52) and (2.1). Thus $z_1(t)$ is a solution of (2.77). Consequently,

$$\begin{aligned} &r(t)W(t)\big(u(t) - x(t)\big)'' - r(t)W'(t)\big(u(t) - x(t)\big)' \\ &\quad + \big(\big(rW'\big)'(t) + q(t)W(t)\big)\big(u(t) - x(t)\big) = 1. \end{aligned} \tag{2.78}$$

Since $z(t)$ is oscillatory, $u(t) - x(t)$ is oscillatory. From Theorem 2.3.10, it follows that $((rW')'(t) + q(t)W(t))x^2(t)$ is bounded. As

$$\begin{aligned} \big[\big(\big(rW'\big)'(t) + q(t)W(t)\big)x(t)\big]^2 = {}& \big(\big(rW'\big)'(t) + q(t)W(t)\big)x^2(t)t^2\big(\big(rW'\big)'(t) \\ & + q(t)W(t)\big)t^{-2}, \end{aligned}$$

from Lemma 2.3.5(iii), we obtain

$$\lim_{t\to\infty}\big[\big(\big(rW'\big)'(t) + q(t)W(t)\big)x(t)\big] = 0.$$

Hence there exists a $T > \sigma$ such that

$$\big|\big(\big(rW'\big)'(t) + q(t)W(t)\big)x(t)\big| < \frac{1}{4}$$

for $t \geq T$. From (2.77) we get, for $t \geq \sigma$,

$$0 < \big(\big(rW'\big)'(t) + q(t)W(t)\big)u(t) < 1.$$

Let $t_0 > T$ be a maximum of $u(t) - x(t)$. So $u(t_0) - x(t_0) \geq 0$ and $u'(t_0) - x'(t_0) = 0$. Now multiplying (2.78) through by $u'(t) - x'(t)$ and integrating the resulting identity from t_0 to t, we obtain

$$\begin{aligned} &\frac{1}{2}r(t)W(t)\big(u'(t) - x'(t)\big)^2 - \frac{1}{2}\int_{t_0}^{t}(rW)'(s)\big(u'(s) - x'(s)\big)^2\,ds \\ &\quad - \int_{t_0}^{t} r(s)W'(s)\big(u'(s) - x'(s)\big)^2\,ds \\ &\quad + \frac{1}{2}\big(\big(rW'\big)'(t) + q(t)W(t)\big)\big(u(t) - x(t)\big)^2 \\ &\quad - \frac{1}{2}\big(\big(rW'\big)'(t_0) + q(t_0)W(t_0)\big)\big(u(t_0) - x(t_0)\big)^2 \end{aligned}$$

$$- \frac{1}{2} \int_{t_0}^{t} p(s)W(s)\big(u(s) - x(s)\big)^2 \, ds$$
$$= (u - x)(t) - (u - x)(t_0),$$

since $W(t)$ is a solution of (2.49). As $(rW)'(t) = r'(t)W(t) + r(t)W'(t) < 0$, we have

$$(u - x)(t_0)\left[1 - \frac{1}{2}\big((rW')'(t_0) + q(t_0)W(t_0)\big)(u - x)(t_0)\right] < (u - x)(t).$$

But

$$\begin{aligned}\big((rW')'(t_0) + q(t_0)W(t_0)\big)(u - x)(t_0) &= \big((rW')'(t_0) + q(t_0)W(t_0)\big)u(t_0) \\ &\quad - \big((rW')'(t_0) + q(t_0)W(t_0)\big)x(t_0) \\ &< 1 + \frac{1}{4} = \frac{5}{4}.\end{aligned}$$

So, for $t > t_0$, $(u - x)(t) > \frac{3}{8}(u - x)(t_0) \geq 0$, which contradicts the fact that $(u - x)$ is oscillatory. Hence our claim holds. This completes the proof of the theorem. □

Theorem 2.3.11 *If* (2.1) *admits an oscillatory solution, then there exist two linearly independent oscillatory solutions u and v of* (2.1) *such that any nontrivial linear combination of u and v is also oscillatory and their zeros separate.*

2.4 Behaviour of Solutions of $x''' + a(t)x'' + b(t)x' + c(t)x = 0$ with $a(t) \geq 0$, $b(t) \leq 0$ and $c(t) < 0$

In this section, we present some results on the oscillation and asymptotic behaviour of solutions of Eq. (2.1) with $a(t) \geq 0$, $b(t) \leq 0$ and $c(t) < 0$. The results presented in this section generalise Proposition 1.2.4 of Chap. 1 with variable coefficients. Results pertaining to the behaviour of nonoscillatory solutions in the presence of an oscillatory solution are also presented.

Lemma 2.4.1 *Let $a(t)b(t) + b'(t) - c(t) \leq 0$. Then Eq.* (2.1) *is of type C_I.*

Proof Let $x(t)$ be a solution of (2.1) with $x(t_0) = x'(t_0) = 0$ and $x''(t_0) > 0$, where $t_0 > \sigma$. From the continuity of $x''(t)$, it follows that there exists a δ, $\sigma < t_0 - \delta$, such that $x''(t) > 0$ for $t \in [t_0 - \delta, t_0]$. We claim that $x''(t) > 0$ for $t \in [\sigma, t_0]$. If not, then there exists a $t_1 \in [\sigma, t_0 - \delta]$ such that $x''(t_1) = 0$ and $x''(t) > 0$ for $t \in (t_1, t_0]$. Thus, $x'(t) < 0$ and $x(t) > 0$ for $t \in (t_1, t_0)$. Integrating (2.2) from t_1 to t_0, we obtain

$$0 < r(t_0)x''(t_0) = q(t_1)x(t_1) + \int_{t_1}^{t_0} \big(q'(s) - p(s)\big)x(s) \, ds < 0,$$

a contradiction. Hence our claim holds. Consequently, $x(t) > 0$, $x'(t) < 0$ for $t \in [\sigma, t_0)$. The proof of the lemma is complete. □

Lemma 2.4.2 *If $x(t)$ is a solution of (2.1) with $x(t_0) \geq 0$, $x'(t_0) \geq 0$ and $x''(t_0) > 0$ for some $t_0 \geq \sigma$, then $x(t) > 0$, $x'(t) > 0$ and $x''(t) > 0$ for some $t > t_0$. Similarly, if $x(t_0) \leq 0$, $x'(t_0) \leq 0$ and $x''(t_0) < 0$ for $t_0 \geq \sigma$, then $x(t) < 0$, $x'(t) < 0$ and $x''(t) < 0$ for $t > t_0$.*

Proof Let $x(t)$ be a solution of (2.1) with $x(t_0) \geq 0$, $x'(t_0) \geq 0$ and $x''(t_0) > 0$ for $t_0 \geq \sigma$. Hence there exists a $\delta > 0$ such that $x''(t) > 0$ for $t \in [t_0, t_0 + \delta)$. If there is a $t_1 \geq t_0 + \delta$ such that $x''(t_1) = 0$ and $x''(t) > 0$ for $t_0 \leq t < t_1$, then $x'(t) > 0$ and $x(t) > 0$ for $t_0 \leq t < t_1$. Multiplying Eq. (2.2) through by $x'(t)$ and integrating the resulting identity from t_0 to t_1, we obtain

$$0 < \int_{t_0}^{t_1} r(t)\big(x''(t)\big)^2\,dt = \int_{t_0}^{t_1} q(t)\big(x'(t)\big)^2\,dt + \int_{t_0}^{t_1} p(t)x(t)x'(t)\,dt < 0,$$

a contradiction. Hence $x''(t) > 0$ for $t \geq t_0$. Then $x(t) > 0$ and $x'(t) > 0$ for $t > t_0$. The other assertion follows similarly. The lemma is proved. □

Corollary 2.4.1 *Equation (2.1) is of Class II.*

Theorem 2.4.1 *Equation (2.1) admits a positive increasing solution which goes to ∞ as $t \to \infty$. Further, if*

$$\int_{\sigma}^{\infty} \frac{dt}{r(t)} = \infty,$$

then the derivative of the solution tends to ∞ as $t \to \infty$.

Proof If $x(t)$ is a solution of (2.1) with $x(t_0) \geq 0$, $x'(t_0) \geq 0$ and $x''(t_0) > 0$, $t_0 \geq \sigma$, then from Lemma 2.4.2 it follows that $\lim_{t\to\infty} x(t) = \infty$. Since $x(t) > 0$ and $x'(t) > 0$ for $t > t_0$, $r(t)x''(t)$ is monotonically increasing in $[t_0, \infty)$. Thus $\int_{\sigma}^{\infty} \frac{dt}{r(t)} = \infty$ implies that $x'(t) \to \infty$ as $t \to \infty$. This completes the proof of the theorem. □

Theorem 2.4.2 *If $b(t) - a'(t) \leq 0$, $b(t) - 2a'(t) \leq 0$, $c(t) - b'(t) + a''(t) < 0$ and*

$$\int_{\sigma}^{\infty} \Big[-\frac{2a^3(t)}{27} + \frac{a(t)b(t)}{3} - c(t) + b'(t) - a''(t) - \frac{2a(t)a'(t)}{3} - \frac{2}{3\sqrt{3}}\Big(\frac{a^2(t)}{3} - b(t) + a'(t)\Big)^{\frac{3}{2}}\Big]\,dt = \infty, \tag{2.79}$$

then (2.1) is oscillatory.

Theorem 2.4.3 *Let* $b(t) - 2a'(t) \leq 0$, $c(t) - b'(t) + a''(t) < 0$, $0 \leq \lim_{t\to\infty} ta(t) < \infty$ *and* $1 - t^2 b(t) + \frac{1}{3}t^2 a^2(t) + t^2 a'(t) \geq 0$. *If* (2.70) *holds*, *then* (2.1) *is oscillatory.*

Remark 2.4.1 Theorem 2.4.3 holds for the Euler equation (2.5) with $a_0 \geq 0$, $b_0 \leq 0$ and $c_0 < 0$. Indeed, if $c_0 + 2b_0 + 2a_0 < 0$, $1 - b_0 + \frac{a_0^2}{3} - a_0 \geq 0$, $b_0 + 2a_0 \leq 0$ and (2.71) hold, then (2.5) is oscillatory.

Example 2.4.1 By Theorem 2.4.3, the equation

$$x''' + \frac{1}{t}x'' - \frac{3}{t^2}x' - \frac{3}{t^3}x = 0, \quad t \geq 1$$

is oscillatory. In particular, $x_1(t) = t^{-1/2}\cos(\sqrt{3}\log t)$ and $x_2(t) = t^{-1/2}\sin(\sqrt{3}\log t)$ are the oscillatory solutions of the equation, and $x_3(t) = t^3$ is a nonoscillatory solution of the equation.

Remark 2.4.2 One may observe in the above example that $x_3(t) \to \infty$ as $t \to \infty$ along with its derivatives, whereas, $x_1(t)$ and $x_2(t) \to 0$ as $t \to \infty$. This indicates that there exists oscillatory solution of Eq. (2.1) converging to zero as $t \to \infty$. However, no such result has been obtained in the literature. This has been left as an open problem to the readers. However, we provide a partial answer to the above problem. See Corollary 2.4.2 and Theorem 2.8.1.

Lemma 2.4.3 *If* $x(t)$ *is a nonoscillatory solution of* (2.1), *then there exists a* $t_0 \geq \sigma$ *such that* $x(t)x'(t) \leq 0$ *or* $x(t)x'(t) > 0$ *for* $t \geq t_0$.

Proof Without any loss of generality, we may assume that $x(t) > 0$ for $t \geq T \geq \sigma$. Let t_1 and t_2 ($T \leq t_1 < t_2$) be two consecutive zeros of $x'(t)$ such that $x'(t) > 0$ for $t \in (t_1, t_2)$. Multiplying (2.1) through by $x'(t)$ and integrating the resulting identity from t_1 to t_2, we obtain the inequality

$$0 < \int_{t_1}^{t_2} r(t)\big(x''(t)\big)^2\,dt = \int_{t_1}^{t_2} q(t)\big(x'(t)\big)^2\,dt + \int_{t_1}^{t_2} p(t)x(t)x'(t)\,dt < 0,$$

a contradiction. Hence there exists a $t_0 > T$ such that $x'(t) > 0$ or ≤ 0 for $t \geq t_0$. Thus the lemma is proved. □

Theorem 2.4.4 *If* (2.1) *has an oscillatory solution*, *then every nonoscillatory solution* $x(t)$ *of* (2.1) *satisfies the following conditions*:

$$x(t)x'(t) \neq 0, \qquad \operatorname{sgn} x(t) = \operatorname{sgn} x'(t), \quad t \geq t_0 \geq \sigma$$

and

$$\lim_{t\to\infty} \big|x(t)\big| = \infty.$$

If, in addition

$$\int_{\sigma}^{\infty} p(t)\,dt = -\infty,$$

then

$$x(t)x'(t)x''(t) \neq 0$$

and

$$\operatorname{sgn} x(t) = \operatorname{sgn} x'(t) = \operatorname{sgn} x''(t), \quad t \geq T_0 \geq t_0.$$

Proof Let $x(t) > 0$ for $t \geq T \geq \sigma$. Let $z(t)$ be an oscillatory solution of (2.1). To prove that $W(t) = x(t)z'(t) - x'(t)z(t)$ must vanish for some value of $t \in [T, \infty)$. If not, then $W(t) \neq 0$ for $t \in [T, \infty)$. If $u(t) = \frac{z(t)}{x(t)}$, then $u'(t) = \frac{W(t)}{x^2(t)} \neq 0$ for $t \geq T$. If t_1 and t_2 $(T \leq t_1 < t_2)$ are consecutive zeros of $z(t)$, then $u(t_1) = 0$, $u(t_2) = 0$ and $u(t) \neq 0$ for $t \in (t_1, t_2)$. This is impossible, since $u'(t) \neq 0$ for $t \geq T$. Thus $W(t)$ must vanish for some value of $t \in [\sigma, \infty)$. Let $W(\alpha) = 0$ for some $\alpha \in [T, \infty)$. It is possible to obtain c_1 and c_2, not both zero, such that

$$c_1 x(\alpha) + c_2 z(\alpha) = 0, \qquad c_1 x'(\alpha) + c_2 z'(\alpha) = 0 \quad \text{and} \quad c_1 x''(\alpha) + c_2 z''(\alpha) = 0,$$

because $x(t)$ and $z(t)$ are linearly independent on $[T, \infty)$. Without any loss of generality, we may assume that $c_1 x''(\alpha) + c_2 z''(\alpha) > 0$. If $v(t) = c_1 x(t) + c_2 z(t)$, then $v(t)$ is a solution of (2.1) with $v(\alpha) = 0$, $v'(\alpha) = 0$ and $v''(\alpha) > 0$. Proceeding as in Lemma 2.4.2, we obtain $v(t) \to \infty$ as $t \to \infty$.

From Lemma 2.4.3, it follows that $x'(t) > 0$ or ≤ 0 for $t \geq t_0 \geq T$. If $x'(t) \leq 0$ for $t \geq t_0$, then $\lim_{t\to\infty} x(t) = \lambda$ exists, where $0 \leq \lambda < \infty$. Clearly, $c_2 = 0$ implies that $\lim_{t\to\infty} v(t) = c_1\lambda < \infty$, a contradiction. Thus $c_2 \neq 0$. If $c_2 > 0$, then

$$\begin{aligned}
\liminf_{t\to\infty} c_2 z(t) &= \liminf_{t\to\infty} \big[v(t) - c_1 x(t)\big] \\
&\geq \liminf_{t\to\infty} v(t) + \liminf_{t\to\infty} \big(-c_1 x(t)\big) \\
&= \liminf_{t\to\infty} v(t) - c_1 \liminf_{t\to\infty} x(t) \\
&= \infty
\end{aligned}$$

implies that $\liminf_{t\to\infty} z(t) = \infty$, and hence $\lim_{t\to\infty} z(t) = \infty$. If $c_2 < 0$, then $\liminf_{t\to\infty} c_2 z(t) = \infty$ implies that $\limsup_{t\to\infty} z(t) = -\infty$, and hence $\lim_{t\to\infty} z(t) = -\infty$. Thus $\lim_{t\to\infty} z(t) = \infty$ or $-\infty$ according to $c_2 > 0$ or < 0. In either case, it is a contradiction since $z(t)$ is oscillatory. Hence $x'(t) > 0$ for $t \geq t_0$. Clearly, $c_1 \neq 0$ because $c_1 = 0$ implies that $c_2 \neq 0$ and $v(t) = c_2 z(t)$ is oscillatory, a contradiction. If $c_2 = 0$, then $\lim_{t\to\infty} c_1 x(t) = \lim_{t\to\infty} v(t) = \infty$. As $c_1 < 0$ implies that $x(t) < 0$ for large t, then $c_1 > 0$, and hence $\lim_{t\to\infty} x(t) = \infty$. Suppose that $c_2 \neq 0$. If $\lim_{t\to\infty} x(t)$ exists finitely, then $\lim_{t\to\infty} z(t) = \pm\infty$, contradicting the oscillatory nature of $z(t)$. Thus $\lim_{t\to\infty} x(t) = \infty$.

Suppose that $\int_\sigma^\infty p(t)\,dt = -\infty$. Since $x(t) > 0$ and $x'(t) > 0$ for $t \geq t_0$, $r(t)x''(t)$ is monotonic increasing, and hence $x''(t)$ has a constant sign for $t \geq T_0 \geq t_0$. If $x''(t) < 0$ for $t \geq T_0$, then integrating (2.2) from T_0 to t we obtain

$$\begin{aligned} r(t)x''(t) &\geq r(T_0)x''(T_0) - \int_{T_0}^t p(s)x(s)\,ds \\ &\geq r(T_0)x''(T_0) - x(T_0)\int_{T_0}^t p(s)\,ds. \end{aligned}$$

Thus $x''(t) > 0$ for large t, a contradiction. Hence $x''(t) > 0$ for $t \geq T_0$. This completes the proof of the theorem. □

Corollary 2.4.2 *If* (2.1) *has an oscillatory solution, then every bounded solution of* (2.1) *oscillates.*

Theorem 2.4.5 *Let* $\int_\sigma^\infty p(t)\,dt = -\infty$. *Then Eq.* (2.1) *has an oscillatory solution, if and only if every nonoscillatory solution* $x(t)$ *of* (2.1) *satisfies the properties*

$$\left.\begin{aligned} &x(t)x'(t)x''(t) \neq 0, \\ &\operatorname{sgn} x(t) = \operatorname{sgn} x'(t) = \operatorname{sgn} x''(t), \quad t \geq T_0 \geq \sigma, \quad \textit{and} \\ &\lim_{t\to\infty} \left|x(t)\right| = \infty. \end{aligned}\right\} \tag{2.80}$$

Proof The necessity follows from Theorem 2.4.4. For sufficiency, assume that (2.80) holds for every nonoscillatory solution $x(t)$ of (2.1). To show that (2.1) admits an oscillatory solution, let z_0, z_1, z_2 be solutions of (2.1) with initial conditions

$$z_k^{(j)}(\sigma) = \begin{cases} 0, & j \neq k, \\ 1, & j = k, \end{cases}$$

$j, k = 0, 1, 2$. Clearly z_0, z_1, z_2 are linearly independent. For each positive integer $n > \sigma$, it is possible to determine the real numbers a_{0n}, a_{2n}, b_{1n} and b_{2n} such that

$$\begin{aligned} a_{0n}z_0(n) + a_{2n}z_2(n) &= 0, \\ b_{1n}z_1(n) + b_{2n}z_2(n) &= 0 \end{aligned}$$

and $a_{0n}^2 + a_{2n}^2 = 1$, $b_{1n}^2 + b_{2n}^2 = 1$. Define, for each positive integer $n > \sigma$,

$$\begin{aligned} u_n &= a_{0n}z_0 + a_{2n}z_2, \\ v_n &= b_{1n}z_1 + b_{2n}z_2. \end{aligned}$$

Thus u_n and v_n are solutions of (2.1) with $u_n(n) = 0$ and $v_n(n) = 0$. Clearly, there exists a sequence $\langle n_j \rangle$ of positive integers $> \sigma$ such that $a_{0n_j} \to a_0$, $a_{2n_j} \to a_2$, $b_{1n_j} \to b_1$ and $b_{2n_j} \to b_2$ as $n_j \to \infty$ and hence $a_0^2 + a_2^2 = 1$ and $b_1^2 + b_2^2 = 1$. If

$u = a_0 z_0 + a_2 z_2$ and $v = b_1 z_1 + b_2 z_2$, then u and v are nontrivial solutions of (2.1) and

$$\lim_{n_j \to \infty} u_{n_j}^{(k)} = u^{(k)}, \qquad \lim_{n_j \to \infty} v_{n_j}^{(k)} = v^{(k)},$$

$k = 0, 1, 2$, uniformly on any compact subinterval of $[\sigma, \infty)$. To show that each of u and v is an oscillatory solution of (2.1). If u is nonoscillatory, then there exists a $T_0 \ge \sigma$ such that

$$u(t)u'(t)u''(t) \neq 0, \qquad \operatorname{sgn} u(t) = \operatorname{sgn} u'(t) = \operatorname{sgn} u''(t), \quad t \ge T_0$$

and $\lim_{t\to\infty} |u(t)| = \infty$. In particular,

$$u(T_0)u'(T_0)u''(T_0) \neq 0, \qquad \operatorname{sgn} u(T_0) = \operatorname{sgn} u'(T_0) = \operatorname{sgn} u''(T_0).$$

Hence there exists a positive integer N such that

$$u_{n_j}(T_0)u'_{n_j}(T_0)u''_{n_j}(T_0) \neq 0,$$
$$\operatorname{sgn} u_{n_j}(T_0) = \operatorname{sgn} u'_{n_j}(T_0) = \operatorname{sgn} u''_{n_j}(T_0)$$

for $n_j \ge N$. From Lemma 2.4.2, it follows that $u_{n_j} \neq 0$ for $n_j \ge N$ and $t \ge T_0$. Thus $u_{n_j}(n_j) \neq 0$ for all $n_j > \max\{N, T_0\}$. This contradicts the fact that $u_n(n) = 0$ for every positive integer $n > \sigma$. Hence $u(t)$ is oscillatory. Similarly, it may be shown that $v(t)$ is oscillatory. Thus the theorem is proved. □

Theorem 2.4.6 *Let* $\int_\sigma^\infty p(t)\,dt = -\infty$. *If* (2.1) *admits an oscillatory solution, then there exist two linearly independent oscillatory solutions* u *and* v *of* (2.1) *such that any nontrivial linear combination of* u *and* v *is also oscillatory and the zeros of* u *and* v *separate.*

Remark 2.4.3 Theorems 2.4.5 and 2.4.6 are similar to Theorems 6.23 and 6.25, respectively, in [14]. While $\int_\sigma^\infty p(t)\,dt = -\infty$ is assumed in Theorem 6.23, the disconjugacy of $x'' + a(t)x' + b(t)x = 0$ is assumed in Theorem 6.25 in [14]. The proof of Theorems 6.23 and 6.25 may be found in Gera [12]. Note that the assumption $\int_\sigma^\infty p(t)\,dt = -\infty$ is not needed in the sufficiency part of Theorem 2.1.6. Moreover, this condition is satisfied if a, b and c are constants.

Theorem 2.4.7 *Suppose that* $q'(t) - p(t) \le 0$ *but* $\not\equiv 0$ *in any neighbourhood of infinity. Then Eq.* (2.1) *admits a solution* $x(t)$ *with the following properties*:

$$x(t)x'(t)x''(t) \neq 0,$$
$$\operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t), \quad t \ge \sigma,$$
$$\lim_{t\to\infty} x'(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} x(t) = \lambda, \quad -\infty < \lambda < \infty.$$

If, in addition, $\int_\sigma^\infty (q'(t) - p(t))\,dt = -\infty$ *and* $\lim_{t\to\infty} q(t) = k$, $-\infty < k < \infty$, *then* $\lim_{t\to\infty} x(t) = 0$.

Proof For every positive integer $n > \sigma$, let $x_n(t)$ be a solution of (2.1) with initial conditions

$$x_n(n) = 0, \qquad x_n'(n) = 0, \qquad x_n''(n) > 0.$$

Since $q'(t) - p(t) = (a(t)b(t) + b'(t) - c(t))r(t)$, Lemma 2.4.1 yields $x_n(t) > 0$, $x_n'(t) < 0$ and $x_n''(t) > 0$ for $t \in [\sigma, n)$. We may write

$$x_n(t) = c_{1n}u_1(t) + c_{2n}u_2(t) + c_{3n}u_3(t), \quad t \in [\sigma, n),$$

where $c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1$ and $\{u_1(t), u_2(t), u_3(t)\}$ is a basis of the solution space of (2.1). The sequence $\langle c_{in} \rangle$, $i = 1, 2, 3$ has a convergent subsequence $\langle c_{in_j} \rangle$ such that $c_{in_j} \to c_i$ as $n_j \to \infty$ and $i = 1, 2, 3$. Hence $c_1^2 + c_2^2 + c_3^2 = 1$. Setting $x(t) = c_1u_1(t) + c_2u_2(t) + c_3u_3(t)$, we see that $x(t)$ is a solution of (2.1) with

$$\lim_{n_j \to \infty} x_{n_j}^{(k)}(t) = x^{(k)}(t), \quad k = 0, 1, 2$$

uniformly on every compact subinterval of $[\sigma, \infty)$. Thus, $x(t) > 0, x'(t) < 0$ and $x''(t) > 0$ for $t \geq \sigma$. As $\lim_{t\to\infty} x'(t) = L$, $-\infty < L < 0$, which implies that $x(t) < 0$ for large t, then $\lim_{t\to\infty} x'(t) = 0$. Clearly $\lim_{t\to\infty} x(t) = \lambda$, $0 \leq \lambda < \infty$. If $\lim_{t\to\infty} x(t) = \lambda$, $\lambda > 0$, then integrating (2.2) from σ to t and by using the additional conditions, we get

$$\begin{aligned}
0 < r(t)x''(t) &= r(\sigma)x''(\sigma) - q(t)x(t) + q(\sigma)x(\sigma) + \int_\sigma^t \big(q'(s) - p(s)\big)x(s)\,ds \\
&\leq r(\sigma)x''(\sigma) - q(t)x(t) + q(\sigma)x(\sigma) + x(t)\int_\sigma^t \big(q'(s) - p(s)\big)\,ds \\
&< 0,
\end{aligned}$$

for large t, a contradiction. The theorem is proved. □

Theorem 2.4.8 *If $q'(t) - p(t) \leq 0$ but $\not\equiv 0$ in any neighbourhood of infinity, then (2.1) is nonoscillatory.*

Proof If possible, let (2.1) admits an oscillatory solution. From Theorem 2.4.4, it follows that every nonoscillatory solution $x(t)$ of (2.1) satisfies the property $|x(t)| \to \infty$ as $t \to \infty$. On the other hand, Theorem 2.4.7 yields the result that (2.1) has a nonoscillatory solution $u(t)$ such that $\lim_{t\to\infty} u(t) = \lambda$, $-\infty < \lambda < \infty$, a contradiction. This completes the proof of the theorem. □

An alternate proof of Theorem 2.4.8 is given below:

Clearly, (2.1) is of type C_{II}. On the other hand $q'(t) - p(t) \leq 0$ implies that (2.1) is of type C_I. We now prove, since (2.1) is both C_I and C_{II}, (2.1) is nonoscillatory. Since (2.1) is of type C_{II}, the solution $x(t)$ of (2.1) with the initial conditions $x(\sigma) = x'(\sigma) = 0$, $x''(\sigma) > 0$ has the property that $x(t) > 0$ for

$t > \sigma$. If possible, let $z(t)$ be an oscillatory solution of (2.1). Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ $(\sigma < \alpha_1 < \beta_1 < \alpha_2 < \beta_2)$ be the successive zeros of $z(t)$ such that $z(t) > 0$ for $t \in (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2)$. By Lemma 1.5.13, there exist nonzero constants λ_1 and λ_2 such that $z_1(t) = z(t) - \lambda_1 x(t)$ has a double zero at $t_1 \in (\alpha_1, \beta_1)$ and $z_2(t) = z(t) - \lambda_2 x(t)$ has a double zero at $t_2 \in (\alpha_2, \beta_2)$. Since $z(t) > 0$ in (α_1, β_1) and (α_2, β_2) and $x(t) > 0$ for $t > \sigma$, we have $\lambda_1 > 0$ and $\lambda_2 > 0$. If $\lambda_1 > \lambda_2$, then $z_2(t_1) = z(t_1) - \lambda_2 x(t_1) > z(t_1) - \lambda_1 x(t_1) = z_1(t_1) = 0$ and $z_2(\beta_1) = z(\beta_1) - \lambda_2 x(\beta_1) = -\lambda_2 x(\beta_1) < 0$. Thus, $z_2(t)$ is a solution of (2.2) with a zero in (t_1, β_1) and a double zero at t_2, which contradicts the assumption that (2.2) is of type C_I. If $\lambda_1 \leq \lambda_2$, then $z_1(t_2) = z(t_2) - \lambda_1 x(t_2) \geq z_2(t_2) = 0$ and $z_1(\beta_1) = z(\beta_1) - \lambda_1 x(\beta_1) = -\lambda_1 x(\beta_1) < 0$. Hence $z_1(t)$ is a solution of (2.1) with a zero in $(\beta_1, t_2]$ and a double zero at t_1, a contradiction to the assumption that (2.1) is of type C_{II}. Hence (2.1) cannot have an oscillatory solution. Thus, (2.1) is nonoscillatory.

The above argument leads us to the following corollary;

Corollary 2.4.3 *If (2.1) is of type C_I and C_{II}, then it is nonoscillatory.*

In the literature, one may find several conditions for the nonoscillation of (2.1). Below, we state them: We mention the results for Eq. (2.2) without any sign restrictions on $p(t)$ and $q(t)$. We make the assumption that $r(t)$ is sufficiently smooth so that $r(t) > 0$ and $r(t) \neq 0$ in any neighbourhood of infinity.

Theorem 2.4.9 *Let (2.1) be of type C_I. If $r'(t) \leq 0$ and $2p(t) - q'(t) \leq 0$ for $t \geq \sigma$, then (2.1) is nonoscillatory.*

Theorem 2.4.10 *Let (2.1) be of type C_{II}. If $r'(t) \geq 0$ and $2p(t) - q'(t) \geq 0$ for $t \geq \sigma$, then (2.1) is nonoscillatory.*

Remark 2.4.4 One may observe that $r'(t) \leq 0$ and $2p(t) - q'(t) \leq 0$ implies that (2.1) is of type C_{II}. Similarly, $r'(t) \geq 0$ and $2p(t) - q'(t) \geq 0$ for $t \geq \sigma$ implies that (2.1) is of type C_I. Thus, the proof of Theorems 2.4.9 and 2.4.10 follow from Corollary 2.4.3.

On the other hand, when $r'(t) \geq 0$ and $p(t) > 0$, the condition $2p(t) - q'(t) \geq 0$ is better than the condition $p(t) - q'(t) \geq 0$. However, for the case considered in this section, that is, $a(t) \geq 0$, $b(t) \leq 0$ and $c(t) < 0$, the condition $p(t) - q'(t) \geq 2p(t) - q'(t)$ holds. Hence Theorem 2.4.8 provides a better condition than the ones given in Theorem 2.4.10.

Theorem 2.4.11 *Suppose that $q'(t) - p(t) \geq 0$ and*

$$\int_\sigma^\infty \big(q'(t) - p(t)\big)\, dt = \infty.$$

Then (2.1) has an oscillatory solution, if and only if every nonoscillatory solution $x(t)$ of (2.1) satisfies the properties (2.80).

Proof The sufficient part is similar to that of Theorem 2.4.5. For necessity, one may proceed, as in Theorem 2.4.4, to obtain $x(t)x'(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn} x'(t)$ for $t \geq t_0 \geq \sigma$ and $\lim_{t\to\infty} |x(t)| = \infty$. In order to be definite about the sign of $x''(t)$, we may assume that $x(t) > 0$ for $t \geq t_0$. Hence $x'(t) > 0$ for $t \geq t_0$. Since $r(t)x''(t)$ is monotonically increasing, we have $x''(t) > 0$ or < 0 for $t \geq T_0 \geq t_0$. If $x''(t) < 0$ for $t \geq T_0$, then integration of (2.2) from T_0 to t yields

$$\begin{aligned} r(t)x''(t) &\geq r(T_0)x''(T_0) + q(T_0)x(T_0) + \int_{T_0}^{t} \big(q'(s) - p(s)\big)x(s)\,ds \\ &\geq r(T_0)x''(T_0) + q(T_0)x(T_0) + x(T_0)\int_{T_0}^{t} \big(q'(s) - p(s)\big)\,ds. \end{aligned}$$

Hence $x''(t) > 0$ for large t. This contradiction completes the proof of the theorem. □

Example 2.4.2 Consider

$$x''' + \frac{1}{t^2}x'' - \left(1 + \frac{2}{t^3} - \frac{1}{3t^4}\right)x' - \left(e^t + \frac{2}{3\sqrt{3}}\right)x = 0, \quad t \geq 1.$$

Clearly, the conditions of Theorems 2.4.2, 2.4.5 and 2.4.11 are satisfied. Hence the given equation admits an oscillatory solution, and all nonoscillatory solutions of the equation tend to ∞ as $t \to \infty$.

Theorem 2.4.12 *Suppose that* $q'(t) - p(t) \leq 0$,

$$\int_{\sigma}^{\infty} \big(q'(t) - p(t)\big)\,dt = -\infty$$

and $\lim_{t\to\infty} q(t) = k$, $-\infty < k < 0$. *Then every solution* $x(t)$ *of* (2.1) *satisfies*

$$x(t)x'(t)x''(t) \neq 0, \qquad \operatorname{sgn} x(t) = \operatorname{sgn} x'(t) \neq \operatorname{sgn} x''(t), \quad t \geq t_0 \geq \sigma$$

or

$$\begin{aligned} &x(t)x'(t)x''(t) \neq 0, \qquad \operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t), \quad t \geq t_0 \geq \sigma, \\ &\lim_{t\to\infty} x(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} x'(t) = 0. \end{aligned}$$

Proof Let $x(t)$ be any solution of (2.1). From Theorem 2.4.8, it follows that (2.1) is nonoscillatory, and hence $x(t)$ is nonoscillatory. We may assume, without any loss of generality, that $x(t) > 0$ for $t \geq T \geq \sigma$. Lemma 2.4.3 yields $x'(t) > 0$ or < 0 for $t \geq T_0 \geq T$. If $x'(t) > 0$ for $t \geq T_0$, then $r(t)x''(t)$ is monotonically increasing and hence $x''(t) > 0$ or < 0 for large t. As $x''(t) > 0$ for large t yields, due to Theorem 2.4.5, the result that (2.1) has an oscillatory solution, the case $x''(t) > 0$ is not possible. Thus $x''(t) < 0$ for $t \geq t_0 \geq T_0$. Thus, we have

$$\operatorname{sgn} x(t) = \operatorname{sgn} x'(t) \neq \operatorname{sgn} x''(t), \quad \text{for } t \geq t_0.$$

Next, suppose that $x'(t) < 0$ for $t \geq T_0$. If possible, let $x''(t)$ be oscillatory with a sequence of zeros $\langle t_n \rangle$ such that $T_0 < t_1 < t_2 < \cdots$ and $t_n \to \infty$ as $n \to \infty$. Clearly, $\lim_{t\to\infty} x(t) = \alpha$ exists. If $\alpha = 0$, then integrating (2.2) from t_1 to t_n, we obtain

$$0 = r(t_n)x''(t_n) - r(t_1)x''(t_1) + q(t_n)x(t_n) - q(t_1)x(t_1) + \int_{t_1}^{t_n} \big(p(t) - q'(t)\big)x(t)\,dt$$

$$= q(t_n)x(t_n) - q(t_1)x(t_1) + \int_{t_1}^{t_n} \big(p(t) - q'(t)\big)x(t)\,dt.$$

If the zeros of $x''(t)$ and $q(t)$ coincide, then we get a contradiction $0 > 0$ from the above identity. Otherwise, taking limit in

$$0 > q(t_n)x(t_n) - q(t_1)x(t_1)$$

as $n \to \infty$, we obtain $0 \geq -q(t_1)x(t_1) > 0$, a contradiction. If $\alpha > 0$, then taking limit as $n \to \infty$ in

$$0 \geq q(t_n)x(t_n) - q(t_1)x(t_1) + x(t_n)\int_{t_1}^{t_n} \big(p(t) - q'(t)\big)\,dt,$$

we get a contradiction. Thus, $x''(t) > 0$ or < 0 for large t. As $x''(t) < 0$ for large t implies that $x(t) < 0$ for large t, then $x''(t) > 0$ for $t \geq t_0 \geq T_0$. If $\lim_{t\to\infty} x'(t) = \lambda$, $-\infty < \lambda < 0$, then $x(t) < 0$ for large t. Thus, $\lambda = 0$. Let $\alpha > 0$. Integrating (2.2) from t_0 to t, we obtain

$$0 < r(t)x''(t) \leq r(t_0)x''(t_0) - q(t)x(t) + \int_{t_0}^{t} \big(q'(s) - p(s)\big)x(s)\,ds$$

$$\leq r(t_0)x''(t_0) - q(t)x(t) + x(t)\int_{t_0}^{t} \big(q'(s) - p(s)\big)\,ds$$

and hence $x''(t) < 0$ for large t, a contradiction. Thus $x(t)x'(t)x''(t) \neq 0$, $\operatorname{sgn} x(t) = \operatorname{sgn} x''(t) \neq \operatorname{sgn} x'(t)$, $t \geq t_0$, and $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} x'(t) = 0$. This completes the proof of the theorem. □

Theorem 2.4.13 *Suppose that* $r(t) \equiv 1$, $p(t) > 0$, *and* $q(t) \leq 0$. *If* $\int_\alpha^\infty p(t)\,dt < \infty$ *for large* α, $2p(t) - q'(t) \geq 0$, $\lim_{t\to\infty} q(t) = 0$ *and the second-order differential equation*

$$z'' + \left[q(t) + \frac{3}{2}\int_t^\infty p(s)\,ds\right]z = 0$$

is nonoscillatory, then (2.2) *is nonoscillatory.*

Theorem 2.4.14 *Suppose that* $r(t) \equiv 1$, $p(t) > 0$, *and* $q(t) \leq 0$. *If* $\int_\alpha^\infty (2p(t) - q'(t))\,dt < \infty$ *for large* α, $2p(t) - q'(t) \geq 0$ *and the second-order differential equa-*

tion

$$z'' + \left[\frac{1}{4}q(t) + \frac{3}{4}\int_t^\infty \big(2p(s) - q'(s)\big)\,ds\right]z = 0$$

is nonoscillatory, then (2.2) *is nonoscillatory.*

2.5 Behaviour of Solutions of $x''' + a(t)x'' + b(t)x' + c(t)x = 0$ with $a(t) \geq 0$, $b(t) \geq 0$ and $c(t) > 0$

In this section, we consider (2.1) with $a(t) \geq 0$, $b(t) \geq 0$ and $c(t) > 0$. Sufficient conditions are given in terms of the coefficient functions $a(t)$, $b(t)$ and $c(t)$ for the oscillation of (2.1). Asymptotic behaviour of nonoscillatory solutions of (2.1) in the presence of an oscillatory solution are also presented.

For most of the results given in this section, we assume

$$2c(t) - a(t)b(t) - b'(t) \geq 0. \tag{2.81}$$

Lemma 2.5.1 *If* (2.81) *holds, then* (2.1) *is of type* C_I.

Proof Suppose that $x(t)$ is a solution of (2.1) with $x(t_0) = 0$, $x'(t_0) = 0$ and $x''(t_0) > 0$, $t_0 > \sigma$. From the continuity of $x''(t)$ at $t = t_0$, it follows that there exists a $\delta > 0$ such that $x''(t) > 0$ for $t \in (t_0 - \delta, t_0]$. Hence $x'(t) < 0$ and $x(t) > 0$ for $t \in (t_0 - \delta, t_0]$. We claim that $x(t) > 0$ for $t \in [\sigma, t_0)$. If possible, suppose that $x(t_1) = 0$ for some $t_1 \in [\sigma, t_0)$. Multiplying (2.2) throughout by $x(t)$ and integrating the resulting identity from t_1 to t_0, we obtain

$$-r(t_1)\big(x'(t_1)\big)^2 \geq \int_{t_1}^{t_0} \big(2p(t) - q'(t)\big)x^2(t)\,dt,$$

that is,

$$0 \geq -r(t_1)\big(x'(t_1)\big)^2 \geq \int_{t_1}^{t_0} \big(2c(t) - a(t)b(t) - b'(t)\big)r(t)x^2(t)\,dt > 0,$$

a contradiction. Hence $x(t) > 0$ for $t \in [\sigma, t_0)$. This completes the proof of the lemma. □

Theorem 2.5.1 *Let* (2.81) *hold. If* $ta(t) \geq 3$, $2ta(t) - t^2b(t) + t^2a'(t) > 2$ *and* (2.17) *holds, then* (2.1) *is oscillatory.*

Proof We may note that $ta(t) \geq 3$ and $2ta(t) - t^2b(t) + t^2a'(t) > 2$ imply that

$$1 + \frac{t^2a^2(t)}{3} + t^2a'(t) - t^2b(t) > 0$$

and

$$\left(1-\frac{ta(t)}{3}\right)+\frac{1}{\sqrt{3}}\left(1+\frac{t^2a^2(t)}{3}+t^2a'(t)-t^2b(t)\right)^{1/2}>0.$$

If possible, let (2.1) be nonoscillatory. From Lemma 1.5.5, it follows that (2.1) admits a nonoscillatory solution $x(t)$ such that $x(t)x'(t)>0$ for $t\ge t_0>\sigma$. We may assume, without any loss of generality, that $x(t)>0$ for $t\ge t_0$ and $t_0>\sigma$. Setting $u(t)=t^2\frac{x'(t)}{x(t)}$ for $t\ge t_0$, we observe that $u(t)$ is a positive solution of the second-order Riccati equation (2.15), where $G(u(t),t)$ is defined as in (2.16). Then the rest of the proof is similar to Theorem 2.1.4. The proof is complete. □

Theorem 2.5.2 *Let* (2.81) *and* (2.17) *hold. If* $ta(t)\le 3$ *and*

$$1+\frac{t^2a^2(t)}{3}+t^2a'(t)-t^2b(t)\ge 0$$

holds, then (2.1) *is oscillatory.*

Remark 2.5.1 Applying Theorems 2.5.1 and 2.5.2 to the Euler equation (2.5), we observe the following:

(i) Let $a_0>0$, $b_0\ge 0$ and $c_0>0$ hold. Further, suppose that $a_0\ge 3$, $a_0-b_0>2$, $2c_0-a_0b_0+2b_0\ge 0$ and (2.18) hold. Then (2.5) is oscillatory.
(ii) Let $a_0\ge 0$, $b_0\ge 0$ and $c_0>0$ hold. If $0\le a_0\le 3$, $2c_0-a_0b_0+2b_0\ge 0$, $1+\frac{a_0^3}{3}-a_0-b_0\ge 0$ and (2.18) hold, then (2.5) is oscillatory.

Theorem 2.5.3 *Let* (2.81) *hold. If* $\frac{a^2(t)}{3}-b(t)+a'(t)\ge 0$ *and* (2.14) *holds, then* (2.1) *is oscillatory.*

Remark 2.5.2 Theorem 2.5.3 holds for Eq. (1.5). Indeed, if $2c-ab\ge 0$, $a^2-3b\ge 0$ and (1.8) hold, then (1.5) is oscillatory. However, the theorem fails to hold for the Euler equation (2.5). From Proposition 1.2.5(iv) of Chap. 1, it is clear that the assertions $a^2-3b\ge 0$ and (1.8) are superfluous for oscillation of (1.5). Further, $a^2-3b\ge 0$ and $9c\ge ab$ imply that

$$\frac{2a^3}{27}-\frac{ab}{3}+c\ge\frac{2a}{9}\left(\frac{a^2}{3}-b\right)^{3/2}\ge 0$$

and hence (1.8) holds and (1.12) reduces to (1.8). Thus if the condition $2c\ge ab$ is weakened to $9c\ge ab$, then the condition $a^2\ge 3b$ becomes relevant for the oscillation of Eq. (1.5). Indeed, it is clear from Proposition 1.2.5(iii) in Chap. 1 that $a^2-3b\ge 0$ and $9c\ge ab$ imply that (1.5) is oscillatory. This observation is strengthened by Example 2.5.1.

Example 2.5.1 Consider

$$x''' + 3x'' + 2x' + \frac{15}{8}x = 0, \quad t \geq 0.$$

Clearly, $a^2 - 3b = 3 > 0$ and $9c - ab > 0$. However, $2c - ab = -\frac{9}{4} < 0$. The equation is oscillatory because it admits oscillatory solutions

$$x_1(t) = e^{-t/4}\cos\left(\frac{\sqrt{11}}{4}t\right) \quad \text{and} \quad x_2(t) = e^{-t/4}\sin\left(\frac{\sqrt{11}}{4}t\right).$$

The above example suggests that the assumption (2.81) could be weakened to

$$9c(t) - a(t)b(t) - b'(t) \geq 0. \tag{2.82}$$

However, it has not been established yet. Theorem 2.5.3 may be viewed as a generalisation of the observation as in Proposition 1.2.5(iii) in Chap. 1.

Theorem 2.5.4 *Suppose that* (2.81) *holds. If* $a'(t) \leq 0$ *and* $\int_\sigma^\infty c(t)\,dt = \infty$, *then* (2.1) *is oscillatory.*

Proof On the contrary, let (2.1) be nonoscillatory. By Lemma 1.5.5, Eq. (2.1) admits a nonoscillatory solution $x(t)$ such that $x(t)x'(t) > 0$, $t \geq t_0 > \sigma$. If $z(t) = \frac{x'(t)}{x(t)}$, then it satisfies

$$z''(t) + 3z(t)z'(t) + a(t)z'(t) = -\left[z^3(t) + a(t)z^2(t) + b(t)z(t) + c(t)\right] \leq -c(t).$$

Integrating the above inequality from t_0 to t yields

$$\begin{aligned} z'(t) &\leq z'(t) + \frac{3}{2}z^2(t) + a(t)z(t) - \int_{t_0}^t a'(s)z(s)\,ds \\ &\leq -\int_{t_0}^t c(s)\,ds + z'(t_0) + \frac{3}{2}z^2(t_0) + a(t_0)z(t_0). \end{aligned}$$

Then $z'(t) \to -\infty$ as $t \to \infty$ and hence $z(t) < 0$ for large t, a contradiction. This completes the proof of the theorem. □

Theorem 2.5.5 *Let* (2.81) *hold. If*

$$\int_\sigma^\infty \exp\left(-\int_\sigma^t a(s)\,ds\right)dt = \infty$$

and

$$\int_\sigma^\infty \big(2c(t) - a(t)b(t) - b'(t)\big)\exp\left(\int_\sigma^t a(s)\,ds\right)dt = \infty,$$

then (2.1) *is oscillatory.*

Example 2.5.2 Consider

$$x''' + \frac{2}{t}x'' + \frac{1}{t^2}x' + \frac{1}{t}x = 0, \quad t \geq 1.$$

By Theorem 2.5.4, this example has an oscillatory solution. However, $ta(t) = 2 < 3$ implies that Theorem 2.5.1 cannot be applied to this example. Again, $1 + \frac{t^2a^2(t)}{3} + t^2a'(t) - t^2b(t) = -\frac{2}{3} < 0$ implies that Theorem 2.5.2 cannot be applied to this example. Theorem 2.5.3 fails to apply to this example because $\frac{a^2(t)}{3} - b(t) + a'(t) = -\frac{5}{3t^2} < 0$. Further, Theorem 2.5.5 cannot be applied to this example.

Remark 2.5.3 Theorem 2.5.4 fails to hold for the Euler equation (2.5) because

$$\int_\sigma^\infty c(t)\,dt < \infty,$$

where $c(t) = \frac{c_0}{t^3}$, for any $c_0 > 0$. Theorem 2.5.6 can be applied to the Euler equation (2.5).

Theorem 2.5.6 *Suppose that* (2.81) *holds,* $a(t) \geq \frac{3}{t}$, $2 - 2ta(t) - t^2a'(t) + t^2b(t) \geq 0$. *If* $\int_\sigma^\infty t^2c(t)\,dt = \infty$, *then* (2.1) *is oscillatory.*

Proof Suppose that (2.1) is nonoscillatory. Then from Lemma 1.5.5, it follows that (2.1) admits a nonoscillatory solution $x(t)$ such that $x(t)x'(t) > 0$ for $t \geq t_0 > \sigma$. If $z(t) = \frac{t^2x'(t)}{x(t)}$ for $t \geq t_0$, then $z(t) > 0$ is a solution of the second-order Riccati equation (2.15). Since

$$\begin{aligned} G\big(z(t), t\big) &= \frac{z^3(t)}{t^4} + \frac{z^2(t)}{t^2}\left(a(t) - \frac{3}{t}\right) \\ &\quad + \frac{z(t)}{t^2}\big(2 - 2ta(t) - t^2a'(t) + t^2b(t)\big) + t^2c(t) \\ &\geq t^2c(t), \end{aligned}$$

(2.15) reduces to

$$\left[z'(t) - \frac{4}{t}z(t) + \frac{3}{2t^2}z^2(t) + a(t)z(t)\right]' \leq -t^2c(t).$$

Integration of the above inequality from t_0 to t yields

$$\left[z'(t) - \frac{4}{t}z(t) + \frac{3}{2t^2}z^2(t) + a(t)z(t)\right] \leq K - \int_{t_0}^t s^2c(s)\,ds,$$

where $K = z'(t_0) - \frac{4}{t_0}z(t_0) + \frac{3}{2t_0^2}z^2(t_0) + a(t_0)z(t_0)$. Since $\frac{3}{2t^2}z^2(t) - \frac{4}{t}z(t)$ attains the minimum $-\frac{8}{3}$ for $z(t) > 0$ at $z(t) = \frac{4t}{3}$, we have

$$z'(t) \le \frac{8}{3} + K - \int_{t_0}^{t} s^2 c(s)\, ds.$$

Thus $z'(t) \to -\infty$ as $t \to \infty$ and hence $z(t) < 0$ for large t, a contradiction. This completes the proof of the theorem. □

Example 2.5.3 Consider

$$x''' + \frac{3}{t}x'' + \frac{2}{t^2}x' + \frac{2}{t^3}x = 0, \quad t \ge 1.$$

Since all the conditions of Theorem 2.5.6 are satisfied, this example has an oscillatory solution. In particular, $x_1(t) = e^{t/2}\cos(\frac{\sqrt{7}}{2}\log t)$ and $x_2(t) = e^{t/2}\sin(\frac{\sqrt{7}}{2}\log t)$ are the oscillatory solutions of this example. Note that $x_3(t) = \frac{1}{t}$ is a nonoscillatory solution of this example. Since $\int_1^\infty c(t)\, dt < \infty$, Theorem 2.5.4 cannot be applied to this example. Further, $\int_1^\infty \exp(-\int_1^\infty a(s)\, ds)\, dt < \infty$ implies that Theorem 2.5.5 cannot be applied to this example.

Remark 2.5.4 In Theorem 2.5.6, it is assumed that $ta(t) \ge 3$. However, no such result exists for the case $ta(t) < 3$. This has been remain as an open problem.

Theorem 2.5.7 *If* $c(t) - a(t)b(t) - b'(t) \ge 0$,

$$0 \le \int_\sigma^\infty a(t)\, dt < \infty$$

and

$$\int_\sigma^\infty t\big[c(t) - a(t)b(t) - b'(t)\big] \exp\left(\int_\sigma^t a(s)\, ds\right) dt = \infty,$$

then (2.1) *is oscillatory.*

Proof If possible, let (2.1) be nonoscillatory. Since (2.81) holds, it follows from Lemma 1.5.5 that (2.1) admits a nonoscillatory solution $x(t)$ such that $x(t)x'(t) > 0$ for $t \ge t_0 > \sigma$. Without any loss of generality, it may be assumed that $x(t) > 0$ for $t \ge t_0$. Hence $x'(t) > 0$ for $t \ge t_0$. From (2.2), we obtain

$$\big(r(t)x''(t)\big)' = -q(t)x'(t) - p(t)x(t) < 0$$

for $t \ge t_0$. Hence $x''(t) > 0$ or < 0 for $t \ge t_1 > \max\{t_0, \sigma, 0\}$. If $x''(t) < 0$ for $t \ge t_1$, then $r(t)x''(t) < r(t_1)x''(t_1)$ implies that

$$x''(t) < r(t_1)x''(t_1) \exp\left(-\int_\sigma^\infty a(t)\, dt\right).$$

Hence $x'(t) < 0$ for large t, a contradiction. Thus $x''(t) > 0$ for $t \ge t_1$. Clearly, $x'(t) > 0$ and $x''(t) > 0$ for $t \ge t_1$ imply that

$$x(t) > x'(t_1)(t - t_1) \tag{2.83}$$

for $t \ge t_1$. Integrating (2.2) from t_1 to t and using (2.83), we get

$$\begin{aligned} r(t_1)x''(t_1) + q(t_1)x(t_1) &= r(t)x''(t) + q(t)x(t) + \int_{t_1}^{t} \big(p(s) - q'(s)\big)x(s)\,ds \\ &> \int_{t_1}^{t} \big(c(s) - b'(s) - a(s)b(s)\big)r(s)x(s)\,ds \\ &> x'(t_1)\int_{t_1}^{t} (s - t_1)\big(c(s) - a(s)b(s) - b'(s)\big)r(s)\,ds \\ &> x'(t_1)\int_{2t_1}^{t} (s - t_1)\big(c(s) - b'(s) - a(s)b(s)\big)r(s)\,ds \\ &> \frac{1}{2}x'(t_1)\int_{2t_1}^{t} s\big(c(s) - a(s)b(s) - b'(s)\big)r(s)\,ds. \end{aligned}$$

Hence

$$\int_{2t_1}^{\infty} t\big(c(t) - a(t)b(t) - b'(t)\big)\exp\left(\int_{\sigma}^{t} a(s)\,ds\right)dt < \infty,$$

a contradiction. Thus the theorem is proved. □

Example 2.5.4 From Theorem 2.5.7, it follows that the equation

$$x''' + \frac{1}{t(t-1)}x'' + \left(1 - \frac{1}{t}\right)x' + \frac{3}{t^2}x = 0, \quad t \ge 2$$

is oscillatory. Since $\int_2^\infty c(t)\,dt < \infty$, $\frac{a^2(t)}{3} - b(t) + a'(t) < 0$ for large t,

$$1 + \frac{t^2a^2(t)}{3} + t^2a'(t) - t^2b(t) < 0, \quad t \ge 2$$

and $ta(t) < 3$ for $t \ge 2$, then none of the Theorems 2.5.1–2.5.4 can be applied to this example.

Theorem 2.5.8 *Suppose that* $2a^3(t) - 9a(t)b(t) + 27c(t) - 9a''(t) > 0$, $3b(t) - 3a'(t) - a^2(t) \ge 0$, $2a^3(t) - 9a(t)b(t) + 27c(t) + 18a''(t) - 27b'(t) + 18a(t)a'(t) \ge 0$ *and*

$$\int_{\sigma}^{\infty} t\big[2a^3(t) - 9a(t)b(t) + 27c(t) + 18a''(t) - 27b'(t) + 18a(t)a'(t)\big]\,dt = \infty.$$

Then (2.1) *is oscillatory.*

Remark 2.5.5 Theorem 2.5.8 may be viewed as the generalisation of Proposition 1.2.5(ii) in Chap. 1. Indeed, from Theorem 2.5.8, it follows that Eq. (1.5) is oscillatory if $2a^3 - 9ab + 27c > 0$ and $3b - a^2 \geq 0$. However, $3b = a^2$ and $2a^3 - 9ab + 27c > 0$ imply that $9c > ab$. It confirms the earlier observation that $a^2 \geq 3b$ and $9c \geq ab$ imply that

$$x''' + ax'' + bx' + cx = 0$$

is oscillatory, where $a \geq 0$, $b \geq 0$ and $c > 0$ are constants.

Next theorem follows from Theorem 1.3 in [23].

Theorem 2.5.9 *If $a \in C^2([\sigma, \infty), R)$, $3b(t) - 3a'(t) - a^2(t) \leq 0$, $2a^3(t) - 9a(t)b(t) + 27c(t) - 9a''(t) > 0$ and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{a''(t)}{3} - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} + a'(t) - b(t)\right)^{3/2}\right] dt = \infty,$$

then (2.1) *is oscillatory.*

Remark 2.5.6 If $\frac{2a^3}{27} - \frac{ab}{3} + c > 0$, then Theorem 2.5.9 is the generalisation of the observation from Proposition 1.2.5(iii) in Chap. 1.

Theorem 2.5.10 *If Eq.* (2.1) *is of type C_I or C_{II}, then it admits a nonoscillatory solution.*

Proof Suppose that (2.1) is of type C_I. For every integer $n \geq \sigma$, it is possible to construct a solution $x_n(t)$ of (2.1) such that $x_n(n) = x_n'(n) = 0$, $x_n''(n) > 0$. Hence $x_n(t) > 0$ for $t \in [\sigma, n)$. Let $\{u_1(t), u_2(t), u_3(t)\}$ be a basis for the solution space of (2.1). Hence it is possible to write

$$x_n(t) = c_{1n}u_1(t) + c_{2n}u_2(t) + c_{3n}u_3(t),$$

where

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1.$$

Since $\langle c_{in}\rangle$, $i = 1, 2, 3$, is bounded, it admits a convergent subsequence. Without any loss of generality, it may be assumed that $\langle c_{in_j}\rangle$, $i = 1, 2, 3$ is the convergent subsequence. Let $c_{in_j} \to c_i$, $i = 1, 2, 3$, as $n_j \to \infty$. Then $c_1^2 + c_2^2 + c_3^2 = 1$. If $x_0(t) = c_1u_1(t) + c_2u_2(t) + c_3u_3(t)$, $t \geq \sigma$, then $\langle x_n(t)\rangle$ converges to $x_0(t)$ uniformly on every compact subinterval of $[\sigma, \infty)$. Hence $x_0(t) \geq 0$ for $t \in [\sigma, \infty)$. If $x_0(t_0) = 0 = x_0(t_1)$ for $\sigma \leq t_0 < t_1 < \infty$, then $x_0'(t_0) = 0$, $x_0''(t_0) > 0$ and $x_0'(t_1) = 0$, $x_0''(t_1) > 0$. This is impossible, since (2.1) is of type C_I. Hence $x_0(t)$ is nonoscillatory. If (2.1) is of type C_{II}, then obviously it admits a nonoscillatory solution. $\square$

Remark 2.5.7 No sign restrictions on coefficient functions in (2.1) have been used in Theorem 2.5.10.

Corollary 2.5.1 *If (2.81) holds, then (2.1) admits a nonoscillatory solution.*

Theorem 2.5.11 *Suppose that $c(t) \geq d > 0$, $c(t) - a(t)b(t) - b'(t) \geq 0$ and $\int_\sigma^\infty a(t)\,dt = \infty$. If (2.1) is oscillatory, then every nonoscillatory solution $x(t)$ of (2.1) satisfies the property*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0. \tag{2.84}$$

Proof Since (2.81) holds, from Lemma 1.5.6 it follows that $F[x(t)] < 0$ for $t \geq \sigma$. If $x(t_0) = 0$ for some $t_0 \in [\sigma, \infty)$, then $F[x(t_0)] \geq 0$, a contradiction. Hence $x(t) \neq 0$ for $t \in [\sigma, \infty)$. Without any loss of generality, it may be assumed that $x(t) > 0$ for $t \geq \sigma$. As $r(t) \geq 1$ and $F[x(t)] < 0$ for $t \geq \sigma$, then

$$0 \leq \big(x'(t)\big)^2 \leq r(t)\big(x'(t)\big)^2 < x(t)\big(2r(t)x''(t) + q(t)x(t)\big). \tag{2.85}$$

Then

$$2r(t)x''(t) + q(t)x(t) > 0, \quad t \geq \sigma. \tag{2.86}$$

Consequently, from (2.2) and (2.86), we obtain

$$\begin{aligned} 0 &< r(t)x''(t) + q(t)x(t) \\ &= r(\sigma)x''(\sigma) + q(\sigma)x(\sigma) + \int_\sigma^t \big(q'(s) - p(s)\big)x(s)\,ds \leq K, \end{aligned} \tag{2.87}$$

since $c(t) - a(t)b(t) - b'(t) \geq 0$, where $K = r(\sigma)x''(\sigma) + q(\sigma)x(\sigma)$. Then

$$2r(t)x''(t) + q(t)x(t) \leq 2K. \tag{2.88}$$

Using (2.85) and (2.88), we get

$$\big(x'(t)\big)^2 \leq 2Kx(t). \tag{2.89}$$

Further from (2.54), for $t \geq \sigma$,

$$\begin{aligned} 0 > F\big[x(t)\big] &\geq F\big[x(\sigma)\big] + \int_\sigma^t \big(2p(s) - q'(s)\big)x^2(s)\,ds \\ &\geq F\big[x(\sigma)\big] + \int_\sigma^t \big(2c(s) - a(s)b(s) - b'(s)\big)x^2(s)\,ds \\ &\geq F\big[x(\sigma)\big] + \int_\sigma^t c(s)x^2(s)\,ds \\ &\geq F\big[x(\sigma)\big] + d\int_\sigma^t x^2(s)\,ds. \end{aligned}$$

This inequality implies that $x \in L^2([\sigma, \infty), R)$. By Lemma 1.5.15, $\lim_{t\to\infty} x(t) = 0$. Hence (2.89) yields $\lim_{t\to\infty} x'(t) = 0$. Next, we show that $\lim_{t\to\infty} x''(t) = 0$. From (2.87), we obtain

$$K \geq \int_\sigma^t \big(p(s) - q'(s)\big)x(s)\,ds = \int_\sigma^t \big(c(s) - a(s)b(s) - b'(s)\big)r(s)x(s)\,ds. \quad (2.90)$$

Since

$$\begin{aligned}\big[r(t)x''(t) + q(t)x(t)\big]' &= -\big(p(t) - q'(t)\big)x(t)\\ &= -\big(c(t) - a(t)b(t) - b'(t)\big)r(t)x(t)\\ &\leq 0,\end{aligned}$$

then using (2.87), we get

$$\lim_{t\to\infty}\big[r(t)x''(t) + q(t)x(t)\big] = l, \quad (2.91)$$

where $0 \leq l < \infty$. Integrating (2.2) from t to s $(\sigma < t < s)$ and then taking limit as $s \to \infty$, we have

$$r(t)x''(t) + q(t)x(t) = l + \int_t^\infty \big(p(\theta) - q'(\theta)\big)x(\theta)\,d\theta,$$

that is, using (2.90),

$$\begin{aligned}x''(t) + b(t)x(t) &\leq \left[l + \int_\sigma^\infty \big(c(\theta) - a(\theta)b(\theta) - b'(\theta)\big)r(\theta)x(\theta)\,d\theta\right]r^{-1}(t)\\ &\leq [l + K]r^{-1}(t).\end{aligned}$$

From the given hypothesis, it follows that

$$\lim_{t\to\infty}\big[x''(t) + b(t)x(t)\big] = 0.$$

Since by (2.86),

$$0 < x''(t) + \frac{1}{2}b(t)x(t) \leq x''(t) + b(t)x(t),$$

we have

$$\lim_{t\to\infty}\left[x''(t) + \frac{1}{2}b(t)x(t)\right] = 0$$

and, hence

$$\lim_{t\to\infty}\frac{1}{2}b(t)x(t) = \lim_{t\to\infty}\left[\big(x''(t) + b(t)x(t)\big) - \left(x''(t) + \frac{1}{2}b(t)x(t)\right)\right] = 0.$$

Consequently, $\lim_{t\to\infty} x''(t) = 0$. Thus the theorem is proved. □

In view of Lemma 1.5.7 and Theorem 2.5.11, we obtain the following corollary:

Corollary 2.5.2 *Let the conditions of Theorem 2.5.11 hold. If $b(t) \geq L > 0$, then*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0$$

for every nonoscillatory solution $x(t)$ of (2.1).

Theorem 2.5.12 *Let $c(t) \geq d > 0$, $c(t) - a(t)b(t) - b'(t) \geq 0$ and $b(t)$ be bounded. If (2.1) is oscillatory, then every nonoscillatory solution of (2.1) satisfies the property (2.84).*

Proof From Lemma 1.5.6, it follows that $F[x(t)] < 0$ for $t \geq \sigma$. If $x(t_0) = 0$ for some $t_0 \in [\sigma, \infty)$, then $F[x(t_0)] \geq 0$, a contradiction. Hence $x(t) \neq 0$ for $t \in [\sigma, \infty)$. Without any loss of generality, we may assume that $x(t) > 0$ for $t \geq \sigma$. As $r(t) > 1$ and $F[x(t)] < 0$ for $t \geq \sigma$, then proceeding as in the proof of Theorem 2.5.11, we obtain $\lim_{t\to\infty} x(t) = 0$. Now, we show that $\lim_{t\to\infty} x''(t) = 0$. We consider the following two cases:

$$\int_\sigma^\infty a(t)\,dt = \infty$$

and

$$\int_\sigma^\infty a(t)\,dt < \infty. \tag{2.92}$$

In view of Theorem 2.5.11, it is enough to prove $\lim_{t\to\infty} x''(t) = 0$ when (2.92) holds. Let (2.92) hold. Clearly (2.91) holds, too. Since $b(t)$ is bounded, we have $\lim_{t\to\infty} r(t)x''(t) = \alpha$, $0 \leq \alpha < \infty$. Clearly, (2.92) implies that $\int_\sigma^\infty \frac{1}{r(t)}\,dt = \infty$ and hence $\alpha > 0$ yields $x'(t) \to \infty$ as $t \to \infty$, a contradiction. Hence $\alpha = 0$. Consequently, $\lim_{t\to\infty} x''(t) = 0$. This completes the proof of the theorem. □

Theorem 2.5.13 *Suppose that there exists a positive number m such that $m \leq b(t)$ for $t \geq \sigma$. Suppose, furthermore, that there is a function $h \in C^1([\sigma, \infty), (0, \infty))$ such that $h'(t) \leq 0$ on $[t_0, \infty) \subset [\sigma, \infty)$,*

$$\lim_{t\to\infty} h(t) = 0, \qquad \int_{t_0}^\infty h(t)\,dt = \infty$$

and

$$\big[c(t) - b'(t) - a(t)b(t)\big]r(t) \geq b(t)h(t)$$

hold. Then (2.1) is oscillatory and every nonoscillatory solution $x(t)$ of (2.1) satisfies

$$x(t) \neq 0 \quad \text{for } t \geq \sigma, \quad \text{and} \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0.$$

Theorem 2.5.14 *Assume that there exist constants* $0 < m < M$ *such that* $m \le b(t) \le M$ *and* h *have the same properties as in Theorem* 2.5.13. *If either*

$$\big[c(t) - b'(t) - a(t)b(t)\big]r(t) \ge h(t) \quad \text{for } t \in [\sigma, \infty),$$

or

$$c(t) - b'(t) - a(t)b(t) \ge h(t) \quad \text{for } t \in [\sigma, \infty),$$

then Eq. (2.1) *is oscillatory, and every nonoscillatory solution* $x(t)$ *of* (2.1) *has the properties*

$$x(t) \ne 0 \quad \text{for } t \ge \sigma \quad \text{and} \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0.$$

Theorem 2.5.15 *Assume the hypothesis of Theorem* 2.5.13 *or Theorem* 2.5.14 *hold and* $\int_\sigma^\infty a(t)\,dt < \infty$. *Then every solution of* (2.1) *is either oscillatory except for a solution* x (*unique up to a linear combination*) *that satisfies*

$$x(t) \ne 0 \quad \text{for } t \ge \sigma \quad \text{and} \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0.$$

Examples of functions $h \in C^1([\sigma, \infty), (0, \infty))$ that satisfy the hypothesis of Theorem 2.5.13 include $h(t) = \frac{d}{t}$, $d > 0$, $t > 0$ and $h(t) = \frac{1}{(t+1)\ln(t+1)}$, $t > 0$.

Example 2.5.5 Consider

$$x''' + \frac{1}{t}x'' + tx' + \left(2 + \frac{1}{t^3}\right)x = 0, \quad t \ge 1.$$

From Corollary 2.5.2, it follows that every nonoscillatory solution $x(t)$ of the equation satisfies $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0$. However, Lemma 8 in [15] fails to hold for this example, because one of the assumption of the lemma does not hold.

Remark 2.5.8 The condition $c \ge ab$ implies that $2c \ge ab$ and $\frac{2a^3}{27} - \frac{ab}{3} + c > 0$. Hence Corollary 2.5.2 may be viewed as a partial generalisation of Proposition 1.2.5(v) in Chap. 1. It seems that the condition $c(t) - a(t)b(t) - b'(t) \ge 0$ of Corollary 2.5.2 may be weakened to (2.81). The following example strengthens this observation.

Example 2.5.6 Consider

$$x''' + x'' + \left(1 - \frac{1}{t^2}\right)x' + \left(1 - \frac{1}{t^2}\right)x = 0, \quad t \ge 2. \tag{2.93}$$

Clearly (2.81) holds but $c(t) - a(t)b(t) - b'(t) = -\frac{2}{t^3} < 0$. Further, $b(t) = c(t) = 1 - \frac{1}{t^2} > \frac{1}{2}$ and $\int_2^\infty a(t)\,dt = \infty$. By Theorem 2.5.4, (2.93) is oscillatory. It is expected that every nonoscillatory solution $x(t)$ of (2.93) satisfies $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0$. In fact, $x(t) = e^{-t}$ is such a solution of (2.93).

Theorem 2.5.16 *Let* $3b(t) - a^2(t) - 3a'(t) \le 0$ *and* $2a^3(t) - 9a(t)b(t) + 27c(t) - 9a''(t) > 0$. *If* (2.1) *is oscillatory, then nonoscillatory solutions of* (2.1) *form a one-dimensional subspace of the solution space of* (2.1).

Corollary 2.5.3 *Suppose that the conditions of Theorems* 2.5.4 *and* 2.5.16 *hold. Then nonoscillatory solutions of* (2.1) *form a one-dimensional subspace of the solution space of* (2.1).

Corollary 2.5.4 *Let the conditions of Theorem* 2.5.16 *hold. Suppose that* $b(t) \ge L > 0$, $c(t) \ge d > 0$, $c(t) - a(t)b(t) - b'(t) \ge 0$, $a'(t) \le 0$ *and* $\int_\sigma^\infty a(t)\,dt = \infty$. *Then nonoscillatory solutions of* (2.1) *form a one-dimensional subspace of the solution space of* (2.1) *and every nonoscillatory solution of* (2.1) *tends to zero as* $t \to \infty$ *along with its first and second derivatives.*

This follows from Corollary 2.5.2 and Theorems 2.5.4 and 2.5.16.

Theorem 2.5.17 *Suppose that* $3b(t) - a^2(t) - 3a'(t) \le 0$ *and* $2a^3(t) - 9a(t)b(t) + 27c(t) - 9a''(t) < 0$. *If* (2.1) *is oscillatory, then there exist two linearly independent oscillatory solutions of* (2.1) *whose zeros separate and oscillatory solutions of* (2.1) *form a two-dimensional subspace of the solution space of* (2.1).

Corollary 2.5.5 *Let the conditions of Theorems* 2.5.4 *and* 2.5.17 *be satisfied. Then there exist two linearly independent oscillatory solutions of* (2.1) *whose zeros separate and oscillatory solutions of* (2.1) *form a two-dimensional subspace of the solution space of* (2.1).

Now, we consider Eq. (2.10). In [16], Hanan proved the following theorem.

Theorem 2.5.18 (Hanan, [16]) *If* $c(t) > b'(t)$, *the equation*

$$u'' + b(t)u = 0 \tag{2.94}$$

is nonoscillatory, and if

$$\int_{t_0}^{\infty} t\big[c(t) - b'(t)\big]\,dt = \infty, \qquad t_0 \ge \sigma, \tag{2.95}$$

then (2.10) *is oscillatory.*

On the other hand, Lazer [23] proved the following theorem for the oscillation of (2.10).

Theorem 2.5.19 *If* $2c(t) - b'(t) \ge 0$ *and not identically zero in any subinterval of* $[\sigma, \infty)$, *and there exists a number* $m < \frac{1}{2}$ *such that the second-order differential equation*

$$u'' + \big(b(t) + mtc(t)\big)u = 0, \tag{2.96}$$

is oscillatory, then (2.10) *is also oscillatory. In fact, if x is any nonzero solution of Eq.* (2.10) *with a zero at some $t_0 \geq \sigma$, then x is oscillatory.*

One can interpret the conclusion of the theorem in the following way: If $x(t)$ is a solution of (2.10) with $F[x(t_0)] \geq 0$ for some $t_0 \geq \sigma$, then $x(t)$ is oscillatory, where $F[x(t)]$ is given in (2.54).

Lemma 2.5.2 *If $2c(t) - b'(t) \geq 0$ and not identically zero in any subinterval of $[\sigma, \infty)$, and $x(t)$ is a nonoscillatory solution of* (2.10) *which is eventually nonnegative with $F[x(t_0)] \geq 0$, $t_0 \geq \sigma$, then there exists a number $t_1 \geq t_0$ such that $x(t) > 0$, $x'(t) > 0$, $x''(t) > 0$ and $x'''(t) \leq 0$ for $t \geq t_1$.*

If $a(t) \equiv 0$, then from Theorem 2.5.2 we see that if $2c(t) - b'(t) \geq 0$, $t^2 b(t) \leq 1$ and

$$\int_\sigma^\infty \left[t^2 c(t) + t b(t) - \frac{2}{3\sqrt{3}t} \big(1 - t^2 b(t)\big)^{3/2} \right] dt = \infty, \tag{2.97}$$

then (2.10) is oscillatory. Let the hypothesis of Lemma 2.5.2 hold. Considering additional assumptions, we eliminate positive increasing solutions and so we obtain oscillation criterion. Since $t^2 b(t) > \frac{1}{4}$, $t > \sigma$, (2.96) is oscillatory, by the Sturm comparison theorem and Kneser-criterion, Theorem 2.5.19 is applicable here. Therefore, we shall now concentrate with the case $t^2 b(t) \leq \frac{1}{4}$, $t \geq \sigma$.

It is easy to verify that the following inequality is fulfilled for all $t \geq \sigma$:

$$t b(t) - \frac{2}{3\sqrt{3}t} \big[1 - t^2 b(t)\big]^{3/2} \leq 0 \quad \text{for } t^2 b(t) \leq \frac{1}{4}, \tag{2.98}$$

since $4t^6 b^3(t) + 15t^4 b^2(t) + 12t^2 b(t) - 4 = [4t^2 b(t) - 1][t^2 b(t) + 2]^2$.

Using this inequality, we obtain the assertion needed to prove next Theorem 2.5.20.

Lemma 2.5.3 *Let $0 \leq t^2 b(t) \leq \frac{1}{4}$ for all $t \geq \sigma$. Let G be the polynomial in the variable z,*

$$G(z) = z^3 - 3z^2 + \big(2 + t^2 b(t)\big) z + t^3 c(t), \quad t > \sigma.$$

Then

$$G(z) \geq t^3 c(t) + t^2 b(t) - \frac{2}{3\sqrt{3}} \big(1 - t^2 b(t)\big)^{3/2}, \quad t > \sigma \tag{2.99}$$

for all $z \geq 1 - 2\sqrt{\frac{1 - t^2 b(t)}{3}}$.

We note that the right-hand side of (2.99) is the minimum of G at the point

$$z_0 = 1 + \sqrt{\frac{1 - t^2 b(t)}{3}}.$$

Theorem 2.5.20 *Let the hypothesis of Lemma 2.5.3 hold, and in addition $t^2b(t) \le \frac{1}{4}$ for all $t \ge \sigma$. If (2.97) is satisfied, then (2.10) is oscillatory. In fact, any solution x of (2.10) which satisfies $F[x(t^*)] \ge 0$ for some $t^* \ge \sigma$ is oscillatory.*

Remark 2.5.9 Theorems 2.5.2 and 2.5.20 give the same conclusion. In the former, the weaker condition $t^2b(t) < 1$ is used whereas the condition $t^2b(t) < \frac{1}{4}$ is used in the later. Thus Theorem 2.5.2 is better than Theorem 2.5.20. However, no results are obtained on the asymptotic behaviour of nonoscillatory solutions of (2.10) except Theorem 2.5.11. The condition $c(t) \ge d > 0$ has been used in Theorem 2.5.11, which in turn, implies that $\int_\sigma^\infty t^2c(t)\,dt = \infty$. In the following, we prove the asymptotic behaviour of nonoscillatory solutions of (2.10) with the price that $\int_\sigma^\infty t^2c(t)\,dt = \infty$ is satisfied.

Theorem 2.5.21 *Let $t^2b(t) < \frac{1}{4}$ hold. If (2.97) is satisfied, then every nonoscillatory solution of (2.10) has the property* $\lim_{t\to\infty} x(t) = 0$.

Proof Since $t^2b(t) \le \frac{1}{4}$, from Kneser comparison theorem it follows that Eq. (2.94) is nonoscillatory. So by Theorem 3.6 in [18], it follows that there exists $d \ge \sigma$ such that either $x(t)x'(t) \ge 0$ or $x(t)x'(t) < 0$ for all $t \ge d$. Let $x(t)$ be a nonoscillatory solution of (2.10), and suppose that $x(t) > 0, x'(t) > 0$ for all $t \ge d$. We denote $z(t) = \frac{tx'(t)}{x(t)}$, $t \ge d$. Since (2.97) holds, proceeding as in the proof of Theorem 2.1.4, we get the contradiction that $z(t) < 0$ for large t. Hence $x(t) > 0$ and $x'(t) < 0$ for $t \ge d$. Then $\lim_{t\to\infty} x(t) = L \ge 0$ exists. Let $L > 0$. Multiplying (2.10) by t^2 and integration from d to t yields

$$t^2x''(t) - 2tx'(t) + \frac{9}{4}x(t) \le K - L\int_d^t s^2c(s)\,ds,$$

where K is some constant. By (2.98) and (2.97), we have $\int_\sigma^\infty t^2c(t)\,dt = \infty$. From this and the last inequality, we obtain $x''(t) < 0$ for large t, which contradicts the fact that $x(t) > 0$ and $x'(t) < 0$ for $t \ge d$. The proof is complete. □

Now, we shall show that either Theorem 2.5.2 or Theorem 2.5.20 can be applied even in the case when Theorems 2.5.18 and 2.5.19 are not applicable to the third-order equation

$$x''' + b_0t^\beta x' + c_0t^{-3}x = 0, \quad t > 0, \tag{2.100}$$

where $\beta \le -2$, $b_0 > 0$, $c_0 > \frac{2}{3\sqrt{3}}$, and $b_0 < \frac{1}{4}$ if $\beta = -2$; β, b_0 and c_0 are some constants.

Directly, we see that Theorem 2.5.18 is not applicable to (2.100). For $\beta = -2$, Eq. (2.100) becomes the Euler equation. The necessary and sufficient condition for the oscillation of Euler equation (2.100) is

$$c_0 + b_0 - \frac{2}{3\sqrt{3}}(1 - b_0)^{3/2} > 0. \tag{2.101}$$

It is easy to check that (2.101) is equivalent to (2.97) of Theorem 2.5.20. Equation (2.96) with $\beta = -2$ and $b_0 < \frac{1}{4}$ becomes the Euler equation

$$v'' + (b_0 + mc_0)t^{-2}v = 0. \tag{2.102}$$

Equation (2.102) is oscillatory if and only if $b_0 + mc_0 > \frac{1}{4}$ for some $m < \frac{1}{2}$, that is, $2b_0 + c_0 > 2b_0 + 2mc_0 > \frac{1}{2}$. So it is easy to check that inequality $c_0 + b_0 > \frac{1}{2} - b_0 > \frac{2}{3\sqrt{3}}(1 - b_0)^{3/2}$ holds for some $c_0 > 0$ and every $0 < b_0 < \frac{1}{4}$. From this it follows that Theorem 2.5.21 is better than Theorem 2.5.19 in this case. For example, if $b_0 = 0.06$, $c_0 = 0.3$, then condition (2.101) is fulfilled, while (2.102) is nonoscillatory.

Let $\beta < -2$. So there is a number $\delta > 0$ such that $\beta = -2 - \delta$, and hence $t^2 b(t) = b_0 t^{-\delta} \le \frac{1}{4}$ for $t \ge (4b_0)^{1/\delta}$. If we denote $y = b_0 t^{-\delta}$ for $t \ge \sigma$, then the function $f(y) = y - \frac{2}{3\sqrt{3}}(1 - y)^{3/2}$ is increasing and so for $0 \le y \le \frac{1}{4}$, we have

$$\int_\sigma^\infty \frac{1}{t}\left(c_0 + b_0 t^{-\delta} - \frac{2}{3\sqrt{3}}\left(1 - b_0 t^{-\delta}\right)^{3/2}\right) dt \ge \left(c_0 - \frac{2}{3\sqrt{3}}\right) \int_\sigma^\infty \frac{dt}{t}.$$

Hence we see by Theorem 2.5.21 that for $b_0 > 0$, $c_0 > \frac{2}{3\sqrt{3}}$ and $\beta < -2$, Eq. (2.100) is oscillatory. On the other hand, by Theorem 2.5.19, Eq. (2.100) is oscillatory only if $c_0 > 0.5$.

One may find several oscillation and asymptotic behaviour of nonoscillatory solutions of the third-order linear differential equation

$$x''' + A(t)x' + \left[A'(t) + b(t)\right]x = 0, \tag{2.103}$$

in [13], where A' and b are continuous functions on $[\sigma, \infty)$ and $b(t) \ge 0$ for $[\sigma, \infty)$ such that $b(t) \not\equiv 0$ on each subinterval. The following theorem is an easy consequence of Corollary 2.3 and Theorem 2.17 in [14]. In fact, the following theorem can be obtained if one proceeds as in Theorem 2.5.11 or Theorem 2.5.12.

Theorem 2.5.22 *Let $A(t) \ge m > 0$, $A'(t) \le 0$, $A'(t) + b(t) \ge \frac{d}{t} > 0$ for $t \in [\sigma, \infty)$, $\sigma > 0$. Then every solution of (2.103) is oscillatory in $[\sigma, \infty)$, except a solution $x(t)$ (unique up to linear dependence), which satisfies the property (2.84).*

2.6 Behaviour of Solutions of $x''' + a(t)x'' + b(t)x' + c(t)x = 0$ with $a(t) \le 0$, $b(t) \ge 0$ and $c(t) < 0$

This section is concerned with the study of behaviour of solutions of (2.1) with $a(t) \le 0$, $b(t) \ge 0$ and $c(t) < 0$. An attempt has been made to generalise, as far as possible, Proposition 1.2.8 to the variable coefficients. An explicit sufficient condition in terms of $a(t), b(t)$ and $c(t)$ is given for the strong oscillation of (2.1). An example is also provided to strengthen the result. Throughout this section, we assume that the condition

$$2c(t) - a(t)b(t) - b'(t) \le 0 \tag{2.104}$$

holds.

Theorem 2.6.1 *Let* (2.104) *hold. If* $\int_\sigma^\infty b(t)\,dt = \infty$, *then* (2.1) *is oscillatory.*

Proof If possible, suppose that (2.1) is nonoscillatory. Then by Lemma 1.5.2, there exists a nonoscillatory solution $x(t)$ of (2.1) such that $F[x(t)] > 0$ for $t \geq \sigma$ and $x(t)x'(t) > 0$ for $t \geq t_0 \geq \sigma$, where F is same as in (2.54). Since $F[x(t)] > 0$ for $t \geq \sigma$, we have

$$2\frac{x''(t)}{x(t)} + b(t) \leq \left(\frac{x'(t)}{x(t)}\right)^2, \quad t \geq \sigma. \tag{2.105}$$

If $z(t) = \frac{x'(t)}{x(t)}$, then $z(t) > 0$ for $t \geq t_0$ and

$$\begin{aligned} z'(t) = \frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2 &= \frac{x''(t)}{x(t)} - \frac{1}{2}\left(\frac{x'(t)}{x(t)}\right)^2 - \frac{1}{2}\left(\frac{x'(t)}{x(t)}\right)^2 \\ &< -\frac{1}{2}\left(\frac{x'(t)}{x(t)}\right)^2 - \frac{b(t)}{2} < -\frac{b(t)}{2}. \end{aligned}$$

Integrating the inequality from t_0 to t, we get $z(t) < 0$ for large t, a contradiction. Hence (2.1) is oscillatory. This completes the proof of the theorem. □

Theorem 2.6.2 *Let* (2.104) *hold. If* $t^2 b(t) > 1$ *for* $t \geq \sigma$ *and* $\int_\sigma^\infty (\frac{t^2 b(t) - 1}{t})\,dt = \infty$, *then* (2.1) *is oscillatory.*

Proof Let (2.1) be nonoscillatory. Then, by Lemma 1.5.2, there exists a nonoscillatory solution $x(t)$ of (2.1) such that $x(t)x'(t) > 0$ for $t \geq t_0 \geq \sigma$ and $F[x(t)] > 0$ for $t \geq \sigma$. $F[x(t)] > 0$ for $t \geq \sigma$ implies that the inequality (2.105) holds. If $z(t) = \frac{tx'(t)}{x(t)}$, then $z(t) > 0$ for $t \geq t_0$ and

$$z'(t) = \frac{x'(t)}{x(t)} + t\left[\frac{x''(t)}{x(t)} - \left(\frac{x'(t)}{x(t)}\right)^2\right].$$

Using (2.105), we obtain

$$z'(t) + \frac{1}{2t}z^2(t) - \frac{1}{t}z(t) \leq -\frac{1}{2}tb(t).$$

Since the function $\frac{1}{2t}z^2(t) - \frac{1}{t}z(t)$ attains the minimum $-\frac{1}{2t}$ for $z(t) > 0$ at $z(t) = 1$, the above inequality yields

$$z'(t) \leq \frac{1}{2t}\big(1 - t^2 b(t)\big).$$

From the given hypotheses, it follows that $z(t) < 0$ for large t, a contradiction. Hence (2.1) is oscillatory, and the theorem is proved. □

Remark 2.6.1 Theorem 2.6.1 generalise the assertion (iv) of Proposition 1.2.8. This theorem cannot be applied to the Euler equation (2.5) because $\int_\sigma^\infty \frac{b_0}{t^2}\,dt < \infty$. On the other hand, Theorem 2.6.2 can be applied to the Euler equation (2.5). Indeed, if $2c_0 - a_0b_0 + 2b_0 \le 0$ and $b_0 > 1$, then (2.5) is oscillatory. When $a(t) \equiv a, b(t) \equiv b$ and $c(t) \equiv c$ are constants, then the conditions of Theorem 2.6.2 are satisfied for $t > b^{-1/2}$ provided that $2c \le ab$. Hence Theorem 2.6.2 may be regarded as a generalisation of the assertion (iv) of Proposition 1.2.8.

Using the transformation

$$x(t) = z(t)\exp\left(-\frac{1}{3}\int_\sigma^t a(s)\,ds\right),$$

one can obtain (2.4) from (2.1).

Theorem 2.6.3 *Let $b(t) - \frac{a^2(t)}{3} - a'(t) \le 0$ and $\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{a''(t)}{3} > 0$. If*

$$\int_\sigma^\infty \left[\frac{2t^2a^3(t)}{27} - \frac{t^2a(t)b(t)}{3} + t^2c(t) - \frac{t^2a''(t)}{3} + tb(t) - \frac{ta^2(t)}{3} - ta'(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{3/2}\right]dt = \infty,$$

then (2.1) is oscillatory.

Remark 2.6.2 Theorem 2.6.3 generalises Proposition 1.2.8(iii) of Chap. 1. Indeed, $\frac{a^2}{3} - b > 0$ and $\frac{2a^3}{27} - \frac{ab}{3} + c > \frac{2}{3\sqrt{3}}(\frac{a^2}{3} - b)^{3/2}$ imply that Eq. (1.5) is oscillatory because

$$\frac{2a^3}{27} - \frac{ab}{3} + c - \frac{b}{t} - \frac{a^2}{3t} - \frac{2}{3\sqrt{3}}\left(\frac{1}{t^2} - b + \frac{1}{3}a^2\right)^{3/2} \to \frac{2a^3}{27} - \frac{ab}{3} + c - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0$$

as $t \to \infty$. It holds for the Euler equation (2.5). Indeed (2.5) is oscillatory if

$$b_0 - \frac{a_0^2}{3} + a_0 \le 0, \qquad \frac{2a_0^3}{27} - \frac{a_0b_0}{3} + c_0 - \frac{2a_0}{3} > 0$$

and

$$\frac{2a_0^3}{27} - \frac{a_0b_0}{3} + c_0 - \frac{2a_0}{3} - \frac{2}{3\sqrt{3}}\left(1 - b_0 + \frac{a_0^2}{3} - a_0\right)^{3/2} > 0$$

are satisfied.

Theorem 2.6.4 *Let* $b(t) - \frac{a^2(t)}{3} - a'(t) \le 0$ *and* $\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - b'(t) + \frac{2a(t)a'(t)}{3} + \frac{2a''(t)}{3} < 0$. *If*

$$\int_\sigma^\infty \bigg[-\frac{2t^2a^3(t)}{27} + \frac{t^2a(t)b(t)}{3} - t^2c(t) + t^2b'(t) - \frac{2t^2a(t)a'(t)}{3} - \frac{2t^2a''(t)}{3} + tb(t) - \frac{t^2a(t)}{3} - ta'(t) - \frac{2}{3\sqrt{3}t}\Big(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\Big)^{3/2}\bigg]\,dt = \infty,$$

then (2.1) *is oscillatory.*

Remark 2.6.3 Theorem 2.6.4 can be applied to the Euler equation (2.5). Indeed, (2.5) is oscillatory if

$$b_0 - \frac{a_0^2}{3} + a_0 < 0, \qquad \frac{2a_0^3}{27} - \frac{a_0b_0}{3} + c_0 + 2b_0 - \frac{2a_0^2}{3} + 4a_0 < 0$$

and

$$-\frac{2a_0^3}{27} + \frac{a_0b_0}{3} - \frac{a_0}{3} - b_0 - c_0 + \frac{a_0^2}{3} - \frac{2}{3\sqrt{3}}\Big(1 - b_0 + \frac{a_0^2}{3} - a_0\Big)^{3/2} > 0$$

are satisfied.

Theorem 2.6.5 *Assume that* (2.104) *holds and*

$$\int_\sigma^\infty \big(2c(t) - a(t)b(t) - b'(t)\big)\exp\Big(\int_\sigma^t a(s)\,ds\Big)\,dt = -\infty.$$

Then (2.1) *is oscillatory.*

Example 2.6.1 Consider

$$x''' - \frac{1}{t}x'' + \Big(2 + \frac{2}{t}\Big)x' - \Big(\frac{1}{t} + \frac{4}{t^2}\Big)x = 0, \quad t \ge 1. \tag{2.106}$$

Since $2c(t) - a(t)b(t) - b'(t) = -\frac{4}{t^2} < 0$ and $\int_1^\infty b(t)\,dt = \infty$, by Theorem 2.6.1, (2.106) has an oscillatory solution. Theorem 2.6.2 can be applied to (2.106). However, $\int_1^\infty (2c(t) - a(t)b(t) - b'(t))\exp(\int_\sigma^\infty a(s)\,ds)\,dt = -\int_1^\infty \frac{4}{t^3}\,dt > -\infty$ implies that Theorem 2.6.5 cannot be applied to (2.106).

Example 2.6.2 Consider

$$x''' - \frac{1}{t}x'' + \frac{2}{t^2}x' - \frac{20}{t^3}x = 0, \quad t \ge 1. \tag{2.107}$$

Clearly, $2c(t) - a(t)b(t) - b'(t) = -\frac{34}{t^3} < 0$, $t^2b(t) = 2 > 1$ and $\int_1^\infty (\frac{t^2b(t)-1}{t})\,dt = \int_1^\infty \frac{dt}{t} = \infty$. Hence by Theorem 2.6.2, (2.107) has an oscillatory solution. In particular, $x_1(t) = \cos(\sqrt{5}\log t)$ and $x_2(t) = \sin(\sqrt{5}\log t)$ are oscillatory solutions of (2.107). Theorem 2.6.1 cannot be applied to (2.107), because $\int_1^\infty \frac{dt}{t^2} < \infty$. Further, since $b(t) - \frac{a^2(t)}{3} - a'(t) = \frac{2}{3t^2} > 0$, Theorems 2.6.3 and 2.6.4 fail to hold for (2.107).

Example 2.6.3 Consider

$$x''' - \frac{1}{t}x'' + \frac{1}{t^2}x' - \frac{16}{t^3}x = 0, \quad t \geq 1. \tag{2.108}$$

Since $b(t) - \frac{a^2(t)}{3} - a'(t) = -\frac{1}{3t^2} < 0$,

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - b'(t) + \frac{2a(t)a'(t)}{3} + \frac{2}{3}a''(t) = \frac{-425}{27t^3} < 0$$

and

$$\begin{aligned}
&\int_1^\infty \left[-\frac{2t^2a^3(t)}{27} + \frac{t^2a(t)b(t)}{3} - t^2c(t) + t^2b'(t) - \frac{2t^2a(t)a'(t)}{3} - \frac{2t^2a''(t)}{3} \right.\\
&\qquad \left. + tb(t) - \frac{ta^2(t)}{3} - ta'(t) - \frac{2}{3\sqrt{3}t}\left(1 - t^2b(t) + \frac{1}{3}t^2a^2(t) + t^2a'(t)\right)^{3/2} \right] dt\\
&= \frac{400}{27}\int_1^\infty \frac{dt}{t} = \infty,
\end{aligned}$$

then, by Theorem 2.6.4, (2.108) is oscillatory. In particular, $x_1(t) = \cos(2\log t)$ and $x_2(t) = \sin(2\log t)$ are oscillatory solutions of (2.108). On the other hand,

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{a''(t)}{3} = \frac{-407}{27t^3} < 0$$

implies that Theorem 2.6.3 cannot be applied to (2.108).

Example 2.6.4 Consider

$$x''' - \frac{3}{t}x'' + \frac{6}{t^2}x' - \frac{33}{8t^3}x = 0, \quad t \geq 1. \tag{2.109}$$

Here $a_0 = -3$, $b_0 = 6$ and $c_0 = \frac{-33}{8}$. Clearly, $b_0 - \frac{a_0^2}{3} - a_0 = 0$, $\frac{2a_0^3}{27} - \frac{a_0b_0}{3} + c_0 - \frac{2a_0}{3} = \frac{15}{8} > 0$ and

$$\frac{2a_0^3}{27} - \frac{a_0b_0}{3} + c_0 - \frac{2a_0}{3} - \frac{2}{3\sqrt{3}}\left(1 - b_0 + \frac{a_0^2}{3} - a_0\right)^{3/2} = \frac{15}{8} - \frac{2}{3\sqrt{3}} > 0.$$

Then by Remark 2.6.2, (2.109) is oscillatory. It may be observed that $x_1(t) = t^{11/4}\cos(\frac{\sqrt{407}}{8}\log t)$ and $x_2(t) = t^{11/4}\sin(\frac{\sqrt{407}}{8}\log t)$ are oscillatory solutions of (2.109). Since

$$\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - b'(t) + \frac{2a(t)a'(t)}{3} + \frac{2}{3}a''(t) = \frac{1}{t^3}\left(6 - \frac{33}{8}\right) > 0,$$

Theorem 2.6.4 cannot be applied to Eq. (2.109).

Lemma 2.6.1 *If* (2.104) *holds, then* (2.1) *is of type* C_{II}.

Proof Suppose that $x(t)$ is a solution of (2.1) with $x(t_0) = x'(t_0) = 0$ and $x''(t_0) > 0$, $t_0 \geq \sigma$. From the continuity of $x''(t)$, it follows that there exists a $\delta > 0$ such that $x''(t) > 0$ for $t \in [t_0, t_0 + \delta)$. We claim that $x(t) > 0$ for $t > t_0$. If not, then $x(t_1) = 0$ for some $t_1 > t_0$. Now, multiplying (2.2) through by $x(t)$ and integrating the resulting identity from t_0 to t_1, we obtain

$$r(t_1)\big(x'(t_1)\big)^2 \leq \int_{t_0}^{t_1} \big(2p(t) - q'(t)\big)x^2(t)\,dt,$$

that is,

$$0 < r(t_1)\big(x'(t_1)\big)^2 \leq \int_{t_0}^{t_1} \big(2c(t) - a(t)b(t) - b'(t)\big)r(t)x^2(t)\,dt < 0,$$

a contradiction. Hence $x(t) > 0$ for $t > t_0$. This completes the proof of the theorem. □

Lemma 2.6.2 *If the second-order differential equation*

$$lu = u'' + a(t)u' + b(t)u = 0 \tag{2.110}$$

is disconjugate, then (2.1) *is of type* C_{II}.

Theorem 2.6.6 *If* $(2c(t) - a(t)b(t) - b'(t))r(t) \leq -d < 0$, *then all oscillatory solutions of* (2.1) *tend to zero as* $t \to \infty$.

Proof Since $(2c(t) - a(t)b(t) - b'(t))r(t) \leq -d < 0$, by Theorem 2.6.5, (2.1) is oscillatory. Let $u(t)$ be an oscillatory solution of (2.1). Then there exists a sequence $\langle t_n \rangle$ such that $t_n \to \infty$ as $n \to \infty$ and $u(t_n) = 0$. If

$$F\big[u(t)\big] = r(t)\big(u'(t)\big)^2 - 2r(t)u(t)u''(t) - q(t)u^2(t),$$

then

$$F'\big[u(t)\big] = \big(2p(t) - q'(t)\big)u^2(t) + r'(t)\big(u'(t)\big)^2$$

and hence

$$F\big[u(t)\big] = F\big[u(\sigma)\big] - \int_\sigma^t \big(q'(s) - 2p(s)\big)u^2(s)\,ds + \int_\sigma^t r'(s)\big(u'(s)\big)^2\,ds. \quad (2.111)$$

Clearly, $F[u(t)]$ is a decreasing function of t with $F[u(t_n)] \ge 0$. Put $t = t_n$ in (2.111) to get

$$0 \le F\big[u(t_n)\big] = F\big[u(\sigma)\big] - g(t_n)$$

and hence

$$g(t_n) \le F\big[u(\sigma)\big], \quad (2.112)$$

where

$$g(t) = \int_\sigma^t \big(q'(s) - 2p(s)\big)u^2(s)\,ds - \int_\sigma^t r'(s)\big(u'(s)\big)^2\,ds.$$

Clearly, $g(t)$ is a positive, continuous and increasing function of t. We claim that $g(t) \le F[u(\sigma)]$ for large t, say, for $t \ge T$. If not, then, for every T^*, there exists a $t^* > T^*$ such that $g(t^*) > F[u(\sigma)]$. We can find a $t_n > t^*$. Hence $g(t_n) \ge g(t^*)$. Consequently $g(t_n) > F[u(\sigma)]$, a contradiction to (2.112). Hence $g(t) \le F[u(\sigma)]$ for large t, that is,

$$\int_\sigma^t \big(q'(s) - 2p(s)\big)u^2(s)\,ds - \int_\sigma^t r'(s)\big(u'(s)\big)^2\,ds \le F\big[u(\sigma)\big]$$

for large t. Thus

$$\int_\sigma^t u^2(s)\,ds \le \frac{1}{d}\int_\sigma^t \big(q'(s) - 2p(s)\big)u^2(s)\,ds < \frac{F[u(\sigma)]}{d}$$

implies that

$$\int_\sigma^\infty u^2(s)\,ds < \infty,$$

that is, $u \in L^2[\sigma, \infty)$.

To complete the proof of the theorem, it is enough to show that $u'(t)$ is bounded (see Lemma 1.5.15). Since $F[u(t)]$ is a decreasing function of t, for $t \ge \sigma$, $F[u(t)] \le F[u(\sigma)]$. If $t_1 \in [\sigma, \infty)$ is a zero of $u''(t)$, then

$$\big(u'(t_1)\big)^2 \le \frac{[F[u(\sigma)] + q(t_1)u^2(t_1)]}{r(t_1)} < \infty.$$

Thus the values of $u'(t)$ are bounded at its maxima and minima. Hence $u'(t)$ is bounded. Consequently, by Lemma 1.5.15, $u(t) \to 0$ as $t \to \infty$. Hence the theorem is proved. □

Lemma 2.6.3 *If (2.110) is nonoscillatory, then every nonoscillatory solution $x(t)$ of (2.1) satisfies the property $x(t)x'(t) > 0$ for $t \ge T_x \ge \sigma$.*

Proof Let $x(t)$ be a nonoscillatory solution of (2.1). Hence $x(t) \neq 0$ for $t \geq t_x \geq \sigma$. Without any loss of generality, it may be assumed that $x(t) > 0$ for $t \geq t_x$. Since $x'(t)$ is a solution of the second-order nonhomogeneous equation

$$\big(r(t)z'\big)' + q(t)z = -p(t)x(t),$$

by Lemma 1.5.29, $x'(t)$ is nonoscillatory. Thus there exists a $T_x \geq t_x$ such that $x'(t) > 0$ or < 0 for $t \geq T_x$. Let $x'(t) < 0$ for $t \geq T_x$. Then from Eq. (2.2), $(r(t)x''(t))' \geq 0$. Hence $x''(t)$ is nonoscillatory for large t. If $x''(t) < 0$ for large t, then $x(t) < 0$ for large t, a contradiction. If $x''(t) > 0$ for large t, then from Eq. (2.1), $x'''(t) > 0$ for large t, which in turn implies that $x'(t) > 0$ for large t, a contradiction. Thus, $x'(t) > 0$ for $t \geq T_x$. This completes the proof of the lemma. □

Lemma 2.6.4 *Let* (2.110) *be nonoscillatory with a solution* $u(t)$ *such that* $u(t)u'(t) > 0$ *for* $t \geq t_0 \geq \sigma$. *If* $x(t)$ *is a solution of* (2.1) *with*

$$x(t_1) \geq 0, \qquad x'(t_1) = 0 \quad \textit{and} \quad x''(t_1) > 0, \quad t_1 > t_0,$$

then $x(t) > 0$, $x'(t) > 0$ *and* $x''(t) > 0$ *for* $t > t_1$ *and*

$$\lim_{t\to\infty} x(t) = \infty.$$

Further, if

$$\int_\sigma^\infty p(t)\,dt = -\infty,$$

then

$$\lim_{t\to\infty} x'(t) = \infty.$$

Proof From the continuity of $x''(t)$, it follows that there exists a $\delta > 0$ such that $x''(t) > 0$ for $t \in [t_1, t_1 + \delta)$. We claim that $x''(t) > 0$ for $t > t_1$. If not, then there exists a $t_2 > t_1$ such that $x''(t_2) = 0$ and $x''(t) > 0$ for $t \in [t_1, t_2)$. Consequently, $x'(t) > 0$ and $x(t) > 0$ for $t \in (t_1, t_2]$. Multiplying (2.2) through by $x'(t)$ and integrating the resulting identity from t_1 to t_2 one may obtain, by Lemma 1.5.1,

$$0 = -\int_{t_1}^{t_2} r(t)\big(x''(t)\big)^2\,dt + \int_{t_2}^{t_1} q(t)\big(x'(t)\big)^2\,dt + \int_{t_1}^{t_2} p(t)x(t)x'(t)\,dt < 0,$$

a contradiction. Hence $x''(t) > 0$ for $t \geq t_1$. Consequently $x'(t) > 0$ and $x(t) > 0$ for $t > t_1$ and hence $\lim_{t\to\infty} x(t) = \infty$. Further, let $\int_\sigma^\infty p(t)\,dt = -\infty$ holds. If $\lim_{t\to\infty} x'(t) = k$, $0 < k < \infty$, then integration of (2.2) from t_3 ($\geq t_1$) to t yields

$$\begin{aligned} r(t)x''(t) &= r(t_3)x''(t_3) - \int_{t_3}^t q(s)x'(s)\,ds - \int_{t_3}^t p(s)x(s)\,ds \\ &> r(t_3)x''(t_3) - x'(t)\int_{t_3}^t q(s)\,ds - x(t_3)\int_{t_3}^t p(s)\,ds. \end{aligned}$$

Since $\int_\sigma^\infty \frac{dt}{r(t)} = \infty$, we have $\int_\sigma^\infty q(t)\,dt < \infty$, because (2.110) is nonoscillatory. Thus from above, it follows that $r(t)x''(t) \to \infty$ as $t \to \infty$, which in turn implies that $x'(t) \to \infty$ as $t \to \infty$, a contradiction. Hence $k = \infty$. The lemma is proved. □

Theorem 2.6.7 *Suppose that* (2.110) *is nonoscillatory. If* (2.1) *has an oscillatory solution, then every nonoscillatory solution* $x(t)$ *of* (2.1) *satisfies*

$$x(t)x'(t) \neq 0, \qquad \operatorname{sgn} x(t) = \operatorname{sgn} x'(t) \quad \text{for } t \geq T_0 \geq \sigma$$

and

$$\lim_{t\to\infty} |x(t)| = \infty.$$

Proof Let $u(t)$ be a solution of (2.110). Hence $u(t)$ is nonoscillatory. If $u(t) > 0$ for $t \geq t_0^* \geq \sigma$, then $u'(t) > 0$ for $t \geq t_0 \geq t_0^*$ since $b(t) \geq 0$. Let $x(t)$ be a nonoscillatory solution of (2.1). Without any loss of generality, we may assume that $x(t) > 0$ for $t \geq T \geq t_0$. From Lemma 2.6.3, it follows that $x'(t) > 0$ for $t \geq T_0 \geq T$. Let $z(t)$ be an oscillatory solution of (2.1). We claim that $W(t) = x(t)z'(t) - x'(t)z(t)$ must vanish for some values of $t \in [T_0, \infty)$. If not, then $w(t) \neq 0$ for $t \in [T_0, \infty)$. Setting $y(t) = \frac{z(t)}{x(t)}$, $t \geq T_0$, we see that $y(t)$ is oscillatory. Hence $y'(t)$ is oscillatory. On the other hand, $y'(t) = \frac{W(t)}{x^2(t)}$ implies that $y'(t)$ is nonoscillatory. This contradiction proves our claim. Let $W(b) = 0$ for some $b \in [T_0, \infty)$. Hence it is possible to obtain constants c_1 and c_2, not both zero, such that

$$c_1 x(b) + c_2 z(b) = 0 \quad \text{and} \quad c_1 x'(b) + c_2 z'(b) = 0.$$

Clearly, $c_1 x''(b) + c_2 z''(b) \neq 0$. Indeed, $c_1 x''(b) + c_2 z''(b) = 0$ implies that $c_1 x(t) + c_2 z(t) \equiv 0$, a contradiction to the fact that $x(t)$ and $z(t)$ are linearly independent. Set $v(t) = c_1 x(t) + c_2 z(t)$. Then, $v(t)$ is a solution of (2.1) with $v(b) = 0$, $v'(b) = 0$ and $v''(b) \neq 0$. We may assume that $v''(b) > 0$. Hence $v(t) > 0$, $v'(t) > 0$ and $v''(t) > 0$ for $t > b$ and $\lim_{t\to\infty} v(t) = \infty$. Clearly, $c_1 \neq 0$ because $c_1 = 0$ implies that $c_2 \neq 0$ and hence $v(t) = c_2 z(t)$ is oscillatory, a contradiction. If $c_2 = 0$, then $c_1 \neq 0$ and $\lim_{t\to\infty} c_1 x(t) = \lim_{t\to\infty} v(t) = \infty$. Clearly, $c_1 < 0$ implies that $x(t) < 0$ for large t, a contradiction. Thus, $c_1 > 0$ and $\lim_{t\to\infty} x(t) = \infty$. Suppose that $c_2 \neq 0$. If $\lim_{t\to\infty} x(t)$ exists finitely, then $\lim_{t\to\infty} z(t) = \pm\infty$, depending upon $c_2 > 0$ or < 0, a contradiction to the oscillatory nature of $z(t)$. Hence $\lim_{t\to\infty} x(t) = \infty$. This completes the proof of the theorem. □

Theorem 2.6.8 *Suppose that* $(2c(t) - a(t)b(t) - b'(t))r(t) \leq -d < 0$, *and* (2.110) *is nonoscillatory. If* (2.1) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions* $u_1(t)$ *and* $u_2(t)$ *of* (2.1) *whose zeros separate in* $[\sigma, \infty)$ *and any oscillatory solution of* (2.1) *can be expressed as a linear combination of* $u_1(t)$ *and* $u_2(t)$.

Theorem 2.6.9 *If* $b(t) - \frac{a^2(t)}{3} - a'(t) \le 0$, $\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{a''(t)}{3} > 0$ *and* (2.1) *admits an oscillatory solution, then nonoscillatory solution of* (2.1) *forms a one-dimensional subspace of the solution space of* (2.1).

Theorem 2.6.10 *If* $b(t) - \frac{a^2(t)}{3} - a'(t) \le 0$, $\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + c(t) - \frac{a''(t)}{3} < 0$ *and* (2.1) *has an oscillatory solution, then there exist two linearly independent oscillatory solutions whose zeros separate each other in* $[\sigma, \infty)$ *and they form a basis for the solution space of* (2.1), *that is, oscillatory solutions form a two-dimensional subspace of the solution space of* (2.1).

In [10], Dolan established the following results:

Theorem 2.6.11 *If* (2.1) $\{$(2.3)$\}$ *is weakly oscillatory, then* (2.1) $\{$(2.3)$\}$ *is oscillatory.*

Theorem 2.6.12 *If* S $\{S^*\}$ *contains a nonoscillatory two-dimensional subspace, then* S $\{S^*\}$ *is either nonoscillatory or strongly oscillatory, where* S *and* S^* *are the solution spaces of* (2.1) *and* (2.3), *respectively.*

Dolan [10] raised the following two questions:

(i) Does there exist an example of a linear third-order differential equation with the property that every two-dimensional subspace of the solution space is weakly oscillatory?
(ii) Does there exist an example of a linear third-order differential equation such that the solution space S and S^* are strongly oscillatory?

In [24], Neumann has provided answers to the above two questions. He has shown that there does not exist a linear third-order differential equation of the form (2.1) with the property that every two-dimensional subspace of its solution space is weakly oscillatory. Further, he has constructed an example of a strongly oscillatory equation (2.1) whose adjoint (2.3) is also strongly oscillatory.

Now we give the following explicit sufficient condition for the strong oscillation of (2.1).

Clearly, (2.110) is equivalent to

$$\left(r(t)z'\right)' + q(t)z = 0, \tag{2.113}$$

where $r(t) = \exp(\int_\sigma^t a(s)\,ds)$ and $q(t) = r(t)b(t)$. Since $r(t) > 0$ and $r'(t) \le 0$, we have

$$\int_\sigma^\infty \frac{dt}{r(t)} = \infty.$$

Hence by Lemma 1.5.25, we have

$$\int_\sigma^\infty q(t)\,dt < \infty.$$

Theorem 2.6.13 *Let* (2.113) *be nonoscillatory. If* $-\infty < \liminf_{t\to\infty} a(t) \le 0$, $\frac{1}{3}a^2(t) - b(t) + a'(t) > 0$ *and*

$$\int_\sigma^\infty \left[\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \frac{a(t)a'(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\right] dt = \infty, \quad \sigma > 0 \tag{2.114}$$

then (2.1) *is strongly oscillatory.*

Proof If possible, let (2.1) admits a nonoscillatory solution $x(t)$. Then $x(t)x'(t) > 0$ for $t \ge t_0 > 0$ by Lemma 2.6.3. Clearly, $z(t) = \frac{x'(t)}{x(t)} > 0$, $t \ge t_0$, is a positive solution of the second-order Riccati equation (2.11). Integrating (2.11) from t_0 to t $(t > t_0)$, we obtain

$$z'(t) = z'(t_0) + \frac{3}{2}z^2(t_0) + a(t_0)z(t_0) - \frac{3}{2}z^2(t) - a(t)z(t) - \int_{t_0}^t \left[z^3(s) + a(s)z^2(s) + \big(b(s) - a'(s)\big)z(s) + c(s)\right] ds. \tag{2.115}$$

If

$$H\big(z(t)\big) = z^3(t) + a(t)z^2(t) + \big(b(t) - a'(t)\big)z(t) + c(t),$$

then $H(z(t))$ attains its minimum value for $z(t) > 0$ at

$$z(t) = \frac{1}{3}\left[-a(t) + \big(a^2(t) - 3b(t) + 3a'(t)\big)^{1/2}\right]$$

and the minimum value of $H(z(t))$ is given by

$$\frac{2}{27}a^3(t) - \frac{1}{3}a(t)b(t) + \frac{1}{3}a(t)a'(t) + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}.$$

Further, if

$$F\big(z(t)\big) = \frac{3}{2}z^2(t) + a(t)z(t),$$

then $F(z(t))$ attains its minimum value for $z(t) > 0$ at $z(t) = -\frac{a(t)}{3}$ and the minimum value of $F(z(t))$ is given by $-\frac{a^2(t)}{6}$. Hence (2.115) yields

$$z'(t) \le z'(t_0) + \frac{3}{2}z^2(t_0) + a(t_0)z(t_0) + \frac{a^2(t)}{6} - \int_{t_0}^t \left[\frac{2a^3(s)}{27} - \frac{a(s)b(s)}{3} + \frac{a(s)a'(s)}{3} + c(s) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(s)}{3} - b(s) + a'(s)\right)^{3/2}\right] ds.$$

From (2.114), it follows that $\lim_{t\to\infty} z'(t) = -\infty$. Thus $z(t) < 0$ for large t, a contradiction. The proof of the theorem is complete. □

Theorem 2.6.14 *Suppose that r and $q \in C^1((0,\infty), R)$ such that r is positive and q is nonnegative in $(0,\infty)$ and*

$$\int_1^\infty \frac{dt}{r(t)} = \infty.$$

If $L = \lim_{t\to\infty} r(t)\{[r(t)q(t)]^{-1/2}\}'$ exists and $L > 2$, then (2.113) is nonoscillatory.

Remark 2.6.4 Theorem 2.6.13 does not hold for (1.5), the third-order differential equation with constant coefficients, with $a < 0$, $b > 0$ and $c < 0$ because

$$L = \lim_{t\to\infty} e^{at}\{[e^{2at}b]^{-1/2}\}' > 2$$

if and only if $a^2 > 4b$. Further,

$$\frac{2a^3}{27} - \frac{ab}{3} - \frac{2}{3\sqrt{3}}\left(\frac{a^2}{3} - b\right)^{3/2} > 0,$$

if and only if $a^2 < 4b$. Thus, the condition (2.114) and $L > 0$ do not hold simultaneously.

Example 2.6.5 Consider

$$x''' - x'' + \left(\frac{1}{4.0000004} + \frac{1}{t}\right)x' - \frac{K}{t^2}x = 0, \quad t \ge 12, \tag{2.116}$$

where $K > 0$ is a constant. In this case $L = 2.0000001 > 2$. Hence, by Theorem 2.6.14, (2.113) is nonoscillatory. Further, the calculation shows that

$$\begin{aligned}&\frac{2a^3(t)}{27} - \frac{a(t)b(t)}{3} + \frac{a(t)a'(t)}{3} + c(t) - \frac{2}{3\sqrt{3}}\left(\frac{a^2(t)}{3} - b(t) + a'(t)\right)^{3/2}\\ &\quad = 0.00000005 + \frac{1}{3t} + \frac{0.1666664}{t} + \cdots - \frac{K}{t^2}\end{aligned}$$

and

$$\frac{a^2(t)}{3} - b(t) + a'(t) = \frac{1.0000004}{12.000001} - \frac{1}{t} > 0$$

for $t \ge 12$. Hence the conditions of Theorem 2.6.13 are satisfied. This, in turn, implies that (2.116) is strongly oscillatory.

Remark 2.6.5 A similar strong oscillation criteria for Eq. (2.1) can be obtained when $a(t) > 0$, $b(t) \geq 0$ and $c(t) < 0$. However, one has to assume the additional condition $\int_\sigma^\infty \frac{dt}{r(t)} = \infty$ in Theorem 2.6.13. One may observe in the rest six cases, viz., (i) $a(t) \geq 0, b(t) \leq 0, c(t) > 0$, (ii) $a(t) \leq 0, b(t) \leq 0, c(t) > 0$, (iii) $a(t) \leq 0, b(t) \leq 0, c(t) < 0$, (iv) $a(t) \geq 0, b(t) \leq 0, c(t) < 0$, (v) $a(t) \geq 0, b(t) \geq 0, c(t) > 0$ and (vi) $a(t) \leq 0, b(t) \geq 0, c(t) > 0$, Eq. (2.1) always admits a nonoscillatory solution. Hence strong oscillation of (2.1) cannot be studied in the above six cases.

2.7 Oscillation and Nonoscillation of Third-Order Linear Differential Equations of the Form $(r_2(t)(r_1(t)x')')' + r_3(t)x = 0$

In this section, we shall study the oscillatory and nonoscillatory behaviour of solutions of the third-order linear differential equation

$$\big(r_2(t)\big(r_1(t)x'\big)'\big)' + r_3(t)x = 0, \tag{2.117}$$

where $r_1 > 0, r_2 > 0$ and $r_3 > 0$ are twice continuously differentiable functions with fixed sign on $[\sigma, \infty)$, that is, $r_2(t) \neq 0, r_1(t) \neq 0$ and $r_3(t) \neq 0$ for $t \in [\sigma, \infty)$.

First of all, we present the property "Correspondence Principle" between the solutions of (2.117) and

$$\left(\frac{1}{r_3(t)}\big(r_2(t)y'\big)'\right)' + \frac{1}{r_1(t)}y = 0 \tag{2.118}$$

and

$$\left(r_1(t)\left(\frac{1}{r_3(t)}z'\right)'\right)' + \frac{1}{r_2(t)}z = 0, \tag{2.119}$$

obtained by means of an ordered cyclic permutation on the coefficients r_1, r_2 and r_3 of (2.117). Such a result plays an important role in the study of the classification of solutions as well as in the theory of disconjugacy.

Denote $\mathcal{S}_i, i = 1, 2, 3$, the linear space of solutions of (2.117), (2.118) and (2.119), respectively, and by $\mathcal{O}_i$ and $\mathcal{N}_i, i = 1, 2, 3$, the subsets of $\mathcal{S}_i$ given by oscillatory and nonoscillatory solutions of (2.117), (2.118) and (2.119), respectively.

Definition 2.7.1 The spaces $\mathcal{S}_i, i = 1, 2, 3$ is said to be isomorphic with respect to the oscillation, if there exists an isomorphism $\mathcal{L}_{ij}: \mathcal{S}_i \to \mathcal{S}_j$ which keeps the oscillatory properties of the solutions, that is,

$$g \in \mathcal{O}_i \Rightarrow \mathcal{L}_{ij}(g) \in \mathcal{O}_j, \qquad g \in \mathcal{N}_i \Rightarrow \mathcal{L}_{ij}(g) \in \mathcal{N}_j.$$

In this case, the operator $\mathcal{L}_{ij}$ is said to be an isomorphism of oscillation.

It is obvious to note that if the space $\mathcal{S}_i, \mathcal{S}_j, i, j = 1, 2, 3$ are isomorphic with respect to the oscillation, then $\mathcal{L}_{ij}$ maps $\mathcal{O}_i$ into $\mathcal{O}_j$ and $\mathcal{N}_i$ into $\mathcal{N}_j$. Hence the existence of an isomorphism of oscillation between the spaces $\mathcal{S}_i$ and $\mathcal{S}_j$ enables us to describe the oscillatory and nonoscillatory behaviour of the solutions of (2.117), (2.118) and (2.119) by the oscillatory and nonoscillatory behaviour of the solutions of (2.118), (2.119) and (2.117), respectively.

Lemma 2.7.1 *If $x(t)$ is a solution of* (2.117), *then $y = r_1(t)x'$ is a solution of* (2.118), *$z = r_2(t)(r_1(t)x')'$ is a solution of* (2.119), *and $w = \frac{1}{r_3(t)}(r_2(t)(r_1(t)x')')'$ is a solution of* (2.117).

Let Γ_1, Γ_2 and Γ_3 be the first-order differential operators

$$\begin{aligned} \Gamma_1 &: \mathcal{S}_1 \to \mathcal{S}_2 \quad \text{by } \Gamma_1 x(t) = r_1(t)x'(t) = y(t), \\ \Gamma_2 &: \mathcal{S}_2 \to \mathcal{S}_3 \quad \text{by } \Gamma_2 x(t) = r_2(t)y'(t) = z(t), \\ \Gamma_3 &: \mathcal{S}_3 \to \mathcal{S}_1 \quad \text{by } \Gamma_3 x(t) = \frac{1}{r_3(t)} z'(t). \end{aligned}$$

Then we have the following property.

Correspondence Principle The operators $\Gamma_i, i = 1, 2, 3$ are isomorphisms of oscillations.

As a consequence of the "Correspondence Principle", the linear spaces $\mathcal{S}_i, i = 1, 2, 3$ have the same structures with respect to oscillation or nonoscillation. Hence the oscillation (nonoscillation) criteria for Eqs. (2.118)–(2.119) may immediately be transformed into oscillation (nonoscillation) criteria for Eq. (2.117). As we shall see, this fact enables us to obtain easily some new results about oscillation or nonoscillation of (2.117) that may be hard to prove directly.

Lemma 2.7.2 *The following hold*:

(a) *If* (2.117) *is oscillatory and of type C_I, then any solution of* (2.117) *with a zero is oscillatory.*
(b) *If* (2.117) *is of type C_{II}, then any solution of* (2.117) *with a double zero is nonoscillatory.*
(c) *If* (2.117) *is both of type C_I and C_{II}, then* (2.117) *is nonoscillatory.*
(d) *If* (2.117) *is of type C_I, then it admits a nonoscillatory solution.*
(e) *Equation* (2.117) *is of type C_I if and only if its adjoint equation*

$$\big(r_1(t)\big(r_2(t)u'\big)'\big)' - r_3(t)u = 0 \tag{2.120}$$

is of type C_{II}. Similarly, if (2.117) *is of type C_{II}, then* (2.120) *is of type C_I.*
(f) *Equation* (2.117) *is oscillatory if and only if its adjoint equation* (2.120) *is oscillatory.*

We note that, if x_1 and x_2 are two linearly independent solutions of (2.117), then $u(t) = r_1(t)(x_1'(t)x_2(t) - x_1(t)x_2'(t))$ is a solution of (2.120).

Lemma 2.7.3 *If* $r_3(t) > 0$, *then* (2.117) *is of type* C_I.

Proof Assume that (2.117) does not belongs to C_I. The for some $a > \sigma$, there is a solution $x(t)$ of (2.117) with $x(a) = 0, x'(a) = 0$ and $x''(a) > 0$ vanishes in $[\sigma, a)$. Let $b, \sigma < b < a$ be such that $x(b) = 0, x(t) \neq 0$ for $t \in (b, a)$. Then there exists a $c, b < c < a$ such that $x'(c) = 0, x'(t) \neq 0$ for $t \in (c, a)$. By integrating (2.117) from t to a, we obtain

$$r_2(a).\big(r_1(t)x'(t)\big)'\big|_{t=a} + \int_t^a r_3(s)x(s)\,ds = r_2(t)\big(r_1(t)x'(t)\big)'$$

or

$$\frac{1}{r_2(t)}L_2x(a) + \frac{1}{r_2(t)}\int_t^a r_3(s)x(s)\,ds = \big(r_1(t)x'(t)\big)', \tag{2.121}$$

where $L_0x(t) = x(t)$, $L_1x(t) = r_1(t)x'(t)$, $L_2x(t) = r_2(t)(r_1(t)x'(t))'$ and $L_3x(t) = (r_2(t)L_2x(t))'$. Integrating (2.121) from t to a, we have

$$L_2x(a)\int_t^a \frac{1}{r_2(s)}\,ds + \int_t^a \frac{1}{r_2(s)}\int_s^a r_3(u)x(u)\,du\,ds = r_1(a)x'(a) - r_1(t)x'(t).$$

For $t = c$, we get

$$L_2x(a)\int_c^a \frac{1}{r_2(s)}\,ds + \int_c^a \frac{1}{r_2(s)}\int_s^a r_3(u)x(u)\,du\,ds = r_1(a)x'(a) - r_1(c)x'(c). \tag{2.122}$$

Since $x'(a) = 0$ and $x''(a) > 0$, we have

$$\begin{aligned} L_2x(a) = r_2(a)\big(r_1(t)x'(t)\big)'\big|_{t=a} &= r_2(a).\big(r_1(a)x''(a) + r_1'(a)x'(a)\big) \\ &= r_2(a)r_1(a)x''(a) > 0. \end{aligned}$$

Then, taking into account that $x(t)$, $r_2(t)$ and $r_3(t)$ are positive functions in (c, a), from (2.122), we obtain a contradiction. The lemma is proved. □

If functions r_1, r_2 and r_3 are positive functions on $[\sigma, \infty)$, then Eqs. (2.118) and (2.119) are of type C_I.

Lemma 2.7.3 shows that Eq. (2.117) can never be strongly oscillatory. This is because (2.117) is of type C_I and hence admits a nonoscillatory solution.

Lemma 2.7.4 *The equation*

$$\big(r_2(t)\big(r_1(t)x'\big)'\big)' - r_3(t)x = 0 \tag{2.123}$$

is of type C_{II}.

Theorem 2.7.1 *Let (2.117) be oscillatory. Then the space $\mathcal{S}_1$ contains a two-dimensional subspace $\mathcal{W}_1$ of oscillatory solutions. In addition, there exists a solution $x \in \mathcal{S}_1$ such that $x \in \mathcal{O}_1$ and $x \notin \mathcal{W}_1$.*

The above conclusion remains true for (2.123).

Proof Clearly (2.117) is of type C_I and so its adjoint equation (2.120) is of type C_{II}. From Lemma 2.7.2(f), it follows that (2.120) is oscillatory. By Dolan [10], the linear space of solutions of the adjoint equation (2.120) contains a two-dimensional subspace $\mathcal{W}_1$ of oscillatory solutions. Hence the assertion follows by taking into account that the adjoint of (2.120) is (2.117).

Now, we consider (2.123). Clearly the adjoint of (2.123) given by

$$\big(r_1(t)\big(r_2(t)u'\big)'\big)' + r_3(t)u = 0 \tag{2.124}$$

is of type C_I. Hence from Lemma 2.7.2(f), Eq. (2.124) is oscillatory. By the same argument as above, the first assertion follows.

In order to complete the proof, it is enough to show that there exists $x \in \mathcal{S}_1$ such that $x \in \mathcal{O}_1$ and $x \notin \mathcal{W}_1$ for (2.117). Let $x_1 = x_1(t)$ and $x_2 = x_2(t)$ be a basis for the subspace $\mathcal{W}_1$, and let $\overline{x} = \overline{x}(t)$ be a nonoscillatory solution of (2.117), $\overline{x}(t) \neq 0$ for $t \geq a \geq \sigma$, and let $b \in (a, \infty)$ be such that $x_1(b) \neq 0$. Now, we consider the solution $x_3(t)$ of (2.117) is given by

$$x_3(t) = x_1(t) + \frac{-x_1(b)}{\overline{x}(b)}\overline{x}(t). \tag{2.125}$$

Since $x_3(b) = 0$, $x_3(t)$ is oscillatory. Let us prove, by contradiction, that $x_3(t) \notin \mathcal{W}_1$. If $x_3(t) \in \mathcal{W}_1$, then there exist λ_1 and $\lambda_2 \in R$, $\lambda_1^2 + \lambda_2^2 = 1$, such that $x_3(t) = \lambda_1 x_1(t) + \lambda_2 + x_2(t)$. Then from (2.125), we obtain

$$(\lambda_1 - 1)x_1(t) + \lambda_2 x_2(t) = \frac{-x_1(b)}{\overline{x}(b)}\overline{x}(t), \tag{2.126}$$

which is a contradiction because the right-hand side of (2.126) is a nonoscillatory solution of (2.117) and the left-hand side is an oscillatory solution. The proof is now complete. □

Remark 2.7.1 Consider (2.123). Here, the set of oscillatory solutions $\mathcal{O}_1$ may be exactly a two-dimensional subspace of $\mathcal{S}_1$. That is, it may happen that there do not exist solutions $x(t)$ of (2.117) such that $x \in \mathcal{O}_1$, $x \notin \mathcal{W}_1$, as equations with constant coefficients show.

Theorem 2.7.2 *If (2.117) is oscillatory, then $\mathcal{S}_1$ cannot contain a two-dimensional subspace of nonoscillatory solutions.*

Remark 2.7.2 Lemmas's 2.7.1–2.7.4, and Theorems 2.7.1–2.7.2 are still valid for Eqs. (2.118) and (2.119).

Theorem 2.7.3 *If*

$$\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_1(s)} \int_\sigma^s \frac{1}{r_2(u)} \, du\, ds\, dt < \infty, \tag{2.127}$$

then (2.117) *is nonoscillatory.*

Proof Let $a > \sigma$ be such that

$$\int_a^\infty r_3(t) \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds\, dt < 1,$$

and let $x(t)$ be a solution of (2.117) such that $x(a) = 0$, $L_1 x(a) = 0$ and $L_2 x(a) = 1$, where $L_1 x(t) = r_1(t)x'(t)$ and $L_2 x(t) = r_2(t)(r_1(t)x'(t))'$. Integrating (2.117) three times from a to t, we obtain

$$\begin{aligned} x(t) = x(a) + L_1 x(a) \int_a^t \frac{1}{r_1(s)} ds + L_2 x(a) \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds \\ - \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \int_a^u r_3(v) x(v)\, dv\, du\, ds. \end{aligned} \tag{2.128}$$

Then from (2.128), we have

$$x(t) = \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds - \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \int_a^u r_3(v) x(v)\, dv\, du\, ds. \tag{2.129}$$

Assume that (2.117) is oscillatory. Since (2.117) is of type C_I, $x(t)$ is oscillatory. Let $b > a$ be such that $x(b) = 0$ and $x(t) > 0$ for $t \in (a, b)$. Hence from (2.129), we obtain for $t \in (a, b)$ that

$$x(t) \le \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds. \tag{2.130}$$

Since $x(b) = 0$, from (2.129) and (2.130), we have

$$\begin{aligned} 0 = x(b) &= \int_a^b \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds - \int_a^b \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \int_a^u r_3(v) x(v)\, dv\, du\, ds \\ &= \int_a^b \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \left(1 - \int_a^u r_3(v) x(v)\, dv\right) du\, ds \\ &\ge \int_a^b \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \left(1 - \int_a^u r_3(v) \int_a^v \frac{1}{r_1(\theta)} \int_a^\theta \frac{1}{r_2(\tau)} d\tau\, d\theta\, dv\right) du\, ds \end{aligned}$$

which is a contradiction, since

$$\int_a^b r_3(t) \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds\, dt < \int_a^\infty r_3(t) \int_a^t \frac{1}{r_1(s)} \int_a^s \frac{1}{r_2(u)} \, du\, ds\, dt < 1.$$

Hence (2.117) is nonoscillatory. The theorem is proved. □

Remark 2.7.3 Let $r_1(t) \equiv 1$ and $r_2(t) \equiv 1$ and $r_3(t) = c(t)$. Then (2.117) reduces to (2.6). As an application of Theorem 2.7.3, we see that if $\int^\infty t^2 c(t)\,dt < \infty$, then (2.6) is nonoscillatory, which we may restate as

Corollary 2.7.1 *If Eq.* (2.6) *is oscillatory, then* $\int_\sigma^\infty t^2 c(t)\,dt = \infty$.

Remark 2.7.4 The converse of Corollary 2.7.1 is not necessarily true, that is, $\int^\infty t^2 c(t)\,dt = \infty$ need not imply that (2.6) is oscillatory. The following example strengthens this remark.

Example 2.7.1 Let $c(t) = \frac{1}{t^2}$. then $\int_t^\infty c(s)\,ds = -\frac{1}{s}$. Then the equation $z'' - \frac{3}{2t} z = 0$ is nonoscillatory. Thus, by Theorem 2.1.9, the equation $x''' + \frac{1}{t^2} x = 0$, $t \geq 2$ is nonoscillatory. Observe that $\int^\infty t^2 c(t)\,dt = \infty$.

On the other hand, the equation $x''' + \frac{3}{t^2} x = 0$, $t \geq 2$ is oscillatory by Theorem 2.5.7, and the property $\int^\infty t^2 c(t)\,dt = \infty$ holds.

Theorem 2.7.4 *If*

$$\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s \frac{1}{r_1(u)}\,du\,ds\,dt < \infty, \tag{2.131}$$

then (2.123) *is nonoscillatory.*

Applying the "Correspondence Principle" to (2.117), we obtain the following theorem:

Theorem 2.7.5 (i) *Equation* (2.117) *is nonoscillatory in the case any of the following conditions is satisfied*:

(a) $\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t r_3(s) \int_\sigma^s \frac{1}{r_1(u)}\,du\,ds\,dt < \infty$,
(b) $\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s r_3(u)\,du\,ds\,dt < \infty$,

(ii) *Eq.* (2.123) *is nonoscillatory in the case any of the following is satisfied*:

(a) $\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t \frac{1}{r_1(s)} \int_\sigma^s r_3(u)\,du\,ds\,dt < \infty$,
(b) $\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t r_3(s) \int_\sigma^s \frac{1}{r_2(u)}\,du\,ds\,dt < \infty$.

Definition 2.7.2 A nonoscillatory solution $x(t)$ of (2.117) is said to be a Kneser solution or a completely monotone solution if there exists a $t_0 \geq \sigma$ such that

$$(-1)^i x(t) L_i x(t) > 0, \quad i = 0, 1, 2,\ t \geq t_0, \tag{2.132}$$

where $L_i x(t)$ denotes the ith quasiderivative of x defined as

$$\left.\begin{aligned} L_0x(t) &= x(t),\\ L_1x(t) &= r_1(t)\big(L_0x(t)\big)' = r_1(t)x'(t),\\ L_2x(t) &= r_2(t)\frac{d}{dt}\big(L_1x(t)\big)' = r_2(t)\big(r_1(t)x'(t)\big)',\\ L_3x(t) &= \frac{d}{dt}\big(L_2x(t)\big) = \big(r_2(t)\big(r_1(t)x'(t)\big)'\big)'. \end{aligned}\right\} \tag{2.133}$$

Assuming the conditions

$$\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty, \tag{2.134}$$

Kiguradze [22] obtained the following classical result:

Theorem 2.7.6 (i) *Every nonoscillatory solution of* (2.117) *is either a Kneser solution or satisfies*

$$x(t)L_i x(t) > 0, \quad i = 0, 1, 2 \tag{2.135}$$

for $t \ge t_0 \ge \sigma$.

(ii) *Equation* (2.123) *does not have any Kneser solution.*

Corollary 2.7.2 *The set of all Kneser solutions of* (2.117) *is a nonempty subset of* $\mathcal{S}_1$ *in case any of the following conditions are satisfied*:

(i) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty$, $\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_1(s)} \int_\sigma^s \frac{1}{r_2(u)}\,du\,ds\,dt < \infty$.
(ii) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty r_3(t)\,dt = \infty$, $\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s r_3(u)\,du\,ds\,dt < \infty$.
(iii) $\int_\sigma^\infty r_3(t)\,dt = \int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty$, $\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t r_3(s) \int_\sigma^s \frac{1}{r_1(u)}\,du\,ds\,dt < \infty$.

Theorem 2.7.7 *If*

$$\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty, \qquad \int_\sigma^\infty r_3(t)\int_\sigma^t \frac{1}{r_1(s)}\,ds\,dt = \infty,$$

then (2.117) *is oscillatory.*

Proof If (2.117) is nonoscillatory, then by [8], there exists a nonoscillatory solution $x(t)$ of (2.117) such that $x(t)$ satisfies (2.135) for large t. Without any loss of generality, we may assume that $x(t) > 0$, $L_1x(t) > 0$ and $L_2x(t) > 0$ for some $t \ge t_0 \ge \sigma$, where $L_ix(t)$ is defined in (2.133). Integrating (2.117) from t_0 to t, we obtain

$$r_2(t)\big(r_1(t)x'(t)\big)' - r_2(t)\big(r_1(t)x'(t)\big)'\big|_{t=t_0} + \int_{t_0}^t r_3(s)x(s)\,ds = 0.$$

Since $L_2x(t) > 0$, we have

$$r_2(t)\big(r_1(t)x'(t)\big)'\big|_{t=t_0} > \int_{t_0}^{t} r_3(s)x(s)\,ds. \tag{2.136}$$

Since $L_1x(t)$ is a positive increasing function, we have

$$x(t) > x(t_0) + r_1(t_0)x'(t_0)\int_{t_0}^{t}\frac{1}{r_1(u)}\,du.$$

Hence (2.136) gives

$$\begin{aligned} r_2(t)\big(r_1(t)x'(t)\big)'\big|_{t=t_0} &> \int_{t_0}^{t} r_3(s)\left(x(t_0) + r_1(t_0)x'(t_0)\int_{t_0}^{s}\frac{1}{r_1(u)}\,du\right)ds \\ &> r_1(t_0)x'(t_0)\int_{t_0}^{t} r_3(s)\int_{t_0}^{s}\frac{1}{r_1(u)}\,du\,ds. \end{aligned}$$

The above inequality yields a contradiction by taking the limit as $t \to \infty$. The theorem is proved. □

Theorem 2.7.8 *If*

$$\int_{\sigma}^{\infty}\frac{1}{r_2(t)}\,dt = \int_{\sigma}^{\infty}\frac{1}{r_1(t)}\,dt = \infty \quad \textit{and} \quad \int_{\sigma}^{\infty} r_3(t)\int_{\sigma}^{t}\frac{1}{r_2(s)}\,ds\,dt = \infty,$$

then (2.123) *is oscillatory.*

Applying the "Correspondence Principle" to (2.117), we obtain the following:

Theorem 2.7.9 (i) *Equation* (2.117) *is oscillatory in the case any one of the following conditions is satisfied*:

(a) $\int_{\sigma}^{\infty} r_3(t)\,dt = \int_{\sigma}^{\infty}\frac{1}{r_2(t)}\,dt = \int_{\sigma}^{\infty}\frac{1}{r_1(t)}\int_{\sigma}^{t}\frac{1}{r_2(s)}\,ds\,dt = \infty$,
(b) $\int_{\sigma}^{\infty}\frac{1}{r_1(t)}\,dt = \int_{\sigma}^{\infty} r_3(t)\,dt = \int_{\sigma}^{\infty}\frac{1}{r_2(t)}\int_{\sigma}^{t} r_3(s)\,ds\,dt = \infty$.

(ii) *Equation* (2.123) *is oscillatory in the case any one of the following conditions is satisfied*:

(a) $\int_{\sigma}^{\infty} r_3(t)\,dt = \int_{\sigma}^{\infty}\frac{1}{r_2(t)}\,dt = \int_{\sigma}^{\infty}\frac{1}{r_1(t)}\int_{\sigma}^{t} r_3(s)\,ds\,dt = \infty$,
(b) $\int_{\sigma}^{\infty}\frac{1}{r_1(t)}\,dt = \int_{\sigma}^{\infty} r_3(t)\,dt = \int_{\sigma}^{\infty}\frac{1}{r_2(t)}\int_{\sigma}^{t}\frac{1}{r_1(s)}\,ds\,dt = \infty$.

Corollary 2.7.3 *If any one of the following conditions*:

(i) $\int_{\sigma}^{\infty}\frac{1}{r_2(t)}\,dt = \int_{\sigma}^{\infty}\frac{1}{r_1(t)}\,dt = \int_{\sigma}^{\infty} r_3(t)\int_{\sigma}^{t}\frac{1}{r_1(s)}\,ds\,dt = \infty$,
(ii) $\int_{\sigma}^{\infty} r_3(t)\,dt = \int_{\sigma}^{\infty}\frac{1}{r_2(t)}\,dt = \int_{\sigma}^{\infty}\frac{1}{r_1(t)}\int_{\sigma}^{t}\frac{1}{r_2(s)}\,ds\,dt = \infty$,
(iii) $\int_{\sigma}^{\infty}\frac{1}{r_1(t)}\,dt = \int_{\sigma}^{\infty} r_3(t)\,dt = \int_{\sigma}^{\infty}\frac{1}{r_2(t)}\int_{\sigma}^{t} r_3(s)\,ds\,dt = \infty$

is satisfied, *then*

(a) *any solution* $x(t)$ *of* (2.117) *which vanishes* (*at some* $t_0 \geq \sigma$)

$$x(t_0) = 0 \quad or \quad x'(t_0) = 0 \quad or \quad \big(r_1(t)x'(t)\big)'\big|_{t=t_0} = 0$$

is oscillatory;

(b) *every nonoscillatory solution of* (2.117) *is a Kneser solution on* $[\sigma, \infty)$.

Proof (a) From Theorems 2.7.7 and 2.7.9, it follows that Eqs. (2.117)–(2.119) are oscillatory. Let $x(t)$ be a solution of (2.117) such that $x(t_0) = 0$. Since (2.117) is of type C_I, by Lemma 2.7.2(a), $x(t)$ is oscillatory.

Assume that $x(t)$ is a solution of (2.117) with $x'(t_0) = 0$. Then $y(t) = r_1(t)x'(t)$ is a solution of (2.118). Clearly, $y(t_0) = 0$. Since $r_1(t) > 0$, (2.118) is of type C_I. Hence the "Correspondence Principle" implies that (2.118) is oscillatory. This in turn implies that $y(t)$ is oscillatory. Applying the "Correspondence Principle", once again we obtain the required result. A similar argument may be applied when $(r_1(t)x'(t))'|_{t=t_0} = 0$ holds.

(b) Assume the condition (i) and let $x(t)$ be a nonoscillatory solution of (2.117). Since (2.117) is oscillatory, by [8], there exists a $t_0 \geq \sigma$ such that $x(t)$ is the Kneser solution on $[t_0, \infty)$. If, by contradiction, $x(t_1) = 0$, $t_0 \geq t_1 \geq \sigma$, then $x(t)$ is oscillatory, a contradiction. Hence $x(t)$ is a Kneser solution on $[\sigma, \infty)$.

If the conditions (ii) or (iii) is valid, then the assertion follows from the "Correspondence Principle" and by using arguments similar to that above. The proof is complete. □

It is well known that (2.117) is oscillatory, if and only if its adjoint (2.120) is oscillatory. The same situation does not occur for Eqs. (2.117) and (2.123) even if $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty$, as the following example shows:

Example 2.7.2 Let $\epsilon \in (0, 1)$ and $T > 1$. Consider the equations

$$\left(t \ln t \left(\frac{x'(t)}{\ln t}\right)'\right)' + \frac{1}{t^2(\ln t)^{1+\epsilon}}x(t) = 0, \quad t \in [T, \infty), \tag{2.137}$$

and

$$\left(t \ln t \left(\frac{x'(t)}{\ln t}\right)'\right)' - \frac{1}{t^2(\ln t)^{1+\epsilon}}x(t) = 0, \quad t \in [T, \infty). \tag{2.138}$$

By Theorem 2.7.7, (2.137) is oscillatory, and by Theorem 2.7.4, (2.138) is nonoscillatory.

Lemma 2.7.1 shows that (2.117) is closely related to (2.118) and (2.119) obtained by means of an ordered cyclic permutations of the functions r_2, r_1 and r_3. Similarly, if $u(t)$ is a solution of (2.120), then $v(t) = r_1(t)u'(t)$ is a solution of

$$\left(\frac{1}{r_3(t)}\big(r_1(t)v'\big)'\right)' - \frac{1}{r_2(t)}v = 0, \tag{2.139}$$

and $w(t)=r_2(t)(r_1(t)v'(t))'$ is a solution of

$$\left(r_2(t)\left(\frac{1}{r_3(t)}w'\right)'\right)' - \frac{1}{r_1(t)}w = 0. \tag{2.140}$$

Denote $\aleph$ and $\aleph(a)$ as the set of all nontrivial nonoscillatory solutions of (2.117) and its adjoint equation (2.120). If $x(t)$ is nonoscillatory, then $L_1x(t)$, $L_2x(t)$ and $L_3x(t)$ are nonoscillatory. In view of this, one can divide the set $\aleph$ into the following four classes:

(i) $\aleph_0=\{x\in\aleph, \exists T_x;\ x(t)L_1x(t)<0, x(t)L_2x(t)>0 \text{ for } t\geq T_x\}$,
(ii) $\aleph_1=\{x\in\aleph, \exists T_x;\ x(t)L_1x(t)>0, x(t)L_2x(t)<0 \text{ for } t\geq T_x\}$,
(iii) $\aleph_2=\{x\in\aleph, \exists T_x;\ x(t)L_1x(t)>0, x(t)L_2x(t)>0 \text{ for } t\geq T_x\}$,
(iv) $\aleph_3=\{x\in\aleph, \exists T_x;\ x(t)L_1x(t)<0, x(t)L_2x(t)<0 \text{ for } t\geq T_x\}$,

and the set $\aleph(a)$ into the following four classes:

(i) $\aleph_0(a)=\{u\in\aleph(a), \exists T_u;\ u(t)L_1u(t)<0, u(t)L_2u(t)>0 \text{ for } t\geq T_u\}$,
(ii) $\aleph_1(a)=\{u\in\aleph(a), \exists T_u;\ u(t)L_1u(t)>0, u(t)L_2u(t)<0 \text{ for } t\geq T_u\}$,
(iii) $\aleph_2(a)=\{u\in\aleph(a), \exists T_u;\ u(t)L_1u(t)<0, u(t)L_2u(t)<0 \text{ for } t\geq T_u\}$,
(iv) $\aleph_3(a)=\{u\in\aleph(a), \exists T_u;\ u(t)L_1u(t)>0, u(t)L_2u(t)>0 \text{ for } t\geq T_u\}$.

Obviously, if every nonoscillatory solution $x(t)$ of (2.117) satisfies the property $\lim_{t\to\infty}|x^{(i)}(t)|=0, i=0,1,2$, then $x(t)$ belongs to the class $\aleph_0$. Similarly, if $u\in\aleph(a)$ and satisfies the property $\lim_{t\to\infty}|u^{(i)}(t)|=\infty, i=0,1,2$, then u belongs to the class $\aleph_3(a)$. This means that if (2.117) has property A, then $\aleph=\aleph_0$ and if (2.120) has property B, then $\aleph(a)=\aleph_3(a)$. In addition, if $x\in\aleph_0$, then the quasiderivatives $L_ix(t)$, $i=0,1,2,3$ have eventually an alternate sign which we call a Kneser solution. Similarly, if $u\in\aleph_3(a)$, then the quasiderivatives $L_iu(t)$, $i=0,1,2,3$ have eventually same sign, which we call a strongly monotone solution. Their existence is ensured by the following lemma.

Lemma 2.7.5 *Equation* (2.117) *has always a Kneser solution, and* (2.120) *has always a strongly monotone solution.*

Remark 2.7.5 It is easy to see that if $x\in\aleph_0$, then $x(t)L_1x(t)<0$ and $x(t)L_2x(t)>0$ not only eventually but also for all $t\geq\sigma$. Indeed, let $x(t)>0, L_1x(t)<0, L_2x(t)>0$ for $t\geq T\geq\sigma$ and suppose that there exists $t_1\in[\sigma,T)$ such that $x'(t_1)=0$ and $x(t)>0$ for $t\in(t_1,T)$. Then $L_2x(t)$ is decreasing on (t_1,T). Because $L_2x(T)>0$, we have $L_2x(t)>0$ for $t\in(t_1,T)$, which implies that $L_1x(t)$ is decreasing on (t_1,T). Again, since $L_1x(T)<0$, we obtain $L_1x(t_1)=r_1(t_1)x'(t_1)<0$, which is a contradiction. Thus $x(t)L_1x(t)<0$ for $t\geq\sigma$. By a similar argument, Lemma 2.7.1 implies that $x(t)L_2x(t)>0$ for $t\geq\sigma$.

Lemma 2.7.6 *Let x and y be two linearly independent solutions of* (2.117) [(2.120)], *then*

$$w=L_1x(t)\cdot y(t)-x(t)\cdot L_1y(t) \tag{2.141}$$

is a solution of (2.120) [(2.117)] *and its quasiderivatives satisfy*

$$L_1 w(t) = L_2 x(t) \cdot y(t) - x(t) \cdot L_2 y(t)$$

and

$$L_2 w(t) = L_2 x(t) \cdot L_1 y(t) - L_1 x(t) \cdot L_2 y(t).$$

Lemma 2.7.7 *The following conditions are equivalent*:

(i) $\aleph = \aleph_0$
(ii) $\aleph(a) = \aleph_3(a)$.

Proof First, we prove that (ii) ⇒ (i). By Lemma 2.7.5, we have $\aleph_0 \neq \phi$. Assume that there exists $j \in \{1, 2, 3\}$ such that $\aleph_j \neq \phi$. Let $x \in \aleph_0$ and $y \in \aleph_j$. Without any loss of generality, we may assume that $x(t) > 0$ and $y(t) > 0$, for large t. Then the function w defined by (2.141) is a solution of (2.120) and satisfies the properties

$$w(t) < 0, \qquad L_1 w(t) > 0 \quad \text{if } j = 1;$$
$$w(t) < 0, \qquad L_2 w(t) > 0 \quad \text{if } j = 2;$$

and

$$L_1 w(t) > 0, \qquad L_2 w(t) < 0 \quad \text{if } j = 3$$

for large t, which contradicts the fact that all nonoscillatory solutions of (2.120) are strongly monotone.

The claim (i) ⇒ (ii) can be proved by using a similar argument as given in the first part. □

Lemma 2.7.8 *If there exists* $x \in \aleph_0$ *such that*

$$\lim_{t\to\infty} L_i x(t) = 0, \quad i = 0, 1, 2, \tag{2.142}$$

then

$$\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s \frac{1}{r_1(\tau)}\, d\tau\, ds\, dt = \infty.$$

Proof Suppose that

$$\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s \frac{1}{r_1(\tau)}\, d\tau\, ds\, dt < \infty.$$

Then there exists a $t_0 > \sigma$ such that

$$\int_{t_0}^\infty r_3(t) \int_{t_0}^t \frac{1}{r_2(s)} \int_{t_0}^s \frac{1}{r_1(\tau)}\, d\tau\, ds\, dt < \frac{1}{2} \tag{2.143}$$

holds. Let $x(t)$ be an eventually positive solution of (2.117), which belongs to the class $\aleph_0$ such that $x(t)$ satisfies (2.142) for $t \geq t_0$. From repeated integration of (2.117) from t to ∞, $t \geq t_0$, we obtain

$$x(t) = \int_t^\infty \frac{1}{r_1(s)} \int_s^\infty \frac{1}{r_2(\tau)} \int_\tau^\infty r_3(\theta) x(\theta)\, d\theta\, d\tau\, ds$$
$$\leq x(t) \int_t^\infty \frac{1}{r_1(s)} \int_s^\infty \frac{1}{r_2(\tau)} \int_\tau^\infty r_3(\theta)\, d\theta\, d\tau\, ds.$$

Thus,

$$1 \leq \int_t^\infty \frac{1}{r_1(s)} \int_s^\infty \frac{1}{r_2(\tau)} \int_\tau^\infty r_3(\theta)\, d\theta\, d\tau\, ds.$$

Then, by interchanging the order of integration, we obtain a contradiction to (2.143). The lemma is proved. □

In a similar way, it can be proved that, if $x \in \aleph_0$ satisfies (2.142), then

$$\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t r_3(s) \int_\sigma^s \frac{1}{r_2(\tau)}\, d\tau\, ds\, dt = \infty$$

and

$$\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t \frac{1}{r_1(s)} \int_\sigma^s r_3(\tau)\, d\tau\, ds\, dt = \infty$$

hold.

Lemma 2.7.9 *The following hold*:

(i) *if there exists* $x \in \aleph_0$ *such that* $\lim_{t\to\infty} x(t) \neq 0$, *then* (2.131) *holds*;
(ii) *if there exists* $x \in \aleph_0$ *such that* $\lim_{t\to\infty} L_1 x(t) \neq 0$, *then*

$$\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t r_3(s) \int_\sigma^s \frac{1}{r_2(\tau)}\, d\tau\, ds\, dt < \infty; \tag{2.144}$$

(iii) *if there exists a* $x \in \aleph_0$ *such that* $\lim_{t\to\infty} L_2 x(t) \neq 0$, *then*

$$\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t \frac{1}{r_1(s)} \int_\sigma^s r_3(\tau)\, d\tau\, ds\, dt < \infty. \tag{2.145}$$

Proof (i) Let $x(t)$ be an eventually positive solution of (2.117) in the class $\aleph_0$ such that $\lim_{t\to\infty} x(t) = \lambda > 0$. Hence there exists a $T \geq \sigma$ such that $x(t) > 0$, $L_1 x(t) < 0$ and $L_2 x(t) > 0$ for $t \geq T$. In view of the signs of $\int_\sigma^\infty \frac{1}{r_2(t)}\, dt$ and $\int_\sigma^\infty \frac{1}{r_1(t)}\, dt$, three possible cases may arise:

(I) $\int_\sigma^\infty \frac{1}{r_2(t)}\, dt < \infty$, $\int_\sigma^\infty \frac{1}{r_1(t)}\, dt < \infty$,
(II) $\int_\sigma^\infty \frac{1}{r_2(t)}\, dt = \infty$, $\int_\sigma^\infty \frac{1}{r_1(t)}\, dt < \infty$, and

(III) $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty$.

First consider the case (I), that is, $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt < \infty$ and $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt < \infty$. Integrating (2.117) from t to ∞, $t \ge T$, we obtain

$$\begin{aligned} L_2x(t) &= L_2x(\infty) + \int_t^\infty r_3(s)x(s)\,ds \\ &\ge L_2x(\infty) + x(\infty)\int_t^\infty r_3(s)\,ds. \end{aligned}$$

Then $\int_\sigma^\infty r_3(s)\,ds < \infty$, and so (2.131) holds.

Now assume the case (II), that is, $\int_\sigma^\infty \frac{1}{r_2(s)}\,ds = \infty$ and $\int_\sigma^\infty \frac{1}{r_1(s)}\,ds < \infty$. Clearly, $\int_\sigma^\infty \frac{1}{r_2(s)}\,ds = \infty$ implies that $L_2x(\infty) = 0$. Integrating (2.117) twice from t to ∞, $t > T$, we obtain

$$\begin{aligned} L_1x(t) &= L_1x(\infty) - \int_t^\infty \frac{1}{r_2(s)}\int_s^\infty r_3(\tau)x(\tau)\,d\tau\,ds \\ &\le L_1x(\infty) - x(\infty)\int_t^\infty \frac{1}{r_2(s)}\int_s^\infty r_3(\tau)\,d\tau\,ds \\ &\le L_1x(\infty) - x(\infty)\int_t^\infty r_3(s)\int_t^s \frac{1}{r_2(\tau)}\,d\tau\,ds. \end{aligned}$$

Then $\int_\sigma^\infty r_3(t)\int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt < \infty$, and hence (2.131) holds.

Finally, suppose that the case (III) holds, that is, $\int_\sigma^\infty \frac{1}{r_1(s)}\,ds = \infty$ holds. Proceeding as in case (II), we see that $\int_\sigma^\infty \frac{1}{r_1(s)}\,ds = \infty$ implies that $L_1x(\infty) = 0$. Integrating (2.117) three times from t to ∞, $t > T$, we obtain

$$\begin{aligned} x(t) \ge x(\infty) &+ L_2x(\infty)\int_t^\infty \frac{1}{r_1(s)}\int_s^\infty \frac{1}{r_2(\tau)}\,d\tau\,ds \\ &+ x(\infty)\int_t^\infty \frac{1}{r_1(s)}\int_s^\infty \frac{1}{r_2(\tau)}\int_\tau^\infty r_3(\theta)\,d\theta\,d\tau\,ds \end{aligned}$$

and then by interchanging the order of integration, we get (2.131).

The claims (ii) and (iii) may be proved by applying similar arguments as above by using (2.118) and (2.119) instead of (2.117). □

Theorem 2.7.10 *Equation* (2.117) *has property* A, *if and only if* (2.120) *has property* B.

Proof Assume that (2.117) has property A. Then $\aleph = \aleph_0$ and, by Lemma 2.7.7, $\aleph(a) = \aleph_3(a)$. By Lemma 2.7.5, $\aleph_3(a) \ne \phi$. Let $u \in \aleph_3(a)$, that is, there exists a $T \ge \sigma$ such that $L_iu(t) > 0$ for $t > T$ and $i = 0, 1, 2$. Suppose that (2.120) does not have property B, that is, there exists $i \in \{0, 1, 2\}$ such that $L_iu(t)$ is bounded.

First, let $L_2u(t)$ be bounded. Observe that $w(t) = L_2u(t)$ is a solution of (2.140) and $L_1w(t) = u(t)$, $L_2w(t) = L_1u(t)$. Then $L_iw(t) > 0$ for $t \ge T$, $i = 0, 1, 2$, and $w(t)$ is bounded. Thus, there exists $k_i > 0$, $i = 1, 2$ such that

$$0 < k_1 \le w(t) \le k_2 \quad \text{for } t \ge T. \tag{2.146}$$

Integrating (2.140) three times from T to t, we have

$$\begin{aligned} w(t) &= w(T) + L_1w(T)\int_T^t r_3(s)\,ds + L_2w(T)\int_T^t r_3(s)\int_T^s \frac{1}{r_2(\tau)}\,d\tau\,ds \\ &\quad + \int_T^t r_3(s)\int_T^s \frac{1}{r_2(\tau)}\int_T^\tau \frac{1}{r_1(\theta)}w(\theta)\,d\theta\,d\tau\,ds. \end{aligned} \tag{2.147}$$

Since $L_iw(T) > 0$ for $i = 0, 1, 2$, from (2.146) and (2.147), we obtain

$$k_1\int_T^t r_3(s)\int_T^s \frac{1}{r_2(\tau)}\int_T^\tau \frac{1}{r_1(\theta)}\,d\theta\,d\tau\,ds \le w(t) \le k_2. \tag{2.148}$$

By Lemma 2.7.5, (2.117) has a Kneser solution. Since (2.117) has property A, any Kneser solution of (2.117) satisfies (2.142). Thus, by Lemma 2.7.8, the integral on the left-hand side of (2.148) is divergent, which is a contradiction.

If $L_1u(t)$ is bounded, we consider (2.139) instead of (2.140). Using a similar argument as above and Lemma 2.7.8, we again get a contradiction. Hence property A of (2.117) implies the property B of (2.120).

Now, assume that (2.120) has property B. Then $\aleph(a) = \aleph_3(a)$. From Lemmas 2.7.5 and 2.7.7, we have $\aleph = \aleph_0 \ne \phi$. Assume that there exists a Kneser solution $x(t)$ of (2.117) such that for some $i \in \{0, 1, 2\}$, $\lim_{t\to\infty} L_ix(t) = c \ne 0$. First, suppose that $\lim_{t\to\infty} x(t) = c \ne 0$. Then by Lemma 2.7.9, we obtain (2.131), so also $\int_\sigma^\infty r_3(t)\,dt < \infty$ and $\int_\sigma^\infty r_3(t)\int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt < \infty$.

Let $w(t)$ be a nonoscillatory solution of (2.140). Without any loss of generality, we may assume that $w(t)$ is eventually positive. Clearly, there exists a solution $u(t)$ of (2.120) such that $w(t) = L_2u(t)$. Since (2.120) has property B, $u(t)$ satisfies the property (2.135). Hence $w(t) \to \infty$ as $t \to \infty$ and there exists a $T \ge \sigma$ such that $L_iw(t) > 0$ for all $t \ge T$. In view of (2.131), we can find T large enough such that

$$\int_T^\infty r_3(t)\int_T^t \frac{1}{r_2(s)}\int_T^s \frac{1}{r_1(\tau)}\,d\tau\,ds\,dt < 1. \tag{2.149}$$

Repeated integration of (2.140) from T to t and by using the property w is a nondecreasing function, it follows from (2.147) that

$$w(t) \le f(t) + w(t)g(t),$$

where

$$f(t) = w(T) + L_1w(T)\int_T^t r_3(s)\,ds + L_2w(T)\int_T^t r_3(s)\int_T^s \frac{1}{r_2(\tau)}\,d\tau\,ds$$

and

$$g(t)=\int_T^t r_3(s)\int_T^s \frac{1}{r_2(\tau)}\int_T^\tau \frac{1}{r_1(\theta)}\,d\theta\,d\tau\,ds.$$

Then

$$w(t)\le \frac{f(t)}{1-g(t)}$$

which implies that $w(t)$ is bounded, a contradiction to the property B of (2.120).

Similar contradictions can be obtained if

$$\lim_{t\to\infty} L_1x(t)=c_1\ne 0 \quad \text{and} \quad \lim_{t\to\infty} L_2x(t)=c_2\ne 0.$$

Here one has to use Lemma 2.7.9. The proof is complete. □

Corollary 2.7.4 *If* (2.117) *has property* A, *then* (2.118) *and* (2.119) *have property* A, *and* (2.120), (2.139) *and* (2.140) *has property* B.

Corollary 2.7.5 *Every Kneser solution* x *of* (2.117) *satisfies* $\lim_{t\to\infty} L_ix(t)=0$ *for* $i=0,1,2$ *if and only if every strongly monotone solution* u *of* (2.120) *satisfies* $\lim_{t\to\infty} L_iu(t)=\infty$ *for* $i=0,1,2$.

Theorem 2.7.11 *If* (2.117) *has property* A *or* (2.120) *has property* B, *then it is oscillatory.*

Theorem 2.7.12 *If any one of the following conditions*:

(i) $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt=\int_\sigma^\infty \frac{1}{r_2(t)}\,dt=\int_\sigma^\infty r_3(t)\int_\sigma^t \frac{1}{r_1(s)}\,ds\,dt=\int_\sigma^\infty r_3(t)\int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt=\infty$,
(ii) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt=\int_\sigma^\infty r_3(t)\,dt=\int_\sigma^\infty \frac{1}{r_1(t)}\int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt=\int_\sigma^\infty \frac{1}{r_1(t)}\int_\sigma^t r_3(s)\,ds\,dt=\infty$,
(iii) $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt=\int_\sigma^\infty r_3(t)\,dt=\int_\sigma^\infty \frac{1}{r_2(t)}\int_\sigma^t r_3(s)\,ds\,dt=\int_\sigma^\infty \frac{1}{r_2(t)}\int_\sigma^t \frac{1}{r_1(s)}ds\,dt=\infty$,
(iv) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt=\infty$, $\int_\sigma^\infty r_3(t)\int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt<\infty$, *and*

$$\int_\sigma^\infty \frac{1}{r_1(t)}\left(\int_t^\infty r_3(s)\,ds\right)\left(\int_t^\infty \frac{1}{r_2(s)}\int_s^\infty r_3(\tau)\,d\tau\,ds\right)dt=\infty,$$

(v) $\int_\sigma^\infty r_3(t)\,dt=\infty$, $\int_\sigma^\infty \frac{1}{r_1(t)}\int_\sigma^t r_3(s)\,ds\,dt<\infty$, *and*

$$\int_\sigma^\infty \frac{1}{r_2(t)}\left(\int_t^\infty \frac{1}{r_1(s)}\,ds\right)\left(\int_t^\infty r_3(s)\int_s^\infty \frac{1}{r_1(\tau)}\,d\tau\,ds\right)dt=\infty,$$

is satisfied, then (2.120) *has property* B. *Consequently*, (2.139) *and* (2.140) *also have property* B.

Theorem 2.7.13 *If any one of the following conditions*

(i) $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \int_\sigma^\infty r_3(t)\,dt = \int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t r_3(s)\,ds\,dt = \int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t \frac{1}{r_1(s)}\,ds\,dt = \infty$,
(ii) $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_1(s)}\,ds\,dt = \int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt = \infty$,
(iii) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty r_3(t)\,dt = \int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt = \int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t r_3(s) ds\,dt = \infty$,
(iv) $\int_\sigma^\infty r_3(t)\,dt = \infty$, $\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t r_3(s)\,ds\,dt < \infty$, *and*

$$\int_\sigma^\infty \frac{1}{r_2(t)} \left(\int_t^\infty \frac{1}{r_1(s)}\,ds \right) \left(\int_t^\infty r_3(s) \int_s^\infty \frac{1}{r_1(\tau)}\,d\tau\,ds \right) dt = \infty,$$

(v) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \infty$, $\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt < \infty$, *and*

$$\int_\sigma^\infty \frac{1}{r_1(t)} \left(\int_t^\infty r_3(s)\,ds \right) \left(\int_t^\infty \frac{1}{r_2(s)} \int_s^\infty r_3(\tau)\,d\tau\,ds \right) dt = \infty$$

is satisfied, then (2.117)–(2.119) *are oscillatory and have property A.*

Theorem 2.7.14 *If* $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \infty$, $\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t \frac{1}{r_1(s)}\,ds\,dt < \infty$, *and*

$$\int_\sigma^\infty r_3(t) \left(\int_t^\infty \frac{1}{r_2(s)}\,ds \right) \left(\int_t^\infty \frac{1}{r_1(s)} \int_s^\infty \frac{1}{r_2(\tau)}\,d\tau\,ds \right) dt = \infty,$$

then

(i) (2.117)–(2.119) *are oscillatory and have property A.*
(ii) (2.120), (2.139) *and* (2.140) *have property B.*

Theorem 2.7.15 *The following assertions are equivalent*:

(a) (2.117) *has property A.*
(b) (2.117) *is oscillatory and*

$$\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s \frac{1}{r_1(u)}\,du\,ds\,dt = \infty,$$
$$\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t r_3(s) \int_\sigma^s \frac{1}{r_2(u)}\,du\,ds\,dt = \infty,$$
$$\int_\sigma^\infty \frac{1}{r_2(t)} \int_\sigma^t \frac{1}{r_1(s)} \int_\sigma^s r_3(u)\,du\,ds\,dt = \infty.$$

Corollary 2.7.6 *If the functions $r_1(t)$, $r_2(t)$ and $r_3(t)$ satisfy* (2.131), *and*

$$\int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_1(s)}\,ds\,dt = \infty$$

holds, then there exists an unbounded oscillatory solution of (2.117).

Corollary 2.7.7 *Let* $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt = \int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \infty$. *If* (2.117) *is oscillatory and has no property* A, *then* (2.123) *is nonoscillatory.*

Corollaries 2.7.6 and 2.7.7 are illustrated by the following example.

Example 2.7.3 Let $\epsilon \in (0, 1)$ and $T > 1$. Consider Eqs. (2.137) and (2.138). Clearly

$$\int_T^t \frac{1}{r_1(s)}\,ds = t(\ln t - 1) + c_1 \to \infty,$$

$$\int_T^t \frac{1}{r_2(s)}\,ds = \ln\ln t + c_2 \to \infty \quad \text{as } t \to \infty,$$

$$\int_T^t \frac{1}{r_2(s)} \int_T^s \frac{1}{r_1(s)}\,ds\,dt = t - \int_T^t \frac{ds}{\ln s} - T - \int_T^t \frac{c\,ds}{s \ln s} \to \infty \quad \text{as } t \to \infty,$$

$$\int_T^\infty r_3(t) \int_T^t \frac{1}{r_2(s)} \int_T^s \frac{1}{r_1(\tau)}\,d\tau\,ds\,dt \le \int_T^\infty \frac{dt}{t(\ln t)^{1+\epsilon}} = \int_{\ln T}^\infty \frac{du}{u^{1+\epsilon}} < \infty$$

and

$$\int_T^\infty r_3(t) \int_T^t \frac{1}{r_1(s)}\,ds\,dt \ge \int_T^\infty \frac{\ln t - 1}{t(\ln t)^{1+\epsilon}}\,dt = \int_T^\infty \frac{dt}{t(\ln t)^{\epsilon}} = \int_{\ln T}^\infty \frac{ds}{s^\epsilon} = \infty$$

implies that Corollary 2.7.6 can be applied to (2.137). By Corollary 2.7.6, Eq. (2.137) has an unbounded oscillatory solution and a Kneser solution tending to a nonzero constant. Similarly, by Corollary 2.7.7, Eq. (2.138) is nonoscillatory.

Theorem 2.7.16 *Let one of the following two conditions hold*:

(i) $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt = \infty$, $\int_\sigma^\infty \frac{1}{r_1(t)} \int_\sigma^t \frac{1}{r_2(s)}\,ds\,dt < \infty$ *and*

$$\int_\sigma^\infty r_3(t)\left(\int_t^\infty \frac{1}{r_2(s)}\,ds\right) \int_t^\infty \frac{1}{r_1(s)} \int_s^\infty \frac{1}{r_2(\tau)}\,d\tau\,ds\,dt = \infty,$$

(ii) $\int_\sigma^\infty \frac{1}{r_1(t)}\,dt < \infty$, $\int_\sigma^\infty \frac{1}{r_2(t)}\,dt < \infty$ *and*

$$\int_\sigma^\infty r_3(t)\left(\int_t^\infty \frac{1}{r_1(s)}\,ds\right)\left(\int_t^\infty \mu(s)\,ds\right) \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty \frac{1}{r_1(\tau)}\,d\tau\,ds\,dt = \infty,$$

where $\mu(t) = \left(\frac{1}{\int_t^\infty \frac{1}{r_1(s)}\,ds}\right)' \int_t^\infty \frac{1}{r_2(s)} \int_s^\infty \frac{1}{r_1(\tau)}\,d\tau\,ds$. *Then* (2.117)–(2.119) *are oscillatory and their adjoint equations* (2.120), (2.139) *and* (2.140) *are also oscillatory. In addition, every nonoscillatory solution of* (2.117) *is a Kneser solution or satisfies the property* $x(t)L_1x(t) < 0$ *and* $x(t)L_2x(t) < 0$ *for large* t *and tends to zero as* $t \to \infty$.

Theorem 2.7.17 *Equation* (2.117) *is oscillatory, if and only if every nonoscillatory solution* $x(t)$ *of* (2.117) *satisfies* $x(t)L_1x(t) < 0$ *and* $x(t)L_2x(t) > 0$ *for* $t \ge \sigma$.

Theorem 2.7.18 *Equation (2.123) is oscillatory, if and only if every nonoscillatory solution $u(t)$ of (2.123) satisfies $u(t)L_1u(t) > 0$ and $u(t)L_2u(t) > 0$ for $t \geq \sigma$.*

The following corollary is a consequence of Theorem 2.7.9.

Corollary 2.7.8 *If*

$$\int_\sigma^\infty r_3(t) \int_\sigma^t \frac{1}{r_2(s)} \int_\sigma^s \frac{1}{r_1(\tau)}\, d\tau\, ds\, dt = \infty,$$

then any solution $x(t)$ of (2.117) such that $x(t)L_1x(t) < 0$ and $x(t)L_2x(t) > 0$ for $t \geq \sigma$ satisfies $\lim_{t\to\infty} x(t) = 0$.

Theorems 2.7.3–2.7.5 yield the following corollary:

Corollary 2.7.9 *If (2.127) holds, then (2.120) is nonoscillatory.*

Corollary 2.7.10 *If (2.131) holds, then (2.123) is nonoscillatory and Eq. (2.124) is nonoscillatory.*

Now, we consider the second-order linear homogeneous equation (2.94), where b is a continuous function for $t \geq \sigma$ with either $b(t) > 0$ or $b(t) < 0$ for $t \geq \sigma$. Let $h(t)$ be a positive solution of (2.94) on $[t_0, \infty)$, $t_0 \geq \sigma$. Then the third-order differential equation (2.10) can be written in the disconjugate form

$$\left(h^2(t)\left(\frac{1}{h(t)}x'\right)'\right)' + c(t)h(t)x = 0, \tag{2.150}$$

which is a special case of the linear disconjugate equation (2.117).

In the following, we shall make an attempt to study the oscillation, nonoscillation, property A and property B of (2.10) with the help of Eq. (2.150).

Remark 2.7.6 Assume that $b(t) \leq 0$ for $t \geq \sigma$. Hartman [17] proved that (2.94) is nonoscillatory and every nonoscillatory solution $h(t)$ of (2.94) satisfying either

$$h(t)h'(t) > 0 \quad \text{for large } t \quad \text{and} \quad \lim_{t\to\infty} |h(t)| = \infty$$

or

$$h(t)h'(t) \leq 0 \quad \text{for } t \geq \sigma.$$

Further, (2.94) admits a nondecreasing nonoscillatory solution and a nonincreasing nonoscillatory solution $h(t)$, such that $\lim_{t\to\infty} h(t) = c_h \neq 0$ if and only if $\int_\sigma^\infty t|b(t)|\, dt < \infty$.

Lemma 1.5.3 shows that (2.10) is oscillatory, if and only if every nonoscillatory solution of the equation is a Kneser solution. However, it does not guarantee that this equation has property A. The following result proves the equivalence between the oscillation and property A for (2.10).

Theorem 2.7.19 *Let $b(t) \le 0$ and $c(t) > 0$. Then* (2.10) *is oscillatory if and only if* (2.10) *has property A.*

Proof By Lemma 1.5.4, it is sufficient to prove that if (2.10) is oscillatory, then every Kneser solution of (2.10) tends to zero as $t \to \infty$. Remark 2.7.6 implies that (2.94) has a nonincreasing solution. Let $h(t)$ be the nonincreasing solution of (2.94) for $t \ge \sigma$. Then (2.10) can be written in the disconjugate form (2.150). Clearly, (2.150) is oscillatory and hence by Theorem 2.7.3, we have

$$\int_\sigma^\infty c(t)h(t)\int_\sigma^t h(s)\int_\sigma^s \frac{1}{h^2(\tau)}\,d\tau\,ds\,dt = \infty. \tag{2.151}$$

Since $h(t)$ is a nonincreasing function for $t \ge \sigma$, the function $\frac{1}{h^2}$ is nondecreasing for $t \ge \sigma$. Then

$$\int_\sigma^t \frac{1}{h^2(s)}\int_\sigma^s h(u)\,du\,ds \ge \int_\sigma^t \frac{1}{h^2(s)}\int_\sigma^s h(s)\,du\,ds = \int_\sigma^t \frac{s-\sigma}{h(s)}\,ds,$$

and

$$\int_\sigma^t h(s)\int_\sigma^s \frac{1}{h^2(u)}\,du\,ds \le \int_\sigma^t h(s)\int_\sigma^s \frac{1}{h^2(s)}\,du\,ds \le \int_\sigma^t \frac{s-\sigma}{h(s)}\,ds.$$

The above two inequalities together with (2.151) implies that

$$\int_\sigma^\infty c(t)h(t)\int_\sigma^t \frac{1}{h^2(s)}\int_\sigma^s h(\tau)\,d\tau\,ds\,dt = \infty.$$

Since (2.150) is oscillatory, Corollary 2.7.8 implies that every Kneser solution of the equation tends to zero as $t \to \infty$. This completes the proof of the theorem. □

Remark 2.7.7 Let $b(t) \le 0$ and $c(t) < 0$. Lazer [23] proved that if (2.10) is oscillatory, then every nonoscillatory solution $x(t)$ of (2.10) is a strongly monotone solution and

$$\lim_{t\to\infty} |x(t)| = \lim_{t\to\infty} |x''(t)| = \infty.$$

Gera in [12] proved the converse of the above statement as follows: if every nonoscillatory solution of (2.10) is strongly monotone, then (2.10) is oscillatory. Thus we have the following corollary:

Corollary 2.7.11 *Equation* (2.10) *is oscillatory, if and only if every nonoscillatory solution of* (2.10) *is strongly monotone.*

Theorem 2.4.4 partially improves the above quoted Lazer's result to equation (2.1) with $a(t) \geq 0$. On the other hand, Theorem 2.4.5 extends Corollary 2.7.11. Applying Theorem 2.4.5 to (1.5), we obtain the following corollary:

Corollary 2.7.12 *Let $a \geq 0$, $b \leq 0$ and $c < 0$. Equation* (1.5) *admits an oscillatory solution, if and only if every nonoscillatory solution of* (1.5) *satisfies the property* (2.80).

Now, we give a stronger result which establishes the equivalence between oscillation and property B of (2.10).

Theorem 2.7.20 *Let $b(t) \leq 0$ and $c(t) < 0$. Then* (2.10) *is oscillatory if and only if* (2.10) *has property B.*

Proof In view of Corollary 2.7.11, it is sufficient to show that if (2.10) is oscillatory, then (2.10) has property B. Let $x(t)$ be a nonoscillatory solution of (2.10). Since (2.10) is oscillatory, $x(t)$ is strongly monotone. Without any loss of generality, we may assume that there exists a $t_0 \geq \sigma$ such that $x(t) > 0, x'(t) > 0$ and $x''(t) > 0$ for $t \geq t_0$. Then $x'''(t) > 0$ for $t \geq t_0$. Repeated integration of $x'''(t) > 0$ from t_0 to t gives

$$x'(t) > x''(t_0)(t - t_0)$$

and

$$x(t) \geq x(t_0) + \frac{x''(t_0)}{2}(t - t_0)^2 > \frac{x''(t_0)}{2}(t - t_0)^2$$

for $t \geq t_0$. Integrating (2.10) from t_0 to t, we get

$$\begin{aligned} x''(t) &= x''(t_0) - \int_{t_0}^{t} b(s)x'(s)\,ds - \int_{t_0}^{t} c(s)x(s)\,ds \\ &\geq x''(t_0)\left[1 - \int_{t_0}^{t} (s - t_0)b(s)\,ds - \frac{1}{2}\int_{t_0}^{t} (s - t_0)^2 c(s)\,ds\right]. \end{aligned}$$

Since $x(t)$ is an eventually positive strongly monotone solution of (2.10), we have $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \infty$. Hence, it is sufficient to show that $\lim_{t\to\infty} x''(t) = \infty$. Assume that $\lim_{t\to\infty} x''(t) < \infty$. From the above inequality, we have

$$-\int_{\sigma}^{\infty} t^2 c(t)\,dt < \infty \quad \text{and} \quad -\int_{\sigma}^{\infty} tc(t) < \infty. \tag{2.152}$$

By Remark 2.7.6, there exists a positive nonincreasing solution $h(t)$ of (2.94) such that $\lim_{t\to\infty} h(t) = l_h > 0$. Thus $h(\sigma) \geq h(t) \geq l_h$ and

$$\int_{\sigma}^{t} \frac{1}{h^2(s)} \int_{\sigma}^{s} h(u)\,du\,ds \leq kt^2,$$

where $k = h(\sigma)/2l_h^2$. Consequently, by using (2.152), we obtain

$$\int_\sigma^\infty c(t)h(t)\int_\sigma^t \frac{1}{h^2(s)}\int_\sigma^s h(u)\,du\,ds\,dt$$
$$\le k\int_\sigma^\infty t^2c(t)h(t)\,dt \le k\,h(\sigma)\int_\sigma^\infty t^2c(t)\,dt < \infty,$$

and by Corollary 2.7.10, (2.150) is nonoscillatory, which is a contradiction, because (2.150) is the disconjugate form of (2.10). The theorem is proved. □

Finally, we consider the linear differential equation with quasiderivatives

$$L_3x(t) + q(t)x'(t) + p(t)x(t) = 0, \tag{2.153}$$

where $p(t)$ and $q(t)$ are continuous functions on $[\sigma,\infty)$, $L_3x = (L_2x)'$, $L_2x = r_2(t)(L_1x)'$, $L_1x = r_1(t)x'$, $L_0x = x$, and r_1 and r_2 are continuous and positive functions on $[\sigma,\infty)$.

Lemma 2.7.10 *Let $2p(t) - q'(t) \ge 0$ and $\frac{r_2(t)}{r_1(t)}$ be differentiable on $[\sigma,\infty)$, and $(\frac{r_2(t)}{r_1(t)})' \ge 0$ on $[\sigma,\infty)$. Then every solution of (2.153) with a double zero at some point $t_0 > \sigma$ has no simple zero in $[\sigma, t_0)$.*

Proof At first, we shall say that the solution $x(t)$ of (2.153) has a double zero at a point t_0 if $x(t_0) = 0$, $L_1x(t_0) = 0$ and $L_2x(t_0) \ne 0$. Define a function $w(t,x(t))$ by

$$w\bigl(t,x(t)\bigr) = \frac{1}{2}\left[\bigl(2p(t) - q'(t)\bigr)x^2(t) + \left(\frac{r_2(t)}{r_1(t)}\right)'\bigl(r_1(t)x'(t)\bigr)^2\right].$$

Multiplying (2.153) by $x(t)$, we obtain the identity

$$\left[r_2(t)\bigl(r_1(t)x'(t)\bigr)'x(t) - \frac{1}{2}\frac{r_2(t)}{r_1(t)}\bigl(r_1(t)x'(t)\bigr)^2 + \frac{1}{2}q(t)x^2(t)\right]' = -w\bigl(t,x(t)\bigr). \tag{2.154}$$

Without any loss of generality, we may suppose that $x(t_0) = 0$, $L_1x(t_0) = 0$ and $L_2x(t_0) > 0$. If there exists a $t_1 < t_0$ such that $x(t_1) = 0$, $L_1x(t_1) \ne 0$, then from $L_2x(t_0) > 0$, it follows that $x(t)$ has a minimum at t_0. Suppose that $x(t) > 0$ on (t_1, t_0). Then integrating the identity (2.154) over the interval $[t_1, t_0]$, we get

$$0 < \frac{1}{2}\frac{r_2(t_1)}{r_1(t_1)}\bigl(r_1(t_1)x'(t_1)\bigr)^2 = -\int_{t_0}^{t_1} w\bigl(t,x(t)\bigr)\,dt \le 0,$$

which is a contradiction. Therefore $x(t) > 0$ for $t \le t_0$. The proof is complete. □

Lemma 2.7.11 *Let the conditions of Lemma 2.7.10 hold. Then there exists a nonnegative solution of (2.153).*

Lemma 2.7.12 *Let the conditions of Lemma 2.7.10 hold. If (2.153) is oscillatory, then every solution of (2.153), which vanishes at least once is oscillatory.*

Theorem 2.7.21 *Let the conditions of Lemma 2.7.10 hold and $b(t) \leq 0$. Then there exists a Kneser solution of (2.153).*

Lemma 2.7.13 *If $q(t) \leq 0$, $p(t) > 0$ and $x(t)$ is any solution of (2.153) satisfying the condition $x(t_0) \geq 0$, $L_1 x(t_0) \leq 0$ and $L_2 x(t_0) > 0$ for $t_0 > \sigma$, then $x(t) > 0$, $L_1 x(t) < 0$ and $L_2 x(t) > 0$ for $\sigma \leq t < t_0$.*

Lemma 2.7.14 *If $q(t) \leq 0$, $p(t) > 0$ and (2.153) is oscillatory, then every solution of (2.153), which vanishes at least once is oscillatory.*

Lemma 2.7.15 *If $q(t) \leq 0$, $p(t) > 0$, then there exists a Kneser solution of (2.153).*

From Theorem 2.7.21, it follows that the equation $x''' + p(t)x = 0$ has Kneser-solution if $p(t) \geq 0$. This condition is sufficient for the uniqueness of a Kneser solution of $x''' + p(t)x = 0$ with $x(\sigma) = 1$. However, the conditions of Theorem 2.7.21 are not sufficient for the uniqueness of Kneser solution for (2.153). This is evident from the equation $x''' - 3x' + 2x = 0$, which has two Kneser solutions $x_1(t) = e^{-t}$ and $x_2(t) = te^{-t}$.

Theorem 2.7.22 *Let the conditions of Lemma 2.7.10 hold, $b(t) \leq 0$ and*

$$\int^{\infty} \frac{1}{r_2(t)}\, dt = \infty \tag{2.155}$$

hold, and (2.153) be oscillatory. Then there exists a unique Kneser solution of (2.153) such that

$$x(\sigma) = 1, \qquad x(t) > 0, \qquad L_1 x(t) < 0 \quad \textit{and} \quad L_2 x(t) \geq 0 \quad \textit{for } t \geq \sigma. \tag{2.156}$$

Proof On the contrary, suppose that there exist two solutions $x_1(t)$ and $x_2(t)$ of (2.153) that satisfy (2.156). Then the solution $v(t) = x_1(t) - x_2(t)$ of (2.153) satisfies the condition $v(\sigma) = 0$. Since (2.153) is oscillatory, by Lemma 2.7.12, $v(t)$ is an oscillatory solution of (2.153). Hence there exists a sequence $\{t_n\}$ of zeros of $v(t)$ such that $\lim_{n\to\infty} t_n = \infty$. Integrating (2.154) over $[t_n, t_{n+1}]$, we get

$$r_2(t_{n+1}) r_1(t_{n+1}) \big(v'(t_{n+1})\big)^2 \geq r_2(t_n) r_1(t_n) \big(v'(t_n)\big)^2. \tag{2.157}$$

Now two cases may occur: (i) there exists n_0 such that $v'(n_0) \neq 0$ (and so $v'(t_n) \neq 0$ for $n > n_0$) or (ii) $v'(t_n) = 0$ for all n. Consider the case (i). If $v'(n_0) \neq 0$, then (2.157) implies that

$$\liminf_{t\to\infty} r_2(t_n) r_1(t_n) \big(v'(t_n)\big)^2 \geq c > 0. \tag{2.158}$$

Since $L_1 x_1 \le 0$, then $x_1'(t) \le 0$. Hence $\lim_{t\to\infty} r_2(t)x_1'(t) = 0$. Similarly

$$\lim_{t\to\infty} r_2(t)x_2'(t) = 0 \quad \text{and so} \quad \lim_{t\to\infty} r_2(t)v'(t) = 0.$$

By the second assumption of Lemma 2.7.10, we see that $r_1(t)/r_2(t)$ is nonincreasing. Then

$$\lim_{t\to\infty} r_2(t)r_1(t)\big(v'(t)\big)^2 = \lim_{t\to\infty} \frac{r_1(t)}{r_2(t)}\big(r_2(t)v'(t)\big)^2 = 0,$$

which contradicts (2.158). Now, we consider the case (ii). If $v'(t_n) = 0$ for all $n = 1, 2, 3, \ldots$, then with respect to the initial data, we have $v(t) \ge 0$ or ≤ 0 on $[\sigma, \infty)$. Let us suppose that $v(t) \ge 0$. Let $\{\tau_n\}$ be a sequence of zeros of $v'(t)$ such that $\tau_n < t_n < \tau_{n+1} < t_{n+1}$ and $v'(t) \ne 0$ on (τ_n, t_n). Since $v'(\tau_n) = v'(t_n) = 0$, there exists a number $\xi \in (\tau_n, t_n)$ such that $(r_1(t)v'(t))' = 0$ for $t = \xi$. Now if we integrate the equality (2.154), we shall obtain a contradiction. Therefore $v(t) \equiv 0$, that is, $x_1(t) = x_2(t)$ for $t \ge \sigma$. The theorem is proved. □

Corollary 2.7.13 *Let $r_1(t) = r_2(t) = r(t)$ and (2.155) hold. Let either*

(i) *$q(t) \ge k$ and $2p(t) - q'(t) \ge \frac{\alpha}{r(t)}$, where $\beta \le 0$ and $\alpha > \frac{4}{3\sqrt{3}}(-\beta)^{3/2}$, α, β are constants, or*
(ii) *$q(t) \ge \frac{\beta}{R^2(t)}$ and $2p(t) - q'(t) \ge \frac{\epsilon}{r(t)R^3(t)}$, where $\beta < 1, \epsilon > \frac{4}{3\sqrt{3}}(1-\beta)^3$ are constants and $R(t) = \int^t \frac{1}{r(s)}\,ds$.*

Then there exists a unique Kneser solution of (2.153).

Example 2.7.4 By Corollary 2.7.13(i), the equation

$$\big(t\big(tx'\big)'\big)' - 2x' + \frac{4}{t}x = 0$$

admits a unique Kneser solution. Note that $x(t) = t^{-2}$ on $[1, \infty)$ is the unique Kneser solution of the equation.

Lemma 2.7.16 *If $q(t) \le 0$, $p(t) > 0$ and*

$$\int^\infty \frac{1}{r_1(t)}\,dt = \infty. \tag{2.159}$$

Let $x(t)$ be a nonoscillatory solution of (2.153). Then either

(i) *$x(t)x'(t) \le 0$, $L_2 x(t) \ge 0$ and $x(t) \ne 0$ on $[\sigma, \infty)$, or*
(ii) *there exists a number $t_0 \in [\sigma, \infty)$ such that $x(t)x'(t) \ge 0$ and $x(t) \ne 0$ for $t \ge t_0$.*

Theorem 2.7.23 *Let $q(t) \le 0$, $p(t) \ge 0$ and (2.155), (2.159) hold. Let m be the positive stationary point of the function*

$$F\bigl(x(t)\bigr) = h(t)\frac{1}{r_1(t)}x^3(t) - \frac{1}{2}h'(t)x^2(t) + q(t)\frac{1}{r_1(t)}x(t), \tag{2.160}$$

where $h(t) = \frac{r_1(t)}{r_2(t)}$ and

$$\int^{\infty}\bigl[p(t) + F\bigl(m(t)\bigr)\bigr]\,dt = \infty.$$

Then (2.153) has an oscillatory solution.

Proof Suppose that $u(t)$ is any nonoscillatory solution of (2.153). By Lemma 2.7.16, there exists a number T such that either

$$x(t) = r_1(t)\frac{u'(t)}{u(t)} \ge 0 \quad \text{for } t \ge T \tag{2.161}$$

or

$$x(t) = r_1(t)\frac{u'(t)}{u(t)} \le 0 \quad \text{for } t \ge T. \tag{2.162}$$

We assert that (2.161) is impossible. On the contrary, assume that (2.161) holds. The function $x(t)$ satisfies the second-order nonlinear differential equation

$$\left(r_2(t)x' + \frac{3}{2}h(t)x^2\right)' = -\bigl[F\bigl(x(t)\bigr) + p(t)\bigr], \tag{2.163}$$

where $F(x(t))$ is defined in (2.160). The function $F(x(t)) + p(t)$ has a minimum on the interval $[0, \infty)$ at the point

$$m(t) = \frac{1}{6h(t)}\left(r_1(t)h'(t) + \left(r_1^2(t)\bigl(h'(t)\bigr)^2 - 12h(t)q(t)\right)^{1/2}\right)$$

and

$$F\bigl(m(t)\bigr) = \frac{2}{3}m(t)\frac{q(t)}{r_1(t)} + \frac{1}{18}\frac{h'(t)}{h(t)}\left(q(t) - m(t)r_1(t)\frac{h'(t)}{h(t)}\right). \tag{2.164}$$

Then (2.163) gives

$$\left(r_2(t)x'(t) + \frac{3}{2}h(t)x^2(t)\right)' = -\bigl[F\bigl(x(t)\bigr) + p(t)\bigr] \le -\bigl[F\bigl(m(t)\bigr) + p(t)\bigr], \quad t \ge T.$$

Integrating the last inequality from T to t, we get

$$r_2(t)x'(t) \le r_2(t_0)x'(t_0) + \frac{3}{2}h(t_0)x^2(t_0) - \frac{3}{2}h(t)x^2(t) - \int_{t_0}^{t}\bigl(p(s) + F\bigl(m(s)\bigr)\bigr)\,ds,$$

which tends to ∞. Therefore,

$$\lim_{t\to\infty} r_2(t)\left(r_1(t)\frac{u'(t)}{u(t)}\right)' = -\infty \quad \text{as } t \to \infty$$

and

$$r_1(t)\frac{u'(t)}{u(t)} < -k^2\frac{1}{r_2(t)}, \quad h \neq 0$$

eventually. Since $\int^\infty \frac{1}{r_2(t)}\,dt = \infty$, (2.161) is impossible and hence $u(t)u'(t) \le 0$ and $u(t) \neq 0$ on $[0,\infty)$. Since $u(t)$ was taken to be any nonoscillatory solution, it follows that (2.153) has an oscillatory solution. The proof is complete. □

Theorem 2.7.24 *Let $q(t) \le 0$, $p(t) \ge 0$, the conditions of Lemma 2.7.10 and the integral condition (2.155) hold, and*

$$\int^\infty \bigl(p(t) + F\bigl(m(t)\bigr)\bigr)\,dt = \infty$$

holds. Then (2.153) has an unique Kneser solution.

Corollary 2.7.14 *Let $q(t) \le 0$, $p(t) \ge 0$, $2p(t) - q'(t) \ge 0$, $\int^\infty \frac{1}{r(t)}\,dt = \infty$ and*

$$\int^\infty \left(p(t) - \frac{2}{3\sqrt{3}}\frac{(-q(t))^{3/2}}{r(t)}\right) dt = \infty.$$

Then there exists a unique Kneser solution of the differential equation

$$\bigl(r(t)\bigl(r(t)x'\bigr)'\bigr)' + q(t)x'(t) + p(t)x = 0.$$

2.8 Open Problems and Discussions

- Theorem 2.1.2 gives the sufficient condition that if (2.12) holds, then (2.1) is oscillatory. In view of Proposition 1.2.1(i), we propose the following: If (2.1) is oscillatory, then prove that (2.12) holds when $a(t) \ge 0$, $b(t) \le 0$ and $c(t) > 0$. In a similar way, can we prove that (2.55) holds if (2.1) is oscillatory, when $a(t) \le 0$, $b(t) \le 0$ and $c(t) > 0$? This problem is proposed in view of Proposition 1.2.2(i) and Theorem 2.2.1.
- Let $a(t) \le 0, b(t) \le 0$ and $c(t) < 0$. In view of Theorem 2.3.2 and Proposition 1.2.3(i), it would be interesting to prove the following open problem: If (2.1) is oscillatory, then (2.69) holds.
- Let $a(t) \ge 0, b(t) \le 0$ and $c(t) < 0$. The following assertions are yet to be established:

  (A) If (2.1) admits an oscillatory solution, then (2.79) holds.

(B) If (2.1) admits an oscillatory solution, then all oscillatory solutions of (2.1) tend to zero as $t \to \infty$.

Corollary 2.4.2 provides an indication in this direction.

(C) If (2.1) has an oscillatory solution, then all oscillatory solutions of (2.1) form a two-dimensional subspace of the solution space of (2.1).

We can prove (C) with the assumption that (B) holds.

Theorem 2.8.1 *Suppose that the existence of an oscillatory solution of* (2.1) *implies that all oscillatory solutions of* (2.1) *tends to zero as* $t \to \infty$. *If* $\int_\sigma^\infty p(t)\,dt = -\infty$, *and* (2.1) *admits an oscillatory solution, then all oscillatory solutions of* (2.1) *form a two-dimensional subspace of the solution space of* (2.1).

Proof From Theorem 2.4.6, it follows that (2.1) admits two linearly independent oscillatory solutions u and v whose linear combination is an oscillatory solution of (2.1). Let $x(t)$ be any oscillatory solution of (2.1). Theorem 2.4.1 yields the result that (2.1) admits a positive solution $x_0(t)$ such that $x_0(t) \to \infty$ as $t \to \infty$. Clearly $\{u, v, x_0\}$ is a basis of the solution space of (2.1). If possible, let $x(t) = c_1 u(t) + c_2 v(t) + c_3 x_0(t)$, where c_1, c_2 and c_3 are reals such that $c_3 \neq 0$. Thus $x(t) \to \infty$ or $-\infty$ as $t \to \infty$ according to $c_3 > 0$ or < 0. In either case, we get a contradiction because $x(t)$ is oscillatory. Thus, $x(t)$ can be expressed as a linear combination of u and v, and hence the theorem is proved. □

- Let $a(t) \geq 0$, $b(t) \geq 0$ and $c(t) > 0$. The following assertions are yet to be established:

  (i) Let $9c(t) - a(t)b(t) - b'(t) \geq 0$. If (2.1) is oscillatory, then it admits a nonoscillatory solution $x(t)$ such that $x(t)x'(t) > 0$ for $t \geq t_0 \geq \sigma$.

  (ii) Let $9c(t) - a(t)b(t) - b'(t) \geq 0$, that is, (2.82) holds. If $\frac{a^2(t)}{3} - b(t) + a'(t) \geq 0$, then (2.1) is oscillatory.

  (iii) Suppose that $c(t) \geq d > 0$, $2c(t) - a(t)b(t) - b'(t) \geq 0$ and $\int_\sigma^\infty a(t)\,dt = \infty$. If (2.1) is oscillatory, then every nonoscillatory solution of (2.1) satisfies

  $$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = \lim_{t\to\infty} x''(t) = 0.$$

  (iv) Every oscillatory solution of (2.1) tends to zero as $t \to \infty$ under the assumption of Corollary 2.5.5.

  (v) To generalise the observation in Proposition 1.2.5(vii).

  (vi) To obtain a result similar to Theorem 2.5.6 for $ta(t) < 3$.

- To generalise the observations in Proposition 1.2.6.
- To generalise the observations in Proposition 1.2.7.
- Let $a(t) \leq 0$, $b(t) \geq 0$ and $c(t) < 0$. The following assertions are yet to be established:

  To generalise the assertions (i), (ii), (iii), (v), (vi) and (vii) of Proposition 1.2.8 to Eq. (2.1).

2.9 Notes

Lemmas 2.1.1 and 2.1.2 are the easy consequences of the definitions of Class I. Theorems 2.1.1, 2.1.2, 2.1.11–2.1.13 are due to Parhi and Das [28]. Theorems 2.1.4 and 2.1.5 are taken from Padhi [26], whereas Remarks 2.1.8 and 2.1.9 are adopted from Škerlik [36]. Theorems 2.1.7–2.1.9 are proved in Das [9]. Theorem 2.2.2 is taken from Padhi [26], whereas the rest part of Sect. 2.2 is brought from Parhi and Das [29]. Lemma 2.3.4 follows from Theorem 4.1 due to Hanan [16]. Theorems 2.3.3 and 2.3.4 are taken from Padhi [26]. The proof of Theorem 2.3.5 is as in proof of a theorem due to Jones [21], Theorem 2.3.11 is due to Ahmad and Lazer [1], and the rest part of Sect. 2.3 is taken from Parhi and Das [31]. Theorems 2.4.2 and 2.4.3 are due to Padhi [26], whereas Theorems 2.4.4–2.4.8, 2.4.11 and 2.4.12 are due to Parhi and Padhi [32]. Theorems 2.4.9–2.4.10, 2.8.1 and 2.4.13 are taken from Parhi and Das [30]. Theorems 2.5.1–2.5.11 and Lemma 2.5.1 are due to Parhi and Padhi [34]. Theorem 2.5.12 is taken from Padhi [27]. Theorem 2.5.16 follows from Theorem 4 due to Jones [20], whereas Theorem 2.5.17 follows from Theorem 1 by Jones [21]. Theorems 2.5.20 and 2.5.21 are due to Škerlik [37]. Theorem 2.6.5 is due to Gera [11]. Theorems 2.6.11 and 2.6.12 are taken from Dolan [10], and Theorem 2.6.13 is due to Parhi and Padhi [33], while Theorem 2.6.14 is adopted from Swanson [38]. The rest of Sect. 2.6 are due to Padhi [26]. Some of the results of Sect. 2.7 are adopted from [3–6] and Rovder [35].

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