Preface

Over the last four decades, intensive work has been carried out in the field of theory of nonautonomous third-order ordinary and delay differential equations. However, it is only recently that the global attractivity of third-order equations has been given a serious study. During these years, new investigations were developed and results of principal importance were obtained. In particular, suitable oscillation criteria for third-order linear and nonlinear differential equations were established, with emphasis on the oscillation of third-order nonhomogeneous differential equations with respect to oscillation and nonoscillation of third-order homogeneous differential equations. Existence and nonexistence conditions for oscillatory and nonoscillatory solutions of various types were found and asymptotic formulae for solutions of a sufficiently wide class of linear and nonlinear equations were derived. These studies notwithstanding, one would observe that oscillation and asymptotic behaviour of nonoscillatory solutions of third-order delay differential equations has rarely been studied.

In this book, an attempt is made to sum up these results. The necessity of such an attempt is felt especially as the well-known monograph of M. Greguš, Third Order Linear Differential Equations, Reidel, Dordrecht, The Netherlands, 1982, devoted to related topics, reflects results which are 30 years old.

This book consists of seven chapters. In the first one, we consider the third-order linear differential equation, with constant coefficients, of the form

$$x''' + ax'' + bx' + cx = 0, \tag{1}$$

where a, b and c are constants. Eight different cases of a, b and c were considered while dealing with (1). Further, it is observed that different structures of solution spaces of (1) appears for different cases on a, b and c. In this chapter, an introduction is given to the oscillation theory of the nonhomogeneous equation

$$x''' + ax'' + bx' + cx = f, \tag{2}$$

where a, b, c and f are constants. An apparatus of comparison theorems is developed, which allows us to establish criteria for (2) to have oscillatory or nonoscillatory solutions with the help of oscillation and nonoscillation of (1). An overall idea on oscillation of third-order delay differential equations is also included in this chapter. Additionally, some basic results are incorporated, which are needed in the forthcoming chapters.

Chapter 2 deals with the linear equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = 0, \tag{3}$$

where $a, b \in C^1([\sigma, \infty), R)$ and $c \in C([\sigma, \infty), R), \sigma \in R$. This chapter contains seven sections, where most of the results obtained for (1) have been generalised to (3). Many problems have been proposed in the sections or at the end of the chapter. Apart from these, some new nontrivial sufficient conditions for the oscillation of (3) are given. Further, several sufficient conditions, different from earlier ones, are given which can be applied to the third-order Euler equation

$$x''' + \frac{a_0}{t}x'' + \frac{b_0}{t^2}x' + \frac{c_0}{t^3}x = 0,$$

where a_0, b_0 and c_0 are real constants. Several comparison principles have been used to establish the nonoscillation of (3). We have established criteria for (3) to have a family of solutions asymptotically, equivalent to solutions of (3), and we studied the properties of this family. Finally, existence criteria for solutions vanishing at infinity have been explored in this chapter.

Chapter 3 is concerned with the oscillation of the equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = f(t), \tag{4}$$

where $a(t)$, $b(t)$ and $c(t)$ are as defined earlier and $f \in C([\sigma, \infty), R)$. Some of the eight different cases of $a(t)$, $b(t)$ and $c(t)$ have been considered while the remaining have been retained as open problems. We have applied the results of Chap. 2 to obtain sufficient conditions for the oscillation and nonoscillation of (4).

In Chap. 4, emphasis is given to the oscillation and nonoscillation of solutions and the asymptotic behaviour of the third-order nonlinear differential equations

$$x''' + b(t)x' + c(t)x^\alpha = 0,$$
$$x''' + b(t)x' + c(t)f(x) = 0,$$

and results of more general equations

$$x''' + a(t)x'' + b(t)x' + c(t)x^\alpha = 0,$$

and

$$x''' + a(t)x'' + b(t)x' + c(t)f(x) = 0,$$

where $a(t)$, $b(t)$ and $c(t)$ were defined earlier in (4) and $f : R \to R$ and some other restrictions on f. In this chapter, we solve the Kneser problem on monotone solutions and prove, under certain restrictions on f and α, that monotone solutions tend to zero eventually.

Chapter 5 is quite interesting. We have studied oscillation of solutions of the third-order nonlinear and nonhomogeneous equations of the form

$$\big(r(t)x''\big)' + q(t)\big(x'\big)^\beta + p(t)x^\alpha = f(t), \tag{5}$$

where r, p, q and $f \in C([\sigma, \infty), R)$ and $r(t) > 0$, α and β are quotient of odd integers. Along the way, we have given some interesting results on oscillation of the nonlinear equation

$$x''' + a(t)x'' + b(t)x' + c(t)x = f\left(t, x, x', x''\right), \tag{6}$$

where a, b and $c \in C([\sigma, \infty), R)$ and $f : [\sigma, \infty) \times R^3 \to R$.

In Chap. 6, we apply the results of Chap. 2 to obtain oscillation and asymptotic behaviour of nonoscillation of the third-order delay differential equation

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + c(t)x(g(t)) = 0, \tag{7}$$

where $a(t)$, $b(t)$ and $c(t)$ are as defined earlier and $g : R \to R$ with the property that $0 \le g(t) \le t$ and $\lim_{t\to\infty} g(t) = \infty$. It has been observed that the presence of the delay term $g(t)$ makes it difficult to make a direct study of the asymptotic behaviour of solutions of (7). However, the canonical transformations makes it possible to study the properties of solutions of (7). Equations (3) and (7) have been transformed to equivalent canonical equations and then the asymptotic behaviour of solutions of (7) have been studied by knowing the oscillation and nonoscillation of (3). We have classified the nonoscillatory solutions of (7) as Property A and Property B, according to their asymptotic behaviour.

In the last chapter, we study the global attractivity of solutions of equations

$$x''' + \psi\left(x, x'\right)x'' + f\left(x, x'\right) = 0 \tag{8}$$

and

$$x''' + \psi\left(x, x'\right)x'' + f\left(x, x'\right) = p(t), \tag{9}$$

where $\psi, f, \psi_x, f_x \in C(R \times R, R)$ and $p \in C([0, \infty), R)$. Lyapunov function has been used to obtain the results. A direct step by step method has been used to find the asymptotic stability of solutions of the third-order equation

$$\begin{aligned} x'''(t) = p_1x''(t) + p_2x''(t-\tau) + q_1x'(t) + q_2x'(t-\tau) + r_1x(t) \\ + r_2x(t-\tau), \quad t \ge 0, \end{aligned} \tag{10}$$

with the initial condition

$$x(t) = \phi(t), \quad t \in [-\tau, 0], \tag{11}$$

where p_1, p_2, q_1, q_2, r_1 and r_2 are real constants, $\tau > 0$ is a real number and $\phi \in C([-\tau, 0), R)$ is an initial function. The results are then applied to obtain stability and asymptotic stability of solutions of (11).

Some results are given in the form of open problems to be solved by the readers. We also state some problems whose solutions we do not know.

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