Chapter 2
Application to DC Circuits

In this chapter we use the results obtained in Chap. 1 to develop a new measurement based approach to solve synthesis problems in unknown linear direct current (DC) circuits. We consider two classes of synthesis problems: (1) current control problem, (2) power level control problem. A similar approach can be used for voltage control problems.

2.1 Introduction

Quite often, in a large scale circuit, the detailed model is not available and one may be interested in designing only a small set of, say one, two or three elements, which constitute the design variables. To solve such design problems it is desirable to determine the behavior of the system with respect to these design variables. In this chapter, we provide a new measurement based approach to answer this question by confining ourselves to the domain of linear DC circuits. The approach to be presented can be extended to linear AC circuits, mechanical systems, civil structures, hydraulic networks, transfer functions and parametrized controllers, as we show in later chapters.

Motivated by the question stated above, we pose the problem of determining a circuit signal, such as current, voltage, or power level, in a given branch of an unknown circuit as a function of the design elements located somewhere in the circuit. Note that this functional behavior is nonlinear, in general, even though the underlying circuit is linear.

Of course, the problem of solving the circuit for all the currents can be easily worked out if one knows the circuit model, by applying Kirchhoff's laws to form the linear equations of the system and solving them for the unknown currents. If the circuit model is unavailable, which is often the case in real world design situations, one can resort to experimentally determining this functional dependency by extensive experiments.

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Here, we present an alternative new method which can determine the functional dependency of any circuit variable with respect to any set of design variables directly from a small set of measurements. This has been shown in Chap. 1 to be applicable to any system described by linear equations. The obtained functional dependency can then be used to solve a synthesis problem wherein the circuit variable of interest is to be controlled by adjusting the design variables.

2.2 Current Control

In this section we consider circuit synthesis problems where the current in any branch of an *unknown* linear DC circuit is to be controlled or assigned by adjusting the design elements at arbitrary locations of the circuit. The approach provided here considers several cases where, for example, a single or multiple resistors are used as the design elements. Sources or amplifier gains can also be considered as the design elements.

Let us revisit Example 1.1 (see Fig. 2.1)

In this example, V and I are the ideal voltage and current sources, respectively; R_1, R_2, R_3 are linear branch resistors, and R_4 is a gyrator resistance. In order to make a distinction between a branch resistor and a gyrator resistance, henceforth, we refer to a branch resistor simply as a *resistor* and refer to the latter as a *gyrator resistance*. V_{amp} is the dependent voltage of the amplifier where $V_{\text{amp}} = KI_1$, and K is the amplifier gain. We also introduce the parameter vector and the vector of sources as

$$\mathbf{p} := \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ K \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} \quad \text{and} \quad \mathbf{q} := \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}. \tag{2.1}$$

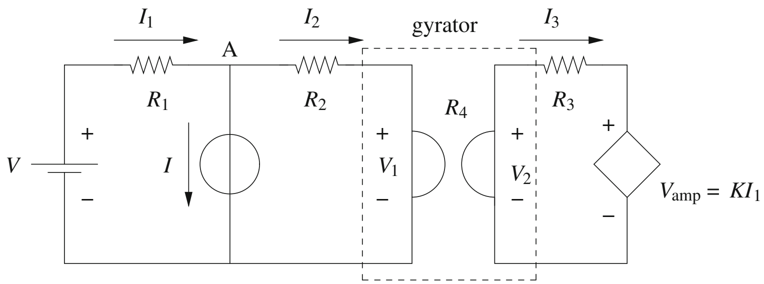


Fig. 2.1 A general circuit

Upon an application of Kirchhoff's current and voltage laws, one can write the equations of the system in the following matrix form,

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ R_1 & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}}_{\mathbf{A}(\mathbf{p})} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} I \\ V \\ 0 \end{bmatrix}}_{\mathbf{b}(\mathbf{q})}. \tag{2.2}$$

The governing equations of a linear DC circuit are represented in the following matrix form

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}), \tag{2.3}$$

where $\mathbf{A}(\mathbf{p})$ is the circuit characteristic matrix, $\mathbf{p}$ is the vector of circuit *design parameters*, including resistors, amplifier gains, gyrators, but excluding independent voltage and current sources, $\mathbf{x}$ is the vector of unknown currents and $\mathbf{q}$ is the vector of independent voltage and current *sources*. The vector $\mathbf{b}(\mathbf{q})$ can be written in the following form

$$\mathbf{b}(\mathbf{q}) = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + \cdots + q_m\mathbf{b}_m, \tag{2.4}$$

where $q_1, q_2, \ldots, q_m$ represent the independent sources. Suppose that our objective is to control the current in the i-th branch of the circuit, denoted by I_i. Applying Cramer's rule to (2.3), I_i can be calculated as

$$x_i = I_i = \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \tag{2.5}$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is the matrix obtained by replacing the i-th column of the characteristic matrix $\mathbf{A}(\mathbf{p})$ by the vector $\mathbf{b}(\mathbf{q})$. We emphasize that in an unknown circuit the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$ are unknown. However, based on Lemma 1.2 and (2.4), if the ranks of the parameters are known, a general rational function for the current I_i, in terms of the design elements can be derived, as given below.

$$I_i = \frac{\displaystyle\sum_{j_m=0}^{1} \cdots \sum_{j_1=0}^{1} \sum_{i_l=0}^{t_l} \cdots \sum_{i_1=0}^{t_1} \alpha_{i_1 \cdots i_l j_1 \cdots j_m}\, p_1^{i_1} \cdots p_l^{i_l} q_1^{j_1} \cdots q_m^{j_m}}{\displaystyle\sum_{i_l=0}^{r_l} \cdots \sum_{i_1=0}^{r_1} \beta_{i_1 \cdots i_l}\, p_1^{i_1} \cdots p_l^{i_l}}. \tag{2.6}$$

In the above formula, the vectors of numerator and denominator coefficients α and β are constants, $t_1, \ldots, t_l$ are the ranks of the coefficient matrices of the parameters $p_1, \ldots, p_l$ in the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$, and $r_1, \ldots, r_l$ are the ranks of the coefficient matrices of the parameters $p_1, \ldots, p_l$ in the matrix $\mathbf{A}(\mathbf{p})$.

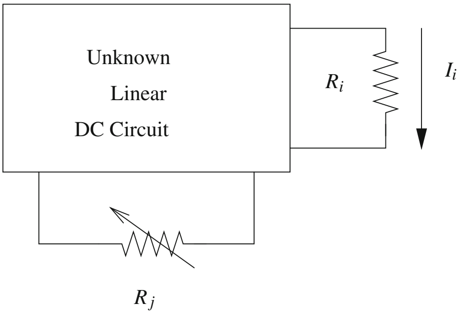


Fig. 2.2 An unknown linear DC circuit

2.2.1 Current Control Using a Single Resistor

Consider the unknown linear DC circuit shown in Fig. 2.2. Suppose that we wish to control the current in the i-th branch, denoted by I_i, by adjusting the resistor R_j at an arbitrary location of the circuit. Following Example 1.1, in general, the resistor R_j will appear in the matrix $\mathbf{A}$ in (2.3) with rank 1 dependency, unless it is a gyrator resistance, in which case the dependency is of rank 2.

Theorem 2.1 *In a linear DC circuit, the functional dependency of any current I_i on any resistance R_j can be determined by at most 3 measurements of the current I_i obtained for 3 different values of R_j.*

Proof. Let us consider two cases: (1) $i \neq j$, and (2) $i = j$.

Case 1: $i \neq j$

In this case, the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (2.5), are both of rank 1 with respect to R_j. According to Lemma 1.1, the functional dependency of I_i on R_j can be expressed as

$$I_i(R_j) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j}{\tilde{\beta_0} + \tilde{\beta_1} R_j}, \tag{2.7}$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\beta_0}, \tilde{\beta_1}$ are constants. If $\tilde{\beta_0} = \tilde{\beta_1} = 0$, then, $I_i \to \infty$, for any value of the resistance R_j, which is physically impossible. Therefore, we rule out this case. Assuming that $\tilde{\beta_1} \neq 0$, one can divide the numerator and denominator of (2.7) by $\tilde{\beta_1}$ and obtain

$$I_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j}, \tag{2.8}$$

where $\alpha_0, \alpha_1, \beta_0$ are constants. In order to determine $\alpha_0, \alpha_1, \beta_0$ one conducts 3 experiments by setting 3 different values to the resistance R_j, namely R_{j1}, R_{j2}, R_{j3}, and measuring the corresponding currents I_i, namely I_{i1}, I_{i2}, I_{i3}. Then, the following set of measurement equations can be formed

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & -I_{i1} \\ 1 & R_{j2} & -I_{i2} \\ 1 & R_{j3} & -I_{i3} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1} R_{j1} \\ I_{i2} R_{j2} \\ I_{i3} R_{j3} \end{bmatrix}}_{\mathbf{m}}. \tag{2.9}$$

The set of Eq. (2.9) can be uniquely solved for the constants $\alpha_0, \alpha_1, \beta_0$, if and only if $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$, the last column of the matrix $\mathbf{M}$ can be expressed as a linear combination of the first two columns because by assigning different values to the resistance R_j, the first two columns of $\mathbf{M}$ become linearly independent. In such a case, the functional dependency of I_i on R_j can be expressed as

$$I_i(R_j) = \alpha_0 + \alpha_1 R_j, \tag{2.10}$$

where α_0, α_1 are constants that can be determined from any two of the experiments conducted earlier. The functional dependency in (2.10) corresponds to the case where $\tilde{\beta_1} = 0$ in (2.7), and the numerator and denominator of (2.7) are divided by $\tilde{\beta_0}$.

Case 2: $i = j$

In this case, the matrix $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_i; however, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_i. According to Lemma 1.1, the functional dependency of I_i on R_i can be expressed as

$$I_i(R_i) = \frac{\tilde{\alpha_0}}{\tilde{\beta_0} + \tilde{\beta_1} R_i}, \tag{2.11}$$

where $\tilde{\alpha_0}, \tilde{\beta_0}, \tilde{\beta_1}$ are constants. Assuming that $\tilde{\beta_1} \neq 0$, and dividing the numerator and denominator of (2.11) by $\tilde{\beta_1}$, gives

$$I_i(R_i) = \frac{\alpha_0}{\beta_0 + R_i}, \tag{2.12}$$

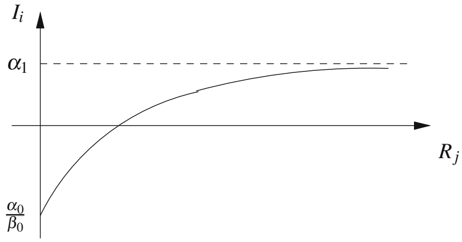
where α_0, β_0 are constants that can be determined by conducting 2 experiments, by setting 2 different values to the resistance R_i, namely R_{i1}, R_{i2}, and measuring the corresponding currents I_i, namely I_{i1}, I_{i2}. The following set of measurement equations can then be formed

$$\underbrace{\begin{bmatrix} 1 & -I_{i1} \\ 1 & -I_{i2} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1} R_{i1} \\ I_{i2} R_{i2} \end{bmatrix}}_{\mathbf{m}}. \tag{2.13}$$

The set of Eq. (2.13) can be uniquely solved for the constants α_0, β_0, provided $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$ in (2.13), it can be concluded that I_i is a constant,

$$I_i(R_i) = \alpha_0, \tag{2.14}$$

Fig. 2.3 Graph of (2.8) for $\beta_0 > 0$, $\alpha_0 < 0$ and $\alpha_1 > 0$



which can be determined from any of the experiments conducted earlier. In this case, the functional dependency in (2.14) corresponds to the situation where $\tilde{\beta_1} = 0$ in (2.11), and the numerator and denominator of (2.11) are divided by $\tilde{\beta_0}$. □

Remark 2.1 Suppose that $i \neq j$ and $|\mathbf{M}| \neq 0$ in (2.9), then the derivative of $I_i(R_j)$ in (2.8), with respect to R_j, can be calculated as

$$\frac{dI_i}{dR_j} = \frac{\alpha_1\beta_0 - \alpha_0}{(\beta_0 + R_j)^2}. \tag{2.15}$$

If $\beta_0 \geq 0$, we have the following:

1. The function (2.8) is monotonic in R_j, i.e. $I_i(R_j)$ monotonically increases or decreases as R_j increases from 0 to large values. The limiting values of this function are: $I_i(0) = \frac{\alpha_0}{\beta_0}$ and $I_i(\infty) = \alpha_1$. If $\frac{\alpha_0}{\beta_0} > \alpha_1$, then (2.8) will monotonically decrease, and if $\frac{\alpha_0}{\beta_0} < \alpha_1$, then (2.8) will monotonically increase.
2. The achievable range for I_i, by varying R_j in the interval $[0, \infty)$, is

$$\min\left\{\frac{\alpha_0}{\beta_0}, \alpha_1\right\} < I_i < \max\left\{\frac{\alpha_0}{\beta_0}, \alpha_1\right\}. \tag{2.16}$$

3. In a current control problem of this type, this monotonic behavior allows us to uniquely determine a range of values of the design parameter R_j, $R_j^- \leq R_j \leq R_j^+$, for which the current I_i lies within a desired prescribed range, $I_i^- \leq I_i \leq I_i^+$, which of course must be within the achievable range (2.16).

   These observations also are clear from the graph of (2.8). For instance, if $\beta_0 > 0$, $\alpha_0 < 0$ and $\alpha_1 > 0$, the graph of (2.8) has the general shape as depicted in Fig. 2.3.

If $\beta_0 < 0$, we have:

1. The function (2.8) is monotonic in R_j, in the intervals $[0, -\beta_0)$ and $(-\beta_0, \infty)$. If $\alpha_1\beta_0 - \alpha_0 > 0$, then I_i starts at $\frac{\alpha_0}{\beta_0}$ and monotonically increases to $+\infty$ as $R_j \to -\beta_0^-$, then, at $R_j \to -\beta_0^+$ it starts from $-\infty$ and monotonically increases to α_1 as $R_j \to \infty$. If $\alpha_1\beta_0 - \alpha_0 < 0$, I_i starts at $\frac{\alpha_0}{\beta_0}$ and monotonically decreases to $-\infty$ as $R_j \to -\beta_0^-$, then, at $R_j \to -\beta_0^+$ it starts from $+\infty$ and monotonically decreases to α_1 as $R_j \to \infty$.

2. The achievable range for I_i, by varying R_j in the interval $[0, \infty)$, is

$$I_i \in \left(-\infty, \min\left\{\frac{\alpha_0}{\beta_0}, \alpha_1\right\}\right) \cup \left(\max\left\{\frac{\alpha_0}{\beta_0}, \alpha_1\right\}, +\infty\right). \tag{2.17}$$

3. Similarly, one can uniquely determine a range of values of the design parameter R_j, $R_j^- \le R_j \le R_j^+$, for which the current I_i lies within a desired prescribed range, $I_i^- \le I_i \le I_i^+$, which of course must be within the achievable range (2.17).

The graph of (2.8) for this case again clearly illustrates (see Fig. 2.4).

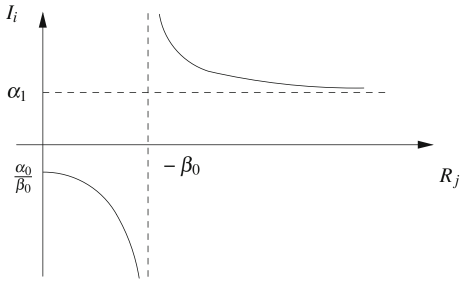
Thevenin's Theorem (the special case $i = j$)
Thevenin's Theorem of circuit theory follows as a special case of the results developed here. To see this, consider the current functional dependency given in (2.12). From this relationship, it is clear that the short circuit current I_{sc} is given by $I_{sc} = \frac{\alpha_0}{\beta_0}$, which is obtained by setting $R_i = 0$. Similarly, the open circuit voltage V_{oc} is obtained by multiplying both sides of (2.12) by R_i and taking the limit as $R_i \to \infty$. This yields $V_{oc} = V_{Th} = \alpha_0$. Thus, the Thevenin resistance is given by $R_{Th} = \frac{V_{oc}}{I_{sc}} = \beta_0$, so that (2.12) becomes

$$I_i(R_i) = \frac{V_{Th}}{R_{Th} + R_i}, \tag{2.18}$$

which is exactly Thevenin's Theorem. We point out that in our approach, it is not necessary to measure short circuit current or open circuit voltage; indeed two *arbitrary* measurements suffice. This has practical and useful implications in circuits where short circuiting and open circuiting may sometimes be impossible.

Remark 2.2 (*Generalization of Thevenin's Theorem*) Theorem 2.1 and the subsequent results in this chapter represent generalizations of Thevenin's Theorem. In Thevenin's Theorem, the current in a resistor/impedance connected to an arbitrary network can be obtained by representing the network by a voltage source and a resistance/impedance and these can be determined from short circuit and open circuit measurements made at these terminals. We have shown that the resistor can be connected at a point different from the point where measurements are made and that the current can be predicted from arbitrary measurements, not necessarily short

Fig. 2.4 Graph of (2.8) for $\beta_0 < 0$, $\alpha_0 > 0$ and $\alpha_1 > 0$



or open circuit. The results given in the subsequent chapters can be thought of as Thevenin-like results for mechanical, hydraulic, truss and other systems.

Current Assignment Problem

Once the functional dependency of interest, (2.8), (2.10), (2.12) or (2.14) is obtained, a synthesis problem can be solved. For instance, suppose that it is required to assign $I_i = I_i^*$, where I_i^* is a desired prescribed value of the current in the i-th branch of the unknown circuit. Let us assume that the design variable is the resistance R_j, and $i \neq j$. How can one find a value of R_j for which $I_i = I_i^*$? Based on the Theorem 2.1, since $i \neq j$, one conducts 3 experiments by setting 3 different values to the resistance R_j, and measuring the corresponding currents I_i. The matrix $\mathbf{M}$ in (2.9) can then be evaluated from the measurements. If $|\mathbf{M}| \neq 0$, then the functional dependency of interest will be of the form given in (2.8), and if $|\mathbf{M}| = 0$, it will be of the form obtained in (2.10). Suppose that $|\mathbf{M}| \neq 0$ is the case; hence, the functional dependency of interest is as the one given in (2.8). In order to determine the value of R_j, for which the desired current I_i^* is attained, one may solve (2.8) for R_j, with $I_i = I_i^*$,

$$R_j(I_i^*) = \frac{\alpha_0 - I_i^* \beta_0}{I_i^* - \alpha_1}. \tag{2.19}$$

Interval Design Problem

Suppose now that the current I_i is to be controlled to stay within the following range (which is inside the achievable range (2.16)),

$$I_i^- \leq I_i \leq I_i^+, \tag{2.20}$$

by adjusting the design resistance R_j, $i \neq j$. Also, assume that after conducting 3 experiments, we found $|\mathbf{M}| \neq 0$ in (2.9) and $\beta_0 \geq 0$. Therefore, the functional dependency of I_i on R_j is of the form (2.8) and is monotonic. Thus, one may find a unique corresponding interval for R_j values where (2.20) is satisfied. Supposing that I_i, in (2.8), monotonically increases as R_j increases, one gets

$$R_j^- \leq R_j \leq R_j^+, \tag{2.21}$$

where

$$R_j^- = \frac{\alpha_0 - I_i^- \beta_0}{I_i^- - \alpha_1}, \qquad R_j^+ = \frac{\alpha_0 - I_i^+ \beta_0}{I_i^+ - \alpha_1}. \tag{2.22}$$

If I_i, in (2.8), monotonically decreases as R_j increases; then, R_j^- and R_j^+ in (2.21) can be calculated from

$$R_j^- = \frac{\alpha_0 - I_i^+ \beta_0}{I_i^+ - \alpha_1}, \qquad R_j^+ = \frac{\alpha_0 - I_i^- \beta_0}{I_i^- - \alpha_1}. \tag{2.23}$$

Following the same strategy, one may solve a synthesis problem for the case $i = j$. The problem of maintaining several currents in the circuit within prescribed intervals can be solved similarly. Also, the discussion above pertains to detection of short and open circuit faults.

2.2.2 Current Control Using Two Resistors

Consider the unknown linear DC circuit shown in Fig. 2.5. Suppose we want to control the current in the i-th branch, denoted by I_i, by adjusting any two resistors R_j and R_k at arbitrary locations of the circuit. Assume, as before, that R_j and R_k are not gyrator resistances.

Theorem 2.2 *In a linear DC circuit, the functional dependency of any current I_i on any two resistances R_j and R_k can be determined by at most 7 measurements of the current I_i obtained for 7 different sets of values (R_j, R_k).*

Proof. Let us consider two cases: (1) $i \neq j, k$ and (2) $i = j$ or $i = k$.

Case 1: $i \neq j, k$

In this case, the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (2.5), are both of rank 1 with respect to R_j and R_k. Based on Lemma 1.2, the functional dependency of I_i on R_j and R_k can be expressed as

$$I_i(R_j, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j + \tilde{\alpha_2} R_k + \tilde{\alpha_3} R_j R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_j + \tilde{\beta_2} R_k + \tilde{\beta_3} R_j R_k}, \tag{2.24}$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\alpha_3}, \tilde{\beta_0}, \tilde{\beta_1}, \tilde{\beta_2}, \tilde{\beta_3}$ are constants. Assuming that $\tilde{\beta_3} \neq 0$ and dividing the numerator and denominator of (2.24) by $\tilde{\beta_3}$, yields

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + \beta_1 R_j + \beta_2 R_k + R_j R_k}, \tag{2.25}$$

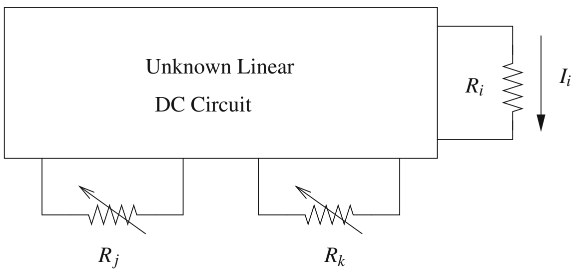


Fig. 2.5 An unknown linear DC circuit

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ are constants. In order to determine these constants, one conducts 7 experiments by assigning 7 different sets of values to the resistances (R_j, R_k), and measuring the corresponding currents I_i. The following set of measurement equations will be obtained

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -I_{i6} & -I_{i6}R_{j6} & -I_{i6}R_{k6} \\ 1 & R_{j7} & R_{k7} & R_{j7}R_{k7} & -I_{i7} & -I_{i7}R_{j7} & -I_{i7}R_{k7} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{j1}R_{k1} \\ I_{i2}R_{j2}R_{k2} \\ I_{i3}R_{j3}R_{k3} \\ I_{i4}R_{j4}R_{k4} \\ I_{i5}R_{j5}R_{k5} \\ I_{i6}R_{j6}R_{k6} \\ I_{i7}R_{j7}R_{k7} \end{bmatrix}}_{\mathbf{m}}. \quad (2.26)$$

This set of equations can be uniquely solved for the constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$, if and only if $|\mathbf{M}| \neq 0$ in (2.26). In the case where $|\mathbf{M}| = 0$, one can follow the same procedure used in Sect. 2.2.1 to derive the corresponding functional dependency of I_i on R_j and R_k. We provide the details of this case in the Appendix.

Case 2: $i = j$ or $i = k$

Suppose that $i = j$ and recall (2.5). In this case, the matrix $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_i and R_k; however, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_i and is of rank 1 with respect to R_k. According to Lemma 1.2 and these rank conditions, the functional dependency of I_i on R_i and R_k can be expressed as

$$I_i(R_i, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_i + \tilde{\beta_2} R_k + \tilde{\beta_3} R_i R_k}, \quad (2.27)$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\beta_0}, \tilde{\beta_1}, \tilde{\beta_2}, \tilde{\beta_3}$ are constants. Assuming that $\tilde{\beta_3} \neq 0$, one can divide the numerator and denominator of (2.27) by $\tilde{\beta_3}$ and obtain

$$I_i(R_i, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + \beta_1 R_i + \beta_2 R_k + R_i R_k}, \quad (2.28)$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$ are constants that can be determined by conducting 5 experiments, by assigning 5 different sets of values to the resistances (R_i, R_k), and measuring the corresponding currents I_i. The following set of measurement equations can then be formed

$$\underbrace{\begin{bmatrix} 1 & R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{j1}R_{k1} \\ I_{i2}R_{j2}R_{k2} \\ I_{i3}R_{j3}R_{k3} \\ I_{i4}R_{j4}R_{k4} \\ I_{i5}R_{j5}R_{k5} \end{bmatrix}}_{\mathbf{m}}. \quad (2.29)$$

Again, this set of equations can be uniquely solved for the constants $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$, provided $|\mathbf{M}| \neq 0$ in (2.29). If $|\mathbf{M}| = 0$, following the same strategy used in Sect. 2.2.1, one can derive the corresponding functional dependency of I_i on R_i and R_k. The details of this case can be found in the Appendix. □

In this problem, the current I_i can be plotted as a surface in a 3D graph. In a synthesis problem of this type, any constraint on the current I_i results in a corresponding region in the R_j–R_k plane, if the solution set for that constraint is not empty.

2.2.3 Current Control Using m Resistors

Consider the unknown linear DC circuit shown in Fig. 2.6.
Suppose that the objective is to control the current in the i-th branch of the circuit, denoted by I_i, by adjusting any m resistors R_j, $j = 1, 2, \ldots, m$, at arbitrary locations of the circuit. Assume that the resistances R_j, $j = 1, 2, \ldots, m$, are not gyrator resistances.

Theorem 2.3 *In a linear DC circuit, the functional dependency of any current I_i on any m resistances R_j, $j = 1, 2, \ldots, m$, can be determined by at most $2^{m+1} - 1$ measurements of the current I_i obtained for $2^{m+1} - 1$ different sets on values of the vector $(R_1, R_2, \ldots, R_m)$.*

Proof. Let us consider two cases: (1) $i \neq j$ for $j = 1, 2, \ldots, m$, and (2) $i = j$ for some $j = 1, 2, \ldots, m$.

Case 1: $i \neq j$, $j = 1, 2, \ldots, m$

In this case, the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (2.5), are both of rank 1 with respect to R_j, $j = 1, 2, \ldots, m$. Hence, based on Lemma 1.2, the functional dependency of I_i on $R_1, R_2, \ldots, R_m$ can be written as

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\alpha}_{i_1 i_2 \cdots i_m} \, R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\beta}_{i_1 i_2 \cdots i_m} \, R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \quad (2.30)$$

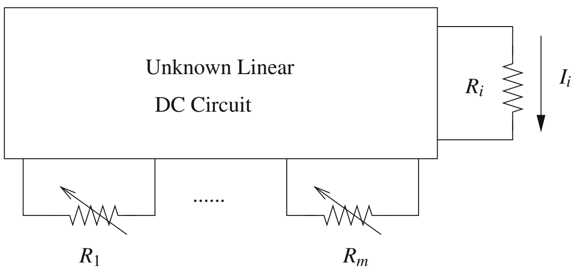


Fig. 2.6 An unknown linear DC circuit

where $\tilde{\alpha}_{i_1 i_2 \cdots i_m}$'s and $\tilde{\beta}_{i_1 i_2 \cdots i_m}$'s are constants. Assuming that $\tilde{\beta}_{11\ldots 1} \neq 0$ and dividing the numerator and denominator of (2.30) by $\tilde{\beta}_{11\ldots 1}$, results in

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \alpha_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \beta_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \quad (2.31)$$

where $\beta_{11\ldots 1} = 1$, and $\alpha_{i_1 i_2 \cdots i_m}$'s and $\beta_{i_1 i_2 \cdots i_m}$'s are $2^{m+1} - 1$ constants. In order to determine these constants, one conducts $2^{m+1} - 1$ experiments, by setting $2^{m+1} - 1$ different sets of values to the resistances $(R_1, R_2, \ldots, R_m)$, and measuring the corresponding currents I_i. The obtained set of measurement equations has a unique solution for the constants if and only if $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$ is the case, then one can follow the same procedure explained for the previous problems to derive the corresponding functional dependency.

Case 2: $i = j$ for some $j = 1, 2, \ldots, m$

Without loss of generality, suppose that $i = m$ and recall (2.5). In this case, the matrix $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_j, $j = 1, 2, \ldots, m$; however, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_m and is of rank 1 with respect to R_j, $j = 1, 2, \ldots, m-1$. According to these rank conditions and based on Lemma 1.2, the functional dependency of I_i on $R_1, R_2, \ldots, R_m$ will be

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_{m-1}=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\alpha}_{i_1 i_2 \cdots i_{m-1}} \; R_1^{i_1} R_2^{i_2} \cdots R_{m-1}^{i_{m-1}}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\beta}_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \quad (2.32)$$

where $\tilde{\alpha}_{i_1 i_2 \cdots i_{m-1}}$'s and $\tilde{\beta}_{i_1 i_2 \cdots i_m}$'s are constants. Supposing $\tilde{\beta}_{11\ldots 1} \neq 0$, one can divide the numerator and denominator of (2.32) by $\tilde{\beta}_{11\ldots 1}$ and get

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_{m-1}=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \alpha_{i_1 i_2 \cdots i_{m-1}} \; R_1^{i_1} R_2^{i_2} \cdots R_{m-1}^{i_{m-1}}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \beta_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \quad (2.33)$$

where $\beta_{11\ldots 1} = 1$, and there are $3\left(2^{m-1}\right) - 1$ constants. These constants can be determined by conducting $3\left(2^{m-1}\right) - 1$ experiments, by assigning $3\left(2^{m-1}\right) - 1$ different sets of values to the resistances $(R_1, R_2, \ldots, R_m)$, and measuring the corresponding currents I_i. The obtained system of measurement equations has a unique solution for the constants if and only if $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$, following the same strategy presented for the previous problems, one can determine the corresponding functional dependency. □

2.2.4 Current Control Using Gyrator Resistance

In this problem we consider the design element to be the resistance of a gyrator. The gyrator resistance appears in the matrix $\mathbf{A}(\mathbf{p})$ with rank 2 dependency. We want to control the current I_i by a gyrator resistance, denoted by R_g, at an arbitrary location of the circuit.

Theorem 2.4 *In a linear DC circuit, the functional dependency of any current I_i on any gyrator resistance R_g can be determined by at most 5 measurements of the current I_i obtained for 5 different values of R_g.*

Proof. Let us consider the following two cases:

(1) the i-th branch is not connected to either port of the gyrator(Fig.2.7a),
(2) the i-th branch is connected to one port of the gyrator (Fig.2.7b).

Case 1: The i-th branch is not connected to either port of the gyrator

In this case, the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (2.5), are both of rank 2 with respect to R_g. Therefore, according to Lemma 1.2, the functional dependency of I_i on R_g can be expressed as

$$I_i(R_g) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_g + \tilde{\alpha_2} R_g^2}{\tilde{\beta_0} + \tilde{\beta_1} R_g + \tilde{\beta_2} R_g^2}, \tag{2.34}$$

where $\tilde{\alpha_0}$, $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\beta_0}$, $\tilde{\beta_1}$, $\tilde{\beta_2}$ are constants. Assuming that $\tilde{\beta_2} \neq 0$, one can divide the numerator and denominator of (2.34) by $\tilde{\beta_2}$ and obtain

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g + \alpha_2 R_g^2}{\beta_0 + \beta_1 R_g + R_g^2}, \tag{2.35}$$

where α_0, α_1, α_2, β_0, β_1 are constants. In order to determine these constants, one conducts 5 experiments by setting 5 different values to the gyrator resistance R_g, and measuring the corresponding currents I_i. In this case, the set of measurement equations will be

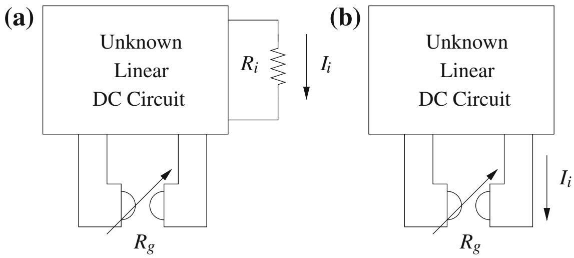


Fig. 2.7 An unknown linear DC circuit

$$\underbrace{\begin{bmatrix} 1 & R_{g1} & R_{g1}^2 & -I_{i1} & -I_{i1}R_{g1} \\ 1 & R_{g2} & R_{g2}^2 & -I_{i2} & -I_{i2}R_{g2} \\ 1 & R_{g3} & R_{g3}^2 & -I_{i3} & -I_{i3}R_{g3} \\ 1 & R_{g4} & R_{g4}^2 & -I_{i4} & -I_{i4}R_{g4} \\ 1 & R_{g5} & R_{g5}^2 & -I_{i5} & -I_{i5}R_{g5} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{g1}^2 \\ I_{i2}R_{g2}^2 \\ I_{i3}R_{g3}^2 \\ I_{i4}R_{g4}^2 \\ I_{i5}R_{g5}^2 \end{bmatrix}}_{\mathbf{m}}, \tag{2.36}$$

which has a unique solution for the constants $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$, if and only if $|\mathbf{M}| \neq 0$ in (2.36). If $|\mathbf{M}| = 0$ is the case, one can use the same procedure presented in Sect. 2.2.1 to derive the corresponding functional dependency of I_i on R_g. The details of this case are provided in the Appendix.

Case 2: The i-th branch is connected to one port of the gyrator

In this case, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 1 with respect to R_g; however, the matrix $\mathbf{A}(\mathbf{p})$ is of rank 2 with respect to R_g. Therefore, using Lemma 1.2, the functional dependency of I_i on R_g can be written as

$$I_i(R_g) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_g}{\tilde{\beta_0} + \tilde{\beta_1} R_g + \tilde{\beta_2} R_g^2}, \tag{2.37}$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\beta_0}, \tilde{\beta_1}, \tilde{\beta_2}$ are constants. Supposing $\tilde{\beta_2} \neq 0$ and dividing the numerator and denominator of (2.37) by $\tilde{\beta_2}$, one gets

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g}{\beta_0 + \beta_1 R_g + R_g^2}, \tag{2.38}$$

where α_0, α_1, β_0, β_1 are constants that can be determined by conducting 4 experiments, by assigning 4 different values to the gyrator resistance R_g, and measuring the corresponding currents I_i. Then, the following set of measurement equations can be formed

$$\underbrace{\begin{bmatrix} 1 & R_{g1} & -I_{i1} & -I_{i1}R_{g1} \\ 1 & R_{g2} & -I_{i2} & -I_{i2}R_{g2} \\ 1 & R_{g3} & -I_{i3} & -I_{i3}R_{g3} \\ 1 & R_{g4} & -I_{i4} & -I_{i4}R_{g4} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{g1}^2 \\ I_{i2}R_{g2}^2 \\ I_{i3}R_{g3}^2 \\ I_{i4}R_{g4}^2 \end{bmatrix}}_{\mathbf{m}}. \tag{2.39}$$

As before, the system of Eq. (2.39) can be uniquely solved for the constants α_0, α_1, β_0, β_1, provided $|\mathbf{M}| \neq 0$. For the situations where $|\mathbf{M}| = 0$, one can follow the same procedure used in Sect. 2.2.1 to find the corresponding functional dependency of I_i on R_g. The details for this case are presented in the Appendix. □

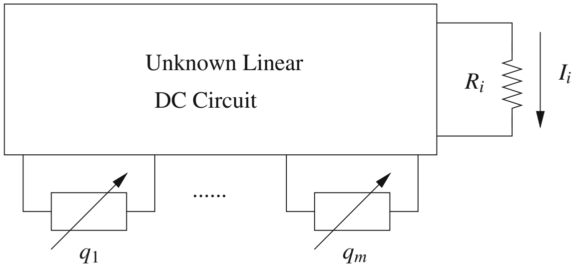


Fig. 2.8 An unknown linear DC circuit

2.2.5 Current Control Using m Independent Sources

Here, we consider the problem of controlling the current in the i-th branch of an unknown linear DC circuit, denoted by I_i, by only using the independent current/voltage sources, denoted by $\mathbf{q} = [q_1, q_2, \ldots, q_m]^T$, at arbitrary locations of the circuit (Fig. 2.8).

Theorem 2.5 *In a linear DC circuit, the functional dependency of any current I_i on the independent sources can be determined by m measurements of the current I_i obtained for m linearly independent sets of values of the source vector* $\mathbf{q}$.

Proof. Recall (2.4),

$$\mathbf{b}(\mathbf{q}) = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + \cdots + q_m\mathbf{b}_m, \tag{2.40}$$

where $q_1, q_2, \ldots, q_m$ represent the independent sources. The matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ in (2.5) can be written as

$$\mathbf{B}_i(\mathbf{p}, \mathbf{q}) = [\mathbf{A}_1(\mathbf{p}), \ldots, \mathbf{A}_{i-1}(\mathbf{p}), \mathbf{b}(\mathbf{q}), \mathbf{A}_{i+1}(\mathbf{p}), \ldots, \mathbf{A}_n(\mathbf{p})]. \tag{2.41}$$

Therefore, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 1 with respect to each of the independent sources $q_1, q_2, \ldots, q_m$ and $|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|$ can be written as a linear combination of the parameters $q_1, q_2, \ldots, q_m$,

$$|\mathbf{B}_i(\mathbf{p}, \mathbf{q})| = q_1|\mathbf{B}_{i1}(\mathbf{p})| + q_2|\mathbf{B}_{i2}(\mathbf{p})| + \cdots + q_m|\mathbf{B}_{im}(\mathbf{p})|, \tag{2.42}$$

where

$$\mathbf{B}_{ij}(\mathbf{p}) = [\mathbf{A}_1(\mathbf{p}), \ldots, \mathbf{A}_{i-1}(\mathbf{p}), \mathbf{b}_j, \mathbf{A}_{i+1}(\mathbf{p}), \ldots, \mathbf{A}_n(\mathbf{p})], \tag{2.43}$$

for $j = 1, 2, \ldots, m$. The matrix $\mathbf{A}(\mathbf{p})$ is of rank 0 with respect to $q_1, q_2, \ldots, q_m$ and thus, according to Lemma 1.1, $|\mathbf{A}(\mathbf{p})|$ is a constant. Hence, the functional dependency of I_i on $q_1, q_2, \ldots, q_m$ can be expressed as

$$\begin{aligned} I_i(\mathbf{q}) &:= I_i(q_1, q_2, \ldots, q_m) \\ &= \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_m q_m, \end{aligned} \tag{2.44}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are constants that can be determined by setting m linearly independent sets of values to the independent sources $(q_1, q_2, \ldots, q_m)$, measuring the corresponding values of the current I_i, and solving the obtained set of measurement equations. □

Remark 2.3

1. Theorem 2.5 is the well-known Superposition Principle of circuit theory.
2. If the independent sources vary in the intervals $q_j^- \leq q_j \leq q_j^+$, $j = 1, 2, \ldots, m$, then the current I_i will vary in an interval whose end values can be computed using the vertices (q_j^-, q_j^+), $j = 1, 2, \ldots, m$. For example suppose that I_i is given as below,

$$I_i(\mathbf{q}) = 2q_1 - q_2 + 5q_3 - 3q_4, \tag{2.45}$$

where $q_j^- \leq q_j \leq q_j^+$, $q_j^- \geq 0$, $j = 1, 2, 3, 4$. One may decompose I_i as

$$I_i(\mathbf{q}) = 2q_1 - q_2 + 5q_3 - 3q_4 = (2q_1 + 5q_3) - (q_2 + 3q_4). \tag{2.46}$$

Then, the maximum and minimum values of I_i, denoted by $I_i^{\max}$ and $I_i^{\min}$, respectively, can be obtained from

$$I_i^{\max} = (2q_1^+ + 5q_3^+) - (q_2^- + 3q_4^-), \tag{2.47}$$

$$I_i^{\min} = (2q_1^- + 5q_3^-) - (q_2^+ + 3q_4^+). \tag{2.48}$$

2.3 Power Level Control

In this section we consider another class of circuit synthesis problems where, in an unknown linear DC circuit, the power level in a resistor is to be controlled by adjusting the design elements at arbitrary locations of the circuit. For the sake of simplicity, suppose that the resistor R_i is located in the i-th branch of the circuit and we wish to control the power level P_i, in the resistor R_i, by some design elements. As in the previous section, we consider several cases of design elements and provide the results for each case.

2.3.1 Power Level Control Using a Single Resistor

In this subsection we show how to control the power level P_i in the resistor R_i, located in the i-th branch of an unknown linear DC circuit, by adjusting any resistor R_j at an arbitrary location of the circuit. Assume that R_j is not a gyrator resistance and recall the results developed in Sect. 2.2.1.

Theorem 2.6 *In a linear DC circuit, the functional dependency of the power level P_i, in the resistor R_i, on any resistance R_j can be determined by at most 3 measurements of the current I_i (passing through R_i) obtained for 3 different values of R_j, and 1 measurement of the voltage across the resistor R_i, corresponding to one of the resistance settings.*

Proof. Let us consider two cases: (1) $i \neq j$, and (2) $i = j$.

Case 1: $i \neq j$

We can write the power level P_i as $P_i = \frac{V_i}{I_i} I_i^2$. The functional dependency of the current I_i, passing through R_i, on any other resistance R_j will be of either forms (2.8) or (2.10). Since the ratio $\frac{V_i}{I_i}$ is the same for each experiment, then only one measurement of the voltage V_i, across the resistor R_i, in addition to the 3 measurements of the current I_i, is required to determine the functional dependency of P_i on R_j. Assuming one measures V_{i1} from the first experiment, then the functional dependency of P_i on R_j can be expressed as one of the following forms:

- If $|\mathbf{M}| \neq 0$ in (2.9):
$$P_i(R_j) = \frac{V_{i1}}{I_{i1}} \left(\frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \right)^2, \tag{2.49}$$
where V_{i1} and I_{i1} are the voltage and current signals, at the resistor R_i, measured from the first experiment, and the constants $\alpha_0, \alpha_1, \beta_0$ are obtained by solving (2.9), as explained in Sect. 2.2.1.
- If $|\mathbf{M}| = 0$ in (2.9):
$$P_i(R_j) = \frac{V_{i1}}{I_{i1}} (\alpha_0 + \alpha_1 R_j)^2, \tag{2.50}$$
where V_{i1} and I_{i1} are the voltage and current signals, at the resistor R_i, measured from the first experiment, and the constants α_0, α_1 can be determined using any two of the conducted experiments, as discussed in Sect.2.2.1.

Case 2: $i = j$

Let us write the power level P_i as $P_i = R_i I_i^2$. Based on the results of Sect.2.2.1, the functional dependency of I_i on R_i will be of either forms given in (2.12) or (2.14). Hence, the functional dependency of P_i on R_i will be of one the following forms:

- If $|\mathbf{M}| \neq 0$ in (2.13):
$$P_i(R_i) = R_i \left(\frac{\alpha_0}{\beta_0 + R_i} \right)^2, \tag{2.51}$$

where the constants α_0, β_0 can be obtained as explained in Sect. 2.2.1.

- If $|\mathbf{M}| = 0$ in (2.13):

$$P_i(R_i) = \alpha_0^2 R_i, \tag{2.52}$$

where α_0 is a constant that can be determined as discussed in Sect. 2.2.1. □

Remark 2.4 Suppose that $i = j$ and $|\mathbf{M}| \neq 0$ in (2.13), then the derivative of P_i, in (2.51), with respect to R_i, can be calculated as

$$\frac{dP_i}{dR_i} = \frac{\alpha_0^2(\beta_0 - R_i)}{(\beta_0 + R_i)^3}. \tag{2.53}$$

We have the following statements:

1. The functional dependency of P_i on R_i, in (2.51), in this case, is *not* monotonic. As $R_i \to 0$, $P_i \to 0$ and when $R_i \to \infty$, $P_i \to 0$. Therefore, as the value of the resistance R_i increases from 0 to ∞, the power P_i increases from 0 to the maximum achievable value of $\frac{\alpha_0^2}{4\beta_0}$, and then decreases to 0 at very large values of R_i. The maximum occurs at $R_i = \beta_0$.
2. The achievable range for the power level P_i, by varying the resistance R_i in the interval $[0, \infty)$, is

$$0 \leq P_i < \frac{\alpha_0^2}{4\beta_0}. \tag{2.54}$$

3. In a power level control problem of this type, for any desired prescribed interval of power P_i, which is within the achievable range (2.54), one may find *two* ranges of values for the design resistance R_i.

2.3.2 *Power Level Control Using Two Resistors*

In this case it is desired to control the power level P_i, by adjusting any two resistors R_j and R_k at arbitrary locations of the circuit. Assuming that R_j and R_k are not gyrator resistances, and based on the results of Sect. 2.2.2, we have the following theorem.

Theorem 2.7 *In a linear DC circuit, the functional dependency of the power level P_i, in any resistor R_i, on any two resistances R_j and R_k can be determined by at most 7 measurements of the currents I_i (passing through R_i) obtained for 7 different sets of values (R_j, R_k), and 1 measurement of the voltage across the resistor R_i, corresponding to one of the resistance settings.*

Proof. The proof is similar to the previous case and thus omitted here. □

In this problem, the power level P_i can be depicted as a surface in a 3D plot. In a synthesis problem of this type, any constraint on the power level P_i results in a corresponding region in the R_j–R_k plane, if the solution set to that constraint is not empty.

Remark 2.5 For the case of m resistors, the functional dependencies can be derived similarly using the results given in Sect. 2.2.3.

2.3.3 Power Level Control Using Gyrator Resistance

Here, we want to control the power level P_i using any gyrator resistance R_g, at an arbitrary location of the circuit. The functional dependency of the current I_i on any gyrator resistance R_g is obtained in Sect. 2.2.4. Applying the same technique, one can find the functional dependency of P_i on any gyrator resistance R_g. We leave the details to the reader.

2.4 Examples of DC Circuit Design

Example 2.1. In this example we show how the method proposed in Sects. 2.2.1 and 2.3.1 can be used toward control design problems in unknown linear DC circuits. Consider the unknown circuit shown in Fig. 2.9.

In this example, it is desired to find the functional dependency of the current I_1 on the resistance R_9. Based on the results given in Sect. 2.2.1, one conducts 3 experiments, by setting 3 different values to R_9, and measuring the corresponding currents I_1. Suppose that experiments are done and let Table 2.1 summarize the numerical values, for this example, obtained from the 3 experiments.

Substituting the numerical values obtained from the experiments into the matrix $\mathbf{M}$ in (2.9) resulted in $|\mathbf{M}| \neq 0$. Therefore, (2.9) can be uniquely solved for the constants

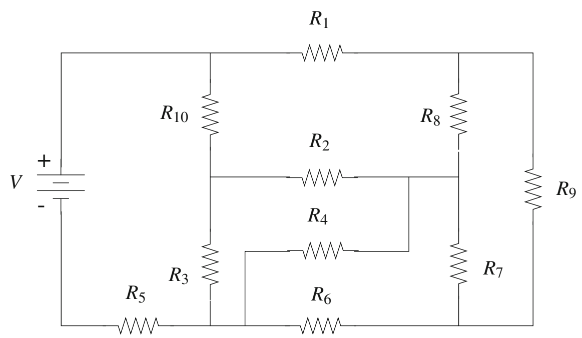


Fig. 2.9 An unknown resistive circuit

Table 2.1 Numerical values of the measurements for the DC circuit Example 2.1

Exp. no.	R_9 (Ω)	I_1 (A)
1	1	0.054
2	5	0.056
3	10	0.058

and yield the following functional dependency which is plotted in Fig. 2.10.

$$I_1(R_9) = \frac{78.4 + 0.66R_9}{181.3 + R_9}. \tag{2.55}$$

Remark 2.6

1. The current I_1 monotonically increases as R_9 increases.
2. By varying R_9 in the range $[0, \infty)$, the achievable range for I_1 becomes $[\frac{\alpha_0}{\beta_0}, \alpha_1] = [0.43, 0.66]$.
3. In a synthesis problem where the current I_1 is to be controlled to stay within an acceptable interval, since I_1 is monotonic in R_9, one can find a corresponding interval for R_9 values for which the current I_1 stays within the acceptable range.

Suppose that we wish to design R_9 such that I_1 lies within the following achievable range

$$0.5 \leq I_1 \leq 0.6\ (A). \tag{2.56}$$

Using (2.55), or Fig. 2.10, the corresponding range for the design resistor R_9 can be obtained as

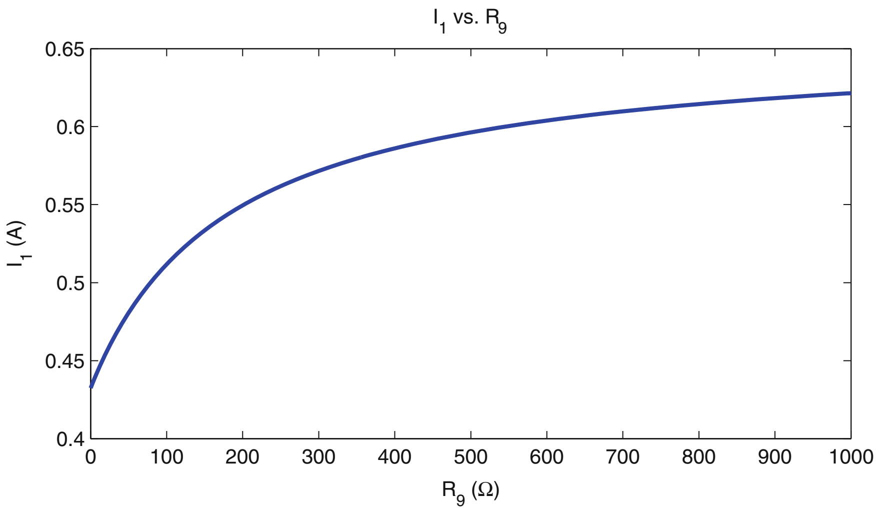


Fig. 2.10 I_1 versus R_9

Table 2.2 Numerical values of the measurements for the DC circuit Example 2.2

Exp. no.	R_9 (Ω)	I_1 (A)	I_3 (A)	I_9 (A)
1	1	0.437	0.964	0.301
2	5	0.438	0.972	0.295
3	10	0.444	0.982	0.287
Exp. no.	R_9 (Ω)	V_1 (V)	V_3 (V)	
1	1	8.67	4.82	

$$79 \leq R_9 \leq 550 \ (\Omega). \tag{2.57}$$

Example 2.2. Consider the same circuit as above (Fig. 2.9). Suppose now that the power levels within R_1, R_3 and R_9, denoted by P_1, P_3 and P_9, respectively, must remain in the following ranges:

$$6 \ (W) \leq P_1 \leq 7 \ (W), \tag{2.58}$$

$$7 \ (W) \leq P_3 \leq 8 \ (W), \tag{2.59}$$

$$3 \ (W) \leq P_9 \leq 3.5 \ (W). \tag{2.60}$$

Assume that the design resistor is R_9. Based on the results of Sect. 2.3.1, one conducts 3 experiments by assigning 3 different values to R_9, and measuring the corresponding currents I_1, I_3 and I_9, passing through the resistors R_1, R_3 and R_9, respectively. In this problem, one also needs to measure the voltage across R_1 and R_3 from one of the experiments. Suppose that the experiments are done and let Table 2.2 summarize the numerical values for this example, obtained from the experiments.

Substituting the numerical values from Table 2.2 into the matrix $\mathbf{M}$ in (2.9), for the currents I_1 and I_3, and into the matrix $\mathbf{M}$ in (2.13), for the current I_9, yields $|\mathbf{M}| \neq 0$, for all cases. Therefore, the functional dependencies of P_1, P_3 and P_9 on R_9 will be

$$P_1(R_9) = \frac{8.67}{0.437}\left(\frac{78.4 + 0.66R_9}{181.3 + R_9}\right)^2, \tag{2.61}$$

$$P_3(R_9) = \frac{4.82}{0.964}\left(\frac{174.4 + 1.34R_9}{181.3 + R_9}\right)^2, \tag{2.62}$$

$$P_9(R_9) = R_9\left(\frac{54.9}{181.3 + R_9}\right)^2. \tag{2.63}$$

Figure 2.11 shows the plots of the power levels P_1, P_3 and P_9 obtained above.

Using (2.61)–(2.63), shown graphically in Fig. 2.11, one imposes the power level constraints (2.58)–(2.60) to find the corresponding ranges of R_9 values. A necessary condition for the existence of a solution is that the constraints (2.58)–(2.60) must be within their corresponding achievable ranges. For this example, the power level constraints are within the achievable ranges; hence, we can find the following ranges for R_9 values:

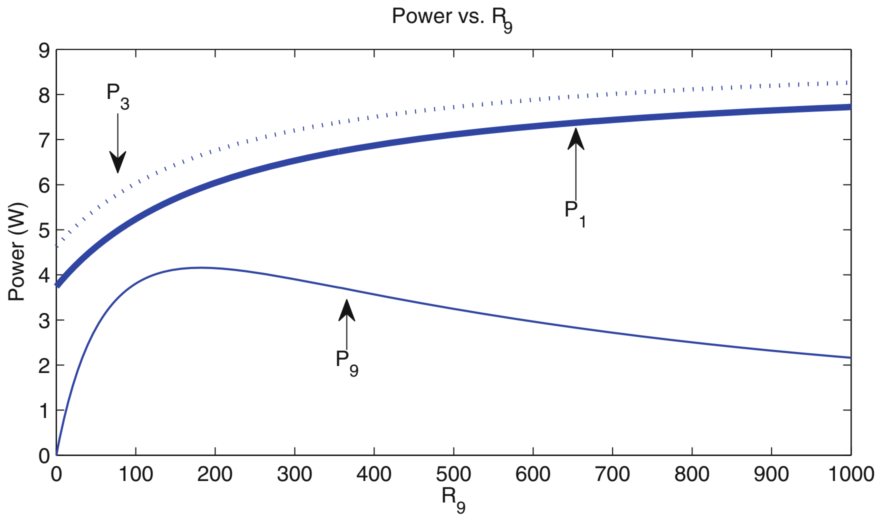


Fig. 2.11 P_1, P_3, P_9 versus R_9

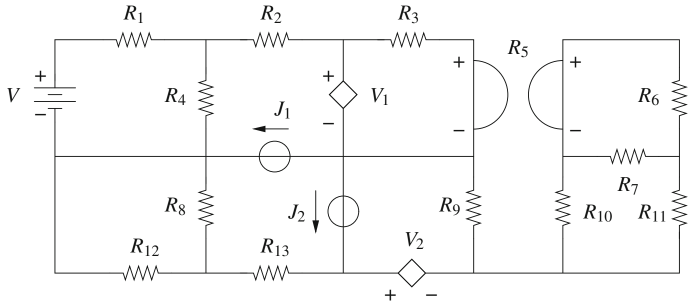


Fig. 2.12 An unknown linear DC circuit

$$190\ (\Omega) \leq R_9 \leq 450\ (\Omega), \tag{2.64}$$

$$250\ (\Omega) \leq R_9 \leq 690\ (\Omega), \tag{2.65}$$

$$60\ (\Omega) \leq R_9 \leq 80\ \cup\ 420\ (\Omega) \leq R_9 \leq 580\ (\Omega), \tag{2.66}$$

corresponding to the power level constraints (2.58), (2.59) and (2.60), respectively. Therefore, the range for R_9 values where (2.58), (2.59) and (2.60) are achieved simultaneously is the intersection of the ranges calculated above, that is

$$420\ (\Omega) \leq R_9 \leq 450\ (\Omega). \tag{2.67}$$

Example 2.3. Consider the unknown linear DC circuit shown in Fig. 2.12.

In this example, R_i, $i = 1, 2, \ldots, 13$, $i \neq 5$ are resistors, R_5 is a gyrator resistance, V, J_1, J_2 are independent sources and V_1, V_2 are dependent sources. Our goal is to control the power levels in R_3, R_6 and R_{11}, denoted by P_3, P_6 and P_{11}, respectively, to be within the following ranges:

$$40\ (W) \leq P_3 \leq 60\ (W), \tag{2.68}$$

$$1\ (W) \leq P_6 \leq 8\ (W), \tag{2.69}$$

$$0.5\ (W) \leq P_{11} \leq 5\ (W). \tag{2.70}$$

Assume that the design elements are the resistances R_1 and R_6. Therefore, we need to find the region in the R_1–R_6 plane where the constraints (2.68), (2.69) and (2.70) are satisfied. Based on the approach presented in Sect. 2.3.2, in order to find the functional dependency of any power level in terms of any two resistances, one needs to do at most 7 measurements of current and one measurement of voltage. Let us treat each power level problem separately as follows:

(a) P_3 versus R_1 and R_6
Based on the results obtained in Sect. 2.3.2, in order to find the functional dependency of P_3 on R_1 and R_6, one needs to conduct 7 experiments by setting 7 different sets of values for the resistances (R_1, R_6), and measuring the corresponding values for current I_3. In addition to the current measurements, one measurement of the voltage, across the resistor R_3, is needed to determine the functional dependency of interest. Suppose that this measurement is taken from the first experiment and denoted as V_{31}. Suppose that experiments are done and let Table 2.3 summarize the numerical values assigned to the resistances R_1 and R_6 along with the corresponding measurements of I_3 and V_{31}. Substituting the numerical values of Table 2.3 into the matrix $\mathbf{M}$, in (2.26), it can be verified that $|\mathbf{M}| \neq 0$. Thus, the functional dependency of interest will be of the form

$$P_3(R_1, R_6) = \frac{V_{31}}{I_{31}} \underbrace{\left(\frac{\alpha_0 + \alpha_1 R_1 + \alpha_2 R_6 + \alpha_3 R_1 R_6}{\beta_0 + \beta_1 R_1 + \beta_2 R_6 + R_1 R_6} \right)^2}_{I_3^2(R_1, R_6)}, \tag{2.71}$$

Table 2.3 Numerical values of the measurements for the DC circuit Example 2.3

Exp. no.	$R_1(\Omega)$	$R_6(\Omega)$	$I_3(A)$
1	7	1	3.33
2	13	8	2.71
3	21	19	2.47
4	35	26	2.57
5	40	32	2.52
6	52	45	2.47
7	59	56	2.44
Exp. no.	$R_1(\Omega)$	$R_6(\Omega)$	$V_{31}(V)$
1	7	1	33.3

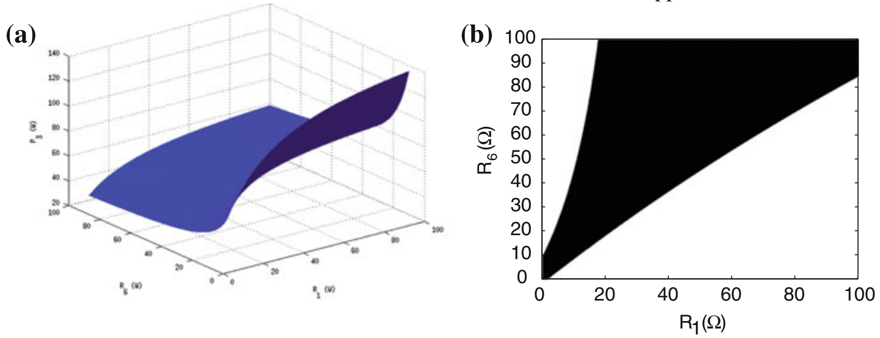


Fig. 2.13 **a** P_3 versus R_1 and R_6. **b** Region (in *black color*) where (2.68) is satisfied

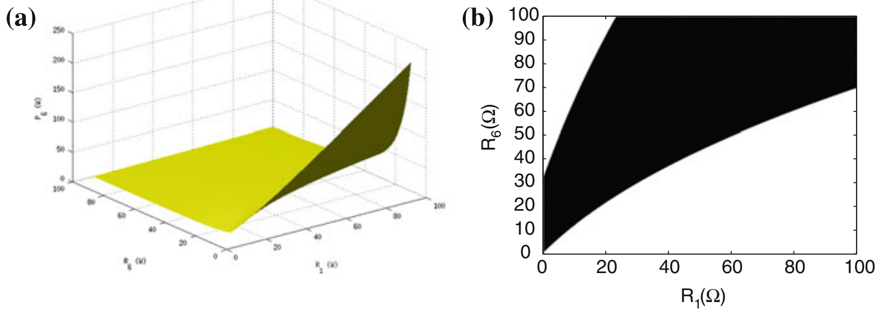


Fig. 2.14 **a** P_6 versus R_1 and R_6. **b** Region (in *black color*) where (2.69) is satisfied

where the constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ can be determined by solving (2.26), using the numerical values of Table 2.3. For this example, the constants are obtained as: $\alpha_0 = 98.4$, $\alpha_1 = 36$, $\alpha_2 = 6.6$, $\alpha_3 = 2.4$, $\beta_0 = 58.5$, $\beta_1 = 5$, $\beta_2 = 11.7$. Hence, the functional dependency of P_3 on R_1 and R_6 will be

$$P_3(R_1, R_6) = \frac{33.3}{3.33}\left(\frac{98.4 + 36R_1 + 6.6R_6 + 2.4R_1R_6}{58.5 + 5R_1 + 11.7R_6 + R_1R_6}\right)^2. \tag{2.72}$$

Figure 2.13(a) shows the plot of the surface P_3 as a function of the design elements R_1 and R_6, obtained in (2.72). Applying constraint (2.68) on P_3, one may obtain the region in the R_1–R_6 plane, shown in black color in Fig. 2.13(b), where this constraint is satisfied.

(b) P_6 versus R_1 and R_6

The functional dependency of P_6 on R_1 and R_6 can be determined by at most 5 measurements of current and one measurement of voltage as discussed in Sect. 2.3.2 (Case 2). The plot of the surface P_6 as a function of R_1 and R_6 is shown in Fig. 2.14(a). Applying constraint (2.69) on P_6, one may obtain the region in the R_1–R_6 plane, shown in black color in Fig. 2.14(b), where this constraint is valid.

(c) P_{11} versus R_1 and R_6

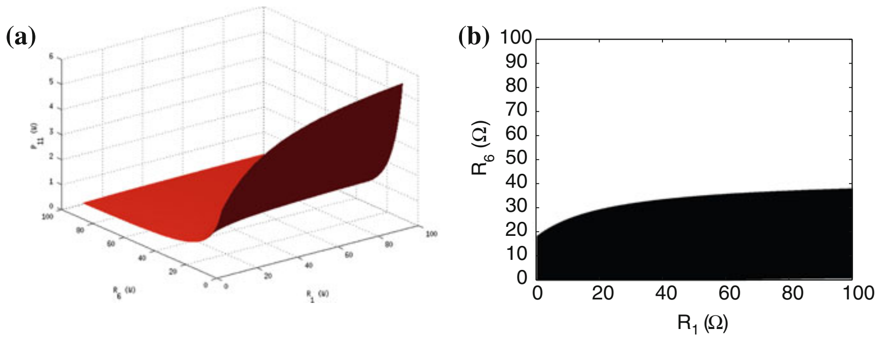


Fig. 2.15 **a** P_{11} versus R_1 and R_6. **b** Region (in *black color*) where (2.70) is satisfied

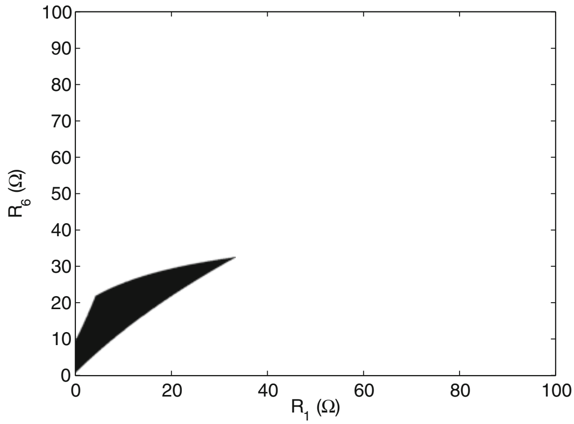


Fig. 2.16 Region (in *black color*) where (2.68), (2.69) and (2.70) are simultaneously satisfied

Following the same procedure used to determine the functional dependency of P_3 on R_1 and R_6, one can determine the dependency of P_{11} on R_1 and R_6. The plot of the surface P_{11} as a function of R_1 and R_6 is shown in Fig. 2.15(a). Applying constraint (2.70) on P_{11}, one finds the region in the R_1–R_6 plane, shown in black color in Fig. 2.15(b), where this constraint is satisfied.

In order to satisfy the constraints given in (2.68), (2.69) and (2.70), simultaneously, one has to intersect the regions shown in Figs. 2.13(b), 2.14(b), 2.15(b). Figure 2.16 shows the region (in black color) in the R_1–R_6 plane where constraints (2.68), (2.69) and (2.70) are satisfied, simultaneously.

2.5 Notes and References

We have shown in this chapter that the design of linear DC circuits can be carried out without knowledge of the circuit model, provided a few measurements can be made. These measurements, strategically processed, yield complete information regarding the functional dependency of the current, voltage or power to be controlled, on the design variables. These relations can then be inverted to extract the design parameters. In the subsequent chapters we show the application of these ideas to AC circuits, mechanical systems, block diagrams and control systems.

The main results of this chapter are taken from [1–5]. A functional dependency of linear fractional form was first introduced in [6]. Also, some related works in the area of symbolic network functions can be found in [7, 8]. This approach can be extended to transfer functions and parametrized controllers [9]. Applications of Kirchhoff's laws in solving circuit analysis problems wherein the circuit model is available is given in [10–12]. Thevenin's theorem of circuit theory can be found in [13–15].

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