

Chapter 2

The Use of Valuations for Classifying Point-Line Geometries

Bart De Bruyn

Abstract A valuation is a map from the point set of a point-line geometry $\mathcal{S}$ to the set $\mathbb{N}$ of nonnegative integers satisfying a number of well-chosen axioms. These axioms are chosen in such a way that these objects are (potentially) useful for classifying certain point-line geometries that contain an isomorphic copy of $\mathcal{S}$ as a full subgeometry. Depending on the list of chosen axioms, different “valuation theories” can be developed. The importance of each such theory depends on the successes that can be achieved in classification problems. Valuations have been successfully used for classifying certain near polygons (in particular generalized polygons). They were also useful tools for obtaining a number of other results. The aim of this paper is to survey some of these results.

Keywords Valuation · Generalized polygon · Near polygon

1 Introduction

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathbf{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *partial linear space* if every two distinct points are incident with at most one line. In this paper, we will always assume that the point set $\mathcal{P}$ and the line set $\mathcal{L}$ of a point-line geometry $\mathcal{S}$ are disjoint. The distance $d(x_1, x_2)$ between two points x_1 and x_2 of $\mathcal{S}$ is measured in its collinearity graph. If $x \in \mathcal{P}$ and $\emptyset \neq X \subseteq \mathcal{P}$, then $d(x, X)$ denotes the minimal distance between x and a point of X . If $x \in \mathcal{P}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance

MSC2000: 51E12, 05B25.

B. De Bruyn (✉)

Department of Mathematics, Ghent University, Gent, Belgium

e-mail: bdb@cage.ugent.be

i from x . A point-line geometry is said to have *order* (s, t) if every line is incident with precisely $s + 1$ points and if every point is incident with precisely $t + 1$ lines.

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is called a *subgeometry* of a point-line geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ if $\mathcal{P} \subseteq \mathcal{P}'$, $\mathcal{L} \subseteq \mathcal{L}'$ and $\mathbf{I} = \mathbf{I}' \cap (\mathcal{P} \times \mathcal{L})$. If $\{x \in \mathcal{P} \mid x \mathbf{I} L\} = \{x \in \mathcal{P}' \mid x \mathbf{I}' L\}$ for every line L of $\mathcal{S}$, then the subgeometry $\mathcal{S}$ of $\mathcal{S}'$ is called *full*. Consider the following problem.

Given a family $\mathcal{F}$ of partial linear spaces and a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$, determine those members $\mathcal{S}'$ of $\mathcal{F}$ that contain an isomorphic copy of $\mathcal{S}$ as a full (isometrically embedded) subgeometry.

A possible way to approach this problem is trying to reconstruct $\mathcal{S}'$ from its subgeometry $\mathcal{S}$. Such a reconstruction can be realized by labeling the points and lines of $\mathcal{S}'$ by certain objects of $\mathcal{S}$ and indicating which points are incident with what lines. The objects of $\mathcal{S}$ suitable for that purposes might depend on the specific situation. The approach followed in the present paper aims to label the points of $\mathcal{S}'$ by certain maps from $\mathcal{P}$ to $\mathbb{N}$, and the lines of $\mathcal{S}'$ by certain sets of those maps.

Suppose $\mathcal{V}$ is a family of maps from $\mathcal{P}$ to $\mathbb{N}$, defined by a certain list of axioms. Suppose the following can be shown: for every point-line geometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ belonging to $\mathcal{F}$ and having $\mathcal{S}$ as a full subgeometry and for every point $x \in \mathcal{P}'$, the map $f_x : \mathcal{P} \rightarrow \mathbb{N}; y \mapsto d(x, y) - \min\{d(x, z) \mid z \in \mathcal{P}\}$ belongs to $\mathcal{V}$. Then information regarding the elements of $\mathcal{V}$ (like for instance a whole classification) can sometimes be turned into information on how $\mathcal{S}$ can be embedded as a full subgeometry into a member of $\mathcal{F}$. For classification purposes, it might often also be worthwhile to study the following problem: for two maps $f_1, f_2 \in \mathcal{V}$, determine a list of conditions for pairs $(f_1, f_2) \in \mathcal{V} \times \mathcal{V}$ such that if x_1 and x_2 are distinct collinear points of $\mathcal{S}'$ then the pair (f_{x_1}, f_{x_2}) satisfies all these conditions. Observe that these conditions may only involve properties of $\mathcal{S}$ not of $\mathcal{S}'$. Indeed, in the classification problem the geometry $\mathcal{S}$ is known while $\mathcal{S}'$ is not ($\mathcal{S}'$ needs to be determined).

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a partial linear space, then a *valuation* of $\mathcal{S}$ can be understood as a certain map from $\mathcal{P}$ to $\mathbb{N}$ satisfying a number of axioms, chosen in such a way that the resulting “valuation theory” is interesting enough to be valuable in actual classifications. Concrete examples will be given later.

Valuations have effectively been used to obtain new classification results for point-line geometries. In the present paper we will give an overview of some of these results. All of them concern a class of point-line geometries called *near polygons*. Valuations have found some other applications as well, and a few of these will also be mentioned here.

2 The Basics of Valuation Theory

A partial linear space is called a *near polygon* if for every point x and every line L there exists a unique point y on L nearest to x . If $d \in \mathbb{N}$ is the maximal distance between two points, then the near polygon is called a *near $2d$ -gon*. Near polygons

were introduced in [33]. The study of near polygons is often restricted to some interesting classes.

(1) A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance two have at least two common neighbors. An important class of dense near polygons are the so-called dual polar spaces with at least three points on each line. These are related to polar spaces. With every polar space Π of rank $n \geq 1$ there is associated a *dual polar space* of rank n . This is the point-line geometry whose points are the maximal singular subspaces of Π (those of dimension $n - 1$) and whose lines are the next-to-maximal singular subspaces of Π (those of dimension $n - 2$), with incidence being reverse containment. By convention, a dual polar space of rank 0 is a point (no lines). The automorphism groups of finite thick (dual) polar spaces provide examples of so-called groups of Lie type.

(2) A near $2d$ -gon with $d \in \mathbb{N} \setminus \{0, 1\}$ is called a *generalized $2d$ -gon* if every point is incident with at least two lines and if for every two points x and y at distance $i \in \{1, 2, \dots, d - 1\}$ from each other there exists a unique point collinear with y at distance $i - 1$ from x . Usually, generalized $2d$ -gons are defined differently, namely as partial linear spaces $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ that have no subgeometries that are ordinary k -gons with $k \in \{3, 4, \dots, 2d - 1\}$, but “sufficiently many” subgeometries that are ordinary $2d$ -gons. With sufficiently many we mean that for every two elements $A, B \in \mathcal{P} \cup \mathcal{L}$ there should exist a subgeometry $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ with $A, B \in \mathcal{P}' \cup \mathcal{L}'$ that is an ordinary $2d$ -gon.

(3) A finite near $2d$ -gon with $d \in \mathbb{N} \setminus \{0, 1\}$ is called *regular* if it has some order (s, t) and if there exist constants $t_i, i \in \{0, 1, \dots, d\}$, such that for every two points x and y at distance i from each other there are $t_i + 1$ lines through y containing a (necessarily unique) point at distance $i - 1$ from x . If this is the case, then $t_0 = -1$, $t_1 = 0$ and $t_d = t$. Finite thick dual polar spaces of rank d and finite generalized $2d$ -gons are examples of regular near $2d$ -gons. The regular near $2d$ -gons are precisely those near $2d$ -gons whose collinearity graph is a so-called *distance-regular graph* [2].

A *semi-valuation* of a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a map $f : \mathcal{P} \rightarrow \mathbb{Z}$ satisfying the property that every line L contains a (unique) point x_L^* such that $f(x) = f(x_L^*) + 1$ for every point x of L distinct from x_L^* . The number $f(x)$ is called the *f-value* of x , or shortly the *value* of x if no confusion is possible. If f is a semi-valuation attaining a maximal value, then the set of all points with non-maximal value is a *hyperplane* H_f of $\mathcal{S}$, that is, a set of points distinct from $\mathcal{P}$ intersecting (the point set of) each line in either a singleton or the whole line. We call H_f the *hyperplane of $\mathcal{S}$ associated with f* . Two semi-valuations f_1 and f_2 of $\mathcal{S}$ are called *neighboring* if there exists an $\epsilon \in \mathbb{Z}$ such that $|f_1(x) - f_2(x) + \epsilon| \leq 1$ for every point $x \in \mathcal{P}$. The notion of (neighboring) semi-valuation(s) was introduced in [11].

Suppose $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is a near $2d$ -gon and $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ is a full subgeometry of $\mathcal{S}'$. For every point x of $\mathcal{S}'$ and every point y of $\mathcal{S}$, we define $f_x(y) := d(x, y) - m_x$, where $m_x = d(x, \mathcal{P})$. Then the map f_x satisfies the following properties.

(V1) There exists a point y of $\mathcal{S}$ for which $f_x(y) = 0$.

(V2) f_x is a semi-valuation of $\mathcal{S}$.

Moreover, the following can be said.

Proposition 2.1. *If x_1 and x_2 are two distinct collinear points of $\mathcal{S}'$, then the semi-valuations f_{x_1} and f_{x_2} are neighboring.*

Proof. Put $\epsilon = m_{x_1} - m_{x_2}$. Then for every point x of $\mathcal{S}$, we have $|f_{x_1}(x) - f_{x_2}(x) + \epsilon| = |(f_{x_1}(x) + m_{x_1}) - (f_{x_2}(x) + m_{x_2})| = |d(x, x_1) - d(x, x_2)| \leq d(x_1, x_2) = 1$.

We have seen that with every semi-valuation f attaining a maximal value there is associated a hyperplane H_f . One can now wonder whether the semi-valuation f can be reconstructed from its associated hyperplane H_f . The following proposition gives sufficient conditions for that to be the case.

Proposition 2.2. *Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a connected partial linear space with finite diameter. Let $f : \mathcal{P} \rightarrow \mathbb{N}$ be a map satisfying the properties (V1) and (V2) mentioned above. Suppose that every point with non-maximal f -value is collinear with a point whose f -value is one more. Let H_f denote the hyperplane associated with f and let M denote the maximal distance from a point of $\mathcal{S}$ to $\mathcal{P} \setminus H_f$. Then $f(x) = M - d(x, \mathcal{P} \setminus H_f)$ for every point x of $\mathcal{S}$.*

Proof. By Property (V2), we must have that $M - f(x) \leq d(x, \mathcal{P} \setminus H_f)$. The fact that every point with non-maximal f -value is collinear with a point whose f -value is one more implies that $d(x, \mathcal{P} \setminus H_f) \leq M - f(x)$.

Suppose $\mathcal{S}$ is a partial linear space. Let $f_i, i \in I$, be a collection of $|I| \geq 2$ mutually distinct maps from $\mathcal{P}$ to $\mathbb{N}$ satisfying the properties (V1) and (V2) mentioned above. We also assume that every $f_i, i \in I$, attains a maximal value M_i . Then we say that the set $\mathcal{F} = \{f_i \mid i \in I\}$ is an L -set if for every point x of $\mathcal{S}$, there exists a (necessary unique) $i \in I$ such that $f_j(x) - M_j = f_i(x) - M_i + 1$ for every $j \in I \setminus \{i\}$. The notion of an L -set was introduced in [14].

Proposition 2.3. *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a partial linear space. Let $f_i, i \in I$, be a collection of mutually distinct maps from $\mathcal{P}$ to $\mathbb{N}$ satisfying the properties (V1) and (V2) mentioned above. Suppose that each $f_i (i \in I)$ attains a maximal value M_i and let H_i denote its associated hyperplane. If $\{f_i \mid i \in I\}$ is an L -set, then each point of $\mathcal{S}$ is contained in either one or all hyperplanes of the collection $H_i, i \in I$.*

Proof. Let $x \in \mathcal{P}$. Since the numbers $f_i(x) - M_i, i \in I$, are nonpositive, the fact that $\{f_i \mid i \in I\}$ is an L -set implies that either one or all these numbers are (strictly) negative, implying that x belongs to either one or all the hyperplanes $H_i, i \in I$.

Proposition 2.4. *Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathbf{I}')$ be a near $2d$ -gon having the partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ as a full subgeometry, and let L' be a line of $\mathcal{S}'$. Suppose there exists a constant δ such that for every point x of L' the maximal distance from a point of $\mathcal{S}$ to x is equal to δ . For every point x of L' and every point y of $\mathcal{S}$, we put $f_x(y) = d(x, y) - m_x$ where $m_x = d(x, \mathcal{P})$. Then the following hold.*

- (1) *If the maps $f_x, x \in L'$, are mutually distinct, then $\{f_x \mid x \in L'\}$ is an L -set.*
- (2) *If there exists a line L in $\mathcal{S}$ at distance $\delta - 1$ from L' , then the maps $f_x, x \in L'$, are mutually distinct and $\{f_x \mid x \in L'\}$ is an L -set.*

Proof. (1) Suppose the maps $f_x, x \in L'$, are mutually distinct. For every point x of L' , the maximal value that f_x attains is equal to $\delta - m_x$. Now, let y be an arbitrary point of $\mathcal{S}$. If x denotes the unique point of L' nearest to y , then for every point $z \neq x$ on L' , we have $f_z(y) - M_z = d(y, z) - m_z - M_z = d(y, z) - \delta = d(y, x) + 1 - \delta = d(y, x) - m_x - M_x + 1 = f_x(y) - M_x + 1$. This implies that $\{f_x \mid x \in L'\}$ is an L-set.

(2) Suppose there exists a line L in $\mathcal{S}$ at distance $\delta - 1$ from L' . In view of (1), it suffices to show that the maps $f_x, x \in L'$, are mutually distinct. So, consider two distinct points x'_1 and x'_2 of L' and let $x_i, i \in \{1, 2\}$, denote the unique point of L nearest to x'_i . Since $d(x'_i, L) \geq \delta - 1$ and $d(x'_i, y) \leq \delta$ for every point y of L , we necessarily have $d(x_i, x'_i) = \delta - 1$. Now, $x_1 \neq x_2$, otherwise the point $x_1 = x_2$ would lie at distance at most $\delta - 2$ from a point of L' since it lies at distance $\delta - 1$ from the points x'_1 and x'_2 . Suppose now that $f_{x'_1} = f_{x'_2}$. Since $d(x'_1, x_2) = \delta$ and $d(x'_2, x_2) = \delta - 1$, the fact that $f_{x'_1}(x_2) = f_{x'_2}(x_2)$ implies that $m_{x'_1} = m_{x'_2} + 1$. On the other hand, since $d(x'_2, x_1) = \delta$ and $d(x'_1, x_1) = \delta - 1$, we would also have $m_{x'_2} = m_{x'_1} + 1$. As this is impossible, we should have $f_{x'_1} \neq f_{x'_2}$ as we needed to prove.

3 Valuations of Dense Near Polygons

3.1 Basic Definitions

Let $\mathcal{S}$ be a dense near polygon. A set X of points of $\mathcal{S}$ is called a *subspace* if every line of $\mathcal{S}$ having two of its points in X has all its points in X . A *subspace* X of $\mathcal{S}$ is called *convex* if every point on a shortest path between two points of X is also contained in X . If X is a convex subspace of $\mathcal{S}$, then the lines of $\mathcal{S}$ that have all their points in X define a point-line geometry $\tilde{X}$ on the set X which itself is also a near polygon. The maximal distance between two points of a convex subspace X is called the *diameter* of X . By results of [33, Proposition 2.5] and [3, Theorem 4], every two points of a dense near polygon at distance δ from each other are contained in a unique convex subspace of diameter δ . These convex subspaces are called *quads* if $\delta = 2$ and *hexes* if $\delta = 3$. The fundamental property of dense near polygons regarding the existence of convex subspaces allows to build a typical valuation theory for dense near polygons.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ be a dense near polygon. A map $f : \mathcal{P} \rightarrow \mathbb{N}$ is called a *valuation* if the following three properties are satisfied.

- (V1) There exists at least one point x with $f(x) = 0$.
- (V2) f is a semi-valuation.
- (V3) Every point x of $\mathcal{S}$ is contained in a convex subspace F_x that satisfies the following properties:
 - $f(y) \leq f(x)$ for every point y of F_x ;
 - every point z of $\mathcal{S}$ which is collinear with a point y of F_x and which satisfies $f(z) = f(y) - 1$ also belongs to F_x .

Valuations of dense near polygons were introduced in [18]. It can be shown, see [18, Proposition 2.5], that the convex subspace F_x mentioned in Property (V3) is unique. For several classes of dense near polygons, it can be shown that Property (V3) is a consequence of Property (V2), see [10, Corollary 1.4].

That the above definition of valuation can serve as a suitable basis for a “valuation theory of dense near polygons” follows from the following two results. The first result was proved in [18, Proposition 2.6].

Proposition 3.1. *Let $S' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ be a dense near $2d$ -gon and let $S = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a dense near polygon which is isometrically embedded as a full subgeometry in S' . (So, $d_{S'}(x_1, x_2) = d_S(x_1, x_2)$ for any two points x_1 and x_2 of S .) Let x be a point of S' and put $m_x := d(x, \mathcal{P})$. For every point $y \in \mathcal{P}$, we define $f_x(y) := d(x, y) - m_x$. Then f_x is a valuation of S , the so-called valuation of S induced by x .*

Proposition 3.2. *Let $S' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ be a dense near $2d$ -gon and let $S = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ be a dense near $2d$ -gon which is fully and isometrically embedded into S' . Then for every line L' of S' , there exists a line L in S at distance $d - 1$ from L' . As a consequence the lines of S' will induce L -sets of valuations of S .*

Proof. We give a sketch of the proof. We implicitly make use of some known properties of dense near polygons.

Let y be a point of S at maximal distance from L' and let x denote the point of L' nearest to y . Then L' and y are contained in a unique convex subspace F_1 of diameter δ of S' where $\delta = d(x, y) + 1$. The points x and y are contained in a unique convex subspace F_2 of diameter $\delta - 1$ of S' .

We show that $\delta = d$. Suppose to the contrary that $\delta < d$. Since $F_1 \cap \mathcal{P}$ is a convex subspace of S of diameter at most $\delta < d$, there exists a line K of S through y not contained in F_1 . Any point of $K \setminus \{y\}$ would lie at distance $d(y, L') + 1$ from L' which is impossible. Hence, $\delta = d$ and $d(x, y) = d - 1$. Since $F_2 \cap \mathcal{P}$ is a convex subspace of S of diameter at most $d - 1$, there exists a line L of S through y not contained in F_2 . This line L lies at distance $d - 1$ from L' .

By Proposition 2.4, it now follows that the lines of S' will induce L -sets of valuations of S .

The first claim in the following proposition is precisely Lemma 4.2 of [9].

Proposition 3.3. *Suppose f is a valuation of a dense near $2d$ -gon S . Then every point with non-maximal value is collinear with a point whose value is one more. As a consequence, the valuation f can be reconstructed from its associated hyperplane H_f in the sense of Proposition 2.2.*

3.2 The Known Valuations

The aim of this subsection is to describe all known classes of valuations of dense near polygons. Nine classes have been described in the literature. With exception of one

class, all known classes were already mentioned in the paper [18] where the notion of valuation was introduced. The remaining class of the so-called semi-diagonal valuations was introduced and studied in [7]. The SDPS-valuations have been studied in [19] (finite case) and [6, Chap. 5] (general case). Throughout this subsection, we suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ is a dense near $2d$ -gon with $d \geq 2$.

(1) **The classical valuations.** Let x be a fixed point of $\mathcal{S}$ and put $f_x(y) := d(x, y)$ for every point y of $\mathcal{S}$. Then f_x is a so-called *classical valuation* of $\mathcal{S}$.

(2) **The ovoidal valuations.** Let O be an *ovoid* of $\mathcal{S}$, that is, a set of points having a unique point on each line of $\mathcal{S}$. If we put $f_O(x) := 0$ for every $x \in O$ and $f_O(x) := 1$ for every $x \in \mathcal{P} \setminus O$, then f_O is a so-called *ovoidal valuation* of $\mathcal{S}$.

(3) **The hybrid valuations.** Suppose x is a point of $\mathcal{S}$ and $\delta \in \{2, 3, \dots, d\}$. Then let $\mathcal{S}'$ be the subgeometry of $\mathcal{S}$ whose points consist of all the points of $\mathcal{S}$ at distance at least δ from x and whose lines consist of all lines of $\mathcal{S}$ at distance at least $\delta - 1$ from x . For every ovoid O' of $\mathcal{S}'$, let $f_{x, O'} : \mathcal{P} \rightarrow \mathbb{N}$ be the map which maps a point $y \in \mathcal{P}$ to $d(x, y)$ if $d(x, y) \leq \delta - 1$, to $\delta - 2$ if $d(x, y) \geq \delta$ and $y \in O'$ and to $\delta - 1$ if $d(x, y) \geq \delta$ and $y \notin O'$. Then $f_{x, O'}$ is a so-called *hybrid valuation of $\mathcal{S}$ with index δ* . A hybrid valuation of index 2 is just an ovoidal valuation. Hybrid valuations of index d are also called *semi-classical valuations*. We can regard classical valuations as “hybrid valuations of index $d + 1$ ”.

(4) **The distance- j -ovoidal valuations.** Let $j \in \{2, 3, \dots, d\}$. A *distance- j -ovoid* of $\mathcal{S}$ is a set X of points satisfying the following properties:

- $|X| \geq 2$ and the minimal distance between two distinct points of X is equal to j ;
- for every point a of $\mathcal{S}$, there exists a point $x \in X$ such that $d(a, x) \leq \frac{j}{2}$;
- for every line L of $\mathcal{S}$, there exists a point $x \in X$ such that $d(x, L) \leq \frac{j-1}{2}$.

A distance-2-ovoid is just an ovoid. Distance- j -ovoids (of generalized polygons) were introduced in [30]. If X is a distance- j -ovoid of $\mathcal{S}$ with $j \in \{2, 3, \dots, d\}$ even, then the map $f : \mathcal{P} \rightarrow \mathbb{N}; x \mapsto d(x, X)$ is a valuation of $\mathcal{S}$. Any valuation of $\mathcal{S}$ which can be obtained in this way is called a *distance- j -ovoidal valuation*. The distance-2-ovoidal valuations are precisely the ovoidal valuations.

(5) **The extended valuations.** Suppose $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', I')$ is a convex subspace of $\mathcal{S}$ such that for every point x of $\mathcal{S}$ there exists a (unique) point x' of $\mathcal{S}'$ such that $d(x, y) = d(x, x') + d(x', y)$ for every point y of $\mathcal{S}'$. Suppose f' is a valuation of $\mathcal{S}'$. Then the map $f : \mathcal{P} \rightarrow \mathbb{N}$ defined by $x \mapsto d(x, x') + f'(x')$ is a valuation of $\mathcal{S}$, the so-called *extension of f'* .

(6) **The SDPS-valuations.** Suppose $\mathcal{S}$ is a dual polar space of rank $2n$ where $n \in \mathbb{N}$. A nonempty set X of points of $\mathcal{S}$ is called an *SDPS-set* if it satisfies the following properties.

- No two distinct points of X are collinear.
- If a quad Q contains two points of X then it intersects X in an ovoid.
- The partial linear space Δ whose points are the elements of X and whose lines are the quad intersections of size at least 2 (natural incidence) is a dual polar space of rank n .

- If $x_1, x_2 \in X$, then $d_S(x_1, x_2) = 2 \cdot d_\Delta(x_1, x_2)$.
- Every line of $\mathcal{S}$ meeting X is contained in a quad which intersects X in at least two points.

An SDPS-set of the near 0-gon consists of the unique point of that geometry. An SDPS-set of a generalized quadrangle is just an ovoid of that geometry. The word SDPS is an abbreviation of *Sub Dual Polar Space* and refers to the fact that Δ can be regarded as a sub dual polar space of $\mathcal{S}$. If X is an SDPS-set of $\mathcal{S}$, then the map $f_X : \mathcal{P} \rightarrow \mathbb{N}; x \mapsto d(x, X)$ is a valuation of $\mathcal{S}$. Any valuation of the dual polar space $\mathcal{S}$ which can be obtained in this way is called an *SDPS-valuation*.

(7) **The product valuations.** Suppose $\mathcal{S}$ is the direct product $\mathcal{S}_1 \times \mathcal{S}_2$ of two other dense near polygons $\mathcal{S}_1 = (\mathcal{P}_1, \mathcal{L}_1, \mathcal{I}_1)$ and $\mathcal{S}_2 = (\mathcal{P}_2, \mathcal{L}_2, \mathcal{I}_2)$ whose diameters are at least 1. Then the point set $\mathcal{P}$ of $\mathcal{S}$ is equal to the cartesian product $\mathcal{P}_1 \times \mathcal{P}_2$, see [6, Sect. 1.6] for more details. Suppose $f_i, i \in \{1, 2\}$, is a valuation of $\mathcal{S}_i$. For every point $x = (x_1, x_2)$ of $\mathcal{S}$, we define $f(x) = f_1(x_1) + f_2(x_2)$. Then f is a valuation of $\mathcal{S}$. Any valuation of $\mathcal{S}$ which can be obtained in this way is called a *product valuation*.

(8) **The diagonal valuations.** Suppose $\mathcal{S} = \mathcal{S}' \times \mathcal{S}'$ where $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is some dense near polygon. Define $X := \{(x, x) \mid x \in \mathcal{P}'\}$. Then the map from $\mathcal{P} = \mathcal{P}' \times \mathcal{P}'$ to $\mathbb{N}$ mapping a point x of $\mathcal{S}$ to $d(x, X)$ is a valuation of $\mathcal{S}$. Any valuation of $\mathcal{S}$ which can be obtained in this way is called a *diagonal valuation*.

(9) **The semi-diagonal valuations.** Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two generalized quadrangles and $\mathcal{S}$ is a glued near hexagon of type $\mathcal{Q}_1 \otimes \mathcal{Q}_2$ as discussed in [6, Chap. 4]. Then there exists a partition T_1 of the point set $\mathcal{P}$ in quads isomorphic to $\mathcal{Q}_1$ and a partition T_2 of $\mathcal{P}$ in quads isomorphic to $\mathcal{Q}_2$ such that the following properties are satisfied: (a) every point of $\mathcal{S}$ is contained in a unique quad of T_1 and a unique quad of T_2 ; (b) every quad of T_1 intersects every quad of T_2 in a line. Suppose X is a set of points of $\mathcal{S}$ satisfying the following properties:

- (1) every element of $T_1 \cup T_2$ has a unique point in common with X ;
- (2) any two distinct points of X lie at distance 2 from each other.

Then the map $f : \mathcal{P} \rightarrow \mathbb{N}; x \mapsto d(x, X)$ is a valuation of $\mathcal{S}$. Any valuation of $\mathcal{S}$ which can be obtained in this way is called a *semi-diagonal valuation* of $\mathcal{S}$.

3.3 Some Properties of Valuations

The following characterization of classical and ovoidal valuations was proved in [18, Proposition 2.10].

Proposition 3.4. *Suppose f is a valuation of a dense near $2d$ -gon $\mathcal{S}$ and let M_f denote the maximal value attained by f . Then f is an ovoidal valuation if and only if $M_f = 1$, and f is a classical valuation if and only if $M_f = d$.*

The following is a consequence of Proposition 3.4.

Corollary 3.5 *Every valuation of a generalized quadrangle with at least three points on each line is either classical or ovoidal.*

We now give two results regarding valuations of dense near hexagons. The first proposition was proved in [18, Propositions 4.1 and 4.2].

Proposition 3.6. *Let f be a valuation of a dense near hexagon $\mathcal{S}$, and let O_f denote the set of points with f -value 0. If f is not classical, semi-classical nor ovoidal, then $|O_f| \geq 2$ and every two distinct points of O_f lie at distance 2 from each other.*

The next proposition is taken from [19, Proposition 3].

Proposition 3.7. *Every valuation of a thick dual polar space of rank 3 is either classical, semi-classical, ovoidal or the extension of an ovoidal valuation of a quad.*

Suppose f is a valuation of a dense near polygon $\mathcal{S}$ and X is a subspace of $\mathcal{S}$ such that the point-line geometry $\tilde{X}$ induced on X is a dense near polygon that is isometrically embedded into $\mathcal{S}$. If m is the minimal value that f attains on the set X , then the map $f' : X \rightarrow \mathbb{N}; x \mapsto f(x) - m$ is a valuation of $\tilde{X}$ by [18, Proposition 2.8]. This valuation is called the *valuation of $\tilde{X}$ induced by f* . The following proposition was proved in [19, Theorems 5 and 6] (for finite dual polar spaces), but its proof easily extends to the infinite case, see [6, Theorems 5.30 and 5.32].

Proposition 3.8. *Let f be a valuation of a thick dual polar space. Then the following are equivalent:*

- *every induced hex valuation is either classical or the extension of an ovoidal valuation of a quad;*
- *there exists a convex subspace X of even diameter in Δ and an SDPS-valuation f' of $\tilde{X}$ such that f is the extension of f' .*

3.4 Applications of Valuations

We give a number of instances where valuations have been used to obtain new classification results regarding dense near polygons.

(1) All dense near octagons with three points per line have been classified in [23]. By [3, Theorem 4], we know that every such dense near octagon must contain convex subspaces of diameter 3, the hexes. Each such hex is a dense near hexagon with three points per line. By [1], we know that there are eleven such near hexagons. The valuations of the eleven dense near hexagons with three points per line have been classified in the papers [20, 21]. This classification of the valuations was subsequently used in [23] to obtain a complete classification of all dense near octagons with three points per line.

(2) A classification of the dense near octagons with four points per line was obtained in [13]. This classification is not complete in the sense that besides the 28

known examples a few open cases still remain. Every hex of a dense near octagon with four points per line must be a dense near hexagon with four points per line. A partial classification of such dense near hexagons was obtained in [5]. In [5], ten examples were described along with four open cases. In [9], the valuations of the dense near hexagons with four points per line were studied, and these results were subsequently used in [13] to obtain the above-mentioned classification for the octagons. Very recently, in [27, Proposition 4.7], one of the four open cases in the classification of the near hexagons was killed. In view of this, there remain now three open cases in the classification of the hexagons and also three open cases in the classification of the octagons.

Valuations have also found applications in the study of hyperplanes of near polygons, in particular, of dual polar spaces. It was the study of certain valuations in dual polar spaces which led to a new interesting class of hyperplanes, the so-called *SDPS-hyperplanes* associated with SDPS-valuations. In the case of rank 4 these hyperplanes were independently discovered by Pralle and Shpectorov in [31]. In [8], the (extensions of) SDPS-hyperplanes were characterized. Valuations played a crucial role in the proof of that characterization result.

Valuations are (potentially) useful for studying isometric embeddings between dense near polygons. They provide information on how such embeddings (can) look like.

Further classification results regarding valuations of dense near polygons were obtained in [11, 12, 15, 22, 24, 25]. We hope that these results will be useful for obtaining new classification results regarding dense near polygons.

4 Valuations of Generalized Polygons

4.1 Basic Definitions

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized $2d$ -gon with $d \in \mathbb{N} \setminus \{0, 1\}$. A map $f : \mathcal{P} \rightarrow \mathbb{N}$ is called a *polygonal valuation* if the following three conditions are satisfied.

- (PV1) There exists a point x whose f -value $f(x)$ equals 0.
- (PV2) f is a semi-valuation.
- (PV3) If x is a point of $\mathcal{S}$ with non-maximal f -value, then there is at most one line through x containing a (necessary unique) point with f -value $f(x) - 1$.

The notion of a polygonal valuation was introduced in [14]. Polygonal valuations seem to be suitable objects to build a valuation theory for generalized polygons.

If f is a polygonal valuation of $\mathcal{S}$, then we denote by $\mathcal{O}_f$ the set of points with f -value 0 and by $\mathcal{M}_f$ the set of all points x of $\mathcal{S}$ that are not collinear with a point having f -value $f(x) - 1$. We denote by M_f the maximal value attained by f . Clearly, $\mathcal{O}_f \subseteq \mathcal{M}_f$ and $M_f \in \{1, 2, \dots, d\}$.

We now describe three classes of polygonal valuations which were introduced in [14].

(1) Let x be a fixed point of $\mathcal{S}$ and put $f_x(y) := d(x, y)$ for every $y \in \mathcal{P}$. Then f_x is a so-called *classical polygonal valuation* of $\mathcal{S}$.

(2) Suppose $x \in \mathcal{P}$ and $O \subseteq \Gamma_d(x)$ such that every line of $\mathcal{S}$ at distance $d - 1$ from x has a unique point in common with O . If y is a point of $\mathcal{S}$ at distance at most $d - 1$ from x , then we define $f_{x,O}(y) := d(x, y)$. If y is a point of $\mathcal{S}$ at distance d from x , then $f_{x,O}(y) := d - 2$ if $y \in O$ and $f_{x,O}(y) := d - 1$ otherwise. Then $f_{x,O}$ is a so-called *semi-classical polygonal valuation* of $\mathcal{S}$.

(3) Let O be a set of points of $\mathcal{S}$ having a unique point in common with every line. We put $f_O(x) = 0$ if $x \in O$ and $f_O(x) = 1$ if $x \in \mathcal{P} \setminus O$. Then f_x is a so-called *ovoidal polygonal valuation* of $\mathcal{S}$.

(4) Let $j \in \{2, 3, \dots, d\}$ and let X be a *distance- j -ovoid* of $\mathcal{S}$ with j even (as defined by the three properties mentioned in Sect. 3.2). The map $\mathcal{P} \rightarrow \mathbb{N}; x \mapsto d(x, X)$ is a so-called *distance- j -ovoidal polygonal valuation* of $\mathcal{S}$. The distance-2-ovoidal polygonal valuations are precisely the *ovoidal polygonal valuations*.

4.2 Properties of Polygonal Valuations

In this subsection, we suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathbf{I})$ is a generalized $2d$ -gon with $d \in \mathbb{N} \setminus \{0, 1\}$. The next proposition says that polygonal valuations satisfy a property that is similar to the one mentioned in Proposition 3.3.

Proposition 4.1. *If f is a polygonal valuation of $\mathcal{S}$, then every point with non-maximal value is collinear with a point whose value is one more. As a consequence, the valuation f can be reconstructed from its associated hyperplane H_f in the sense of Proposition 2.2.*

Proof. The first claim is an immediate consequence of Property (PV3) in the definition of polygonal valuation.

The following proposition was proved in Sect. 3 of [14].

Proposition 4.2. *Suppose f is a polygonal valuation of $\mathcal{S}$. Then $M_f = 1$ if and only if f is ovoidal, $M_f = d - 1$ if and only if f is semi-classical and $M_f = d$ if and only if f is classical. Also, $\mathcal{O}_f = \mathcal{M}_f$ if and only if f is either classical or distance- j -ovoidal for some even j .*

The following is a consequence of Proposition 4.2.

Corollary 4.3 (1) *Every polygonal valuation of a generalized quadrangle is either classical or ovoidal.*

(2) *Every polygonal valuation of a generalized hexagon is either classical, semi-classical or ovoidal.*

Let $f_i, i \in I$, be a collection of mutually distinct polygonal valuations of $\mathcal{S}$, where I is some index set of size at least two. Put $\mathcal{F} = \{f_i \mid i \in I\}$. If $\mathcal{F}$ is an L -set, then by [14, Proposition 4.8], there exists a line of $\mathcal{S}$ containing precisely $|\mathcal{F}| = |I|$ points. The set $\mathcal{F}$ is called *admissible* if the following holds for all $i_1, i_2 \in I$ with $i_1 \neq i_2$, for every $x \in \mathcal{M}_{f_{i_1}}$ and every $y \in \mathcal{M}_{f_{i_2}}$:

- if f_{i_1} and f_{i_2} are classical, then $d(x, y) = 1$;
- if $x = y$, then $(f_{i_1}(x) - M_{f_{i_1}}) - (f_{i_2}(x) - M_{f_{i_2}}) \in \{-1, 0, 1\}$;
- if $x \neq y$ and at least one of f_{i_1}, f_{i_2} is not classical, then $d(x, y) + f_{i_1}(x) + f_{i_2}(y) - M_{f_{i_1}} - M_{f_{i_2}} + 1 \geq 0$.

The *valuation geometry* $\mathcal{V}_{\mathcal{S}}$ of $\mathcal{S}$ is the point-line geometry whose points are the polygonal valuations of $\mathcal{S}$ and whose lines are the admissible L -sets of polygonal valuations of $\mathcal{S}$, with incidence being containment. The properties mentioned in the following proposition were proved in Sect. 6 of [14].

Proposition 4.4. *Suppose $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathcal{I}')$ is a generalized $2d$ -gon containing $\mathcal{S}$ as a full subgeometry. Let x be a point of $\mathcal{S}'$ and put $m_x := d(x, \mathcal{P})$. For every point $y \in \mathcal{P}$, we define $f_x(y) := d(x, y) - m_x$. Then:*

- (1) *f_x is a polygonal valuation of $\mathcal{S}$, the so-called polygonal valuation of $\mathcal{S}$ induced by x . Moreover, $M_{f_x} = d - m_x$.*
- (2) *If x_1 and x_2 are two distinct collinear points of $\mathcal{S}'$, then the polygonal valuations f_{x_1} and f_{x_2} are distinct.*
- (3) *The map $\theta : x \mapsto f_x$ between the point-sets of $\mathcal{S}'$ and $\mathcal{V}_{\mathcal{S}}$ maps every line of $\mathcal{S}'$ to a full line of $\mathcal{V}_{\mathcal{S}}$.*

The properties mentioned in the following proposition were proved in Sect. 5 of [14].

Proposition 4.5. *Let $\mathcal{S}$ be a generalized $2d$ -gon having three points on each line. Then:*

- (1) *If $\{f_1, f_2, f_3\}$ is an L -set of polygonal valuations of $\mathcal{S}$, then H_{f_3} equals the complement $\overline{H_{f_1} \Delta H_{f_2}}$ of the symmetric difference $H_{f_1} \Delta H_{f_2}$ of H_{f_1} and H_{f_2} .*
- (2) *Let f_1, f_2 and f_3 be three polygonal valuations of $\mathcal{S}$ and let $H_i, i \in \{1, 2, 3\}$, be the hyperplane associated with f_i . If $H_3 = \overline{H_1 \Delta H_2}$ and $\{f_1, f_2, f_3\}$ is admissible, then $\{f_1, f_2, f_3\}$ is an admissible L -set of $\mathcal{S}$.*
- (3) *The valuation geometry of $\mathcal{S}$ is a partial linear space.*

4.3 Applications of Polygonal Valuations

Polygonal valuations can be used to obtain classification results for generalized polygons containing a given generalized polygon as a full subgeometry. We give three instances where success has been achieved.

(1) In [28], it was shown that every generalized hexagon of order $(3, 3)$ which contains a subhexagon of order $(3, 1)$ must be isomorphic to the split Cayley hexagon $H(3)$. This result was proved without the aid of a computer. The same result can also be easily proved with the aid of polygonal valuations, but by relying on some computer computations using the computer algebra system GAP [29], see [17].

(2) The existence of semi-finite thick generalized polygons is one of the most important open problems in the theory of generalized polygons (see [35, Appendix E, Problem 5]). A generalized polygon is said to be *semi-finite* if it has a finite number of points on each line, but an infinite number of lines through each point. By relying on polygonal valuations and computer computations (with GAP), it will be shown in [17] that there do not exist semi-finite generalized hexagons of order $(2, t)$ containing a subhexagon of order $(2, 2)$. Note that by [4], there are up to isomorphism two generalized hexagons of order $(2, 2)$, the split Cayley hexagon $H(2)$ and its point-line dual $H^D(2)$.

(3) Up to isomorphism, a unique generalized octagon of order $(2, 4)$ is known to exist. It belongs to the class of the so-called *Ree-Tits generalized octagons*. This family of generalized octagons was first constructed by Tits [34] using a family of simple groups discovered by Ree [32]. In [16], it was shown that the Ree-Tits octagon of order $(2, 4)$ is the up to isomorphism unique generalized octagon of order $(2, 4)$ containing a suboctagon of order $(2, 1)$. The idea of the proof consists of considering the valuation geometry $\mathcal{V}$ of the unique generalized octagon of order $(2, 1)$. By studying $\mathcal{V}$, it was shown in [16] that $\mathcal{S}$ can be reconstructed in a unique way from its suboctagon of order $(2, 1)$. This then implies that $\mathcal{S}$ must be isomorphic to the Ree-Tits generalized octagon of order $(2, 4)$.

Remarks. (1) In [4] all finite generalized hexagons of order $(2, t)$ have been classified.

(2) Let $s \in \mathbb{N} \setminus \{0, 1\}$. In [26, Corollary 3.20] it was shown that there exist no semi-finite generalized hexagons of order (s, t) that contain a subhexagon of order (s, s^3) as a full subgeometry.

References

1. Brouwer, A.E., Cohen, A.M., Hall, J.I., Wilbrink, H.A.: Near polygons and Fischer spaces. *Geom. Dedicata* **49**, 349–368 (1994)
2. Brouwer, A.E., Cohen, A.M., Neumaier, A.: Distance-regular graphs. In: *Results in Mathematics and Related Areas* (3), vol. 18. Springer, Berlin (1989)
3. Brouwer, A.E., Wilbrink, H.A.: The structure of near polygons with quads. *Geom. Dedicata* **14**, 145–176 (1983)
4. Cohen, A.M., Tits, J.: On generalized hexagons and a near octagon whose lines have three points. *Eur. J. Comb.* **6**, 13–27 (1985)
5. De Bruyn, B.: Near hexagons with four points on a line. *Adv. Geom.* **1**, 211–228 (2001)
6. De Bruyn, B.: Near polygons. In: *Frontiers in Mathematics*. Birkhäuser Verlag, Basel (2006)
7. De Bruyn, B.: Valuations of glued near hexagons. *J. Comb. Des.* **15**, 35–48 (2007)

8. De Bruyn, B.: A characterization of the SDPS-hyperplanes of dual polar spaces. *Eur. J. Comb.* **28**, 705–714 (2007)
9. De Bruyn, B.: Dense near octagons with four points on each line, II. *Adv. Geom.* **7**, 191–206 (2007)
10. De Bruyn, B.: An alternative definition of the notion valuation in the theory of near polygons. *Electron. J. Comb.* **16**, 14 (2009) (Research paper 16)
11. De Bruyn, B.: The valuations of the near polygon $\mathbb{G}_n$. *Electron. J. Comb.* **16**, 29 (2009) (Research paper 137)
12. De Bruyn, B.: The valuations of the near $2n$ -gon $\mathbb{I}_n$. *Ars Comb.* **98**, 321–336 (2011)
13. De Bruyn, B.: Dense near octagons with four points on each line, III. *Ann. Comb.* **15**, 19–35 (2011)
14. De Bruyn, B.: Polygonal valuations. *Discrete Math.* **313**, 84–93 (2013)
15. De Bruyn, B.: On the valuations of the near polygon $\mathbb{H}_n$. Preprint 2012
16. De Bruyn, B.: The uniqueness of a certain generalized octagon of order $(2, 4)$. Preprint 2012
17. De Bruyn, B.: On semi-finite generalized hexagons of order $(2, t)$ and $(3, t)$ containing subhexagons. In preparation
18. De Bruyn, B., Vandecasteele, P.: Valuations of near polygons. *Glasg. Math. J.* **47**, 347–361 (2005)
19. De Bruyn, B., Vandecasteele, P.: Valuations and hyperplanes of dual polar spaces. *J. Com. Theory, Ser. A* **112**, 194–211 (2005)
20. De Bruyn, B., Vandecasteele, P.: The valuations of the near hexagons related to the Witt designs $S(5, 6, 12)$ and $S(5, 8, 24)$. *J. Comb. Des.* **14**, 214–228 (2006)
21. De Bruyn, B., Vandecasteele, P.: The distance-2-sets of the slim dense near hexagons. *Ann. Comb.* **10**, 193–210 (2006)
22. De Bruyn, B., Vandecasteele, P.: The valuations of the near octagon $\mathbb{I}_4$. *Electron. J. Comb.* **13**, 23 (2006) (Research paper 76)
23. De Bruyn, B., Vandecasteele, P.: The classification of the slim dense near octagons. *Eur. J. Comb.* **28**, 410–428 (2007)
24. De Bruyn, B., Vandecasteele, P.: The valuations of the near octagon $\mathbb{H}_4$. *Graphs and Combinatorics* **23**, 601–623 (2007)
25. De Bruyn, B., Vandecasteele, P.: The valuations of the near octagon $\mathbb{G}_4$. *Discrete Math.* **310**, 755–766 (2010)
26. De Bruyn, B., Vanhove, F.: Inequalities for regular near polygons, with applications to m -ovoids. *Eur. J. Comb.* **34**, 522–538 (2013)
27. De Bruyn, B., Vanhove, F.: On Q -polynomial regular near $2d$ -gons. *Combinatorica* (to appear)
28. De Medts, T., Van Maldeghem, H.: The uniqueness of a generalized hexagon of order 3 containing a subhexagon of order $(1, 3)$. *Discrete Math.* **309**, 714–720 (2009)
29. The GAP Group, GAP—Groups, Algorithms, and Programming, Version 4.4.12. <http://www.gap-system.org> (2008)
30. Offer, A., Van Maldeghem, H.: Distance- j ovoids and related structures in generalized polygons. *Discrete Math.* **294**, 147–160 (2005)
31. Pralle, H., Shpectorov, S.: The ovoidal hyperplanes of a dual polar space of rank 4. *Adv. Geom.* **7**, 1–17 (2007)
32. Ree, R.: A family of simple groups associated with the simple Lie algebra of type (F_4) . *Am. J. Math.* **83**, 401–420 (1961)
33. Shult, E.E., Yanushka, A.: Near n -gons and line systems. *Geom. Dedicata* **9**, 1–72 (1980)
34. Tits, J.: Les groupes simples de Suzuki et de Ree. *Séminaire Bourbaki* 13 (1960/61), No. 210, pp. 18 (1961)
35. Van Maldeghem, H.: *Generalized polygons*. Birkhäuser, Basel (1998)

Groups of Exceptional Type, Coxeter Groups and
Related Geometries

Sastry, N.S.N. (Ed.)

2014, XIV, 304 p. 32 illus., Hardcover

ISBN: 978-81-322-1813-5