Semi-continuity Properties of Metric Projections

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Abstract This chapter presents some selected results regarding semi-continuity of metric projections onto closed subspaces of normed linear spaces. Though there are several significant results relevant to this topic, only a limited coverage of the results is undertaken, as an extensive survey is beyond our scope. This exposition is divided into three parts. The first one deals with results from finite dimensional normed linear spaces. The second one deals with results connecting semi-continuity of metric projection maps and duality maps. The third one deals with subspaces of finite codimension of infinite dimensional normed linear spaces.

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1 Metric Projections onto Finite Dimensional Spaces

In this survey article, we present some selected results regarding semi-continuity of metric projections onto closed subspaces of normed linear spaces. Though there are several significant results relevant to this topic, we undertake only a limited coverage of the results as an extensive survey is beyond our scope.

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The third one deals with subspaces of finite co-dimension of infinite dimensional normed linear spaces.

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Throughout we consider only real normed linear spaces and we assume all subspaces are closed.

If X is a normed linear space, X^* will denote the dual of X, B_X the closed unit ball, $\{x \in X : \|x\| \leq 1\}$ and S_X the unit sphere $\{x \in X : \|x\| = 1\}$, of X. If x is in X and $r > 0$ then the open and closed balls with center x and radius r are denoted by

$$B(x, r) = \{z \in X : \|x - z\| < r\},$$

and

$$B[x, r] = \{z \in X : \|x - z\| \leq r\},$$

respectively. Further, if $A \subseteq X$, $x \in X$ and $\varepsilon > 0$, then we set

$$B(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\},$$

$$d(x, A) = \inf\{\|x - a\| : a \in A\}, \quad \text{for } x \in X,$$

and

$$P_A(x) = \{a \in A : \|x - a\| = d(x, A)\}.$$

Further, we set

$$A^{\perp} = \{f \in X^* : f \equiv 0 \text{ on A}\}.$$

We now have

Definition 1 Let $A \subseteq X$. Then A is said to be *proximinal* in X if $P_A(x)$ is nonempty for each $x \in X$. Any element in $P_A(x)$ is called *a nearest element to x from A* or *a best approximation to x from A*. The set A is said to be *Chebyshev* if $P_A(x)$ is a singleton set for all $x \in X$.

The set valued map P_A defined on X, is called the metric projection from X onto A. The following stronger notion of proximinality figures in an essential way, often in our discussion. Note that if $d = d(x, A)$ then

$$P_A(x) = B[x, d] \cap A.$$

For $\delta > 0$, set

$$\begin{aligned} P_A(x, \delta) &= \{y \in A : \|x - y\| \leq d(x, A) + \delta\} \\ &= B[x, d + \delta] \cap A. \end{aligned}$$

We now have the following definition from Godefroy and Indumathi [14].

Definition 2 Let X be a normed linear space. A proximinal subset A of X is said to be *strongly proximinal* at x in X, if given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(y, P_A(x)) < \varepsilon \text{ for all } y \in P_A(x, \delta)$$

or equivalently

$$P_A(x, \delta) \subseteq B(P_A(x), \varepsilon).$$

If A is strongly proximinal at each x in X, then we say A is strongly proximinal in X.

Let $F : X \to Y$ be a set-valued map. We say F is *lower semi-continuous* (l.s.c.) at x_0 of X if for any open set U of X such that $U \cap F(x_0) \neq \emptyset$, the set $\{x \in X : F(x) \cap U \neq \emptyset\}$ is a neighbourhood of x_0 and F is *upper semi-continuous* (u.s.c.) at x_0 of X if for any open set U of X such that $F(x_0) \subseteq U$, the set $\{x \in X : F(x) \subseteq U\}$ is a neighbourhood of x_0.

The set-valued map F is said to be *Hausdorff lower semi-continuous* (H.l.s.c.) at x_0 of X if given $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in B(x_0, \delta)$ implies $F(x_0) \subseteq B(F(x), \varepsilon)$.

We say F is *Hausdorff upper semi-continuous* (H.u.s.c.) at x_0 in X if for any $\varepsilon > 0$, the set $\{x \in X : F(x) \subseteq B(F(x_0), \varepsilon)\}$ is a neighbourhood of x_0.

The set valued map F is would be called *Hausdorff semi-continuous* if it is both H.u.s.c. and H.l.s.c.

We observe that

$$F \;\; H.l.s.c. \Rightarrow F \;\; l.s.c, \text{ while } F \;\; u.s.c \Rightarrow F \;\; H.u.s.c.$$

Our discussion would involve the above semi-continuity concepts with reference to metric projections.

It is easily verified that if Y is strongly proximinal then the metric projection is H.u.s.c.

Remark 1 A well-known fact, that can be proved using the usual compactness arguement, is that any finite dimensional subspace Y of a normed linear space X is strongly proximinal and hence the metric projection P_Y is upper Hausdorff semi-continuous. We observe that for a single-valued map, all the above four notions of semi-continuity coincide with the usual notion of continuity of a single-valued map. Thus if Y is a finite dimensional Chebychev subspace of X, then P_Y is continuous.

A single-valued map f on X is said to be a selection for F if $f(x) \in F(x)$ for each x in X. The set valued map F is said to have a continuous selection if it has a selection that is continuous. Among the semi-continuity properties of the metric projection, l.s.c. gains prominence because of the following important theorem of Michael.

Theorem 1 (Michael Selection Theorem) [21] *If X is a paracompact, Hausdorff topological space, Y is a Banach space and $F : X \to 2^Y$ is a nonempty closed convex set valued and lower semi-continuous mapping, then F has a continuous selection; that is, there exists a continuous $s : X \to Y$ such that $s(x) \in F(x)$*

for each x in X. In particular if Y is a subspace of a normed linear space X with $P_Y(x) \neq \phi$ for all $x \in X$ and P_Y lower semi-continuous on X then P_Y has a continuous selection.

However, l.s.c. is not a necessary condition for the existence of a continuous selection as the following example of Deutsch and Kenderov from [8] shows.

Example 1 [8] Let B be the convex hull of the circle $\Gamma = \{(x_1, x_2, 0) : x_1^2 + x_2^2 = 1\}$ and the two points $(0, 0, 1)$ and $(0, 0, -1)$ in $\mathbf{R}^3$. (B is double cone formed by placing the two cones with vertices $(0, 0, 1)$ and $(0, 0, -1)$ in such a way that, their common circular base coincides.) Then B is a closed convex, symmetric set with nonempty interior. Let X be the normed linear space $\mathbf{R}^3$, with the norm for which B is the closed unit ball.

Let $Y = sp(1, 0, 1)$. Then Y is the line L through $(0, 0, 0)$ and $(1, 0, 1)$, which is parallel to the line segment l , lying on the unit sphere, joining $(-1, 0, 0)$ in Γ and the vertex $(0, 0, 1)$. If $x = (x_1, x_2, x_3)$ and $x_2 \neq 0$ then $P_Y(x) = \{(x_3, 0, x_3)\}$.

If $x_2 = 0$, then $P_Y(x)$ is a line segment of nonzero length containing the point $(x_3, 0, x_3)$. It is clear that $f(x) = (x_3, 0, x_3)$ is a continuous selection for P_Y but P_Y is not l.s.c.

An important weaker notion than l.s.c, that is also a necessary condition for the existence of continuous selections, is that of approximate lower semi-continuity (a.l.s.c.).

Definition 3 [7, 8] We say F is *approximate lower semi-continuous* (a.l.s.c) at x_0 if for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\bigcap \{B(F(x), \varepsilon) : x \in B(x_0, \delta)\} \neq \phi.$$

Weaker notions than a.l.s.c. can be naturally defined as follows. Let k be a positive integer ≥ 2. The following notion from [8] is weaker than a.l.s.c.

The set valued map F is said to be $k - l.s.c$ at x_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\cap_{i=1}^{k} B(F(x_i), \varepsilon) \neq \phi$ for every choice of $k-$points in $B(x_0, \delta)$. It follows from Helly's theorem that if Y is a subspace of finite dimension n and F is closed convex valued then

$$F \text{ is a.l.s.c} \Leftrightarrow F \text{ is (n+1)}-\text{l.s.c.} \tag{1}$$

Clearly

$$\begin{aligned} F \text{ l.s.c at } x_0 &\Rightarrow F \text{ is a.l.s.c at } x_0 \\ &\Rightarrow F \text{ is k}-\text{l.s.c at } x_0 \text{ for } k \geq 2. \end{aligned}$$

We refer the reader to the papers [5, 6] of Brown, for a thorough discussion about a.l.s.c. of a set valued map and its derived maps, presenting a lucid and overall perspective of these concepts in relation to the existence of continuous selections.

Let $\varepsilon > 0$. The set valued map F is said to have an ε-approximate continuous selection if there is a continuous map $s_\varepsilon : X \to Y$ such that $s_\varepsilon(x) \in B(F(x), \varepsilon)$ for each $x \in X$.

A parallel result to Michael selection theorem, for a.l.s.c, was proved in [8].

Theorem 2 [8] *Let X be a paracompact space and Y be a normed linear space. Let $F : X \to 2^Y$ have closed,convex images. Then F is a.l.s.c if and only if for each $\varepsilon > 0$, F has a continuous ε-approximate selection.*

Examples of a.l.s.c. and u.s.c. maps with no continuous selections have long been known. Zhivkov [27] constructed an example of a space X of dimension five- and a three-dimensional subspace Y of X such that the metric projection P_Y is a.l.s.c. but does not have a continuous selection. However, Deutsch and kenderov [8] showed that if dim $Y = 1$ then

$$\begin{aligned} P_Y \text{ has a continuous selection} &\Leftrightarrow P_Y \text{is a.l.s.c.} \\ &\Leftrightarrow P_Y\ is\ 2.l.s.c. \end{aligned}$$

Brown [2] and Deutsch and Kenderov [8] have independently constructed examples of one dimensional subspace Y of a three-dimensional space X such that P_Y does not have a continuous selection, which in turn implies P_Y is not 2.l.s.c. We observe that by Remark 1 and Corollary 1, given later in Sect. 2, P_Y is u.s.c.

Example 2 [2, 7] Let X be $\mathbf{R}^3$ with norm generated by the unit ball $B = co\ (l \cup D \cup -D)$, where l is the line segment joining (1, 0, 0) and (−1, 0, 0) and D is the semicircle

$$\{(1, y, z) : y \geq 0,\ z\ \geq 0 \text{ and } y^2 + z^2 = 1\}.$$

If Y is the one dimensional subspace sp(1,0,0) and z is a point that moves on the circle $C = \{0, y, z) : y^2 + z^2 = 1\}$, we have $P_Y(z) = \{(-1,\ 0,\ 0)\}$ if $z > 0$ and $P_Y(z) = \{(1,\ 0,\ 0)\}$ if $z < 0$. It is clear that P_Y cannot have a continuous selection.

As observed earlier, a.l.s.c. does not guarantee existence of continuous selections for metric projections. However, we have surprising positive results when the space $C(Q)$ is considered and the following results of Wu Li and T. Fisher are impressive.

Theorem 3 (Li [19]) *Let Q be a compact Hausdorff topological space, $C(Q)$ the space of real valued continuous maps defined on Q with supnorm. If Y is a finite dimensional subspace of $C(Q)$ then the metric projection P_Y has a continuous selection if and only if P_Y is a.l.s.c.*

Theorem 4 (Fisher [18]) *Let Q be a compact Hausdorff topological space, $C(Q)$ the space of real valued continuous maps defined on Q with supnorm. If Y is a finite dimensional subspace of $C(Q)$ then the metric projection P_Y has a continuous selection if and only if P_Y is $2-$l.s.c.*

The proofs of the above two theorems are elaborate and technical in nature. However, the proof of Fisher, via optimization techniques, leads to a stronger conclusion, viz, the sufficiency of 2- l.s.c. for the existence of the continuous selection. We refer the reader to the papers [5, 6] of Brown, for detailed discussion and comparison of the proofs of Wu Li and Fisher.

The above results are extended to $X = C_0(T)$, the space of real continuous functions which vanish at infinity on a locally compact Hausdorff space T, in a later work of Wu Li. He also proved the equivalence of a.l.s.c and existence of continuous selections for finite dimensional subspaces of $L_1(T, \mu)$.

Theorem 5 [20] *Let* (T, μ) *be a positive measure space and* $X = L_1(T, \mu)$*, the space of real integrable functions on* (T, μ) *equipped with the usual norm. If* Y *is a finite dimensional subspace of* X *then,* P_Y *has a continuous selection if and only if* P_Y *is a.l.s.c.*

We now discuss some geometric conditions that play a vital role in the semicontinuity of metric projections. We need the definition of polyhedral spaces in the discussion now and later.

Definition 4 A finite dimensional normed linear space X is said to be *polyhedral* if extreme points of B_X is a finite set. A normed linear space is called polyhedral if every one of its finite dimensional subspace is polyhedral.

In Brown [3], defined property P for a normed linear space (If x and z in X satisfy $\|x + z\| \leq \|x\|$, then there exist positive constants δ and η such that $\|y + \eta z\| \leq \|y\|$ if y is in $B(x, \delta)$) and showed that normed linear spaces with property P are precisely those spaces in which metric projections onto all finite dimensional subspaces are l.s.c. In [1], the equivalence of Property P to metric projection onto every one dimensional subspace being l.s.c, was shown.

Strictly convex spaces and finite dimensional, polyhedral normed linear spaces are examples of spaces with property (P). We recall that Singer [23] if a normed linear space is strictly convex, then every proximinal subspace of X is Chebyshev. Thus if X is strictly convex, every finite dimensional subspace of X is Chebyshev and by Remark 1 above, the metric projection onto every finite dimensional space is single-valued and continuous and equivalently, l.s.c.

A normed linear space X is said to have property (CS1) Brown et al. [7] whenever Y is a one dimensional subspace of X then P_Y has a continuous selection or equivalently P_Y is $2-$a.l.s.c. Clearly property (P) implies property (CS1).

In Brown et al. [7], an example of a three-dimensional space with property (CS1) but not having property (P) was given. Further it was shown in that paper that a normed linear space X has property (CS1) if and only if the metric projection P_Y is a.l.s.c for every finite dimensional subspace Y of X.

We end this section with two comments. Consider the class of those Banach spaces X, for which the following hold: If Y any finite dimensional subspace of X, then P_Y has a continuous selection if and only if P_Y is a.l.s.c. We note that the above class includes $C(Q)$ and $L_1(T, \nu)$. It would be desirable to have more examples of nonstrictly convex spaces in this class.

Given a Banach space X, identifying some finite dimensional subspaces Y of X with $\dim Y \geq 2$, for which P_Y has a continuous selection if P_Y a.l.s.c, would be an interesting problem. Using the notion of derived maps of Brown [6], the subspaces for which the derived map of the metric projection is l.s.c, would have the above property.

2 Pre-duality Maps and Metric Projections

In this section, we present mostly results from [9] connecting semi-continuity properties of metric projections and the pre-duality maps.

We need the following facts about upper semi-continuity of set-valued maps later, for proving Theorem 8. The fact below is from [25].

Fact 1 [25] *Let X and Y be normed linear spaces and $F : X \to 2^Y$ is a set valued map with nonempty closed convex and bounded images. Assume that F is positively homogeneous: that is, $F(\alpha x) = \alpha x$ for $\alpha \geq 0$ and x in X. Then F is u.s.c if and only if F is H.u.s.c and $F(x)$ is compact for each $x \in X$.*

Proof Assume F is H.u.s.c and $F(x)$ is compact for each $x \in X$. Fix $x_0 \in X$. Suppose F is not u.s.c at x_0. Then there exists a sequence $\{x_n\}$ in X converging to x_0, a neighborhood U of $F(x_0)$ and a sequence $\{y_n\}$ in Y such that $y_n \in F(x_n) \setminus U$ for all $n \geq 1$. Since F is H.u.s.c at x_0, $d(y_n, F(x_0)) \to 0$ as $n \to \infty$. Then there exists $\{z_n\} \subseteq F(x_0)$ such that $\|z_n - y_n\| < \frac{1}{n}$ for all $n \geq 1$. Now $F(x_0)$ compact implies $\{z_n\}$ has a convergent subsequence that converge to $z_0 \in F(x_0)$ and hence $\{y_n\}$ has a convergent subsequence converging to z_0. Since $z_0 \in U$ this implies $y_n \in U$ for all large enough n. This contradicts $y_n \notin U$ for all $n \geq 1$. Hence, F is u.s.c.

Conversely assume F is u.s.c. Clearly F is H.u.s.c. We only show that $F(x)$ is compact for each x in X. Fix x_0 in X and assume $\{y_n\} \subseteq F(x_0)$ has no convergent subsequence. Let $\beta_n = \sup\{\beta y_n : \beta y_n \in F(x_0)\}$. Then $\beta_n \geq 1$ and since $F(x_0)$ is closed, $\beta_n y_n \in F(x_0)$. Note that $\lambda \beta_n y_n \notin F(x_0)$ for $\lambda > 1$, for each $n \geq 1$. Let $\{\lambda_n\}$ be a sequence of scalars such that $\lambda_n > 1$ for all n. Clearly, the sequence $\{\lambda_n \beta_n y_n\}$ lies outside the set $F(x_0)$. We claim that the sequence $\{\lambda_n \beta_n y_n\}$ does not have a convergent subsequence.

Select $M > 0$ such that $\sup\limits_{x \in F(x_0)} \|x\| \leq M$. Since $\{y_n\}$ does not have a convergent subsequence, without loss of generality we can and do assume $\min\{\|y_n\| : n \geq 1\} = \delta > 0$. Then

$$\sup_{n \geq 1} \beta_n \leq \frac{M}{\delta} \text{ and } 0 < \frac{\delta}{\lambda_n M} \leq \frac{1}{\lambda_n \beta_n} \leq 1, \quad \text{for all } n \geq 1.$$

Assume $\{\lambda_n \beta_n y_n\}$ has a convergent subsequence, say $\{\lambda_{n_k} \beta_{n_k} y_{n_k}\}$. Then by the above, the sequence of scalars $\left\{\frac{1}{\lambda_{n_k} \beta_{n_k}}\right\}$ has a convergent subsequence that converges to a limit in the open interval (0, 1). This would imply the sequence $\{y_{n_k}\}$ has a convergent subsequence and contradict our assumption. Hence we conclude $\{\lambda_n \beta_n y_n\}$ does not have a convergent subsequence.

Let $w_n = \beta_n y_n$ and $x_n = \frac{n+1}{n} x_0$ for $n \geq 1$. Then $\frac{n+1}{n} w_n \in F(x_n)$ for all $n \geq 1$. Clearly (x_n) converges to x_0 and $A = \left\{\frac{n+1}{n} w_n : n \geq 1\right\}$ is a closed set, since the sequence $\left\{\frac{n+1}{n} w_n\right\}$ does not have a convergent subsequence. Clearly $A \cap F(x_n)$ is nonempty for all $n \geq 1$, while $A \cap F(x_0)$ is an empty set. This contradicts upper semi-continuity of F at x_0 and $F(x_0)$ is compact. □

The following corollary of the above theorem is immediate.

Corollary 1 *Let X be a Banach space and Y a proximinal subspace of X. Then the metric projection P_Y is u.s.c. if and only if P_Y is H.u.s.c. and $P_Y(x)$ is compact for each x in X.*

Let X be a normed linear space and Y be a proximinal subspace of X. Set

$$D_Y = \{x \in X : d(x, Y) = 1\}.$$

Then it is easy to check that the metric projection P_Y is H.u.s.c. (l.s.c.) on X if and only if P_Y is H.u.s.c. (l.s.c.) on the set D_Y.

The following theorem of Morris [22] is needed for proving Theorem 8 below.

Theorem 6 [22] *Let X be a normed linear space and Y be a proximinal subspace of finite codimension in X. Then P_Y is u.s.c if and only if $P_Y^{-1}\{0\}$ is boundedly compact.*

Proof Assume $P_Y^{-1}\{0\}$ is boundedly compact. Fix x in D_Y. Note that $x - P_Y(x) \subseteq P_Y^{-1}\{0\}$, is a bounded set and therefore is compact. By Corollary 1, we only have to show that P_Y is H.u.s.c.

If P_Y is not H.u.s.c at x_0, there exists $\{x_n\} \subseteq D_Y$ and $\varepsilon > 0$ such that $\{x_n\}$ converges to x and a sequence $\{y_n\}$ with $y_n \in P_Y(x_n)$ for all $n \geq 1$ and

$$d(y_n, P_Y(x)) \geq \varepsilon, \quad \text{for all } n \geq 1. \tag{2}$$

Now $x_n - y_n$ is in $P_Y^{-1}\{0\}$ and $\|x_n - y_n\| \leq 1$, for each $n \geq 1$. So $\{x_n - y_n\}$ has a convergent subsequence, say, $x_{n_k} - y_{n_k}$that converges to some $z \in P_Y^{-1}\{0\}$. This implies $\{y_{n_k}\}$ converges to $z - x$ in Y. Now

$$\left\|x_{n_k} - y_{n_k}\right\| \to \|x - (z - x)\| = \|z\| = d(z, Y) = d(x, Y).$$

So $z - x \in P_Y(x)$ and this contradicts (2).

Conversely assume that P_Y is u.s.c. Then P_Y is u.H.s.c and $P_Y(x)$ is compact for each x in X. Let $\{x_n\}$ be any sequence in $P_Y^{-1}\{0\} \cap S_X$. Since X/Y and hence

$Y^{\perp}$ is finite dimensional, and the sequence $\{x_n + Y\}$ is bounded, it has a convergent subsequence that converges to some $x + Y$ in X/Y. W.l.o.g. we can and do assume that the sequence $\{x_n + Y\}$ converges to $x + Y$, that is, $\lim\limits_{n\to\infty} \|x_n - x + Y\| = 0$.

Select y_n in Y such that $\lim\limits_{n\to\infty} \|x_n - x - y_n\| = 0$. Then the sequence $\{x_n - y_n\}$ converges to x and

$$-y_n \in P_Y(x_n) - y_n = P_Y(x_n - y_n), \quad \text{for all } n \geq 1.$$

Since P_Y is u.s.c at x, $d(-y_n, P_Y(x)) \to 0$ as $n \to \infty$. So there exists $\{z_n\} \subseteq P_Y(x)$ such that $\|y_n + z_n\| \to 0$ as $n \to \infty$. Now $\{z_n\}$ has a convergent subsequence that converges to $z \in P_Y(x)$, as $P_Y(x)$ is compact. This implies $\{y_n\}$ has a convergent subsequence, say $\{y_{n_k}\}$, that converges to z and $\{x_{n_k}\} = \{x_{n_k} - y_{n_k} + y_{n_k}\}$ converges to $x - z \in P_Y^{-1}\{0\}$. This implies (x_n) has a convergent subsequence that converges to an element of $P_Y^{-1}\{0\}$ and this completes the proof. □

For $x \in X$, set

$$J_{X^*}(x) = \{f \in X^* : \|f\| = 1 \text{ and } f(x) = \|x\|\}.$$

Note that $J_{X^*}(x)$ is a nonempty subset, by the Hahn-Banach theorem. The map $x \to J_{X^*}(x),\ x \in X$ is called the *duality map* on X. For f in X^*, define

$$J_X(f) = \{x \in S_X : f(x) = \|f\|\}.$$

Note $J_X(f)$ can be empty. If it is nonempty, we recall that f is said to be a *norm attaining functional*. We will denote the class of all norm attaining functionals on X by $NA(X)$. The set-valued map $f \to J_X(f)$ from X^* into X is called the *pre-duality map* on X^*.

Remark 2 Let X be a normed linear space and Y be a subspace of X. If x is in X it is well known that [23]

$$d(x, Y) = \{\max\{f(x) : f \in Y^{\perp} \text{ and } \|f\| = 1\}.$$

Thus for f in $S_{Y^{\perp}}$ and x is in $J_X(f)$, we have

$$1 = \|x\| \geq d(x, y) = 1$$

and hence equality holds. Thus, x is in $P_Y^{-1}\{0\} \cap S_X$. Conversely, if x is in $P_Y^{-1}\{0\} \cap S_X$, then x is in $J_X(f)$ for some f in $S_{Y^{\perp}}$.

Remark 3 Let Y be a proximinal subspace of finite codimension in a normed linear space X. Then an useful corollary of a characterization, of proximinal subspaces of finite codimension of Garkavi (see Godefroy and Indumathi [14]), implies that $Y^{\perp}$ is contained in $NA(X)$. In other words, $J_X(f)$ is nonempty for each f in $Y^{\perp}$.

We now have the following result from [9], relating continuity properties of the metric projection with that of the pre-duality map.

Theorem 7 [9] *Let X be a normed linear space and Y be a proximinal subspace of finite codimension in X. Then the following assertions are equivalent.*

(a) P_Y *is u.s.c.*
(b) P_Y *is H.u.s.c and $P_Y(x)$ is compact for each x in X.*
(c) $P_Y^{-1}\{0\}$ *is boundedly compact.*
(d) $J_X|_{S_{Y^\perp}}$ *is H.u.s.c and $J_X|_{S_{Y^\perp}}(f)$ is compact for each f in $S_{Y^\perp}$.*
(e) $J_X|_{S_{Y^\perp}}$ *is u.s.c.*
(f) $J_X|_{Y^\perp}$ *is u.s.c.*

Proof (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) follows from the above two theorems.
(c) $\Rightarrow$ (d). Note that $J_X(f)$ is a closed subset of $P_Y^{-1}\{0\}$ for f in $S_{Y^\perp}$, so is compact. Pick f in $S_{Y^\perp}$. If J_X is not H.u.s.c at f then there exist $\varepsilon > 0$ and sequences $\{f_n\} \subseteq S_{Y^\perp}$ converging to f and $\{x_n\} \subseteq J_X(f_n)$ such that

$$d(x_n, J_X(f)) \geq \varepsilon, \quad \text{for all } n \geq 1. \tag{3}$$

Now $\{x_n\} \subseteq P_Y^{-1}\{0\} \cap S_X$ for all $n \geq 1$. Hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ converging to, say x_0. Clearly

$$\|x_0\| = 1 = f(x_0) = \lim_{k\to\infty} f_{n_k}(x_{n_k}).$$

So, $x_0 \in J_X(f)$. This contradicts (3).
(d) $\Leftrightarrow$ (e) Follows from Fact 1.
(e) $\Leftrightarrow$ (f). This is so since $J_x(\alpha f) = \alpha J_X(f)$ for $\alpha > 0$ and f in X^*.
(d) $\Rightarrow$ (c). Suppose $P_Y^{-1}\{0\}$ is not boundedly compact. Then there exists a sequence $\{x_n\} \subseteq P_Y^{-1}\{0\} \cap S_X$, that has no convergent subsequence. Pick f_n in $S_{Y^\perp} \cap J_{X^*}(x_n)$ for each $n \geq 1$. Since codim $Y = \dim Y^\perp < \infty$, without loss of generality assume $\{f_n\}$ converges to $f \in S_{Y^\perp}$. Now (d) implies $d(x_n, J_X(f)) \to 0$ as $n \to \infty$. So there exists $\{v_n\} \subseteq J_X(f)$ such that $\|x_n - v_n\| < \frac{1}{n}$ for all $n \geq 1$. Now $\{v_n\}$ has a convergent subsequence, so $\{x_n\}$ also has a convergent subsequence. This gives a contradiction and completes the proof. $\square$

We observe that u.s.c can not be replaced by l.s.c in Theorem 7, as shown by the following example from [9]. Let $X = \mathbb{R}^3$ with l_∞-norm and $Y = \{(\alpha, 0, 0) : \alpha \in \mathbb{R}\}$. Then P_Y is l.s.c since X is a finite dimensional polyhedral space (See comments at the end of Sect. 1) but it is easy to verify that $J_X|_{S_{Y^\perp}}$ is not l.s.c in this case. In general if $3 \leq \dim X < \infty$ and Y be a subspace of X with $\dim Y \leq \dim X - 2$ then P_Y is l.s.c but $J_X|_{S_{Y^\perp}}$ is not. However, the implication in the reverse direction holds and we describe the details below. We need some facts in the sequel. All the results given below in this section are from [9].

Theorem 8 *Let X be reflexive. If $J_X|_{S_{X^*}}$ is l.s.c then X is strictly convex.*

Proof Pick f_0 in S_{X^*}. Suppose $J_X(f_0)$ has two distinct elements. Since X is reflexive, by a result of Lindenstrauss there exists $\{f_n\} \subseteq S_{X^*}$ such that $\|.\|_{X^*}$ is smooth at f_n and $\{f_n\} \to f_0$. Now J_X is l.s.c at f_0, $J_X(f_n)$ is a singleton set for each n but $J_X(f_0)$ is not a singleton set, gives a contradiction. $\square$

Fact 2 *Let X be Banach and Y be a factor reflexive subspace. Then $C = \{f \in S_{Y^\perp} : J_X(f) - J_X(f) \subseteq Y\}$ is dense in $S_{Y^\perp}$. Further if J_X is l.s.c then $C = S_{Y^\perp}$.*

Proof Since X/Y is reflexive, by a result of Lindenstrauss

$$\{f \in S_{(X/Y)^*} : f \text{ is a smooth point}\}$$

is dense in $S_{(X/Y)^*}$. Recall that $(X/Y)^* \simeq Y^\perp$ and note that if $f \in S_{Y^\perp}$ is a smooth point with $f(x) = f(z) = 1$ then $x + Y = z + Y$ or equivalently $x - z \in Y$. Hence C is dense in $S_{Y^\perp}$.

Now assume J_X is l.s.c. Since $X \setminus Y$ is open, the set

$$U = \{f \in S_{Y^\perp} : [J_X(f) - J_X(f)] \cap (X \setminus Y) \neq \emptyset\}$$

is open, as the map $f \to (J_X(f) - J_X(f))$ is l.s.c on X^*. Clearly $C \cap U = \emptyset$, U is open in $S_{Y^\perp}$, and C is dense in $S_{Y^\perp}$ gives a contradiction if U is nonempty. Hence $U = \emptyset$. $\square$

Corollary 2 *If X is Banach and Y is a factor reflexive subspace of X then for every f in $S_{Y^\perp}$ and x in $J_X(f)$ we have $J_X(f) \subseteq x + Y$ and $x - P_Y(x) = J_X(f)$.*

Proof Since x is in $J_X(f)$ and by Remark 2, $\|x\| = \|x + Y\| = 1$. Let z be in $J_X(f)$. Then by Fact 2, $z - x \in Y$ and so $z = x - y$ for some y in Y. Clearly, $\|x - y\| = \|z\| = 1 = d(x, Y)$ and $y \in P_Y(x)$. So, $z \in x - P_Y(x)$ and $J_X(f) \subseteq x - P_Y(x)$.

Now for any y in $P_Y(x)$, we have

$$f(x - y) = f(x) = 1 \quad \text{and} \quad \|x - y\| = \|x + Y\| = 1.$$

Thus $x - P_Y(x) \subseteq J_X(f)$. $\square$

We can now prove the main result. For a normed linear space X and x in X, we denote by $\hat{x}$, the image of x under the canonical embedding of X into X^{**}.

Theorem 9 *Let X be a Banach space and Y be a proximinal subspace of finite codimension in X. If $J_X|_{S_{Y^\perp}}$ is l.s.c., then P_Y is l.s.c.*

Proof We first observe that it is enough to prove P_Y is l.s.c. on $D_Y = \{x \in X : d(x, Y) = 1\}$. Pick any x in X with $d(x, Y) = \|x + Y\| = 1$. It suffices to show that P_Y is l.s.c. at x or equivalently $I - P_Y$ is l.s.c. at x. Let $\phi = \widehat{x}|_{Y^\perp}$. Then $\phi \in S_{(Y^\perp)^*}$. If $f \in J_{Y^\perp}(\phi)$ then $f(x) = \widehat{x}(f) = \phi(f) = 1$ and $x \in J_X(f)$. By the above corollary $x - P_Y(x) = J_X(f)$. Thus we have

$$J_X(f) = x - P_Y(x), \quad \text{for all } f \in J_{Y^\perp}(\phi). \tag{4}$$

Hence to show $I - P_Y$ is l.s.c. at x, it is enough to prove the following: Given y_0 in $P_Y(x)$ and $\varepsilon > 0$, there exists $\eta > 0$ such that if $z \in D_Y$ and $\|x - z\| < \eta$ then we have $B(x - y_0, \varepsilon) \cap (z - P_Y(z)) \neq \emptyset$.

Pick any f in $J_{Y^\perp}(\phi)$. Since J_X is l.s.c. at f and (4) holds, there exists $\delta_f > 0$ such that $g \in S_{Y^\perp}$, $\|f - g\| < \delta_f$ implies $B(x-y_0, \varepsilon) \cap J_X(g) \neq \emptyset$. Since $J_{Y^\perp}(\phi) \subseteq S_{Y^\perp}$ is closed and compact, the open cover $\left\{B\left(f, \frac{\delta_f}{2}\right) : f \in S_{Y^\perp}\right\}$ has a finite subcover, say, $\left\{B\left(f_i, \frac{\delta_{f_i}}{2}\right) : 1 \leq i \leq k\right\}$. If

$$0 < 2\delta < \min\{\delta_{f_i} : 1 \leq i \leq k\},$$

then for any $f \in J_{Y^\perp}(\phi)$ and $g \in S_{Y^\perp}$ satisfying $\|f - g\| < \delta$ we have

$$B(x - y_0, \varepsilon) \cap J_X(g) \neq \emptyset. \tag{5}$$

Now $\dim Y^\perp < \infty$. Using usual compactness arguments, it is easily shown that the map $J_{Y^\perp}$ is H.u.s.c. on $(Y^\perp)^*$. In particular, $J_{Y^\perp}$ is H.u.s.c. at ϕ. So there exists $\eta > 0$ such that $\psi \in S_{(Y^\perp)^*}$, $\|\phi - \psi\| < \eta$ and $g \in J_{Y^\perp}(\psi)$ implies $\|f - g\| < \delta$ for some f in $J_{Y^\perp}(\phi)$. Consequently, (5) holds.

Now pick any $z \in D_Y$ satisfying $\|x - z\| < \eta$. Let $\psi = \widehat{z}|_{Y^\perp}$. Then $\psi \in S_{(Y^\perp)^*}$, $\|\phi - \psi\| < \eta$. Pick any $g \in J_{Y^\perp}(\psi)$. We have $J_X(g) = z - P_Y(z)$ and this with (5) implies

$$B(x - y_0, \varepsilon) \cap (z - P_Y(z)) \neq \emptyset. \qquad \square$$

Note that P_Y and J_X are single valued if X is strictly convex. Hence, we have the following Corollary of the above theorem.

Corollary 3 *Let X be reflexive and assume $J_X|_{S_{X^*}}$ is l.s.c. Then every closed linear subspace Y of finite codimension has a continuous metric projection.*

3 Metric Projection onto Subspaces of Finite Codimension

In this section,we list some recent results which derive continuity properties of metric projections onto subspaces of finite codimension, using "polyhedral" related geometric conditions. However, we begin with a well known result from [12], giving a sufficient condition for continuity of metric projection in reflexive, strictly convex spaces and then describe a striking negative result of P. D. Morris regarding continuity of metric projections onto Chebeychev subspaces of finite codimension in the space $C(Q)$.

Theorem 10 (Glicksberg [12]) *Let X be reflexive and strictly convex Banach space and every $f \in X^*$ is Fréchet smooth. Then the metric projection onto every closed subspace of X is continuous.*

Proof Let Y be a closed subspace of X. Then Y is Chebyshev. Pick x in D_Y. Then there exists $f \in S_{Y^\perp}$ such that $f(x) = 1 = d(x, Y) = \left\|\widehat{x}|_{Y^\perp}\right\|$. Let z in X^{**} be the unique norm preserving extension of $\widehat{x}|_{Y^\perp}$ to X^*. As X is reflexive, z is in X and $\{x - z\} = P_Y(x)$, equivalently $Q_Y(x) = \{z\}$. We will show that Q_Y is continuous at x.

Since X is reflexive and $\|.\|_{X^*}$ is Fréchet smooth at f, given $\varepsilon > 0$ there exists $\delta > 0$ such that $w \in S_X$ and $f(w) > 1 - \delta$ imply $\|z - w\| < \delta$. Let $u \in D_Y$ and assume $\|x - u\| < \delta$. If $\{v\} = Q_Y(u)$, then $f(v) = f(u) > 1 - \delta$ and therefore $\|z - v\| < \varepsilon$ and Q_Y is continuous at x. □

Let H be a hyperplane or a subspace of codimension 1 in X. It is well known that $H = \ker f$, for some f in X^* and H is proximinal in X if and only if the set $J_X(f)$ is nonempty.

We recall that if X is a Banach space, then using the famous James Theorem, we have X is reflexive if and only if every hyperplane is proximinal.

Also, $P_H(x) = \left\{x - \frac{f(x)}{\|f\|} J_X(f)\right\}$ for any x in X and it can be thus easily shown that P_H is Hausdorff metric continuous.

The situation is dramatically different if we consider proximinal subspaces of codimension ≥ 2. Below, we describe a striking negative result of P. D. Morris, which says that if $X = C(Q)$ and Y is a Chebychev subspace of codimension ≥ 2, then P_Y is not continuous on X.

We need the following facts about Chebyshev subspaces in the sequel.

Fact 3 [15] *Let Y be a Chebyshev subspace of finite codimension of a normed linear space X. If the metric projection P_Y is continuous on X, then $S_{X/Y}$ is homeomorphic to $S_X \cap P_Y^{-1}\{0\}$.*

Proof Let $W : X \to X/Y$ be the quotient map, I the identity map on X and $V : X/Y \to P_Y^{-1}\{0\}$ given by $V(x + Y) = x - P_Y(x)$, $x \in X$. It is easy to check that V is a well defined bijective map on X/Y. We now observe that the following diagram commutes.

Since W is open and continuous, V is continuous on X/Y if P_Y is continuous. Note that the inverse V^{-1} of V, is the restriction of the quotient map W to the set $P_Y^{-1}\{0\}$ and hence continuous. Thus, V is a homeomorphism if and only if P_Y is continuous. Further, V is an isometry and therefore, $V(S_{X/Y})$ is the set $S_X \cap P_Y^{-1}\{0\}$. This completes the proof. □

The characterizations, given below, of semi-Chebyshev subspaces is from [23].

Proposition 1 *Let Y be a subspace of X. Then Y is semi-Chebyshev if and only if there do not exist $f \in S_{Y^\perp}$, $x \in X$ and $y_0 \in Y$ such that $f(x_0) = \|x_0\| = \|x_0 - y_0\|$.*

The following result [23, Theorem 2.1] for a subspace of codimension n shows that semi-Chebyshevity of Y restricts the dimension of the set $J_X(f)$, for each nonzero f in $Y^\perp$. If A is a set, aff A denotes the affine hull of A and rel.int A denotes the relative interior of A (interior of A with respect to aff A).

Proposition 2 *Let X be a Banach space and Y be a subspace of codimension n in X. Then Y is semi-Chebyshev implies that for every f in $Y^\perp \setminus \{0\}$ the set $J_X(f)$ is of dimension $r \leq n-1$.*

Proof Assume that there exists f in $Y^\perp \setminus \{0\}$ with $J_X(f)$ having dimension $\geq n$. Then $J_X(f)$ contains $n+1$ affinely linearly independent elements, say, $\{w_1, w_2, \ldots, w_{n+1}\}$. Let $A = co\{w_1, w_2, \ldots, w_{n+1}\}$. Then A is a compact and convex subset of $J_X(f)$ and dim $A = n$. Recall $rel.int A \neq \emptyset$ and pick x_0 in $rel.int A$. Then $0 \in rel.int(A - x_0)$. We have $Y_1 = sp(A - x_0) = aff(A) - x_0$. Then dim $Y_1 = n$ and $0 \in int(A - x_0)$, considered as a subset of Y_1. Thus every z in Y_1 is a positive multiple of an element in $A - x_0$.

Now if $Y_1 \cap Y = \{0\}$ then $X = Y \oplus Y_1$, since codim $Y = n =$ dim Y_1. Now $f \equiv 0$ on $A - x_0$ and therefore $f \equiv 0$ on Y_1 and $f \equiv 0$ on Y. This is a contradiction to $\|f\| = 1$. Hence there exists y_0 in $Y_1 \cap Y \setminus \{0\}$. Choose $\delta > 0$ small so that $-\delta y_0 \in A - x_0$. That is, $x_0 - \delta y_0 \in A$. Note that $\delta y_0 \neq 0$ and $x_0 - \delta y_0 \in A \subseteq J_X(f)$. Since $x_0 \in J_X(f)$, using Proposition 1 we get a contradiction to Y being Chebyshev. $\square$

We now apply the above result to the space $C(Q)$. Let Y be a proximinal subspace of finite codimension in $C(Q)$. Assume $\mu \in S_{Y^\perp}$ and $Q \setminus S(\mu)$ has r points say $\{q_1, \ldots, q_r\}$. Define x and x_i, $1 \leq i \leq r$ in $C(Q)$, with norm one and satisfying

$$x(t) = \begin{cases} 1 & \text{if } t \in S(\mu^+), \\ -1 & \text{if } t \in S(\mu^-), \\ 0 & \text{Otherwise,} \end{cases}$$

$$\begin{tikzcd} X \arrow[rr, "W"] \arrow[dr, "I-P_Y"'] & & X/Y \arrow[dl, "V"] \\ & P_Y^{-1}\{0\} & \end{tikzcd}$$

and

$$x_i(t) = \begin{cases} 1 & \text{if } t \in S(\mu^+) \cup \{q_i\}, \\ -1 & \text{if } t \in S(\mu^-), \\ 0 & \text{Otherwise.} \end{cases}$$

Then $\{x, x, \ldots, x_r\} \subseteq J_{C(Q)}(\mu)$ is a linearly independent set. If Y is Chebyshev then $r \leq$ dim $J_{C(Q)}(\mu) \leq n-1$. Hence $Q \setminus S(\mu)$ has at most $n-2$ points. Since this set is open, it is contained in the set of isolated points of Q.

We are now in a position to prove the result of Morris mentioned earlier.

Theorem 11 (Morris [22]) *Let Y be a Chebyshev subspace of finite codimension in $C(Q)$. Then P_Y is continuous if and only if $codim\ Y = 1$.*

Proof Since Q is infinite compact, Q has a limit point, say q_0. For any $\mu \in Y^\perp$, $Q \setminus S(\mu)$ consist of isolated points. So $q_0 \in S(\mu)$. Since $\mu \in Y^\perp$ was chosen arbitrarily $q_0 \in \cap\{S(\mu) : \mu \in Y^\perp\}$.

Pick x in $P_Y^{-1}\{0\} \cap S_X$. Then there exists a $\mu \in S_{Y^\perp}$ such that $\mu(x) = 1$. Hence $|x(q_0)| = 1$. Set $A = \{x \in P_Y^{-1}\{0\} \cap S_X : x(q_0) = 1\}$. Thus both A and $-A$ are non empty closed sets, $A \cap -A = \phi$. Further $P_Y^{-1}\{0\} \cap S_X = A \cup -A$ is disconnected.

Now assume P_Y is continuous. Then by Fact 3, $S_X \cap P_Y^{-1}\{0\}$ is homeomorphic to $S_{X/Y}$ and so $S_{X/Y}$ is disconnected. This implies dim $X/Y = 1$. □

It was thought that negative results like the one above, may not occur in "nice" spaces. For instance, it was conjectured that if X is reflexive and strictly convex, the scenario may be different and the metric projections onto subspaces of X would be continuous. However, in [4], Brown constructed an example of a proximinal subspace of codimension 2 in a reflexive, strictly convex space with a discontinuous metric projection. However, some of the recent results show that there are indeed a large class of spaces, for which the metric projection onto subspaces of finite codimension have strong continuity properties. We describe them below.

We recall that in a finite dimensional polyhedral space X, the metric projection P_Y is l.s.c. for every subspace Y of X. The following series of results show that polyhedral condition plays a crucial role in the continuity properties of metric projections onto proximinal subspaces of finite codimension too. As far as we are aware, the first result of this kind was given in [14]. Recall that $NA(X)$ denotes the set of norm attaining functionals on X and by $NA_1(X)$, we denote those functionals in $NA(X)$ of norm one.

Theorem 12 [14] *Let X be a Banach space and Y be a subspace of finite codimension in X with $Y^\perp$ polyhedral. Assume that $Y^\perp \subseteq NA(X)$. Then Y is proximinal and the metric projection P_Y has a continuous selection.*

The geometric notion of *QP-points* [26] was utilized in the same paper to get a sufficient condition for strong proximinality and hence for H.u.s.c. of metric projections. To describe the relevant results from this paper, we need the following two definitions that are central to this part of the discussion.

Definition 5 Let X be a Banach space and $F : X \to \mathbb{R}$ be a convex function. We say F is *strongly subdifferentiable* (SSD) at $x \in X$, if the one sided limit $\lim\limits_{t \to 0^+} \frac{F(x+ty)-F(x)}{t}$ exists uniformly for $y \in S_X$.

Definition 6 [26] Let X be a Banach space. An element x in S_X is called a *Quasi-polyhedral point (QP-point)* if there exists $\delta > 0$ such that $J_{X^*}(y) \subseteq J_{X^*}(x)$ for every y in $B(x, \delta) \cap S_X$. If every element of S_X is a *QP-point* of X, then X is said to be a *QP-space*.

We now have the following result linking the *QP-points* and SSD points.

Lemma 1 [14] *Let X be a Banach space and x in S_X be a QP-point. Then the norm of X is SSD at x.*

Proof Since x is a *QP-point*, there exists $\delta > 0$ such that if $\omega \in B(x, 2\delta) \cap S_X$ then $J_{X^*}(\omega) \subseteq J_{X^*}(x)$. Select any $y \in S_X$ and fix $0 < t < \delta$. If $\omega = \frac{x+ty}{\|x+ty\|}$ then $\omega \in S_X$ and $\|x - \omega\| < 2\delta$. Thus $J_{X^*}(x + ty) = J_{X^*}(\omega) \subseteq J_{X^*}(x)$.

Pick any s such that $0 < s < t$ and let $\lambda = s/t$. Then $x+sy = \lambda(x+ty)+(1-\lambda)x$. Now, for any f in $J_{X^*}(x + ty)$,

$$\|x + sy\| \geq f(x + sy) = \lambda\|x + ty\| + (1 - \lambda)\|x\| \geq \|x + sy\|.$$

Hence $f(x + sy) = \|x + sy\|$ and

$$\frac{\|x + sy\| - \|x\|}{s} = \frac{f(x + sy) - f(x)}{s} = f(y).$$

It is now clear that for all y in S_X, we have

$$\lim_{s\to 0^+} \frac{\|x + sy\| - \|x\|}{s} = \frac{\|x + ty\| - \|x\|}{t} = \frac{f(x + ty) - f(x)}{t} = f(y)$$

and the norm of X is SSD at x. □

The following useful characterization of SSD points of the norm of the dual space leads to a characterization of strongly proximinal hyperplanes.

Theorem 13 [14] *Let X be a Banach space and $f \in S_{X^*}$. Then the norm of X^* is SSD at f if and only if $f \in NA_1(X)$ and given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$, such that for $x \in B_X$ satisfying $f(x) > 1 - \delta_\varepsilon$, we have $d(x, J_X(f)) < \varepsilon$.*

Corollary 4 [14] *Let X be a Banach space and $f \in X^*$. Then $H = \ker f$ is strongly proximinal in X if and only if the norm of X^* is SSD at f.*

We now give a characterization of *QP-points*, that helps to visualize *QP-points* on the unit sphere.

Fact 4 *Let X be a Banach space and $x \in S_X$. Then x is a QP-point of X if and only if there exists $\varepsilon > 0$ such that if $y \in S_X$ and $\|x - y\| < \varepsilon$, then the line segment $[x, y]$ lies on the sphere S_X.*

Proof Since x is a *QP-point*, there exists $\varepsilon > 0$ such that $J_{X^*}(y) \subseteq J_{X^*}(x)$ for all $y \in S_X \cap B(x, \varepsilon)$. Select any $y \in S_X \cap B(x, \varepsilon)$ and $f \in J_{X^*}(y)$. Then $f \in J_{X^*}(x)$ and if $\omega \in [x, y]$ then $f(x) = f(y) = f(\omega) = 1$. Since $\|\omega\| \leq 1$, this implies $\|\omega\| = 1$ and $[x, y] \subseteq S_X$.

For the converse, let $x \in S_X$ and select $\varepsilon > 0$ so that the given condition holds. Let $\alpha = \varepsilon/2$ and $z \in S_X \cap B(x, \alpha)$. Considering the 2-dimensional subspace generated by x and z and using the given condition, we can easily get $y \in B(x, \varepsilon) \cap S_X$ such that $z \in (x, y)$. So there exists $\lambda, 0 < \lambda < 1$, such that $z = \lambda x + (1 - \lambda)y$. If $f \in J_{X^*}(z)$ then

$$f(z) = 1, \ f(x) \leq 1 \text{ and } f(y) \leq 1,$$

and so $1 = f(z) = \lambda f(x) + (1 - \lambda) f(y) \leq 1$. This implies $f(x) = f(y) = 1$ and $f \in J_{X^*}(x)$. Thus $J_{X^*}(z) \subseteq J_{X^*}(x)$ for all $z \in S_X \cap B(x, \alpha)$ and x is a *QP-point* of X. □

For a norm attaining functional f in X^*, the functional f being a *QP-point* can be characterized in terms of the sets $J_X(.)$, instead of the sets $J_{X^{**}}(.)$, as the following result shows.

Fact 5 *Let X be a Banach space and $f \in NA_1(X)$. Then f is a QP-point of X^* if there exists $\alpha > 0$ such that $J_X(g) \subseteq J_X(f)$, for all $g \in B(f, \alpha) \cap NA_1(X)$.*

Proof The necessity follows the above Fact. To prove sufficiency, let $f \in NA_1(X)$ satisfy the condition of the lemma. If $g \in NA_1(X)$ and $\|f - g\| < \varepsilon$, then by assumption $J_X(g) \subseteq J_X(f)$. Pick any z in $J_X(g)$. Then $f(z) = g(z) = 1$ and so $(f + g)(z) = 2$. Since $\|z\| = 1$, this implies $\|f + g\| = 2$. Hence

$$\|f + g\| = 2 \ \text{ for all } g \in B(f, \varepsilon) \cap NA_1(X).$$

By the Bishop-Phelps theorem, the set $B(f, \varepsilon) \cap NA_1(X)$, is dense in $B(f, \varepsilon) \cap S_{X^*}$ and this with the continuity of the norm function yields

$$\|f + g\| = 2 \text{ for all } g \in B(f, \varepsilon) \cap S_{X^*}.$$

It is now easy to verify the above equality implies $[f, g] \subseteq S_{X^*}$ if $g \in B(f, \varepsilon/2) \cap S_{X^*}$. By Fact 4, f is a *QP-point* of X^*. □

We now fix some notation, used hereafter. Let X be a normed linear space and $\{f_1 \dots f_n\} \subseteq X^*$. We define subsets $J_X(f_1, \dots f_i)$ for $1 \leq i \leq n$ inductively as follows.

$$J_X(f_1) = \{x \in B_X : f_1(x) = \|f_1\|\}.$$

Having defined $J_X(f_1)$ we define

$$J_X(f_1, \dots f_i) = \{x \in J_X(f_1 \dots f_{i-1}) : f_i(x) = \sup\{f_i(y) : y \in J_X(f_1 \dots f_{i-1})\}\}$$

for $2 \leq i \leq n$. Note that $J_X(f_1) \neq \emptyset \Leftrightarrow f_1 \in NA(X)$. The sets $J_X(f_1 \dots f_i)$ can be empty and if nonempty, they are faces of B_X.

However, if X is finite dimensional then the sets $J_X(f_1 \dots f_i)$ are nonempty for $1 \leq i \leq n$. Further if $\dim X = n$ and $(f_1 \dots f_n)$ is a basis of X^*, then $J_X(f_1 \dots f_n)$ is a singleton set. We set

$$\alpha_i = \sup\{f_i(x) : x \in J_X(f_1, \ldots, f_{i-1})\}, \quad \text{for } 2 \le i \le n.$$

Clearly,

$$\begin{aligned} J_X(f_1 \ldots f_i) &= \{x \in J_X(f_1 \ldots f_{i-1}) : f_i(x) = \alpha_i\} \\ &= \bigcap_{j=1}^{i} \left\{x \in B_X : f_j(x) = \alpha_j\right\} \end{aligned}$$

for $2 \le i \le n$. Further if $x_0 \in J_X(f_1 \ldots f_i)$ then,

$$J_X(f_1 \ldots f_i) = \left\{x \in B_X : f_j(x) = f_j(x_0) \text{ for } 1 \le j \le i\right\}, \quad \text{for } 1 \le i \le n.$$

Theorem 14 *Let X be a Banach space and Y be a proximinal subspace of finite codimension n in X. Then Y is strongly proximinal if and only if for every basis $f_1, \ldots, f_n$ of $Y^\perp$*

$$\lim_{\varepsilon \to 0} \sup\{d(y, J_X(f_1 \ldots f_i)) : y \in J_X(f_1 \ldots, f_i, \varepsilon)\} = 0$$

for $1 \le i \le n$.

Remark 4 It can be shown that [14] if X is a Banach space and Y is a subspace of finite codimension in X such that each functional in $Y^\perp \cap S_{X^*}$ is a *QP-point* of X^*, then Y is proximinal in X.

We are now in a position to prove the following theorem.

Theorem 15 [14] *Let X be a Banach space such that every f in $NA(X) \cap S_{X^*}$ is a QP-point of X^*. Then every subspace Y of finite codimension $Y^\perp \subseteq NA(X)$ is strongly proximinal and P_Y is H.u.s.c.*

Proof By the above Remark, Y is proximinal in X. By Lemma 1, every $f \in NA_1(X)$ is a SSD point of X^*.

Let $(f_1, \ldots, f_n)$ be a basis of $Y^\perp$. We now show that we can select positive scalars λ_i, $1 \le i \le n$, such that

$$J_X(f_1, \ldots, f_i) = J_X\left(\sum_{j=1}^{i} \lambda_j f_j\right), \text{ for } 1 \le i \le n. \tag{6}$$

We use induction on n. We take $\lambda_1 = 1$ and note that the case $n = 1$ is trivial. Inductively assume that $\lambda_j > 0$ for $1 \le j \le i-1$ have been chosen so that if $g_{i-1} = \sum_{j=1}^{i-1} \lambda_j f_j$ then $J_X(g_{i-1}) = J_X(f_1, f_2, \ldots, f_{i-1})$. Now $g_{i-1} \in Y^\perp$ and so is a *QP-point* of X^*, by assumption. Using Fact 5, choose $\lambda_i > 0$ small enough so that

$$J_X(g_{i-1} + \lambda_i f_i) \subseteq J_X(g_{i-1}).$$

By induction assumption,

$$J_X(g_{i-1}, f_i) = J_X(f_1, f_2, \ldots, f_{i-1}, f_i).$$

We have $J_X(g_{i-1} + \lambda_i f_i) \subseteq J_X(g_{i-1})$ and $\lambda_i > 0$. It is now easy to verify that $J_X(g_{i-1}, f_i) = J_X(g_{i-1} + \lambda_i f_i)$ and we have

$$J_X\left(\sum_{j=1}^{i} \lambda_j f_j\right) = J_X(g_{i-1} + \lambda_i f_i) = J_X(g_{i-1}, f_i) = J_X(f_1, f_2, \ldots, f_{i-1}, f_i). \quad (7)$$

This completes the induction and (6) holds. It now follows that

$$x \in J_X\left(\sum_{j=1}^{i} \lambda_j f_j\right) \Rightarrow f_i(x) = \alpha_i \text{ for } 1 \leq i \leq n,$$

where $\alpha_1 = \|f_1\|$ and $\alpha_i = \sup\{f_i(y) : y \in J_X(f_1, f_2, \ldots, f_{i-1})\}$, for $2 \leq i \leq n$.

We now proceed to show that the condition of Theorem 13 holds for the basis $(f_1, f_2, \cdots, f_n)$. Recall that

$$J_X(f_1 \ldots, f_i, \varepsilon) = \bigcap_{j=1}^{i} \{x \in B_X : f_j(x) > \alpha_j - \varepsilon\}.$$

Now $\sum_{j=1}^{i} \lambda_j f_j \in Y^{\perp} \subseteq NA(X) \subseteq$ *QP-points* of X^*, for $1 \leq i \leq n$. Thus the norm of X^* is SSD at $\sum_{j=1}^{i} \lambda_j f_j$ for $1 \leq i \leq n$. So by Theorem 13

$$\lim_{\varepsilon \to 0} \sup \left\{ d\left(y, J_X\left(\sum_{j=1}^{i} \lambda_j f_j\right)\right) : y \in J_X\left(\sum_{j=1}^{i} \lambda_j f_j, \varepsilon\right)\right\} = 0$$

for $1 \leq i \leq n$. It is easy to check that this with (7) implies

$$\lim_{\varepsilon \to 0} \sup \{d(y, J_X(f_1 \ldots f_i)) : y \in J_X(f_1 \ldots, f_i, \varepsilon)\} = 0$$

for $1 \leq i \leq n$. By Theorem 14, Y is strongly proximinal in X. $\square$

It is a natural question to ask whether a stronger conclusion in Theorem 12 is possible. More precisely, can we conclude P_Y is l.s.c. under the conditions of Theorem 12?

The proof of Theorem 12 is rather short and essentially makes use of a property of finite dimensional polyhedral space that allows continuous selection for measures supported on extreme points. However, an imitation of the same proof did not seem to carry further. Relatively long and elaborate proofs were used to show that P_Y is l.s.c. in Theorem 12 if

(i) X is a subspace of c_0. Indumathi [17]
(ii) X is a separable Banach space with Property (*) (V. P. Fonf and J. Lindenstrauss, Pre-print 2003. See also Fonf et al. [13])

where Property (*) as in Definition 9, below.

We observe that (ii) is a generalization of the earlier result (i). However, it was shown in [16] that no additional condition on the Banach space X is , in fact , needed. More precisely,

Theorem 16 [16] *Let X be a Banach space and Y be a proximinal subspace of finite codimension in X with $Y^{\perp}$ polyhedral. Then P_Y is l.s.c.*

We need some definitions and preliminary results to prove the above theorem. Let X be a Banach space, Y be a closed subspace of finite codimension in X. Set

$$Q_Y(x) = x - P_Y(x), x \in X$$

and for a finite subset $\{f_1, \dots, f_k\}$ of $Y^{\perp}$, let

$$Q_{f_1,\dots,f_k}(x) = \bigcap_{i=1}^{k} \{y \in B_X : f_i(y) = f_i(x)\}.$$

Clearly, $Q_{f_1,\dots,f_k}(x)$ is either empty or convex and the domain of the set valued map $Q_{f_1,\dots,f_k}$ will be taken as the set $D_Y = \{x \in X : d(x, Y) = 1\}$ in the sequel. Note that if $\{f_1, \dots, f_k\} \subseteq Y^{\perp}$ then $Q_{f_1,\dots,f_k}(x) \subseteq Q_Y(x)$ and equality holds if $\{f_1, \dots, f_k\}$ is a basis of $Y^{\perp}$. Further if Y is proximinal, $Q_Y(x)$ is nonempty for each x and hence $Q_{f_1,\dots,f_k}(x)$ is nonempty in this case. For $k > 1$ and $x \in D_Y$, define

$$\alpha_{x,k} = \inf\{f_k(z) : z \in Q_{f_1,\dots,f_{k-1}}(x)\}$$

and

$$\beta_{x,k} = \sup\{f_k(z) : z \in Q_{f_1,\dots,f_{k-1}}(x)\}.$$

We now have the following Proposition from [17].

Proposition 3 *Let X be a Banach space, Y be proximinal in X and $x \in D_Y$. Assume that there exists a finite subset $\{f_1, \dots, f_{k+1}\}$, $1 \le k < n$, of $Y^{\perp}$ such that the map $Q_{f_1,\dots,f_k}$ is H.l.s.c at x and further*

$$\alpha_{x,k+1} < f_{k+1}(x) < \beta_{x,k+1}.$$

Then $Q_{f_1,\dots,f_{k+1}}$ *is H.l.s.c at* x.

Proof Let $2\eta = \min\{\beta_{x,k+1} - f_{k+1}(x), f_{k+1}(x) - \alpha_{x,k+1}\}$. Then $\eta > 0$. Since $Q_{f_1,\dots,f_k}$ is H.l.s.c at x, given $\varepsilon > 0$, there exists $\delta > 0$ such that for any z in $Q_{f_1,\dots,f_k}(x)$ and y in D_Y with $\|x - y\| < \delta$, there exists w in $Q_{f_1,\dots,f_k}(y)$ such that $\|z - w\| < \frac{\eta\varepsilon}{8}$. Without loss of generality we assume that $0 < \delta < \frac{\eta\varepsilon}{8}$, $0 < \varepsilon < 1$, and $\|f_i\| = 1$ for $1 \leq i \leq n$. Now, if $y \in D_Y$ and $\|x - y\| < \delta$, it follows easily that

$$\beta_{y,k+1} > \beta_{x,k+1} - \frac{\eta}{8} \ , \ \ \alpha_{y,k+1} < \alpha_{x,k+1} + \frac{\eta}{8} \tag{8}$$

$$\alpha_{y,k+1} < f_{k+1}(y) < \beta_{y,k+1}. \tag{9}$$

Fix $z \in Q_{f_1,\dots,f_{k+1}}(x)$. We have to show that there exists v in $Q_{f_1,\dots,f_{k+1}}(y)$ such that $\|z - v\| < \varepsilon$.

Since $Q_{f_1,\dots,f_{k+1}}(x) \subseteq Q_{f_1,\dots,f_k}(x)$, there exists w in $Q_{f_1,\dots,f_k}(y)$ such that $\|z - w\| < \frac{\eta\varepsilon}{8}$. We have

$$f_{k+1}(z) = f_{k+1}(x), \quad \|w - z\| < \frac{\eta}{8}, \quad \|x - y\| < \frac{\eta\varepsilon}{8} < \frac{\eta}{8}.$$

This together with (8) and (9) implies

$$\begin{aligned}\beta_{y,k+1} - f_{k+1}(w) &= \beta_{y,k+1} - \beta_{x,k+1} + \beta_{x,k+1} - f_{k+1}(x) \\ &\quad + f_{k+1}(x) - f_{k+1}(z) + f_{k+1}(z) - f_{k+1}(w) > 2\eta - \frac{\eta}{8} + \frac{\eta}{8} > \eta.\end{aligned} \tag{10}$$

Similarly we can show that

$$f_{k+1}(w) - \alpha_{y,k+1} > \eta. \tag{11}$$

Also,

$$\begin{aligned}|f_{k+1}(y) - f_{k+1}(w)| &\leq |f_{k+1}(w) - f_{k+1}(z)| + |f_{k+1}(z) - f_{k+1}(x)| \\ &\quad + |f_{k+1}(x) - f_{k+1}(y)| \\ &< \frac{\eta\varepsilon}{8} + \frac{\eta\varepsilon}{8} = \frac{\eta\varepsilon}{4} < \frac{\eta}{4}.\end{aligned} \tag{12}$$

If $f_{k+1}(w) = f_{k+1}(y)$, then $w \in Q_{k+1}(y)$ and $\|w - z\| < \varepsilon$. Take $v = w$ in this case. Otherwise, we slightly perturb w to get an element of $Q_{f_1,\dots,f_{k+1}}(y)$ as follows. Note that using (10)–(12), we can get w_1 in $Q_{f_1,\dots,f_k}(y)$ such that

$$|f_{k+1}(w_1) - f_{k+1}(w)| > \eta, \tag{13}$$

and $f_{k+1}(y)$ lies in between $f_{k+1}(w)$ and $f_{k+1}(w_1)$. Choose $0 < \lambda < 1$ such that

$$f_{k+1}(\lambda w + (1 - \lambda)w_1) = f_{k+1}(y)$$

and take $v = \lambda w + (1-\lambda)w_1$. Since w and w_1 are in $Q_{f_1,\ldots,f_k}(y)$, v is in $Q_{f_1,\ldots,f_{k+1}}(y)$. Also,

$$(1-\lambda)[f_{k+1}(w_1) - f_{k+1}(w)] = f_{k+1}(y) - f_{k+1}(w).$$

This together with (12) and (13) gives

$$1 - \lambda < \frac{\eta\varepsilon}{4\eta} = \frac{\varepsilon}{4}.$$

Hence

$$\|w - v\| = (1-\lambda)\,\|w - w_1\| \le 2(1-\lambda) < \frac{2\varepsilon}{4} = \frac{\varepsilon}{2},$$
$$\|z - v\| \le \|z - w\| + \|w - v\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence the proof is complete. □

Definition 7 Let Y be a proximinal subspace of codimension n in a Banach space X and x, an element of D_Y. We say x is a *k-corner point*, $1 \le k \le n$, with respect to a linearly independent set of functionals $\{f_1, \ldots, f_k\}$ in $Y^\perp$ if $Q_{f_1,\ldots,f_k}(x) = \cap_{i=1}^k J_X(f_i)$.

We now require some well known facts about finite dimensional convex sets. Let E be a finte dimensional normed linear space. For $C \subseteq A \subseteq E$, where C is convex and A is affine, by "interior of C with respect to A", we mean the interior of C, considered as a subset of the affine space A. The set of all extreme points of C would be denoted by $extC$. A subset D of C is called *extremal* if D contains an interior point of a line segment l in C then D contains l. Clearly, a singleton extremal set is an extreme point.

We now state the following result from [16].

Lemma 2 *Let X be a Banach space, Y be a proximinal subspace of finite codimension n in X with* $Y^\perp$ *polyhedral. For each* x_0 *in* D_Y*, there is a basis* $\{f_1, \ldots, f_n\}$ *of* $Y^\perp$ *such that* $Q_{f_1,\ldots,f_n}(x_0) = \bigcap_{i=1}^k J_X(f_i)$ *either with* $k = n$*, or with* $1 \le k \le n$*. In the later case we have*

$$\alpha_{x_0,j} < f_j(x_0) < \beta_{x_0,j}, \quad \text{for } k+1 \le j \le n.$$

The following theorem is an immediate from Proposition 3 and Lemma 2.

Theorem 17 [16] *Let X be a Banach space, Y a proximinal subspace of finite codimension n in X with* $Y^\perp$ *polyhedral. Assume that, whenever x in* D_Y *is a k-corner point with respect to a set of linearly independent functionals* $\{f_1, \ldots, f_k\}$ *in* $Y^\perp$ *for some positive integer* k, $1 \le k \le n$*, then the map* $Q_{f_1,\ldots,f_k}$ *is H.l.s.c at x. Then the metric projection* P_Y *is H.l.s.c on X.*

The result below shows that the conditions of the above theorem hold under natural assumptions. More specifically, we have

Lemma 3 *Let X be a Banach space and Y be a proximinal subspace of finite codimension in X, with $Y^{\perp}$ polyhedral. If x_0, in D_Y, is a k-corner point with respect to a set of linearly independent functionals $\{f_1, \ldots, f_k\}$ in $Y^{\perp}$, then the set valued map $Q_{f_1,\ldots,f_k}$ is H.l.s.c at x_0.*

It is now easily seen that Theorem 16 follows from Theorem 17 and Lemma 3 above. Hence to complete the proof of Theorem 16, it suffices to prove Lemma 3.

To prove Lemma 3, we need the result quoted below. By $(\mathbb{R}^+)$, we denote the set of non-negative real numbers.

Proposition 4 *Let E be a finite dimensional polyhedral space and let ext $B_E = \{e_1, \ldots, e_m\}$. Then there exists a continuous map $A : B_E \to (\mathbb{R}^+)^m$ such that, if $A(x) = (\mu_i(x))_{i=1}^m$, then*

$$\sum_{i=1}^{m} \mu_i(x) = 1, \ \ and \ \ x = \sum_{i=1}^{m} \mu_i(x) e_i$$

for all x in B_E.

We now continue the proof of Lemma 3.

Proof The finite dimensional space $Y^{\perp}$ is polyhedral and so is its dual, $(Y^{\perp})^*$. Thus $B_{(Y^{\perp})^*}$ has only finite number of extreme points. As

$$S_{(Y^{\perp})^*} = \{\phi_x : x \in D_Y\},$$

there exists a finite subset $\{x_1, \ldots, x_m\}$ of D_Y such that

$$ext \ B_{(Y^{\perp})^*} = \{\phi_{x_1}, \ldots, \phi_{x_m}\}.$$

Let x be in D_Y. Then ϕ_x is in $S_{(Y^{\perp})^*}$. Taking $E = (Y^{\perp})^*$ in Proposition 4, let $A(\phi_x) = (\mu_i(\phi_x))_{i=1}^m$. Since the map $x \to \phi_x$ is continuous, the map $x \to A(\phi_x))$ is continuous from D_Y into $(\mathbb{R}^+)^m$. We abbreviate, $\mu_i(\phi_x)$ as $\mu_i(x)$ for $1 \leq i \leq m$. Then $\sum_{i=1}^m \mu_i(x) = 1$ and

$$\phi(x) = \sum_{i=1}^{m} \mu_i(x) \phi_{x_i}. \tag{14}$$

By assumption, x_0 is in D_Y and is a k-corner point with respect to a set of linearly independent functionals $\{f_1, \ldots, f_k\}$ in $Y^{\perp}$. We need to show that the set valued map $Q_{f_1,\ldots,f_k}$ is H.l.s.c at x_0. For this purpose, we first define a set valued map T_k on D_Y as follows:

$$T_k(x) = \sum_{i=1}^{m} \mu_i(x) Q_{f_1,\ldots,f_k}(x_i), \quad \text{for } x \in D_Y.$$

We now claim that T_k is H.l.s.c on D_Y.

To see this, fix x in D_Y and $\varepsilon > 0$. Since the map $x \to (\mu_i(x))_{i=1}^m$ is continuous from D_Y into $(\mathbb{R}^+)^m$, there exists $\delta > 0$ such that for z in $D_Y \cap B(x, \delta)$, we have

$$\sum_{i=1}^{m} |\mu_i(z) - \mu_i(x)| < \varepsilon.$$

For any v in $T_k(x)$, there is a v_i in $Q_{f_1,\ldots,f_k}(x_i)$, for $1 \le i \le m$, such that $v = \sum_{i=1}^m \mu_i(x) v_i$. Let $w = \sum_{i=1}^m \mu_i(z) v_i$. Clearly w is in $T_k(z)$ and $\|v - w\| < \varepsilon$ as $\|v_i\| \le 1$, for $1 \le i \le m$. It now follows that

$$T_k(x) \subseteq T_k(z) + B(0, \varepsilon),$$

for any z in $B(x, \delta)$. Hence the map T_k is H.l.s.c at x and hence, on D_Y.

We now proceed to show that $Q_{f_1,\ldots,f_k}$ is l.s.c at x_0. In order to do this, we first show that

$$T_k(x_0) = Q_{f_1,\ldots,f_k}(x_0) \text{ and } T_k(x) \subseteq Q_{f_1,\ldots,f_k}(x), \quad \text{for all } x \in D_Y.$$

To begin with, note that

$$Q_{f_1,\ldots,f_k}(x_0) = \cap_{j=1}^m J_X(f_j). \tag{15}$$

Now for x in D_Y, by Eq. (14), we have

$$f_j(x) = \sum_{i=1}^{m} \mu_i(x) f_j(x_i), \quad \text{for } 1 \le j \le k.$$

Select any z in $T_k(x)$. Then there are elements $z_i \in Q_{f_1,\ldots,f_k}(x_i)$ for $1 \le i \le m$, such that $z = \sum_{i=1}^m \mu_i(x) z_i$. Note that $\|z\| \le 1$, as $\|z_i\| \le 1$ for $1 \le i \le m$. Also for $1 \le j \le k$,

$$f_j(z) = \sum_{i=1}^{m} \mu_i(x) f_j(z_i) = \sum_{i=1}^{m} \mu_i(x) f_j(x_i) = f_j(x).$$

Since $\|z\| \le 1$, this implies z is in $Q_{f_1,\ldots,f_k}(x)$ and we have

$$T_k(x) \subseteq Q_{f_1,\ldots,f_k}(x), \quad \forall x \in D_Y.$$

Now $\phi_{x_0} = \sum_{i=1}^m \mu_i(x_0) \phi_{x_i}$, and by Eq. (15),

$$f_j(x_0) = \phi_{x_0}(f_j) = \|f_j\| = \sum_{i=1}^{m} \mu_i(x_0)\phi_{x_i}(f_j), \quad \text{for } 1 \le j \le k.$$

Since ϕ_{x_i} are norm one elements of $(Y^{\perp})^*$, we must have

$$f_j(x_i) = \phi_{x_i}(f_j) = \|f_j\|, \quad \text{for } 1 \le j \le k,$$

whenever $\mu_i(x_0) \neq 0$. Hence

$$Q_{f_1,\ldots,f_k}(x_i) = \bigcap_{j=1}^{k} J_X(f_j) = Q_{f_1,\ldots,f_k}(x_0),$$

whenever $\mu_i(x_0) \neq 0$, $1 \le i \le m$. It is now easy to see that

$$T_k(x_0) = \sum_{i=1}^{m} \mu_i(x_0) Q_{f_1,\ldots,f_k}(x_i) = Q_{f_1,\ldots,f_k}(x_0).$$

Thus

$$T_k(x_0) = Q_{f_1,\ldots,f_k}(x_0) \text{ and } T_k(x) \subseteq Q_{f_1,\ldots,f_k}(x), \quad \text{for all } x \in D_Y. \tag{16}$$

Since the map T_k is H.l.s.c at x_0, it now easily follows from Eq. (16) that the map $Q_{f_1,\ldots,f_k}$ is also H.l.s.c at x_0. This completes the proof of the Lemma. □

Dutta and Narayana [10] proved that if Y is a strongly proximinal subspace of finite codimension in $C(Q)$ then P_Y is Hausdorff metric continuous. Here too, polyhedral condition plays an important role. They in fact show that $Y^{\perp}$ is polyhedral in this case and use it to prove their conclusion.

Dutta and Shanmugaraj [11] quantified strong proximinality through

$$\varepsilon(x,t) = \inf\{r > 0 : P_Y(x,t) \subseteq P_Y(x) + rB_Y\}$$

for $x \in X \backslash Y$ and $t \ge 0$. They have proved that if Y is a strongly proximinal subspace of finite codimension, P_Y is Hausdorff semi-continuous at x in X if and only if $\varepsilon(t)$ is continuous at x for every $t > 0$.

Recently, in 2011, a long and comprehensive paper "Best Approximation in polyhedral spaces" by Fonf et al. [13] presents significant results, linking geometric properties of a Banach space X with that of the continuity properties metric projection onto subspaces of finite codimension. We need the following definitions to state the results.

Definition 8 [13] A set $\mathscr{B} \subseteq S_{X^*}$ is a *boundary* for X if for each x in X there exists $f \in \mathscr{B}$ with $f(x) = \|x\|$.

Definition 9 [13] A Banach Space X satisfies a *property* $(*)$ if there exists a boundary $\mathscr{B} \subseteq S_{X^*}$ such that $\mathscr{B}' \cap NA(X) \neq \emptyset$, where $\mathscr{B}'$ is the set of all w^*-accumulation points of $\mathscr{B}$.

Definition 10 [13] A Banach Space X satisfies a *property* (Δ) if there exists a boundary $\mathscr{B} \subseteq S_{X^*}$ such that the set

$$\{f \in \mathscr{B} : f(x) = 1\} = J_{X^*}(x) \cap \mathscr{B}$$

is finite for each $x \in S_X$.

It is known that if X is a QP-space then X is polyhedral. The result below from [13] explains the relation between the above geometric conditions.

Fact 6 [13] *Let X be a Banach space. Then*

$$\begin{aligned} X \text{ has Property } (*) &\Rightarrow X \text{ is } \ QP \text{ with } \Delta. \\ &\Leftrightarrow X \text{ is polyhedral with } \Delta. \end{aligned}$$

Definition 11 Let X be a Banach space and Y be a closed subspace of X. Then the *effective domain* of P_Y, denoted by $\text{dom} P_Y$, is the set $\{x \in X : P_Y(x) \neq \phi\}$.

Theorem 18 [13] *Let Y be a closed subspace of X. Then*

(a) *If X is polyhedral with (Δ), then P_Y is H.l.s.c on* $\text{dom} P_Y$. *In particular, P_Y restricted to* $\text{dom} P_Y$ *admits a continuous selection by Michael's selection theorem.*
(b) *If X is polyhedral with (Δ), P_Y is not necessarily H.u.s.c on* $\text{dom} P_Y$, *even when Y is proximinal with a finite codimension.*
(c) *If X satisfies $(*)$, then P_Y is Hausdorff continuous on* $\text{dom} P_Y$.

Remark 5 We would like to mention here that Theorem 5.1 of [13], which says that a proximinal subspace Y of a Banach space X is strongly proximinal if and only if the metric projection P_Y is H.u.s.c, is incorrect. While it is easy to prove that strong proximinality of Y implies P_Y is H.u.s.c., the implication in the reverse direction is not true.

To see this, let X be a Banach space and H =ker f, where f is in $NA(X)$. Then H is a proximinal hyperplane and it is easily shown that P_H is H.u.s.c. However, many examples of proximinal hyperplanes which are not strongly proximinal are known [14]. Thus the conclusion of Theorem 5.1 of [13] does not hold.

Before concluding the article, we make two observations. The notion of a.l.s.c. has not been discussed much in the context of metric projections onto subspaces of infinite dimension and in particular, onto subspaces finite codimension. It is desirable to characterize the class of Banach spaces X such that for every proximinal subspace Y of finite codimension in X, the metric projection map P_Y is a.l.s.c.

We are not aware of any characterization of a proximinal subspace of finite codimension Y in $C(Q)$ with the metric projection P_Y having a continuous selection or P_Y is a.l.s.c. It would be desirable to obtain some results in that direction.

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