

Chapter 2

Matrix Transformations

The theory of matrix transformations deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y . This is a natural generalization of the problem to characterize all summability methods given by infinite matrices that preserve convergence.

If A is an infinite matrix with complex entries a_{nk} ($n, k \in \mathbb{N}$), then we may write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the n th row of A , i.e., $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in w$, then we define the A -transform of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k \quad (n \in \mathbb{N})$$

provided the series on the right converges for each $n \in \mathbb{N}$. Further, the sequence x is said to be A -summable to the complex number ℓ if $A_n(x) \rightarrow \ell$ ($n \rightarrow \infty$); we shall write $x \rightarrow \ell(A)$ where ℓ is called the A -limit of x .

Let X and Y be subsets of w and A an infinite matrix. Then, we say that A defines a *matrix mapping* from X into Y if Ax exists and is in Y for every $x \in X$. By (X, Y) , we denote the class of all infinite matrices that map X into Y . Thus, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

The study of sequence spaces is much more profitable when we consider them equipped with certain topologies. In this chapter, we shall study various duals of some sequence spaces and characterize several matrix classes (c.f. [1–3]).

2.1 Continuous Duals

Let X and Y be normed linear spaces. Then, $\mathcal{B}(X, Y)$ denotes the set of all bounded linear operators $L: X \rightarrow Y$. If Y is complete, then $\mathcal{B}(X, Y)$ is a Banach space with the operator norm defined by $\|L\| = \sup_{x \in S_X} \|L(x)\|$ for all $L \in \mathcal{B}(X, Y)$.

By $X' = \mathcal{B}(X, \mathbb{C})$, we denote the *continuous dual of X* , that is, the set of all continuous linear functionals on X . If X is a Banach space, then we write X^* for X' with its norm given by $\|f\| = \sup_{x \in S_X} |f(x)|$ for all $f \in X'$.

We shall write $X^* \simeq Y$ if Y is the continuous dual of X , i.e. X^* is isomorphically isometric to Y .

Theorem 2.1 ([3, pp. 92]) $\ell_1^* \simeq \ell_\infty$.

Proof Let $f \in \ell_1^*$. Since $e^{(k)}$ is a Schauder basis for ℓ_1 , each $x = (x_k) \in \ell_1$ can be represented as $x = \sum_k x_k e^{(k)}$. Let $f \in \ell_1^*$ and define $\alpha_k = f(e^{(k)})$. Then, for every $x = (x_k) \in \ell_1$

$$f(x) = \sum_k x_k f(e^{(k)}) = \sum_k x_k \alpha_k. \quad (2.1)$$

Let us define $x = (x_k)$ by

$$x_k = \begin{cases} \operatorname{sgn} \alpha_n, & k = n, \\ 0, & k \neq n. \end{cases}$$

Then $\|x\| \leq 1$ and $f(x) = |\alpha_n|$ by (2.1). Therefore

$$|\alpha_n| \leq \|f\| \|x\| \leq \|f\|$$

which implies that $\alpha = (\alpha_n) \in \ell_\infty$ and

$$\|\alpha\|_\infty \leq \|f\|. \quad (2.2)$$

Conversely, let $\alpha = (\alpha_n)$ be given such that $\alpha \in \ell_\infty$. Define $T: \ell_1^* \rightarrow \ell_\infty$ by $Tf = u = (f(e^{(k)}))$. Clearly, T is linear and from (2.1), if $Tf = 0$ then $f = 0$, i.e., T is one-one. To prove T is onto, define $f: \ell_1 \rightarrow \mathbb{K}$ by $f(x) = \sum_k \alpha_k x_k$, $x = (x_k) \in \ell_1$. Then f is linear, and

$$|f(x)| \leq \sum_k |\alpha_k x_k| \leq \left(\sup_k |\alpha_k| \right) \left(\sum_k |x_k| \right) = \|\alpha\|_\infty \|x\|_1 \leq M \|x\|_1,$$

where $M = \sup_k |\alpha_k|$. Hence f is bounded and $f \in \ell_1^*$. Thus T is onto. Finally, by (2.1) we get

$$|f(x)| \leq \sum_k |x_k f(e^{(k)})| \leq \left(\sup_k |f(e^{(k)})| \right) \left(\sum_k |x_k| \right) = \left(\sup_k |f(e^{(k)})| \right) \|x\|_1$$

which implies that

$$\|f\| \leq \sup_k |f(e^{(k)})| = \|\alpha\|_\infty. \quad (2.3)$$

Combining (2.2) and (2.3), we have $\|f\| = \|\alpha\|_\infty = \|Tf\|_\infty$. Therefore, T is an isometric isomorphism from ℓ_1^* onto ℓ_∞ , that is, $\ell_1^* \simeq \ell_\infty$.

This completes the proof of the theorem.

Theorem 2.2 ([3, pp. 92]) $c^* \simeq \ell_1$.

Proof To prove our theorem we have to show that c^* is isometrically isomorphism to ℓ_1 . Let $f \in c^*$. Since (e, e_1, e_2, \dots) is a basis for c , any element $x = (x_n) \in c$ can be written as

$$x = \ell e + \sum_k (x_k - \ell) e^{(k)}, \quad \ell = \lim_n x_n.$$

Therefore

$$f(x) = \ell f(e) + \sum_k (x_k - \ell) f(e^{(k)}), \quad \text{for all } x \in c.$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} \operatorname{sgn} f(e^{(k)}), & 1 \leq k \leq n, \\ 0, & k > n. \end{cases}$$

Then $x \in c_0$, $\|x\| = 1$ and

$$|f(x)| = \sum_{k=1}^n |f(e^{(k)})| \leq \|f\|.$$

Hence the series

$$\sum_k f(e^{(k)})$$

is absolutely convergent.

Put

$$f(x) = a\ell + \sum_k a_k x_k, \quad (2.4)$$

where

$$a = f(e) - \sum_k f(e^{(k)}), \quad a_k = f(e^{(k)}).$$

We have

$$|f(x)| \leq \left(|a| + \sum_k |a_k| \right) \|x\|_\infty,$$

since $|\lim x_n| \leq \|x\|_\infty$. Therefore

$$\|f\| \leq |a| + \sum_k |a_k|. \quad (2.5)$$

Also, we have

$$|f(x)| \leq \|f\| \text{ for } \|x\|_\infty = 1. \quad (2.6)$$

For any $r \geq 1$, define $x = (x_n)$ by

$$x_n = \begin{cases} \operatorname{sgn} a_n, & 1 \leq n \leq r, \\ \operatorname{sgn} a, & n > r. \end{cases}$$

Then $x \in c$, $\|x\|_\infty = 1$, $\lim x_n = \operatorname{sgn} a$ and so by (2.4), we have

$$\begin{aligned} |f(x)| &= \left| a \operatorname{sgn} a + \sum_{n=1}^r a_n \operatorname{sgn} a_n + \sum_{n=r+1}^{\infty} a_n \operatorname{sgn} a \right| \\ &= \left| |a| + \sum_{n=1}^r |a_n| + \sum_{n=r+1}^{\infty} a_n \operatorname{sgn} a \right| \leq \|f\| \text{ by (2.6).} \end{aligned} \quad (2.7)$$

Since $a = (a_n) \in \ell_1$, we obtain

$$\sum_{n=r+1}^{\infty} a_n \rightarrow 0 \quad (r \rightarrow \infty).$$

Therefore letting $r \rightarrow \infty$ in (2.7), we get

$$|a| + \sum_n |a_n| \leq \|f\|. \quad (2.8)$$

Combining (2.5) and (2.8), we derive the equality

$$\|f\| = |a| + \sum_n |a_n|. \quad (2.9)$$

Let $T: c^* \rightarrow \ell_1$ be defined by $T(f) = (a, a_1, a_2, \dots)$. Then by (2.9), we have

$$\|T(f)\|_1 = |a| + |a_1| + |a_2| + \dots = \|f\|.$$

$\|T(f)\|_1$ is the ℓ_1 -norm. Thus T is norm preserving. Clearly T is surjective and linear, and so T is isometrically isomorphism whence $c^* \simeq \ell_1$.

This completes the proof.

Remark 2.1 The dual of c_0 can also be identified with ℓ_1 .

In the next theorem, we deal with the dual of ℓ_∞ . It is to be noted that its dual is totally different from other sequence spaces. To deal with, we first introduce some notations and definitions (c.f. [1]).

Definition 2.1 For a non-empty set X , let R denote a ring of subsets of X . The symbol σ_E ($E \in R$) denotes a partition E_1, \dots, E_n of E such that $E_i \in R$, $E_i \cap E_j = \emptyset$, $\cup_{i=1}^n E_i = E$. By a *charge* μ on R , we mean a \mathbb{K} -valued finitely additive set function with $|\mu(E)| < \infty$ for each $E \in R$, and triplet (X, R, μ) is called a *charged space*; further (X, R, μ) is called a *completely charged space*, if $X \in R$ and $|\mu(x)| < \infty$. Let $ba(X, R)$ denote the space of all charges on a ring R , equipped with the norm

$$\|\mu\| = \sup_{E \in R} \sup_{\sigma_E} \sum_{i=1}^n |\mu(E_i)|.$$

If $X \in R$, then

$$\|\mu\| = \sup_{\sigma_X} \sum_{i=1}^n |\mu(E_i)|.$$

Note that $ba(X, R)$ is a Banach space with $\|\mu\|$.

Let ϕ_∞ denote the class of all subsets of \mathbb{N} . Then, $ba(\mathbb{N}, \phi_\infty)$ is a Banach space with the norm

$$\|\mu\| = \sup_{\sigma_{\mathbb{N}}} \sum_{i=1}^n |\mu(E_i)|.$$

Definition 2.2 For a partition $\sigma_X := \{E_i: 1 \leq i \leq N\}$, choose arbitrary points $n_i \in E_i$ and let

$$f_\sigma = f(\sigma; n_1, n_2, \dots, n_N) = \sum_{i=1}^N f(n_i) \mu(E_i).$$

If the net (f_σ) converges in \mathbb{K} , say to I , then f is said to be μ -integrable over X , where $f: X \rightarrow \mathbb{K}$ is μ -measurable and bounded, and $I = \int f d\mu$.

Note that

$$\left| \int f d\mu \right| \leq \sup_{x \in X} |f(x)| \sup_{\sigma_X} \sum_{i=1}^n |\mu(E_i)| \leq \|\mu\| \sup_{x \in X} |f(x)|$$

and if X is a normed space and $f \in X^*$, then

$$\left| \int f d\mu \right| \leq \|f\| \cdot \|\mu\|.$$

For our convenience, we will write \mathcal{M} for $ba(\mathbb{N}, \phi_\infty)$, i.e., the space of bonded finitely additive set functions (or measures) μ defined on subsets of the set of positive integers \mathbb{N} . It is to be remarked that the continuous dual of ℓ_∞ is not isomorphic to a sequence spaces ([3, Example 6.4.8, pp. 93-94]. This can be attributed to the fact that ℓ_∞ has no Schauder basis (it is not separable).

Theorem 2.3 ([1, Theorem 7.22; pp. 130]) *Let $F: \ell_\infty^* \rightarrow \mathcal{M}$ be such that*

$$F(f) = \mu_f, \quad \mu_f(E) = f(\chi_E), \quad E \in \phi_\infty, \quad (2.10)$$

$$F^{-1}(f) = f_\mu, \quad f_\mu(x) = \int_{\mathbb{N}} x d\mu, \quad x \in \ell_\infty. \quad (2.11)$$

Then F is isometric isomorphism from ℓ_∞^ onto \mathcal{M} and*

$$\|F(f)\| = \|f\|, \quad f \in \ell_\infty^*.$$

Proof Let μ_f be defined as in (2.10) and $f \in \ell_\infty^*$. Then μ_f is a complete charge on ϕ_∞ and so there is a map $F: \ell_\infty^* \rightarrow \mathcal{M}$. If $F(f) = 0$, then by (2.10), $f = 0$. Thus F is injective. Let $\mu \in \mathcal{M}$. Then for each $x \in \ell_\infty$, the integral $\int_{\mathbb{N}} x d\mu$ exists and so it defines a linear functional f_μ on ℓ_∞ .

$$|f_\mu(x)| = \left| \int_{\mathbb{N}} x d\mu \right| \leq \|\mu\| \sup_{i \geq 1} |x(i)| = \|\mu\| \cdot \|x\|$$

this implies that $\|f_\mu\| \leq \|\mu\|$, and hence

$$f_\mu \in \ell_\infty^* \quad (2.12)$$

i.e., F is surjective.

Choose a $\sigma_{\mathbb{N}} := \{E_1, E_2, \dots, E_N\}$ for $\epsilon > 0$ such that

$$\sum_{i=1}^N |\mu_f(E_i)| > \|\mu_f\| - \epsilon,$$

where $\mu_f = F(f)$, $f \in \ell_\infty^*$. Define $x \in \ell_\infty$ by

$$x_n = \begin{cases} \frac{\overline{\mu_f(E_i)}}{|\mu_f(E_i)|} & \text{if } n \in E_i \text{ and } \mu_f(E_i) \neq 0, \\ 0 & \text{if } n \in E_i \text{ and } \mu_f(E_i) = 0. \end{cases}$$

Then $\|x\|_\infty \leq 1$ and

$$f(x) = \int x d\mu_f = \sum_{i=1}^N \int_{E_i} x d\mu_f = \sum_{i=1}^N |\mu_f(E_i)|$$

which implies that

$$\|f\| \geq |f(x)| \geq \sum_{i=1}^n |\mu_f(E_i)| \geq \|\mu_f\| - \epsilon.$$

Hence $\|F(f)\| \leq \|f\|$. Now each f corresponds to some $\mu = F(f)$, and so by (2.12), $\|f\| \leq \|F(f)\|$.

Therefore $\|F(f)\| = \|f\|$. Finally we have that the dual of ℓ_∞ is \mathcal{M} whose elements have the representation $\int_{\mathbb{N}} d\mu$.

The proof is complete.

Remark 2.2 Note that the dual of bs is also \mathcal{M} .

2.2 Köthe–Toeplitz Duals

For any two sequences x and y , let $xy = (x_k y_k)_{k=1}^\infty$. If $z = (z_k)_{k=1}^\infty$ is any sequence and Y any subset of ω , then we shall write

$$z^{-1} * Y := \{x \in \omega : zx \in Y\}.$$

Definition 2.3 Let X be a sequence space. Then

$$\begin{aligned} X^\alpha &:= \left\{ a \in \omega : \sum_k |a_k x_k| < \infty \text{ for all } x \in X \right\} \\ &:= \bigcap_{x \in X} (x^{-1} * \ell_1), \end{aligned}$$

$$\begin{aligned}
X^\beta &:= \left\{ a \in \omega: \sum_k a_k x_k \text{ converges for all } x \in X \right\} \\
&:= \bigcap_{x \in X} (x^{-1} * cs),
\end{aligned}$$

and

$$\begin{aligned}
X^\gamma &:= \left\{ a \in \omega: \sup_n \left| \sum_{k=1}^n a_k x_k \right| < \infty \text{ for all } x \in X \right\} \\
&:= \bigcap_{x \in X} (x^{-1} * bs)
\end{aligned}$$

are called the *Köthe–Toeplitz dual* (or α -dual), *generalized Köthe–Toeplitz* (or β -dual), and *bounded dual* (or γ -dual) of X , respectively.

Theorem 2.4 ([2, Theorem 2.5.3]) *We have*

- (i) $\ell_\infty^\alpha = c_0^\alpha = c^\alpha = \ell_1$,
- (ii) $\ell_1^\alpha = \ell_\infty$.

Proof (i) Let $a \in \ell_1$ and $x \in \ell_\infty$. Then $\|x\|_\infty = \sup_k |x_k| < \infty$. Therefore

$$\sum_k |a_k x_k| \leq (\sup_k |x_k|) \sum_k |a_k| = \|x\|_\infty \sum_k |a_k| < \infty,$$

i.e., $a \in \ell_\infty^\alpha$. Hence $\ell_1 \subseteq \ell_\infty^\alpha$.

Conversely, let $a \in \ell_\infty^\alpha$. Then $\sum_k |a_k x_k| < \infty$, since $x = e \in \ell_\infty$,

$$\sum_k |a_k x_k| = \sum_k |a_k| < \infty,$$

this implies that $a \in \ell_1$. Hence, $\ell_\infty^\alpha \subseteq \ell_1$.

Therefore $\ell_\infty^\alpha = \ell_1$. similarly, we can prove $c_0^\alpha = c^\alpha = \ell_1$.

- (ii) Let $a \in \ell_\infty$ and $x \in \ell_1$. Then $\sum_k |x_k| < \infty$. Now, we have

$$\sum_k |a_k x_k| \leq \|a\|_\infty \sum_k |x_k| < \infty,$$

and hence $a \in \ell_1^\alpha$. Therefore $\ell_\infty \subseteq \ell_1^\alpha$.

Conversely, suppose that $a \in \ell_1^\alpha$. Therefore

$$\sum_k |a_k x_k| < \infty \text{ for all } x \in \ell_1.$$

Let $a \notin \ell_\infty$. Then there exists a strictly increasing sequence (n_i) with $|a_{n_i}| > i^3$. If we define $x = (x_n)$ by

$$x_n = \begin{cases} i^{-2}, & n = n_i, \quad i = 1, 2, \dots \\ 0, & n \neq n_i, \end{cases}$$

then $x \in \ell_1$ but $\sum_k |a_k x_k| = \infty$. Thus $a \notin \ell_1^\alpha$ and so $\ell_1^\alpha \subseteq \ell_\infty$.

Hence $\ell_1^\alpha = \ell_\infty$ and this completes the proof.

Theorem 2.5 ([2, Theorem 2.5.4]) *Let $1 < p, q < \infty$, with $p^{-1} + q^{-1} = 1$. Then $\ell_p^\alpha = \ell_q$.*

Proof Suppose that $a \in \ell_p^\alpha$ and $x \in \ell_q$. Then $\sum_k |a_k x_k| < \infty$ for all $x \in \ell_p$. Also let $a \notin \ell_q$. Then we can find a sequence $n_1 < n_2 < \dots$ such that

$$|a_{n_k}|^q > k^2, \quad k = 1, 2, \dots$$

Take

$$x_n = \begin{cases} 1/a_{n_k}, & n = n_k, \quad k = 1, 2, \dots \\ 0, & n \neq n_k. \end{cases}$$

Then $x \in \ell_p$ but

$$\sum_k |a_k x_k| = \sum_k |a_{n_k}| |a_{n_k}|^{-1} = 1 + 1 + \dots = \infty.$$

Hence contradiction. Therefore $a \in \ell_q$ and $\ell_p^\alpha \subseteq \ell_q$.

Conversely, suppose that $x \in \ell_p$ and $a \in \ell_q$. Then by Hölder's inequality

$$\sum_k |a_k x_k| \leq \left(\sum_k |a_k|^q \right)^{1/q} \left(\sum_k |x_k|^p \right)^{1/p} = \|a\|_q \|x\|_p < \infty,$$

and so $a \in \ell_p^\alpha$. Therefore $\ell_q \subseteq \ell_p^\alpha$.

Hence $\ell_p^\alpha = \ell_q$ and the proof is complete.

Theorem 2.6 ([2, Theorem 2.5.9]) $cs^\alpha = bv^\alpha = bv_0^\alpha = bs^\alpha = \ell_1$.

Proof We prove the case $cs^\alpha = \ell_1$, and the rest can be obtained similarly.

Let $x \in cs$ and $a \in \ell_1$. Then

$$\sum_k |a_k x_k| \leq \|x\|_{bs} \sum_k |a_k| < \infty,$$

where

$$\|x\|_{bs} = \sup_n \left| \sum_{k=1}^n x_k \right|.$$

Therefore $a \in cs^\alpha$, and we get $\ell_1 \subseteq cs^\alpha$.

Conversely, let $a \in cs^\alpha \setminus \ell_1$. Then to every positive integer i we can find an odd n_i with $n_i < n_{i+1}$ and

$$\sum_{n_i+1}^{n_{i+1}} |a_k| > 2^i, \quad i = 1, 2, \dots$$

Define

$$x_k = \begin{cases} (-1)^k 2^{-i/2}, & n_i < k \leq n_{i+1}, i \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Then $x = (x_k) \in cs$ but $\sum_k |a_k x_k| = \infty$. This contradicts that $a \in cs^\alpha$, and so $cs^\alpha \subseteq \ell_1$.

Hence $cs^\alpha = \ell_1$ and we finish the proof.

Remark 2.3 Note that $\omega^\alpha = \phi$ and $\phi^\alpha = \omega$.

Theorem 2.7 ([2, Theorem 2.5.6]) *We have*

- (i) $\ell_\infty^\beta = c_0^\beta = c^\beta = \ell_1$,
- (ii) $\ell_1^\beta = \ell_\infty$.

Proof (i) We prove the case $c_0^\beta = \ell_1$, and the other statements can be proved similarly.

Let $a \in c_0^\beta$. Then $\sum a_k x_k$ converges for every $x \in c_0$. To show $a \in \ell_1$, suppose that $a \notin \ell_1$. Then we can find a sequence $n_1 < n_2 < \dots$ such that

$$|a_{n_k}| > k, \quad k = 1, 2, \dots$$

Put

$$x_k = \begin{cases} 1/a_{n_k}, & k = n_k, \quad k = 1, 2, \dots \\ 0, & k \neq n_k. \end{cases}$$

Then $x \in c_0$ but

$$\sum_k a_k x_k = \sum_k a_{n_k} \frac{1}{a_{n_k}} = 1 + 1 + \dots$$

diverges to ∞ .

This contradiction shows that we must have $a \in \ell_1$, and so $c_0^\beta \subseteq \ell_1$.
Conversely, suppose that $x \in c_0$ and $a \in \ell_1$. Then

$$\left| \sum_k a_k x_k \right| \leq \|x\|_\infty \sum_k |a_k| < \infty.$$

This implies that $a \in c_0^\beta$, and so $\ell_1 \subseteq c_0^\beta$.

Hence $c_0^\beta = \ell_1$.

(ii) Let $a \in \ell_1^\beta$. Then $\sum_k a_k x_k$ converges for every $x \in \ell_1$. We have to show that $a \in \ell_\infty$. For, we may define

$$f_n(x) = \sum_{k=1}^n a_k x_k$$

which for each n is clearly a bounded linear functional on ℓ_1 . By hypothesis there exists $\lim_n f_n(x) = \sum_k a_k x_k = f(x)$, say, for every $x \in \ell_1$. Banach–Steinhaus theorem yields

$$|f(x)| \leq M \|x\|_1 = M \sum_k |x_k|.$$

Now put

$$x_k = \begin{cases} \operatorname{sgn} a_n, & k = n \\ 0, & k \neq n. \end{cases}$$

Then $\|x\| = 1$ and $f(x) = |a_n| \leq M$, $n = 1, 2, \dots$. Therefore $a \in \ell_\infty$, and $\ell_1^\beta \subseteq \ell_\infty$.

Conversely, let $a \in \ell_\infty$ and $x \in \ell_1$. Now

$$\left| \sum_k a_k x_k \right| \leq \|a\|_\infty \|x\|_1 < \infty.$$

Therefore $a \in \ell_1^\beta$ and so $\ell_\infty \subseteq \ell_1^\beta$.

Hence $\ell_1^\beta = \ell_\infty$. This completes the proof.

Theorem 2.8 ([2, Theorem 2.5.6 (a)]) *Let $1 < p, q < \infty$, with $p^{-1} + q^{-1} = 1$. Then $\ell_p^\beta = \ell_q$.*

Proof It is similar to that of Theorem 2.5.

Theorem 2.9 ([3, Theorem 7.3.5]) (i) $cs^\beta = bv$, (ii) $bv^\beta = cs$, (iii) $bv_0^\beta = bs$, (iv) $bs^\beta = bv_0$.

Proof We prove here (i), the other statements can be proved similarly.

Let $a = (a_k) \in cs^\beta$ and $z = (z_k) \in c_0$. Then the sequence $x = (x_k)$ defined by $x_k = z_k - z_{k-1}$, $k \geq 1$, where $z_0 = 0$, belongs to cs . Therefore, $\sum_k a_k x_k$ converges, but

$$\sum_{k=1}^n (z_k - z_{k-1})a_k = \sum_{k=1}^{n-1} z_k(a_k - a_{k-1}) + z_n a_n$$

and $a \in cs^\beta \subset \ell_1^\beta = \ell_\infty$ (since $cs \supset \ell_1$) imply that

$$\sum_{k=1}^{\infty} (z_k - z_{k-1})a_k = \sum_{k=1}^{\infty} z_k(a_k - a_{k+1}).$$

Hence $(a_k - a_{k+1})_k \in c_0^\beta = c_0^\alpha = \ell_1$, i.e., $a \in bv$. Therefore $cs^\beta \subseteq bv$.

Conversely, let $a \in bv$. Then $(a_k - a_{k+1})_k \in \ell_1$. Further, if $x \in cs$, the sequence (ω_n) , $\omega_n = \sum_{k=1}^n x_k$, $n \geq 1$, is an element of c . As $c^\alpha = \ell_1$, the series $\sum_{k=1}^{\infty} \omega_k(a_k - a_{k+1})$ is absolutely convergent. Also, we have

$$\left| \sum_{k=m}^n (\omega_k - \omega_{k-1})a_k \right| \leq \left| \sum_{k=m}^{n-1} \omega_k(a_k - a_{k+1}) \right| + |\omega_n a_n - \omega_{m-1} a_m|.$$

As $(\omega_k) \in c$ and $(a_k) \in bv \subset c$, the right-hand side of the above inequality converges to zero as $m, n \rightarrow \infty$. Hence the series

$$\sum_k (\omega_k - \omega_{k-1})a_k \text{ or } \sum_k a_k x_k$$

converges and so $bv \subseteq cs^\beta$. Thus $cs^\beta = bv$.

The proof is complete.

Theorem 2.10 ([2, Theorem 2.5.7]) *We have*

- (i) $\ell_\infty^\gamma = c_0^\gamma = c^\gamma = \ell_1$,
- (ii) $\ell_1^\gamma = \ell_\infty$,
- (iii) $\ell_p^\gamma = \ell_q$.

These statements can be proved on the same lines as α -, β -duals.

Theorem 2.11 ([2, Theorem 2.5.12]) (i) $cs^\gamma = bv$, (ii) $bv^\gamma = bs$, (iii) $bv_0^\gamma = bs$, (iv) $bs^\gamma = bv$.

Proof We prove (i) only, the other parts can be proved along similar lines.

By Theorem 2.9, we have $bv \subset cs^\beta$ and since $cs^\beta \subset cs^\gamma$, so $bv \subset cs^\gamma$. We need to show that $cs^\gamma \subset bv$. Let $a \in cs^\gamma$ and $z \in c_0$. Then for the sequence $(\omega_n) \in cs$ defined by $\omega_n = z_n - z_{n-1}$, $n \geq 1$, $z_0 = 0$, we can find a constant $K > 0$ such that

$$\left| \sum_{k=1}^n a_k \omega_k \right| \leq K \text{ for all } n \geq 1.$$

Since $(z_n) \in c_0$ and $(a_n) \in cs^\gamma \subset \ell_\infty$, there exists a constant M such that $|a_n z_n| \leq M$ for all $n \geq 1$. Now

$$\left| \sum_{k=1}^n z_k (a_k - a_{k+1}) \right| \leq \left| \sum_{k=1}^{n+1} (z_k - z_{k-1}) a_k \right| + |z_{n+1} a_{n+1}| \leq K + M.$$

Hence $(a_k - a_{k+1}) \in c_0^\gamma = \ell_1$ (by Theorem 2.10), i.e., $(a_k) \in bv$. Therefore $cs^\gamma = bv$.

Remark 2.4 (i) $\phi \subset X^\alpha \subset X^\beta \subset X^\gamma$, for any $X \in \omega$.

(ii) If for any two set $X, Y \in \omega$

$$X \subset Y, \text{ then } Y^\dagger \subset X^\dagger,$$

where $\dagger = \alpha, \beta$ or γ .

(iii) $X^{\dagger\dagger\dagger} = X^\dagger$.

Definition 2.4 Let X be a sequence space. Then, X is called

(a) *Normal* or *solid* if $y \in X$ whenever $|y_k| \leq |x_k|$, $k \geq 1$, for some $x \in X$. Note that X is normal if and only if $\ell_\infty X \subset X$.

(b) *Monotone* if and only if $M_0 X \subset X$, where $M_0 = sp\{A\}$, A is the set of all sequences of zeros and ones.

(c) \dagger -*Perfect sequence space* if and only if $X^{\dagger\dagger} = X$, where $\dagger = \alpha, \beta, \gamma$. α -perfect sequence space is known as a *Köthe space* or simply a *perfect space*.

Example 2.1 (i) c is not monotone and hence not normal, since $(1, 0, 1, 0, \dots) \notin c$.

(ii) c_0 is normal but not perfect, since $c_0^{\alpha\alpha} = \ell_1^\alpha = \ell_\infty$.

(iii) M_0 is monotone but not normal, since we can find $\{1/n\} \notin M_0$ although $1/n \leq 1$ for $n \geq 1$, and $(1, 1, 1, \dots) \in M_0$.

(iv) The spaces ϕ, ω, ℓ_p ($1 \leq p < \infty$) and ℓ_∞ all are perfect.

Theorem 2.12 ([1, 4]) *Let X be a sequence space. Then*

(i) $X^\alpha = X^\beta$, if X is monotone,

(ii) $X^\alpha = X^\gamma$, if X is normal.

Proof (i) Let $a \in X^\beta$ and so $a \in (M_0 X)^\beta$, since X is monotone. Thus $\sum_k a_k x_k y_k$ converges for each $y \in M_0$ and $x \in X$. In particular $\sum_k a_k x_k$ is a convergent subseries and thus $\sum_k |a_k x_k| < \infty$ for each $x \in X$. Hence $a \in X^\alpha$, i.e., $X^\beta \subset X^\alpha$. Finally we have $X^\alpha = X^\beta$, since $X^\alpha \subset X^\beta$.

(ii) Let $a \in X^\gamma$. Then

$$\sup_n \left| \sum_{k=1}^n a_k x_k \right| < \infty \quad \text{for all } x \in X.$$

Observe that

$$\left(\frac{\bar{a}_k \bar{x}_k}{a_k x_k} x_k \right) \in X \quad \text{for any } x \in X.$$

Hence $\sum_k |x_k y_k| < \infty$ for each $x \in X$, so that $a \in X^\alpha$, i.e., $X^\gamma \subset X^\alpha$. But $X^\alpha \subset X^\gamma$, so that $X^\alpha = X^\gamma$, where \bar{x}_k denotes the canonical preimage of x_k .

This completes the proof.

Theorem 2.13 ([3, Theorem 7.2.7]) *Let $X \supset \phi$ be an FK space. Then $X^\beta = X^\gamma$ if X has AD.*

Proof Let $u \in X^\gamma$ and define $f_n(x) = \sum_{k=1}^n u_k x_k$ for $x \in X$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous. Since $\lim_n f_n(x)$ exists for all $x \in \phi$, it must exist for all $x \in X$, i.e., $u \in X^\beta$. Hence $X^\gamma \subset X^\beta$. The reverse inclusion $X^\beta \subset X^\gamma$ is trivial, so finally $X^\beta = X^\gamma$. The proof is complete.

Theorem 2.14 ([5, Theorem 1.34]) *Let $X \supset \phi$ be an FK space. Then there is a linear one-to-one map $T: X^\beta \rightarrow X'$, and we denote this by $X^\beta \subset X'$. If X has AK, then T is onto.*

Proof We define the map T on X^β as follows. For every $a \in X^\beta$, let $T_a: X \rightarrow X'$ be defined by $(T_a)(x) = \sum_{k=1}^\infty a_k x_k$ for all $x \in X$. Since $a \in X^\beta$, the series $\sum_k a_k x_k$ converges for all $x \in X$, and obviously, T_a is linear. Further, since X is an FK space, $T_a \in X'$ for each $a \in X^\beta$. Therefore $T: X^\beta \rightarrow X'$. Further it is easy to see that T is linear.

To show that T is one-to-one, we assume $a, b \in X^\beta$ with $T_a = T_b$. This means $(T_a)(x) = (T_b)(x)$ for all $x \in X$. Since $\phi \subset X$, we may choose $x = e^{(k)}$ for each k and obtain $(T_a)(e^{(k)}) = a_k = b_k = (T_b)(e^{(k)})$ for $k = 1, 2, \dots$, and so $a = b$.

Now we assume that X has AK and $f \in X'$. We put $a_k = f(e^{(k)})$ for $k = 1, 2, \dots$. Let $x \in X$ be given. Then $x = \sum_{k=1}^\infty x_k e^{(k)}$, since X has AK and $f \in X'$ implies

$$f(x) = \sum_{k=1}^\infty x_k f(e^{(k)}) = \sum_{k=1}^\infty a_k x_k = (T_a)(x).$$

As $x \in X$ was arbitrary and the series converges, $a \in X^\beta$ and $f = T_a$. This shows that T is onto X' and completes the proof.

2.3 Other Duals

For the sake of completeness, we also include here some other type of duals. But our emphasis is given on continuous duals and β -duals because of their applications toward matrix transformations, in particular β -duals are very helpful to study the convergence of the series $\sum_k a_{nk}x_k$, that is, for the existence of the A -transformed sequence.

Definition 2.5 Let X be a sequence space. Then X is called *symmetric* if $x_\sigma = (x_{\sigma_k}) \in X$ whenever $x \in X$ and $\sigma \in \pi$, where π is the set of all permutations of \mathbb{N} , i.e., one-to-one and onto maps of \mathbb{N} .

Definition 2.6 Let X be a sequence space. Then

$$X^\delta := \left\{ a \in \omega : \sum_k |a_k x_{\sigma_k}| < \infty \text{ for all } x \in X \text{ and } \sigma \in \pi \right\}$$

$$:= \bigcap_{\substack{x \in X \\ \sigma \in \pi}} (x_\sigma * \ell_1)$$

is called the *symmetric dual* (or δ -dual) of X .

The *functional dual* (or f -dual) of X is defined as

$$X^f := \{(g(e^{(k)})) : g \in X'\}.$$

If X is a K space then $\phi \subset X^f$.

Note that $\phi \subset X^\delta \subset X^\alpha$.

Theorem 2.15 ([1, Proposition 2.7]) *For a sequence space X , $X^\alpha = X^\delta$ if X is symmetric.*

Proof Let X be symmetric and $a \in X^\alpha$. Then for each $x \in X$ and $\sigma \in \pi$, $x_{\sigma^{-1}} \in X$, so that

$$\sum_k |a_k x_{\sigma^{-1}(k)}| = \sum_j |a_{\sigma(j)} x_j| < \infty$$

and thus $x \in X^\delta$, i.e., $X^\alpha \subset X^\delta$. Hence $X^\alpha = X^\delta$.

Remark 2.5 It might be expected from $X \subset X^{\dagger\dagger}$ that X is contained in X^{ff} , but this is not the case in general (see following example). However, $X \subset X^{ff}$ if X is a BK space with AD .

Example 2.2 Let $X = c_0 \oplus z$ with $z \in \ell_\infty$. Then X is a BK space, $X^f = \ell_1$ and $X^{ff} = \ell_\infty$, so $X \not\subset X^{ff}$.

Now, the question arises whether $f \rightarrow (f(e^{(k)}))$, $f \in X'$ gives an isomorphism from X' to X^f so that we can identify X' and X^f . In general, it does not work (see example below), however, we have the following result about X^f .

Theorem 2.16 ([3, Theorem 7.2.10 and 7.2.12]) *If $X \supset \phi$ is an FK space, then*

- (i) *the map $q: X' \rightarrow X^f$ given by $q(f) = (f(e^{(k)}))_{k=0}^\infty$ is onto. Moreover, if $T: X^\beta \rightarrow X'$ denotes the map of Theorem 2.14, then $q(Ta) = a$ for all $a \in X^\beta$.*
- (ii) *$X' \cong X^f$, that is, the map q of Part (i) is one-to-one if and only if X has AD.*

Proof (i) Let $a \in X^f$ be given. Then there is $f \in X'$ such that $a_k = (f(e^{(k)}))$ for all k , and so $q(a) = (f(e^{(k)}))_{k=0}^\infty = a$, which shows that q is onto.

Now let $a \in X^\beta$ be given. We put $f = Ta \in X'$ and obtain $q(Ta) = q(f) = (f(e^{(k)}))_{k=0}^\infty = ((Ta)(e^{(k)}))_{k=0}^\infty = (a_k)_{k=0}^\infty = a$.

(ii) First we assume that X has AD. Then $q(f) = 0$ implies $f = 0$ on ϕ , hence $f = 0$, since X has AD. This shows that q is one-to-one.

Conversely we assume that X does not have AD. By the Hahn–Banach theorem, there exists an $f \in X'$ with $f \neq 0$ and $f = 0$ on ϕ . Then we have $q(f) = 0$, and q is not one-to-one.

This completes the proof.

Example 2.3 We have $c^\beta = c^f = \ell_1$. The map T of Theorem 2.14 is not onto. We consider $\lim \in X'$. If there were $a \in X^f$ with $\lim a = \sum_{k=0}^\infty a_k x_k$, then it would follow that $a_k = \lim e^{(k)}$, hence $\lim x = 0$ for all $x \in c$, contradicting $\lim e = 1$. Also the map $q: X' \rightarrow X^f$ of Theorem 2.16 is not onto, since $q(\lim) = 0$.

Theorem 2.17 ([3, Theorem 7.2.4 & 7.2.6]) (a) *Let $X \supset \phi$ be an FK space. Then we have $X^f = (cl_X(\phi))^f$.*

(b) *Let $X, Y \supset \phi$ be FK spaces. If $X \subset Y$ then $X^f \supset Y^f$. If X is closed in Y then $X^f = Y^f$.*

Proof (a) We write $Z = cl_X(\phi)$. First, we assume that $a \in X^f$, that is, $a_n = f(e^{(n)})$ ($n = 0, 1, \dots$) for some $f \in X'$. We write $g = f|_Z$ for the restriction of f to Z . Then $a_n = g(e^{(n)})$ for all $n = 0, 1, \dots$, $g \in Z'$ and so $a \in Z^f$.

Conversely, let $a \in Z$, then $a_n = g(e^{(n)})$ ($n = 0, 1, \dots$) for some $g \in Z'$. By the Hahn–Banach theorem, g can be extended to $f \in X'$, and we have $a_n = f(e^{(n)})$ for $n = 0, 1, \dots$, hence $a \in X^f$.

(b) We assume that $a \in Y^f$. Then $a_n = f(e^{(n)})$ ($n = 0, 1, \dots$) for some $f \in Y'$. Since $X \subset Y$, we have $g = f|_X \in X'$ by Theorem 1.7. If X is closed in Y , then the FK topologies are the same by Theorem 1.7, and we obtain $X^f = (cl_X(\phi))^f = (cl_Y(\phi))^f = Y^f$ from Part (a).

This completes the proof.

2.4 Multiplier Spaces

Definition 2.7 Let X and Y be subsets of ω . The set

$$Z = M(X, Y) = \bigcap_{x \in X} x^{-1} * Y := \{a \in \omega : ax \in Y \text{ for all } x \in X\}$$

is called the *multiplier space* of X and Y .

In the special cases, $M(X, \ell_1) = X^\alpha$, $M(X, cs) = X^\beta$ and $M(X, bs) = X^\gamma$.

Theorem 2.18 ([5, Lemma 1.25]) *Let $X, Y, Z \subset \omega$ and $\{X_\eta: \eta \in A\}$ be any collection of subsets of ω , where A is an arbitrary index set. Then*

- (i) $X \subset M(M(X, Y), Y)$
- (ii) $X \subset Z$ implies $M(Z, Y) \subset M(X, Y)$,
- (iii) $M(X, Y) = M(M(M(X, Y), Y), Y)$,
- (iv) $M(\cup_{\eta \in A} X_\eta, Y) = \cap_{\eta \in A} M(X_\eta, Y)$,
- (v) $Y \subset Z$ implies $M(X, Y) \subset M(X, Z)$

Proof (i) If $x \in X$, then $ax \in Y$ for all $a \in M(X, Y)$, and consequently $x \in M((M(X, Y), Y))$.

(ii) Let $X \subset Z$. If $a \in M(Z, Y)$, then $ax \in Y$ for all $x \in Z$, hence $ax \in Y$ for all $x \in X$, since $X \subset Z$. Thus $a \in M(X, Y)$.

(iii) We apply (i) with X replaced by $M(X, Y)$ to obtain

$$M(X, Y) \subset M(M(M(X, Y), Y), Y).$$

Conversely, by (i), $X \subset M(M((X, Y), Y))$, and so (ii) with $Z = M(M((X, Y), Y))$ yields $M(M(M(X, Y), Y), Y) \subset M(X, Y)$.

(iv) First $X_\eta \subset \cup_{\eta \in A} X_\eta$ for all $\eta \in A$ implies

$$M\left(\bigcup_{\eta \in A} X_\eta, Y\right) \subset \bigcap_{\eta \in A} M(X_\eta, Y) \text{ by part (ii).}$$

Conversely, if $a \in \cap_{\eta \in A} M(X_\eta, Y)$, then $a \in M(X_\eta, Y)$ for all $\eta \in A$, and so we have $ax \in Y$ for all $\eta \in A$ and for all $x \in X_\eta$. This implies $ax \in Y$ for all $x \in \cup_{\eta \in A} X_\eta$, hence $a \in M(\cup_{\eta \in A} X_\eta, Y)$. Thus $\cap_{\eta \in A} M(X_\eta, Y) \subset M(\cup_{\eta \in A} X_\eta, Y)$.

(v) It is trivial.

As an immediate consequence of the above theorem we have

Corollary 2.19 *If \dagger denotes α, β or γ , then*

- (i) $X \subset X^{\dagger\dagger}$, (ii) $X \subset Y$ implies $Y^\dagger \subset X^\dagger$, (iii) $X^\dagger = X^{\dagger\dagger\dagger}$, (iv) $(\cup_{\eta \in A} X_\eta)^\dagger = \cap_{\eta \in A} X_\eta^\dagger$.

Example 2.4 We have

- (i) $M(c_0, c) = \ell_\infty$, (ii) $M(c, c) = c$, (iii) $M(\ell_\infty, c) = c_0$.

Proof (i) If $a \in \ell_\infty$, then $ax \in c$ for all $x \in c_0$, and so $\ell_\infty \subset M(c_0, c)$. Conversely, let $a \notin \ell_\infty$. Then there is a subsequence $(a_{k_j})_{j=0}^\infty$ of the sequence a such that $|a_{k_j}| > j + 1$ for all $j = 0, 1, \dots$. We define the sequence x by

$$x_k = \begin{cases} (-1)^j / a_{k_j} & \text{for } k = k_j \\ 0 & \text{for } k \neq k_j \end{cases} \quad (2.13)$$

for $j = 0, 1, \dots$. Then $x \in c_0$ and $a_{k_j}x_{k_j} = (-1)^j$ for all $j = 0, 1, \dots$, hence $ax \notin c$. This gives $M(c_0, c) \subset \ell_\infty$.

(ii) If $a \in c$, then $ax \in c$ for all $x \in c$, and so $c \subset M(c, c)$. Conversely, let $x \notin c$. Then we have $a \notin M(c, c)$, since $e \in c$ and $ae = a \notin c$. Hence $M(c, c) \subset c$.

(iii) If $a \in c_0$, then $ax \in c$ for all $x \in \ell_\infty$, and so $c_0 \subset M(\ell_\infty, c)$. Conversely, let $a \notin c_0$. Then there is a real number $b > 0$ and a subsequence $(a_{k_j})_{j=0}^\infty$ of the sequence a such that $|a_{k_j}| > b$ for all $j = 0, 1, \dots$. We define the sequence x as in (2.13). Then $x \in \ell_\infty$ and $a_{k_j}x_{k_j} = (-1)^j$ for $j = 0, 1, \dots$, hence $a \notin M(\ell_\infty, c)$. This shows $M(\ell_\infty, c) \subset c_0$ and completes the proof.

Example 2.5 Let \dagger denote any of the symbols α, β or γ . Then $w^\dagger = \phi$, $\phi^\dagger = w$, $c_0^\dagger = c^\dagger = \ell_\infty^\dagger = \ell_1$, $\ell_1^\dagger = \ell_\infty$, and $\ell_p^\dagger = \ell_q$ ($1 < p < \infty$; $q = p/(p-1)$).

Theorem 2.20 ([3, Theorem 7.2.7, p. 106]) *Let $X \supset \phi$ be an FK space.*

- (a) *We have $X^\gamma \subset X^f$*
- (b) *If X has AK then $X^\beta = X^f$.*

Proof Let $a \in X^\beta$. We define the linear functional f by $f(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$. Then $f \in X'$, and we have $f(e^{(n)}) = a_n$ for all n , hence $a \in X^f$. Thus

$$X^\beta \subset X^f. \quad (2.14)$$

(b) Now suppose that X has AK , and $a \in X^f$. Let $x \in X$ be given. Then $x = \sum_{k=0}^\infty x_k e^{(k)}$, since X has AK , and since $f \in X'$, we have $f(x) = \sum_{k=0}^\infty x_k f(e^{(k)}) = \sum_{k=0}^\infty x_k a_k$, hence $a \in X^\beta$. Thus $X^f \subset X^\beta$, which together with (2.14) gives $X^\beta = X^f$.

(a) First we observe that $\bar{\phi} \subset X$ implies $X^\gamma \subset (\bar{\phi})^\gamma$ by Theorem 2.18 (ii). Furthermore, we have $(\bar{\phi})^\gamma = (\bar{\phi})^\beta \subset (\bar{\phi})^f = X^f$ by Theorem 2.13, (2.14) and Theorem 2.17 (a). Thus we have shown $X^\gamma \subset X^f$.

This completes the proof.

It turns out that the multiplier spaces and the functional duals of BK spaces are again BK spaces. These results do not extend to FK spaces, in general.

Theorem 2.21 ([5, Theorem 1.30]) *Let $X \supset \phi$ and Y be BK spaces. Then $Z = M(X, Y)$ is a BK space with $\|z\| = \sup_{x \in S_X} \|xz\|$ for $z \in Z$.*

Proof Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the BK norms of X and Y . Every $z \in Z$ defines a diagonal matrix map $\hat{z}: X \rightarrow Y$ where $\hat{z}(x) = xz = (x_k z_k)_{k=0}^\infty$ for all $x \in X$, and then $\hat{z} \in \mathcal{B}(X; Y)$ (since $(X; Y) \subset \mathcal{B}(X; Y)$ which is proved in the next chapter). This embeds Z in $\mathcal{B}(X, Y)$, for if $\hat{z} = 0$ then $(\hat{z}(e^{(n)}))_n = z_n = 0$ for all n , hence $z = 0$.

To see that the coordinates are continuous, we fix n and put $u = 1/\|e^{(n)}\|_X$ and $v = \|e^{(n)}\|_Y$. Then we have $\|ue^{(n)}\|_X = 1$ and $uv \mid z_n \mid = u \|z_n e^{(n)}\|_Y = u \|e^{(n)}z\|_Y = \|ue^{(n)}z\|_Y \leq \|z\|$.

Now we have to show that Z is a closed subspace of the Banach space $\mathcal{B}(X, Y)$. Let $(\hat{z}^{(m)})_{m=0}^\infty$ be a sequence in $\mathcal{B}(X, Y)$ with $\hat{z}^{(m)} \rightarrow T \in \mathcal{B}(X, Y)$ ($m \rightarrow \infty$). For

every fixed $x \in X$, we obtain $\hat{z}^{(m)}(x) \rightarrow T(x) \in Y$ ($m \rightarrow \infty$), and since Y is a BK space, this implies $x_k z_k^{(m)} = (\hat{z}^{(m)}(x))_k \rightarrow (T(x))_k$ ($m \rightarrow \infty$) for every fixed k . If we choose $x = e^{(k)}$ then we obtain $z_k^{(m)} \rightarrow t_k = (T(e^{(k)}))_k$. Thus we have $x_k z_k^{(m)} \rightarrow t_k$ and $x_k z_k^{(m)} \rightarrow (T(x))_k$ ($m \rightarrow \infty$), hence $T(x) = xt$, and so $T = t$. This shows that Z is closed.

This completes the proof.

We obtain as an immediate consequence of Theorem 2.21 the following corollary.

Corollary 2.22 *The α -, β -, and γ -duals of a BK space X are BK spaces with $\|a\|_\alpha = \|a\|_{X,\alpha} = \sup_{x \in S_X} \|ax\|_1 = \sup_{x \in S_X} (\sum_{k=0}^{\infty} |a_k x_k|)$ for all $a \in X^\alpha$, and $\|a\|_\beta = \|a\|_{X,\beta} = \sup_{x \in S_X} \|a\|_{bs} = \sup_{x \in S_X} (\sup_n |\sum_{k=0}^n a_k x_k|)$ for all $a \in X^\beta$, X^γ . Furthermore, X^β is a closed subspace of X^γ .*

Proof The first part is an immediate consequence of Theorem 2.21. Since the BK norms on X^β and X^γ are the same and $X^\beta \subset X^\gamma$ by Remark 2.4 (i), the second part follows from Theorem 1.7.

Remark 2.6 Let X be any of the spaces ℓ_∞ , c , c_0 , and ℓ_p ($1 \leq p < \infty$). Then, the norms $\|\cdot\|_{X^\beta}$, $\|\cdot\|_{X^*}$, $\|\cdot\|_{X,\alpha}$ and $\|\cdot\|_{X,\beta}$ are equivalent on X^β , where $\|a\|_{X^*} = \sup_{x \in S_X} |\sum_{k=0}^{\infty} a_k x_k|$.

Remark 2.7 Theorem 2.21 fails to hold for FK spaces, in general.

Example 2.6 The space w is an FK space, and $w^\alpha = w^\beta = w^\gamma = \phi$, but ϕ has no Fréchet metric.

We give the following result without proof.

Theorem 2.23 ([3], Theorem 7.2.14, p. 108) *Let $X \supset \phi$ be BK space. Then X^f is a BK space.*

Theorem 2.24 ([3], Theorem 7.2.15, p. 108) *Let $X \supset \phi$ be BK space. Then $X^{ff} \supset cl_X(\phi)$. Hence, if X has AD , then $X \subset X^{ff}$.*

Proof First we have to show $\phi \subset X^f$ in order for X^{ff} to be meaningful. This is true because $P_k \in X'$ for all k where $P_k(x) = x_k$ ($x \in X$) since X is a BK space, and $q(P_k) = e^{(k)}$ (Theorem 2.16 (i)). Since the second part is equivalent to the first part by Theorem 2.17 (b), we assume that X has AD and have to show $X \subset X^{ff}$.

Let $x \in X$ be given. We define the functional $f: X' \rightarrow \mathbb{C}$ by $f(\psi) = \psi(x)$ for all $\psi \in X'$. Then we have $|f(\psi)| = |\psi(x)| \leq \|\psi\| \|x\|$, and consequently $f \in X''$. Let $g: X' \rightarrow X^f$ be the map of Theorem 2.16 (i) which is an isomorphism by Theorem 2.16 (ii), since X has AK . Thus the inverse map $q^{-1}: X^f \rightarrow X'$ exists. We define the map $g: X^f \rightarrow \mathbb{C}$ by $g(b) = \psi(x)$ ($b \in X^f$) where $x = q^{-1}(b)$. It follows that

$$|g(b)| = |\psi(x)| = |f(\psi)| \leq \|f\| \|\psi\| = \|f\| \|q^{-1}(b)\| \leq \|f\| \|q^{-1}\| \|b\|,$$

and the Open Mapping Theorem yields $\|q^{-1}\| < 1$. Thus we have $g \in (X^f)'$. Finally it follows that $x_k = P_k(x) = g(q(P_k)) = g(e^{(k)})$ for all k , hence $x \in X^{ff}$. Thus we have $X \subset X^{ff}$.

This completes the proof.

Remark 2.8 The condition that X has AD is not necessary for $X \subset X^{ff}$, in general. For example, in Example 2.2, X does not have AD .

2.5 Matrix Classes of Some FK and BK Spaces

In this section, we apply the results of the theory of FK and BK spaces to characterize the matrix classes.

Let (X, d) be a metric space, $\delta > 0$, and $x_0 \in X$. Then, we write $B_\delta[x_0] = \{x \in X: d(x, x_0) \leq \delta\}$ for the closed ball of radius δ with its center in x_0 . If $X \subset w$ is a linear metric space and $a \in w$, then we write

$$\|a\|_\delta^* = \|a\|_{\delta, X}^* = \sup_{x \in B_\delta[x_0]} \left| \sum_{k=0}^{\infty} a_k x_k \right| \quad (2.15)$$

provided the expression on the right-hand side exists and is finite which is the case whenever $a \in X^\beta$; if X is a normed space, we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right| \quad (2.16)$$

where S_X is the unit sphere in X .

Let A be an infinite matrix, D a positive real, and X an FK space. Then, we put

$$M_{A,D}^*(X, \ell_\infty) = \sup_n \|A_n\|_D^*$$

and, if X is a BK space, then we write

$$M_A^*(X, \ell_\infty) = \sup_n \|A_n\|^*.$$

Remark 2.9 Let $1 < p < \infty$ and $q = p/(p-1)$. Then, we have $\ell_\infty^\beta = c^\beta = c_0^\beta = \ell_1$ and $\ell_p^\beta = \ell_q$. Furthermore, let X denote any of the spaces $\ell_\infty, c, c_0, \ell_1$, or ℓ_p . Then, we have $\|a\|_X^* = \|a\|_{X^\beta}$ for all $a \in X^\beta$, where $\|\cdot\|_{X^\beta}$ is the natural norm on the dual space X^β .

Now we give here results on matrix transformations using the theory of FK and BK spaces.

Theorem 2.25 ([6, Theorem 5]) *An FK space X contains ℓ_1 if and only if*

$$\{e^{(k)}: k = 0, 1, 2, \dots\} \quad (2.17)$$

is a bounded subset of X .

Proof Let X contain ℓ_1 . Then the inclusion map $\iota: \ell_1 \rightarrow X$ is continuous. Since $\{e^{(k)}: k = 0, 1, 2, \dots\}$ is bounded in ℓ_1 , it is bounded in X .

Conversely, suppose that (2.17) holds. Let $x = (x_k) \in \ell_1$ and let q be a continuous seminorm on X . Then

$$q\left(\sum_{k=m}^n x_k e^{(k)}\right) \leq \sum_{k=m}^n |x_k| q(e^{(k)}).$$

Since $x = (x_k) \in \ell_1$, i.e., $\sum_{k=0}^{\infty} |x_k|$ is convergent, we have $\sum_{k=m}^n |x_k| q(e^{(k)}) \rightarrow 0$ ($m, n \rightarrow \infty$). Thus $\left(\sum_{k=0}^{n-1} x_k e^{(k)}\right)_n$ is a Cauchy sequence convergent to x in X . Moreover, $x \in X$ since X is complete. Hence $\ell_1 \subset X$.

This completes the proof.

We derive the following corollary:

Corollary 2.26 *Let $A = (a_{nk})$ be an infinite matrix and X an FK space. Then, $A \in (\ell_1, X)$ if and only if the columns of A belong to X and form a bounded subset of X .*

Now from this corollary, we easily deduce the following classical results of summability theory.

If we put $X = c$, then we get:

Corollary 2.27 *$A \in (\ell_1, c)$ if and only if (i) $\sup_{n,k} |a_{nk}| < \infty$, and (ii) $\lim_n a_{nk}$ exists for all k .*

If we put $X = \ell_1$, then we get:

Corollary 2.28 *$A \in (\ell_1, \ell_1)$ if and only if $\sup_k \sum_n |a_{nk}| < \infty$.*

If we put $X = \ell_p$, then we get:

Corollary 2.29 *$A \in (\ell_1, \ell_p)$ if and only if $\sup_k (\sum_n |a_{nk}|^p)^{1/p} < \infty$.*

If we put $X = \ell_\infty$ then we get:

Corollary 2.30 *$A \in (\ell_1, \ell_\infty)$ if and only if $\sup_{n,k} |a_{nk}| < \infty$.*

The following result is one of the most important in matrix transformations:

Theorem 2.31 [3, Theorem 4.2.8] *Any matrix map between FK spaces is continuous.*

Proof Let X and Y be FK spaces, $A \in (X, Y)$ and the map $f_A: X \rightarrow \mathbb{C}$ be defined by $f_A(x) = Ax$ for all $x \in X$. Since the maps $P_n \circ f_A: X \rightarrow \mathbb{C}$ are continuous for all $n \in \mathbb{N}_0$ by Theorem 1.6, the linear map f_A is continuous by Corollary 1.5.

This completes the proof.

Theorem 2.32 [5, Theorem 1.23(b)] *Let X be an FK space. Then, we have $A \in (X, \ell_\infty)$ if and only if*

$$\|A\|_\delta^* = \sup_n \|A_n\|_\delta^* < \infty \text{ for some } \delta > 0, \quad (2.18)$$

where $A_n = (a_{nk})_{k=1}^\infty$ denotes the sequence in the n -th row of the matrix A .

Proof First, we assume that (2.18) is satisfied. Then the series $A_n(x)$ converge for all $x \in B_\delta[0]$ and for all n , and $Ax \in \ell_\infty$ for all $x \in B_\delta[0]$. Since $B_\delta[0]$ is absorbing by Remark 1.3, we conclude that the series $A_n(x)$ converge for all n and all $x \in X$, and $Ax \in \ell_\infty$ for all $x \in X$, i.e., $A \in (X, \ell_\infty)$.

Conversely, suppose that $A \in (X, \ell_\infty)$. Then the map $L_A: X \rightarrow \ell_\infty$ defined by $L_A(x) = Ax$ for all $x \in X$ is continuous by Theorem 2.31. Hence there exist a neighborhood U of 0 in X and a real $\delta > 0$ such that $B_\delta[0] \subset U$ and $\|L_A(x)\|_\infty < 1$ for all $x \in X$. This implies (2.18).

This completes the proof.

Theorem 2.33 [7, Theorem 3.20] *Let X and Y be BK spaces. Then, we have*

(a) $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$.

(b) If X has AK , then $\mathcal{B}(X, Y) \subset (X, Y)$, that is, for every operator $L \in \mathcal{B}(X, Y)$ there exists a matrix $A \in (X, Y)$ such that $L(x) = Ax$ for all $x \in X$.

(c) $A \in (X, \ell_\infty)$ if and only if

$$\|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* < \infty. \quad (2.19)$$

if $A \in (X, \ell_\infty)$, then

$$\|A\|_{(X, \ell_\infty)} = \|L_A\|. \quad (2.20)$$

Proof (a) This is Theorem 2.31.

(b) Let $L: X \rightarrow Y$ be a continuous linear operator. We write $L_n = P_n \circ L$ for all n and put $a_{nk} = L_n(e^{(k)})$ for all n and k . Let $x = (x_k)_{k=1}^\infty$ be given. Since X has AK , we have $x = \sum_{k=1}^\infty x_k e^{(k)}$, and since Y is a BK space, it follows that L_n is continuous linear functional on X for all n . Hence we obtain $L_n(x) = \sum_{k=1}^\infty x_k L_n(e^{(k)}) = \sum_{k=1}^\infty a_{nk} x_k = A_n(x)$ for all n , and so $L(x) = Ax$.

(c) This follows immediately from Theorem 2.32 and the definition of $\|A\|_{(X, \ell_\infty)}$.

That is, if X is a BK space, then $L_A \in \mathcal{B}(X, Y)$ implies

$$\|A(x)\|_\infty = \sup_n |A_n(x)| = \|L_A(x)\|_\infty \leq \|L_A\|$$

for all $x \in X$ with $\|x\| = 1$. Thus $|A_n(x)| \leq \|L_A\|$ for all n and for all $x \in X$ with $\|x\| = 1$, and, by the definition of the norm $\|A\|_{(X, \ell_\infty)}$,

$$\|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* \leq \|L_A\|. \quad (2.21)$$

Further, given $\varepsilon > 0$, there is $x \in X$ with $\|x\| = 1$, $\|A(x)\|_\infty \geq \|L_A\| - \varepsilon/2$, and there is $n(x) \in \mathbb{N}_0$ with $|A_{n(x)}(x)| \geq \|A(x)\|_\infty - \varepsilon/2$, consequently $|A_{n(x)}(x)| \geq \|L_A\| - \varepsilon$. Therefore $\|A\|_{(X, \ell_\infty)} = \sup_n \|A_n\|_X^* \geq \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|A\|_{(X, \ell_\infty)} \geq \|L_A\|$ which together with (2.21) gives $\|A\|_{(X, \ell_\infty)} = \|L_A\|$.

This completes the proof.

Theorem 2.34 [3, 8.3.6 and 8.3.7, pp. 123] *We have*

(a) *Let Y and Y_1 be FK spaces with Y_1 a closed subspace of Y . If $(b^{(k)})_{k=0}^\infty$ is a Schauder basis for X , then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $Ab^{(k)} \in Y_1$ for all $k \in \mathbb{N}$.*

(b) *Let X be an FK space, $X_1 = X \oplus e = \{x_1 = x + \lambda e : x \in X, \lambda \in \mathbb{C}\}$, and Y be a linear subspace of w . Then $A \in (X_1, Y)$ if and only if $A \in (X, Y)$ and $Ae \in Y$.*

Proof (a) The necessity of the conditions for $A \in (X, Y_1)$ is trivial.

Conversely, if $A \in (X, Y)$, then $L_A \in \mathcal{B}(X, Y)$. Since Y_1 is a closed subspace of Y , the FK metrics of Y_1 and Y are the same by Theorem 1.7. Consequently, if S is any subset in Y_1 , then, for its closures $\text{clos}_{Y_1}(S)$ and $\text{clos}_{Y|_{Y_1}}(S)$ with respect to the metrics d_{Y_1} and $d_{Y|_{Y_1}}$, we have

$$\text{clos}_{Y_1}(S) = \text{clos}_{Y|_{Y_1}}(S). \quad (2.22)$$

Let $x \in X$ and $E = \{\sum_{k=0}^m \lambda_k b^{(k)} : m \in \mathbb{N}_0, \lambda_k \in \mathbb{C} (k = 0, 1, 2, \dots)\}$ denote the span of $\{b^{(k)} : k = 0, 1, 2, \dots\}$. Since $L_A(b^{(k)}) \in Y_1$ for all $k = 0, 1, \dots$ and the metrics d_{Y_1} and $d_{Y|_{Y_1}}$ are equivalent, the map $L_{A|_E} : (X, d_X) \rightarrow (Y_1, d_{Y_1})$ is continuous. Further, since $(b^{(k)})_{k=0}^\infty$ is a basis of X , we have $\bar{E} = X$. Therefore, by (2.22) and the continuity of L_A , we have

$$L_A(X) = L_A(\bar{E}) = \text{clos}_{Y_1}(L_{A|_E}(E)) = \text{clos}_{Y|_{Y_1}}(L_{A|_E}(E)) \subset \text{clos}_{Y|_{Y_1}}(Y_1) = Y_1.$$

Hence $A \in (X, Y_1)$.

(b) First, we assume $A \in (X_1, Y)$. Then $X \subset X_1$ implies $A \in (X, Y)$, and $e \in X_1$ implies $Ae \in Y$.

Conversely, we assume $A \in (X, Y)$ and $Ae \in Y$. Let $x_1 \in X_1$ be given. Then there are $x \in X$ and $\lambda \in \mathbb{C}$ such that $x_1 = x + \lambda e$, and it follows that $Ax_1 = A(x + \lambda e) = Ax + \lambda Ae \in Y$.

This completes the proof.

Theorem 2.35 *Let $X \supset \phi$ be a BK space. Then, $A \in (X, \ell_1)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and*

$$\sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} A_n \right\|_X^* < \infty. \quad (2.23)$$

If $A \in (X, \ell_1)$, then

$$\|A\|_{(X, \ell_1)} \leq \|L_A\| \leq 4 \cdot \|A\|_{(X, \ell_1)}, \quad (2.24)$$

where

$$\|A\|_{(X, \ell_1)} = \sup_{N \in \mathcal{F}} \left\| \sum_{n \in N} A_n \right\|_X^*,$$

and \mathcal{F} denotes the collection of all non-empty and finite subsets of \mathbb{N} .

Proof For (2.23), we refer to [8].

To show (2.24), let $A \in (X, \ell_1)$ and $m \in \mathbb{N}_0$ be given. Then, for all $N \subset \{1, 2, \dots, m\}$ and for all $x \in X$ with $\|x\| = 1$,

$$\left| \sum_{n \in N} A_n(x) \right| \leq \sum_{n=0}^m |A_n(x)| \leq \|L_A\|,$$

and this implies that

$$\|A\|_{(X, \ell_1)} \leq \|L_A\|. \quad (2.25)$$

Furthermore, given $\varepsilon > 0$, there is $x \in X$ with $\|x\| = 1$ such that

$$\|A(x)\|_1 = \sum_{n=0}^{\infty} |A_n(x)| \geq \|L_A\| - \frac{\varepsilon}{2},$$

and there is an integer $m(x)$ such that

$$\sum_{n=0}^{m(x)} |A_n(x)| \geq \|A(x)\|_1 - \frac{\varepsilon}{2}.$$

Consequently

$$\sum_{n=0}^{m(x)} |A_n(x)| \geq \|L_A\| - \varepsilon.$$

By Lemma 3.9 of [5],

$$4 \left[\max_{N \subset \{0, 1, \dots, m(x)\}} \left| \sum_{n \in N} A_n(x) \right| \right] \geq \sum_{n=0}^{m(x)} |A_n(x)| \geq \|L_A\| - \varepsilon,$$

and so $4\|A\|_{(X, \ell_1)} \geq \|L_A\| - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have $4\|A\|_{(X, \ell_1)} \geq \|L_A\|$ which together with (2.25) this yields (2.24).

This completes the proof.

Consequently we have the following:

Corollary 2.36 *We have $(c_0, \ell_1) = (c, \ell_1) = (\ell_\infty, \ell_1)$. Further, $A \in (c_0, \ell_1)$ if and only if*

$$\sup_{K \in \mathcal{F}} \left(\sum_{n=0}^{\infty} \left| \sum_{k \in K} a_{nk} \right| \right) < \infty.$$

Remark 2.10 Since the BK spaces c_0 and c are closed subspaces of ℓ_∞ , the matrix classes (X, c_0) and (X, c) can be characterized by combining Theorem 2.33 (c) and Theorem 2.34 (a), where X is a BK space with Schauder basis. On the other hand, we may note that if X , in Theorem 2.33 (c) or Theorem 2.34 (a), is any of the classical sequence spaces, then any of (2.19), (2.20), or (2.23) implies the condition ‘ $A_n \in X^\beta$ for all $n \in \mathbb{N}$ ’ by Remark 2.9. Thus, this condition is redundant in such cases. Also, if X is a BK space with AK , then we obtain the following result which is immediate by Propositions 3.2 and 3.3 of [9].

Theorem 2.37 *Let X be a BK space with AK . Then, we have*

- (a) $A \in (X, \ell_\infty)$ if and only if (2.19) holds.
- (b) $A \in (X, c)$ if and only if (2.19) and $\lim_{n \rightarrow \infty} a_{nk}$ exists for every $k \in \mathbb{N}$, hold.
- (c) $A \in (X, c_0)$ if and only if (2.19) and $\lim_{n \rightarrow \infty} a_{nk} = 0$ for all $k \in \mathbb{N}$, hold.
- (d) $A \in (X, \ell_1)$ if and only if (2.23) holds.

2.6 Conservative, Regular, and Schur Matrices

Remark 2.11 The results of the previous sections yield the characterizations of the classes (X, Y) where X and Y are any of the spaces ℓ_p ($1 \leq p < \infty$), c_0 , c with the exceptions of (ℓ_p, ℓ_r) where both $p, r \neq 1, \infty$ (the characterizations are unknown), and of (ℓ_∞, c) (Schur’s theorem) and (ℓ_∞, c_0) for which no functional analytic proofs seem to be known. The class (ℓ_2, ℓ_2) was characterized in [10].

Definition 2.8 A matrix A is called a *conservative matrix* if $Ax \in c$ for all $x \in c$. If in addition $\lim Ax = \lim x$ for all $x \in c$, then A is called a *regular matrix*. The class of conservative matrices will be denoted by (c, c) and of regular matrices by $(c, c; P)$ or $(c, c)_{\text{reg}}$.

A matrix A is called a *Schur matrix* or *coercive matrix* if $Ax \in c$ for all $x \in \ell_\infty$. The class of Schur matrices will be denoted by (ℓ_∞, c) .

The well-known Silverman–Toeplitz conditions for the regularity of A are (c.f. [2, 11, 12]):

- (i) $\|A\|_{(\ell_\infty, \ell_\infty)} = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$,
- (ii) $\lim_{n \rightarrow \infty} a_{nk} = 0$ exists for every k , and
- (iii) $\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} = 1$.

Next, two results concern the transpose A^T of a matrix A which shall be very useful in obtaining some matrix characterizations.

Theorem 2.38 [3, Theorem 8.3.8, pp. 124] *Let X be an FK space and Y be any set of sequences. If $A \in (X, Y)$ then $A^T \in (Y^\beta, X^f)$. If X and Y are BK spaces and Y^β has AD then we have $A^T \in (Y^\beta, cl_{X^f}(X^\beta))$.*

Proof Let $A \in (X, Y)$ and $z \in Y^\beta$ be given. We define the functional $f: X \rightarrow \mathbb{C}$ by $f(x) = \sum_{n=0}^{\infty} z_n A_n(x)$ ($x \in X$). Since X is an FK space, $Ax \in Y$ by assumption and $z \in Y^\beta$, we have $f \in X'$ by Theorem 1.6. Furthermore it follows that $f(e^{(k)}) = \sum_{n=0}^{\infty} z_n a_{nk}$ for all k , hence $A^T z \in X^f$, i.e., $A^T \in (Y^\beta, X^f)$.

Now we assume that X and Y are BK spaces and Y has AD . Then $Y^\beta \subset X^f$ by Theorem 2.20 (a), and X^f is a BK space by Theorem 2.23. Also $cl_{X^f}(X^\beta)$ is a closed subspace of X^f . Since $A \in (X, Y)$, we have $A_n = (a_{nk})_{k=0}^{\infty} \in X^\beta$ for all n , but $A^T e^{(n)} = (\sum_{j=0}^{\infty} a_{jk} e_j^{(k)})_{k=0}^{\infty} = (a_{nk})_{k=0}^{\infty} = A_n \in X^\beta$ for all n . So we have $A^T \in (Y^\beta, cl_{X^f}(X^\beta))$ by Theorem 2.34 (a).

This completes the proof.

Theorem 2.39 [3, Theorem 8.3.9, pp. 124] *Let X and Z be BK spaces with AK and $Y = Z^\beta$. Then we have $(X, Y) = (X^{\beta\beta}, Y)$; furthermore $A \in (X, Y)$ if and only if $A^T \in (Z, X^\beta)$.*

Proof Since X is a BK space with AK , X^β is a BK space by Corollary 2.22, and $X^\beta = X^f$ by Theorem 2.20 (b).

First we assume $A \in (X, Y)$. Then it follows by Theorem 2.38 and since $Z^{\beta\beta} \supset Z$ that $A^T \in (Y^\beta, X^\beta) = (Z^{\beta\beta}, X^\beta) \subset (Z, X^\beta)$.

Conversely, if $A^T \in (Z, X^\beta)$ then it follows by Theorem 2.38 and since $X^{\beta\beta} \supset X$ that $A \in (X^{\beta\beta}, Z^\beta) \subset (X, Z^\beta) = (X, Y)$. This proves the second part.

To prove the first part, we first observe that $X \subset X^{\beta\beta}$ implies $(X^{\beta\beta}, Y) \subset (X, Y)$.

Conversely we assume $A \in (X, Y)$. Then we have $A^T \in (Z, X^\beta)$ as proved above, and Theorem 2.38 implies $A = A^{TT} \in (X^{\beta\beta}, Z^\beta) = (X^{\beta\beta}, Y)$.

This completes the proof.

Theorem 2.40 [7, Example 5.4] (a) *We have $(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$; furthermore $A \in (\ell_\infty, \ell_\infty) = \mathcal{B}(\ell_\infty, \ell_\infty)$ if and only if*

$$\|A\|_{(\ell_\infty, \ell_\infty)} = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty. \quad (2.26)$$

If A is in any of the above classes then $\|L_A\| = \|A\|_{(\ell_\infty, \ell_\infty)}$.

(b) *We have $A \in (c_0, c)$ if and only if (2.26) holds and*

$$\lim_{n \rightarrow \infty} a_{nk} = \alpha_k \text{ exists for every } k. \quad (2.27)$$

If $A \in (c_0, c)$ then

$$\lim_{n \rightarrow \infty} A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k. \quad (2.28)$$

(c) $A \in (c_0, c_0)$ if and only if (2.26) and (2.27) with $\alpha_k = 0$ for all k hold.

Proof (a) We have $A \in (c_0, \ell_\infty)$ if and only if (2.26) holds by Theorem 2.33, and since $c_0^\beta = \ell_1$ and c_0^* and ℓ_1 are norm isomorphic.

Furthermore $c_0 \subset c \subset \ell_\infty$ implies $(c_0, \ell_\infty) \supset (c, \ell_\infty) \supset (\ell_\infty, \ell_\infty)$. Also $(c_0, \ell_\infty) = (c_0^{\beta\beta}, \ell_\infty) = (\ell_\infty, \ell_\infty)$ by the first part of Theorem 2.39. Also for all $x \in \ell_\infty$, we have

$$\|A(x)\|_{\ell_\infty} \leq \|A\|_{(\ell_\infty, \ell_\infty)} \|x\|_{\ell_\infty},$$

i.e., A is a bounded linear operator. The last part is obvious from Theorem 2.33.

(b) Since c is a closed subspace of ℓ_∞ , the characterization of the class (c_0, c) is an immediate consequence of Theorem 2.34 (a) and Part (a).

Now we assume $A \in (c_0, c)$ and write $\|A\| = \|A\|_{(\ell_\infty, \ell_\infty)}$, for short. Let m be a given non-negative integer. Then it follows from (2.27) and (2.26) that $\sum_{k=0}^m |\alpha_k| = \lim_{n \rightarrow \infty} \sum_{k=0}^m |a_{nk}| \leq \|A\|$. Since m was arbitrary, we have $(\alpha_k)_{k=0}^\infty \in \ell_1$,

$$\sum_{k=0}^\infty |\alpha_k| \leq \|A\| \quad \text{and} \quad \sum_{k=0}^\infty |\alpha_k x_k| \leq \|A\| \|x\|_\infty \quad \text{for all } x \in c. \quad (2.29)$$

Now let $x \in c_0$ and $\varepsilon > 0$ be given. Then we can choose an integer $k(\varepsilon)$ such that $|x_k| \leq \varepsilon/(4\|A\| + 1)$ for all $k > k(\varepsilon)$, and by (2.27) we can choose an integer $n(\varepsilon)$ such that $\sum_{k=0}^{k(\varepsilon)} |a_{nk} - \alpha_k| |x_k| < \varepsilon/2$. Let $n > n(\varepsilon)$. Then (2.26) and (2.29) imply

$$\begin{aligned} |A_n(x) - \sum_{k=0}^\infty \alpha_k x_k| &\leq \sum_{k=0}^{k(\varepsilon)} |a_{nk} - \alpha_k| |x_k| + \sum_{k=k(\varepsilon)+1}^\infty (|a_{nk}| + |\alpha_k|) |x_k| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4\|A\| + 1} \left(\sum_{k=0}^\infty |a_{nk}| + \sum_{k=0}^\infty |\alpha_k| \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence (2.28) holds.

(c) It directly follows from Part (b).

This completes the proof.

Theorem 2.41 [7, Example 5.5] *We have $(\ell_1, \ell_1) = \mathcal{B}(\ell_1, \ell_1)$ and $A \in (\ell_1, \ell_1)$ if and only if*

$$\|A\|_{(\ell_1, \ell_1)} = \sup_k \sum_{n=0}^\infty |a_{nk}| < \infty. \quad (2.30)$$

If A is in any of the classes above then $\|L_A\| = \|A\|_{(\ell_1, \ell_1)}$.

Proof Since ℓ_1 has AK, Theorem 2.33 (b) yields the first part.

We apply the second part of Theorem 2.39 with $X = \ell_1$, $Z = c_0$, BK spaces with AK , and $Y = Z^\beta = \ell_1$ to obtain $A \in (\ell_1, \ell_1)$ if and only if $A^T \in (\ell_\infty, \ell_\infty)$; by Theorem 2.40 (a), this is the case if and only if (2.30) is satisfied.

Furthermore, if $A \in (\ell_1, \ell_1)$ then

$$\|L_A(x)\|_{\ell_1} = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| \leq \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_{nk} x_k| \leq \|A\|_{(\ell_1, \ell_1)} \|x\|_1$$

implies $\|L_A\| \leq \|A\|_{(\ell_1, \ell_1)}$. Also $L_A \in \mathcal{B}(\ell_1, \ell_1)$ implies $\|L_A(x)\|_1 = \|Ax\|_1 \leq \|L_A\| \|x\|_1$, and it follows from $\|e^{(k)}\|_1 = 1$ for all k that $\|A\|_{(\ell_1, \ell_1)} = \sup_k \sum_{n=0}^{\infty} |a_{nk}| = \sup_k \|L(e^{(k)})\|_1 \leq \|L_A\|$. Hence $\|L_A\| = \|A\|_{(\ell_1, \ell_1)}$.

This completes the proof.

Theorem 2.42 (Kojima–Schur) [2, Theorem 3.3.3] (a) A is conservative, i.e., $A \in (c, c)$ if and only if (2.26) and (2.27) hold, and

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \text{ exists.} \quad (2.31)$$

If $A \in (c, c)$ and $x \in c$ then

$$\lim_{n \rightarrow \infty} A_n(x) = \left(\alpha - \sum_{k=0}^{\infty} \alpha_k \right) \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} \alpha_k x_k. \quad (2.32)$$

(b) (Silverman–Toeplitz) A is regular, i.e., $A \in (c, c; P)$ if and only if (2.26), (2.27), and (2.31) hold with $\alpha_k = 0$ ($k = 0, 1, 2, \dots$) and $\alpha = 1$. In this case A is also known as the Toeplitz matrix.

Remark 2.12 The characterization of the class (c, c) is an immediate consequence of Theorem 2.40 (a), and Theorem 2.34 (a) and (b). But we prove here something more as follows:

Theorem 2.43 [7, Theorem 6.11] We have $L \in \mathcal{B}(c, c)$ if and only if there exists a matrix $A \in (c_0, c)$ and a sequence $b \in \ell_\infty$ with

$$\lim_{n \rightarrow \infty} \left(b_n + \sum_{k=0}^{\infty} a_{nk} \right) = \tilde{\alpha} \text{ exists} \quad (2.33)$$

such that

$$L(x) = b \lim_{k \rightarrow \infty} x_k + Ax \quad \text{for all } x \in c. \quad (2.34)$$

Furthermore, we have

$$\|L\| = \sup_n \left(|b_n| + \sum_{k=0}^{\infty} |a_{nk}| \right). \quad (2.35)$$

Proof First we assume that $L \in \mathcal{B}(c, c)$. We write $L_n = P_n \circ L$ ($n = 0, 1, \dots$) where P_n is the n^{th} coordinate with $P_n(x) = x_n$ ($x \in w$). Since c is a BK space, we have $L_n \in c^*$ for all n ,

$$L_n(x) = b_n \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} a_{nk} x_k \quad (x \in c) \quad (2.36)$$

with

$$b_n = L_n(e) - \sum_{k=0}^{\infty} L_n(e^{(k)}) \text{ and } a_{nk} = L_n(e^{(k)}) \text{ for } k = 0, 1, \dots$$

and

$$\|L_n\| = |b_n| + \sum_{k=0}^{\infty} |a_{nk}|. \quad (2.37)$$

Now (2.36) yields (2.34). Since $L(x_0) = Ax_0$ for all x_0 , we have $A \in (c_0, c)$, and so $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$ by Theorem 2.40 (b). Also $L(e) = b + Ae$ implies (2.33), and we obtain $\|b\|_{\infty} \leq \|L(e)\|_{\infty} + \|A\|$, that is, $b \in \ell_{\infty}$. Consequently we have $C = \sup_n (|b_n| + \sum_{k=0}^{\infty} |a_{nk}|) < \infty$. Now $\|L(x)\|_{\infty} = \sup_n |b_n \lim_{k \rightarrow \infty} x_k + \sum_{k=0}^{\infty} a_{nk} x_k| \leq (\sup_n (|b_n| + \sum_{k=0}^{\infty} |a_{nk}|)) \|x\|_{\infty} < \infty$ implies $\|L\| \leq C$. We also have $|L_n(x)| \leq \|L(x)\|_{\infty} \leq \|L\|$ for all $x \in \bar{B}_c$ and all n , and so $\sup_n \|L_n\| = C \leq \|L\|$. Thus (2.35) is proved.

Conversely we assume that $A \in (c_0, c)$ and $b \in \ell_{\infty}$ satisfy (2.33). Since $A \in (c_0, c)$ and $b \in \ell_{\infty}$, we obtain $C < \infty$ by (2.26), and so $L \in \mathcal{B}(c, \ell_{\infty})$. Finally let $x \in c$ be given and $\lim_{k \rightarrow \infty} x_k = \xi$. Then we have $x - \xi e \in c_0$, $L_n(x) = b_n \xi + \sum_{k=0}^{\infty} a_{nk} x_k = (b_n + \sum_{k=0}^{\infty} a_{nk}) \xi + A_n(x - \xi e)$ for all n , and it follows from (2.33) and $A \in (c_0, c)$ that $\lim_n L_n(x)$ exists. Since $x \in c$ was arbitrary, we have $L \in \mathcal{B}(c, c)$.

This completes the proof.

First we state the following lemma which is needed in proving Schur's theorem. To the best of our knowledge it seems that the functional analytic proof of Schur's theorem does not exist yet.

Lemma 2.44 [2, Theorem 3.3.7] *Let $B = (b_{nk})_{n,k}$ be an infinite matrix such that $\sum_k |b_{nk}| < \infty$ for each n and $\sum_k |b_{nk}| \rightarrow 0$ ($n \rightarrow \infty$). Then $\sum_k |b_{nk}|$ converges uniformly in n .*

Proof $\sum_k |b_{nk}| \rightarrow 0$ ($n \rightarrow \infty$) implies that $\sum_k |b_{nk}| < \infty$ for $n \geq N(\varepsilon)$. Since $\sum_k |b_{nk}| < \infty$ for $0 \leq n \leq N(\varepsilon)$, there exists $m = M(\varepsilon, n)$ such that

$\sum_{k \geq M} |b_{nk}| < \infty$ for all n , which means that $\sum_k |b_{nk}|$ converges uniformly in n .

This completes the proof.

Theorem 2.45 (Schur) [2, Theorem 3.3.8] $A \in (\ell_\infty, c)$ if and only if (2.27) holds and

$$\sum_k |b_{nk}| \text{ converges uniformly in } n. \quad (2.38)$$

Proof Suppose that the conditions (2.27) and (2.38) hold, and $x \in \ell_\infty$. Then, $\sum_k a_{nk}x_k$ is absolutely and uniformly convergent in $n \in \mathbb{N}$. Hence, $\sum_k a_{nk}x_k \rightarrow \sum_k \alpha_k x_k$ ($n \rightarrow \infty$) which gives that $A \in (\ell_\infty, c)$.

Conversely, suppose that $A \in (\ell_\infty, c)$ and $x \in c$. Then necessity of (2.27) follows easily by taking $x = e^{(k)}$ for each k . Define $b_{nk} = a_{nk} - \alpha_k$ for all $k, n \in \mathbb{N}$. Since $\sum_k |\alpha_k| < \infty$, $(\sum_k b_{nk}x_k)_n$ converges whenever $x = (x_k) \in \ell_\infty$. Now if we can show that this implies

$$\lim_n \sum_k |b_{nk}| = 0, \quad (2.39)$$

then by using Lemma 2.44, we shall get the desired result. Suppose to the contrary that $\lim_n \sum_k |b_{nk}| \neq 0$. Then, it follows that $\lim_n \sum_k |b_{nk}| = l > 0$ through some subsequence of the positive integers. Also we have $b_{mk} \rightarrow 0$ as $m \rightarrow \infty$ for each $k \in \mathbb{N}$. Hence we may determine $m(1)$ such that

$$\left| \sum_k |b_{m(1),k}| - l \right| < l/2 \text{ and } b_{m(1),1} < l/2.$$

Since $\sum_k |b_{m(1),k}| < \infty$ we may choose $k(2) > 1$ such that

$$\sum_{k=k(2)+1}^{\infty} |b_{m(1),k}| < l/2.$$

It follows that

$$\left| \sum_{k=2}^{k(2)} |b_{m(1),k}| - l \right| < l/2.$$

For our convenience we use the notation $\sum_{k=p}^q |b_{mk}| = B(m, p, q)$.

Now we choose $m(2) > m(1)$ such that $|B(m(2), 1, \infty) - l| < l/10$ and $B(m(2), 1, k(2)) < l/10$. Then choose $k(3) > k(2)$ such that $|B(m(2), k(3) + 1, \infty) - l| < l/10$. It follows that $|B(m(2), k(2) + 1, k(3)) - l| < 3l/10$. Continuing in this way and find $m(1) < m(2) < \dots$, $1 = k(1) < k(2) < \dots$ such that

$$\begin{cases} B(m(r), 1, k(r)) < l/10 \\ B(m(r), k(r) + 1, \infty) < l/10 \\ |B(m(r), k(r) + 1, k(r+1)) - l| < 3l/10 \end{cases} \quad (2.40)$$

Let us define $x = (x_k) \in \ell_\infty$ such that $\|x\| = 1$ by

$$x_k = \begin{cases} 0, & \text{if } k = 1, \\ (-1)^r \operatorname{sgn}(b_{m(r),k}), & \text{if } k(r) < k \leq k(r+1), \end{cases} \quad (2.41)$$

for $r = 1, 2, \dots$. Then write $\sum_k b_{m(r),k} x_k$ as $\sum_1 + \sum_2 + \sum_3$, where \sum_1 is over $1 \leq k \leq k(r)$, \sum_2 is over $k(r) < k \leq k(r+1)$ and \sum_3 is over $k > k(r+1)$. It follows immediately from (2.40) with the sequence x given by (2.41) that

$$\left| \sum_k b_{m(r),k} - (-1)^r l \right| < l/2.$$

Consequently, it is clear that the sequence $Bx = (\sum_k b_{nk} x_k)$ is not a Cauchy sequence and so is not convergent. Thus we have proved that Bx is not convergent for all $x \in \ell_\infty$ which contradicts the fact that $A \in (\ell_\infty, c)$. Hence, (2.39) must hold. Now, it follows by Lemma 2.44 that $\sum_k |b_{nk}|$ converges uniformly in n . Therefore, $\sum_k |a_{nk}| = \sum_k |b_{nk} + \alpha_k|$ converges uniformly in n .

This completes the proof.

We get the following corollary:

Corollary 2.46 $A \in (\ell_\infty, c_0)$ if and only if

$$\lim_n \sum_k |b_{nk}| = 0. \quad (2.42)$$

Definition 2.9 The characteristic $\chi(A)$ of a matrix $A = (a_{nk}) \in (c, c)$ is defined by

$$\chi(A) = \lim_{n \rightarrow \infty} \sum_k a_{nk} - \sum_k \left(\lim_{n \rightarrow \infty} a_{nk} \right)$$

which is a multiplicative linear functional. The numbers $\lim_{n \rightarrow \infty} a_{nk}$ and $\lim_{n \rightarrow \infty} \sum_k a_{nk}$ are called the *characteristic numbers* of A . A matrix A is called *coregular* if $\chi(A) \neq 0$ and is called *conull* if $\chi(A) = 0$.

Remark 2.13 The Silverman–Toeplitz theorem yields for a regular matrix A that $\chi(A) = 1$ which leads us to the fact that regular matrices form a subset of coregular matrices. One can easily see for a Schur matrix A that $\chi(A) = 0$ which says us that coercive matrices form a subset of conull matrices. Hence, we have the following result which is known as Steinhaus's theorem.

Theorem 2.47 (Steinhaus) [2, Theorem 3.3.14] *For every regular matrix A , there is a bounded sequence which is not summable by A .*

Proof We assume that a matrix $A \in (c, c; P) \cap (\ell_\infty, c)$. Then it follows from Theorem 2.42 (b) and Theorem 2.45 that $1 = \lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{nk} = \sum_{k=0}^{\infty} (\lim_{n \rightarrow \infty} a_{nk}) = 0$, a contradiction.

This completes the proof.

We observe the following application of Corollary 2.46.

Theorem 2.48 [12, Corollary, pp. 225] *Weak and strong convergence coincide in ℓ_1 .*

Proof We assume that the sequence $(x^{(n)})_{n=0}^\infty$ is weakly convergent to x in ℓ_1 , that is, $\|f(x^{(n)}) - f(x)\| \rightarrow 0$ ($n \rightarrow \infty$) for every $f \in \ell_1^*$. Since ℓ_1^* and ℓ_∞ are norm isomorphic, to every $f \in \ell_1^*$ there corresponds a sequence $a \in \ell_\infty$ such that $f(y) = \sum_{k=0}^\infty a_k y_k$. We define the matrix $B = (b_{nk})_{n,k=0}^\infty$ by $b_{nk} = x_k^{(n)} - x_k$ ($n, k = 0, 1, \dots$). Then we have $f(x^{(n)}) - f(x) = \sum_{k=0}^\infty a_k (x_k^{(n)} - x_k) = \sum_{k=0}^\infty b_{nk} a_k \rightarrow 0$ ($n \rightarrow \infty$) for all $a \in \ell_\infty$, that is, $B \in (\ell_\infty, c_0)$, and it follows from Corollary 2.46 that $\|x^{(n)} - x\|_1 = \sum_{k=0}^\infty |x_k^{(n)} - x_k| = \sum_{k=0}^\infty |b_{nk}| \rightarrow 0$ ($n \rightarrow \infty$).

This completes the proof.

2.7 Matrix Transformations for Matrix Domains

In this section, we characterize the classes (X, Y_T) and $(X, Y_{[T]})$ for triangles T .

Theorem 2.49 [5, Theorem 3.8] *Let T be a triangle.*

(a) *Then, for arbitrary subsets X and Y of w , $A \in (X, Y_T)$ if and only if $B = TA \in (X, Y)$.*

(b) *If X and Y are BK spaces and $A \in (X, Y_T)$, then*

$$\|L_A\| = \|L_B\|. \quad (2.43)$$

Proof (a) It is straightforward, recall that $Ax \in Y_T$ if and only if $Bx = (TA)x = T(Ax) \in Y$.

(b) Let $A \in (X, Y_T)$. Since Y is a BK space and T a triangle, Y_T is a BK space with

$$\|y\|_{Y_T} = \|T(y)\|_Y \quad (y \in Y_T), \quad (2.44)$$

by Theorem 1.11. Therefore A is continuous by Theorem 2.31 and

$$\|L_A\| = \sup \{\|L_A(x)\|_{Y_T} : \|x\| = 1\} = \sup \{\|A(x)\|_Y : \|x\| = 1\} < \infty. \quad (2.45)$$

Further, since B is continuous, we have

$$\|L_B\| = \sup \{\|L_B(x)\|_Y : \|x\| = 1\} = \sup \{\|B(x)\|_Y : \|x\| = 1\} < \infty. \quad (2.46)$$

Let $x \in X$. Since $A_n \in X^\beta$ for all $n = 0, 1, \dots$, we have $x \in w_A$. Further $T_n \in \phi$ ($n = 0, 1, \dots$), since T is a triangle. Hence $B(x) = (TA)(x) = T(A(x))$ (cf. [3, Theorem 1.4.4]), and (2.43) follows from (2.44)–(2.46).

This completes the proof.

For the characterization of the class $(X, Y_{[T]})$, we need the following lemma due to Peyerimhoff [13].

Lemma 2.50 *Let $a_0, a_1, \dots, a_n \in \mathbb{C}$. Then*

$$\sum_{k=0}^n |a_k| \leq 4 \cdot \max_{N \subset \{0, 1, \dots, n\}} \left| \sum_{k \in N} a_k \right|.$$

Proof First we consider the case for $a_0, a_1, \dots, a_n \in \mathbb{R}$. Put $N^+ = \{k \in \{0, 1, \dots, n\} : a_k \geq 0\}$ and $N^- = \{k \in \{0, 1, \dots, n\} : a_k < 0\}$. Then

$$\sum_{k=0}^n |a_k| = \sum_{k \in N^+} a_k + \left| \sum_{k \in N^-} a_k \right| \leq 2 \cdot \max_{N \subset \{0, 1, \dots, n\}} \left| \sum_{k \in N} a_k \right|. \quad (2.47)$$

Now let $a_0, a_1, \dots, a_n \in \mathbb{C}$. We put $a_k = \alpha_k + i\beta_k$ ($k = 0, 1, \dots, n$). For any $N \subset \{0, 1, \dots, n\}$, let us write

$$x_N = \sum_{k \in N} \alpha_k, \quad y_N = \sum_{k \in N} \beta_k, \quad z_N = x_N + iy_N = \sum_{k \in N} a_k.$$

Now we choose subsets N_r, N_i , and N_* of $\{0, 1, \dots, n\}$ such that

$$|x_{N_r}| = \max_{N \subset \{0, 1, \dots, n\}} |x_N|, \quad |y_{N_i}| = \max_{N \subset \{0, 1, \dots, n\}} |y_N|, \quad |z_{N_*}| = \max_{N \subset \{0, 1, \dots, n\}} |z_N|.$$

Then, for all $N \subset \{0, 1, \dots, n\}$, we have $|x_{N_r}|, |y_{N_i}| \leq |z_{N_*}|$ and $|x_{N_r}| + |y_{N_i}| \leq 2|z_{N_*}|$. Therefore, by (2.47),

$$\begin{aligned} \sum_{k=0}^n |a_k| &\leq \sum_{k \in N^+} |\alpha_k| + \sum_{k \in N^-} |\beta_k| \leq 2(|x_{N_r}| + |y_{N_i}|) \\ &\leq 4 \cdot |z_{N_*}| = 4 \cdot \max_{N \subset \{0, 1, \dots, n\}} \left| \sum_{k \in N} a_k \right|. \end{aligned}$$

This completes the proof.

Theorem 2.51 [5, Theorem 3.10] *Let A be an infinite matrix and B a positive triangle. For each $m \in \mathbb{N}_0$, let N_m be a subset of the set $\{0, 1, \dots, n\}$, $N = (N_m)_{m=0}^\infty$ the sequence of the subsets N_m and \mathcal{N} the set of all such sequences N . Furthermore, for each $N \in \mathcal{N}$, define the matrix $S^N = S^N(A)$ by $s_{mk}^N = \sum_{n \in N_m} b_{mn} a_{nk}$ ($m, k = 0, 1, \dots$). Then, for arbitrary subsets X of w and any normal set Y of sequences, $A \in (X, Y_{[B]})$ if and only if $S^N(A) \in (X, Y)$ for all sequences $N \in \mathcal{N}$.*

Proof Assume that $A \in (X, Y_{[B]})$. Then $A_n \in X^\beta$ ($n = 0, 1, \dots$) implies $S_m^N \in X^\beta$ for all m and all $N \in \mathcal{N}$. For each $x \in X$, we put $y = B(|A(x)|)$. Then $A(x) \in Y_{[B]}$, that is, $y \in Y$, and

$$|S_m^N(x)| \leq \left| \sum_{k=0}^{\infty} s_{mk}^N x_k \right| = \left| \sum_{n \in N_m} b_{mn} \sum_{k=0}^{\infty} a_{nk} x_k \right| \leq |y_m| \quad (m = 0, 1, \dots)$$

for all $N \in \mathcal{N}$ together imply $S^N(x) \in Y$ for all $N \in \mathcal{N}$, since Y is normal. Hence $S^N \in (X, Y)$ for all $N \in \mathcal{N}$.

Conversely, let $S^N \in (X, Y)$ for all $N \in \mathcal{N}$. Then $S_m^N \in X^\beta$ for all m and for all $N \in \mathcal{N}$, in particular, for $N = (m)_{m=0}^\infty$, $S_m^N = b_{mm} A_m \in X^\beta$, hence $A_m \in X^\beta$, since $b_{mm} \neq 0$. Further, let $x \in X$ be given. For every $m = 0, 1, \dots$, choose the set $N_m^{(0)} \subset \{0, \dots\}$ such that

$$\left| \sum_{n \in N_m^{(0)}} b_{mn} A_n(x) \right| = \max_{N_m \subset \{0, 1, \dots, m\}} b_{mn} A_n(x) \quad |.$$

Then, by Lemma 2.50, we have

$$|y_m| \leq 4 \left| \sum_{n \in N_m^{(0)}} b_{mn} A_n(x) \right| = 4 |S^{N^{(0)}}(x)|.$$

Hence by hypothesis, $S^{N^{(0)}}(x) \in Y$, and the normality of Y implies $y = B(|A(x)|) \in Y$, that is, $A \in (X, Y_{[B]})$.

This completes the proof.

Lemma 2.52 ([14, Lemma 28]) *Let X be an FK space with AK and $Z = X_T$. We write $R = S^t$ for the transpose of S . Then, we have*

$$(X_T)^\beta \subset (X^\beta)_R.$$

Theorem 2.53 ([14, Theorem 29]) (a) *Let X be an FK space with AK and $Z = X_T$. We write $R = S^t$ for the transpose of S . Then, $a \in (X_T)^\beta$ if and only if*

$$a \in (X^\beta)_R \text{ and } W \in (X, c_0), \quad (2.48)$$

where the matrix W is defined by

$$w_{mk} = \begin{cases} \sum_{j=m}^{\infty} a_j s_{jk}, & (0 \leq k \leq m), \\ 0, & (k > m). \end{cases} \quad (m = 0, 1, 2, \dots).$$

Moreover, if $a \in (X_T)^\beta$ then, we have

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) \quad (2.49)$$

for all $z \in Z$, where $R_k(a) = \sum_{j=k}^{\infty} a_j s_{jk}$.

(b) The statement of Part (a) also holds when $X = \ell_\infty$.

Theorem 2.54 ([14, Remark 30]) We have $a \in (c_T)^\beta$ if and only if $a \in (\ell_1)_R$ and $W \in (c, c)$. Moreover, if $a \in (c_T)^\beta$, then we have

$$\sum_{k=0}^{\infty} a_k z_k = \sum_{k=0}^{\infty} R_k(a) T_k(z) - \xi \alpha \quad \text{for all } z \in c_T, \quad (2.50)$$

where $\xi = \lim_{k \rightarrow \infty} T_k(z)$ and $\alpha = \lim_{m \rightarrow \infty} \sum_{k=0}^m w_{mk}$.

Theorem 2.55 ([14, Theorem 31]) (a) Let X be an FK space with AK , Y be an arbitrary subset of w , and T be a triangle and $R = S^t$. Then, $A \in (X_T, Y)$ if and only if $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all $n = 0, 1, \dots$, where \hat{A} is the matrix with the rows $\hat{A}_n = R(A_n)$ for $n = 0, 1, \dots$, and the triangles $W^{(n)}$ are defined by

$$w_{mk}^{(n)} = \sum_{j=m}^{\infty} a_{nj} s_{jk}.$$

Moreover, if $A \in (X_T, Y)$ then,

$$A(z) = \hat{A}(T(z)) \quad \text{for all } z \in Z = X_T. \quad (2.51)$$

(b) The statement of Part (a) also holds for $X = \ell_\infty$.

Proof (a) First, we assume $A \in (Z, Y)$. Then, $A_n \in Z^\beta$ for all n , hence $W^{(n)} \in (X, c_0)$ and $\hat{A}_n \in X^\beta$ for all n by Theorem 2.53. Let $x \in X$ be given, hence $z = S(x) = T^{-1}(x) \in Z$. Since $A_n \in Z^\beta$ implies $A_n(z) = \hat{A}_n(T(z)) = \hat{A}_n(x)$ for all n by (2.49), and $A(z) \in Y$ for all $z \in Z$ implies $\hat{A}(x) = A(z) \in Y$. Hence $\hat{A} \in (X, Y)$ and (2.51) holds.

Conversely, we assume $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all n . Then, we have $\hat{A}_n \in X^\beta$ for all n , and this and $W^{(n)} \in (X, c_0)$ together imply $A_n \in Z^\beta$ by Theorem 2.53. Now, let $z \in Z$ be given, hence $x = T(z) \in X$, and again we have $A_n(z) = \hat{A}_n(x)$ for all n by (2.49), and $\hat{A}(x) \in Y$ for all $x \in X$ implies $A(z) = \hat{A}(x) \in Y$. Hence we have $A \in (X, Y)$.

(b) It is obvious from Part (a) and the proof of Theorem 2.53.

This completes the proof.

Theorem 2.56 [14, Remark 32] *Let Y be a linear subspace of ω . Then, we have $A \in (c_T, Y)$ if and only if*

$$\hat{A} \in (c_0, Y) \quad \text{and} \quad W^{(n)} \in (c, c) \quad \text{for all } n \quad (2.52)$$

and

$$\hat{A}(e) - (\alpha_n)_{n=0}^\infty \in Y, \quad \text{where } \alpha_n = \lim_{m \rightarrow \infty} \sum_{k=0}^m w_{mk}^{(n)} \quad \text{for all } n. \quad (2.53)$$

Moreover, if $A \in (c_T, Y)$ then, we have

$$A(z) = \hat{A}(T(z)) - \xi((\alpha_n)_{n=0}^\infty) \quad \text{for all } z \in c_T, \quad \text{where } \xi = \lim_{k \rightarrow \infty} T_k(z). \quad (2.54)$$

Proof First we assume $A \in (c_T, Y)$. Then, it follows that $A \in ((c_0)_T, Y)$ and so $\hat{A} \in (c_0, Y)$ by Theorem 2.53. Also by Theorem 2.54, $A_n \in (c_T)^\beta$ for all n implies $W^{(n)} \in (c, c)$ for all n . Furthermore, we obtain (2.53) from (2.50). If $A \in (c_T, Y)$, then (2.54) is an immediate consequence of (2.50).

Conversely, we assume that the conditions in (2.52) and (2.53) are satisfied. Then, $\hat{A}_n = R(A_n) \in c_0^\beta = \ell_1$ and $W^{(n)} \in (c, c)$ together imply $A_n \in (c_T)^\beta$ by Theorem 2.54. Let $z \in c_T$ be given. Then, we have $x = T(z) \in c$. We put $x^{(0)} = x - \xi e$, where $\xi = \lim_{k \rightarrow \infty} x_k$. Then, $x^{(0)} \in c_0$ and it follows from (2.50) that

$$A(z) = \hat{A}(T(z)) - \xi((\alpha_n)_{n=0}^\infty) = \hat{A}(x^{(0)}) + \xi \left(\hat{A}(e) - (\alpha_n)_{n=0}^\infty \right) \in Y$$

since $\hat{A} \in (c_0, Y)$, $\hat{A}(e) - (\alpha_n)_{n=0}^\infty \in Y$ and Y is a linear space.

This completes the proof.

Analogous to Theorem 2.49 (b) we give the operator norm for $A \in (X_T, Y)$.

Theorem 2.57 [14, Theorem 33] *Let T be a triangle. Let X and Y be BK spaces and X have AK . If $A \in (X_T, Y)$, then*

$$\|L_A\| = \|\hat{L}_{\hat{A}}\|, \quad (2.55)$$

where \hat{A} is the matrix defined in Theorem 2.55.

Proof Suppose that $A \in (X_T, Y)$. Since X is a BK space, so is $Z = X_T$ with the norm $\|\cdot\|_Z = \|T(\cdot)\|$ by [3, Theorem 4.3.12, p. 63]. This also means that $x \in B_X(0, 1)$ if and only if $z = S(x) \in B_Z(0, 1)$. Since matrix maps between BK spaces are continuous, it follows that $L_A \in \mathcal{B}(Z, Y)$, and so $\hat{L}_{\hat{A}} \in \mathcal{B}(X, Y)$ by Theorem 2.55. We have by (2.51)

$$\begin{aligned}
\|L_{\hat{A}}\| &= \sup_{x \in B_X(0,1)} \|L_{\hat{A}}(x)\| = \sup_{x \in B_X(0,1)} \|\hat{A}(x)\| \\
&= \sup_{z \in B_Z(0,1)} \|A(z)\| = \sup_{z \in B_Z(0,1)} \|L_A(z)\| = \|L_A\|
\end{aligned}$$

which yields (2.55).

This completes the proof.

Exercises

1. Prove that $\ell_p^* \simeq \ell_q$, i.e., the continuous dual of ℓ_p is ℓ_q for $1 < p < \infty$, $p^{-1} + q^{-1} = 1$.
2. Show that (i) $cs^* \simeq bv$, (ii) $bv^* \simeq bs$.
3. Show that (i) $\Gamma^\alpha = \Lambda$, (ii) $\Lambda^\alpha = \Gamma$, where

$$\Gamma := \left\{ x \in \omega : \lim_n |x_n|^{1/n} = 0 \right\},$$

$$\Lambda := \left\{ x \in \omega : \sup_n |x_n|^{1/n} < \infty \right\}.$$

4. Prove (i) $bv^\beta = cs$, (ii) $bv_0^\beta = bs$, and (iii) $bs^\beta = bv_0$.
5. Show that $m_0^\alpha = \ell_1 = m_0^\beta = m_0^\gamma$, where $m_0 = sp\{E\}$, E is the set of sequences of zeros and ones.
6. Show that $P^\alpha = \ell_1$, $P^\beta = cs$, $P^\gamma = bs$, where

$$P := \{x \in \omega : x_{k+1} = x_k, k \geq 0, \text{ for some } k_0 \in \mathbb{N}\}.$$

7. Show that (i) $m_0^* = \mathcal{M}$, (ii) $\Gamma^* = \Lambda$, (iii) $P^* = \ell_1 \oplus \mathbb{K}$.
8. Show that Γ and Λ are perfect.
9. If $X \supset \phi$ is an FK space, then show that $X^f = (\bar{\phi})^f$ and $(\bar{\phi})' \cong X^f$ by virtue of $f \rightarrow (f(e^{(k)}))$.
10. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be BK spaces with $X \supset \phi$ and $Z = M(X, Y)$. Then prove that Z is a BK space with $\|\cdot\|$ defined by

$$\|z\| = \|z\|_X^* = \sup\{\|xz\|_Y : \|x\|_X = 1\}$$

for all $z \in Z$.

11. Prove that $\sum_{k=1}^\infty x_k y_k$ converges, whenever $\sum_{k=1}^\infty x_k$ has a bounded partial sums, if and only if $(y_k) \in bv \cap c_0$.
12. Let $1 < p < \infty$ and suppose that $A \in (\ell_1, \ell_1) \cap (\ell_\infty, \ell_\infty)$. Then show that $A \in (\ell_p, \ell_p)$.

13. Prove that

- (a) $(c, c; P) \subset (\ell_\infty, \ell_\infty)$,
- (b) $(\ell_\infty, c) \subset (c, c)$,
- (c) $(c, c; P) \subset (c_0, c_0)$.

14. Show that $(c, c; P) \cap (\ell_\infty, c) = \emptyset$.

15. Prove that every matrix in (ℓ_∞, c) is conull.

16. Prove that the Cesàro matrix of order r is a Toeplitz (regular) matrix if $r \geq 0$.

17. Prove that the Euler matrix of order r is a Toeplitz (regular) matrix.

18. Prove that the Riesz matrix R^t is a Toeplitz matrix if and only if $T_n \rightarrow 0$ ($n \rightarrow \infty$).

19. Prove that the Nörlund matrix N^q is a Toeplitz matrix if and only if $\frac{q_n}{Q_n} \rightarrow 0$ ($n \rightarrow \infty$).

20. Prove that the Borel matrix $B = (b_{nk})_{n,k=1}^\infty$ which is defined by $b_{nk} = e^{-n} n^k / k!$ is a Toeplitz matrix.

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