

Chapter 2

Positive Periodic Solutions of Nonlinear Functional Differential Equations with a Parameter λ

In this chapter¹, we provide several different sets of sufficient conditions for the existence of at least three positive periodic solutions to first-order functional differential equations. We will begin by considering the equation

$$x'(t) = -a(t)x(t) + \lambda f(t, x(h(t))) \quad (2.1)$$

in Sect. 2.1. Here, we assume that $R = (-\infty, \infty)$, $R_+ = [0, \infty)$, $T > 0$ is a constant, $\lambda > 0$ is a parameter, $h \in C(R, R)$, $a \in C(R, R_+)$, $a(t) \neq 0$, $a(t + T) = a(t)$, and $f \in C(R \times R_+, R_+)$ is periodic with respect to the first variable with period T .

Let X be a Banach space consisting of all positive T -periodic functions equipped with the sup norm and let K be a positive cone in X . Assume that for any $M > 0$ and $\epsilon > 0$, there exists $\delta > 0$ such that for $u, v \in K$ with $\|u\| \leq M$, $\|v\| \leq M$, and $\|u - v\| < \delta$, we have

$$\|f(t, u) - f(t, v)\| < \epsilon \quad (2.2)$$

uniformly in t .

In Sect. 2.2, we establish the existence of three positive periodic solutions of the differential equation

$$x'(t) = a(t)x(t) - \lambda f(t, x(h(t))) \quad (2.3)$$

by changing the bounds on the Green's kernel. In Sect. 2.3, we present sufficient conditions for the existence of at least three positive periodic solutions of the functional differential equation

$$x'(t) = a(t)x(t) - \lambda b(t)f(t, x(h(t))), \quad (2.4)$$

where λ , a , and f are as above, $b \in C(R, R_+)$ is T -periodic, and $\int_0^T b(t) dt > 0$.

¹ Some of the results in this chapter are based on papers [5–8].

If, in particular, $h(t) = t - \tau(t)$, $\tau \in C(R, R_+)$, $0 \leq \tau(t) \leq t$, then (2.1), (2.3) and (2.4) take the forms

$$x'(t) = -a(t)x(t) + \lambda f(t, x(t - \tau(t))), \quad (2.5)$$

$$x'(t) = a(t)x(t) - \lambda f(t, x(t - \tau(t))), \quad (2.6)$$

and

$$x'(t) = a(t)x(t) - \lambda b(t)f(t, x(t - \tau(t))), \quad (2.7)$$

respectively.

The results obtained in this chapter can be extended to equations with multiple delays such as

$$x'(t) = -a(t)x(t) + \lambda f(t, x(t - \tau_1(t)), \dots, x(t - \tau_m(t))) \quad (2.8)$$

and

$$x'(t) = a(t)x(t) - \lambda f(t, x(t - \tau_1(t)), \dots, x(t - \tau_m(t))), \quad (2.9)$$

where $0 \leq \tau_i(t) \leq t$, $\tau_i(t + T) = \tau_i(t)$, $i = 0, 1, \dots, m$, $f \in C(R \times R_+^m, R_+)$, and $f(t + T, x_1, x_2, \dots, x_m) = f(t, x_1, x_2, \dots, x_m)$.

Functional differential equations of the form (2.5) include many mathematical, ecological, and population models (either directly or after a transformation). For example:

(i) Lasota-Ważewska model

$$x'(t) = -a(t)x(t) + b(t)e^{-\gamma(t)x(t-\tau(t))}; \quad (2.10)$$

(ii) Nicholson's blowflies model

$$x'(t) = -a(t)x(t) + b(t)x^m(t - \tau(t))e^{-\gamma(t)x^n(t-\tau(t))}; \quad (2.11)$$

(iii) Model for red blood cell production

$$x'(t) = -a(t)x(t) + b(t) \frac{x^m(t - \tau(t))}{1 + x^n(t - \tau(t))}. \quad (2.12)$$

Note that Eqs. (2.11) and (2.12) include (1.11) and (1.10) as special cases.

In this chapter, we prove some results on the existence of at least three positive periodic solutions to (2.1), (2.3), and (2.4) by using the Leggett-Williams multiple fixed point theorem (Theorem 1.2.2). We then apply our results to obtain some new criteria for the existence of at least three positive periodic solutions to the models (2.10)–(2.12).

Some explicit intervals on the parameter λ for the existence of solutions are given. We should also point out that the interval on λ changes according to changes in upper

bound on f , that is, an increase in the upper bound on f decreases the range on λ and vice-versa.

To study the existence of periodic solutions, we transform the given equation into an equivalent integral operator. This means that the existence of a positive periodic solution of the differential equation is equivalent to the existence of a fixed point of the operator (see Lemma 2.1.2).

The following notations are used in this chapter:

$$f^h = \limsup_{x \rightarrow h} \max_{0 \leq t \leq T} \frac{f(t, x)}{x}$$

and

$$\tilde{f}^h = \limsup_{x \rightarrow h} \max_{0 \leq t \leq T} \frac{f(t, x)}{a(t)x}.$$

2.1 Positive Periodic Solutions of the Equation

$$x'(t) = -a(t)x(t) + \lambda f(t, x(h(t)))$$

In this section, we present sufficient conditions for the existence of three positive periodic solutions to Eq. (2.1) by using the Leggett-Williams fixed point theorem (Theorem 1.2.2 above). First we observe that Eq. (2.1) is equivalent to

$$x(t) = \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) ds, \quad (2.13)$$

where

$$G(t, s) = \frac{e^{\int_t^s a(\theta) d\theta}}{e^{\int_0^T a(\theta) d\theta} - 1}$$

is the Green's function which satisfies the property

$$0 < \alpha = \frac{1}{\delta - 1} \leq G(t, s) \leq \frac{\delta}{\delta - 1} = \beta, \quad \text{for all } s \in [t, t + T],$$

and $\delta = e^{\int_0^T a(\theta) d\theta}$. Since $a(t) > 0$ for $t \in [0, T]$, we have $\delta > 1$.

Let

$$X = \{x(t) : x(t) \in C(R, R), x(t) = x(t + T)\} \quad (2.14)$$

with $\|x\| = \sup_{t \in [0, T]} |x(t)|$. Then X is a Banach space with the norm $\|\cdot\|$. We define a cone K in X by

$$K = \left\{ x(t) \in X : x(t) \geq \frac{\|x\|}{\delta}, \quad t \in [0, T] \right\} \quad (2.15)$$

and an operator A_λ on X by

$$(A_\lambda x)(t) = \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds. \quad (2.16)$$

Lemma 2.1.1 *The operator A_λ satisfies $A_\lambda(K) \subset K$ and $A_\lambda : K \rightarrow K$ is compact and completely continuous.*

Proof To show $A_\lambda x \in K$, we see that

$$\begin{aligned} (A_\lambda x)(t+T) &= \lambda \int_{t+T}^{t+2T} G(t+T, s) f(s, x(h(s))) \, ds \\ &= \lambda \int_t^{t+T} G(t+T, z+T) f(z+T, x(h(z+T))) \, dz \\ &= \lambda \int_t^{t+T} G(t, z) f(z, x(h(z))) \, dz \\ &= (A_\lambda x)(t). \end{aligned}$$

Hence, $A_\lambda x \in K$. Notice that for $x \in K$, we have

$$\|A_\lambda x\| \leq \lambda \beta \int_0^T f(s, x(h(s))) \, ds \quad (2.17)$$

and

$$(A_\lambda x)(t) \geq \lambda \alpha \int_0^T f(s, x(h(s))) \, ds.$$

The above inequalities imply

$$(A_\lambda x)(t) \geq \frac{\alpha}{\beta} \|A_\lambda x\| \geq \frac{\|A_\lambda x\|}{\delta}.$$

This shows that $A_\lambda x \in K$, that is, $A_\lambda(K) \subset K$.

Next, we show that A_λ is completely continuous. From assumption (2.2) on f , if $x, y \in K$ with $\|x\| \leq M$, $\|y\| \leq M$, and $\|x - y\| < \delta$, then

$$\sup_{0 \leq s \leq T} |f(s, u(h(s))) - f(s, v(h(s)))| < \frac{\epsilon}{\lambda\beta T}.$$

Thus,

$$\begin{aligned} |(A_\lambda x)(t) - (A_\lambda y)(t)| &\leq \lambda \int_t^{t+T} |G(t, s)| |f(s, x(h(s))) - f(s, y(h(s)))| \, ds \\ &\leq \lambda\beta \int_0^T |f(s, x(h(s))) - f(s, y(h(s)))| \, ds \\ &< \lambda\beta \frac{\epsilon}{\lambda\beta T} T < \epsilon. \end{aligned}$$

Hence, A_λ is continuous. Next, we show that f maps bounded sets into bounded sets. Let $\epsilon = 1$. Then again from (2.2), we have

$$|f(s, x(h(s))) - f(s, y(h(s)))| < 1.$$

Choose a positive integer N so that $\frac{M}{N} < \delta$. Let $x \in X$ and define $x_l(t) = x(t)(\frac{l}{N})$ for $l = 0, 1, \dots, N$. If $\|x\| \leq M$, then

$$\|x_l - x_{l-1}\| = \sup_{t \in \mathbb{R}} \left| x(t) \frac{l}{N} - x(t) \frac{l-1}{N} \right| = \frac{\|x\|}{N} \leq \frac{M}{N} < \delta.$$

Hence, $|f(s, x_l(h(s))) - f(s, x_{l-1}(h(s)))| < 1$ for $s \in [0, T]$, and this gives

$$\begin{aligned} |f(s, x(h(s)))| &\leq \sum_{l=1}^N |f(s, x_l(h(s))) - f(s, x_{l-1}(h(s)))| + |f(s, 0)| \\ &\leq N + \|f(s, 0)\| =: M_1. \end{aligned}$$

Now, from (2.17) and the fact that $t \in [0, T]$, we have

$$\|A_\lambda x\| \leq \lambda\beta \int_0^T f(s, x(h(s))) \, ds \leq \lambda\beta T M_1. \quad (2.18)$$

Finally, we have

$$\begin{aligned} \frac{d}{dt}(A_\lambda x)(t) &= \lambda[G(t, t+T)f(t+T, x(h(t+T))) - G(t, t)f(t, x(h(t)))] \\ &\quad - a(t)(A_\lambda x)(t). \end{aligned}$$

From (2.18) and the choice of M_1 , we obtain

$$\left| \frac{d}{dt}(A_\lambda x)(t) \right| \leq \lambda \beta M_1 [\|a\|T + 2].$$

Hence, $\{(A_\lambda x) : x \in K, \|x\| < M\}$ is a family of uniformly bounded and equicontinuous function on $[0, T]$. Thus the operator A_λ is completely continuous by the Ascoli-Arzelà theorem (see Royden [9, p. 169]). This completes the proof of the Lemma. \square

Lemma 2.1.2 *The existence of a positive periodic solution of (2.1) is equivalent to the existence of a fixed point of the operator A_λ in K .*

Proof First, for $x \in K$ and $A_\lambda x = x$, we have

$$\begin{aligned} x'(t) &= \frac{d}{dt} \left(\lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) ds \right) \\ &= \lambda G(t, t+T) f(t+T, x(h(t+T))) - \lambda G(t, t) f(t, x(h(t))) \\ &\quad + \lambda \int_t^{t+T} \frac{\partial}{\partial t} G(t, s) f(s, x(h(s))) ds \\ &= \lambda [G(t, t+T) - G(t, t)] f(t, x(h(t))) - a(t)(A_\lambda x)(t) \\ &= \lambda f(t, x(h(t))) - a(t)x(t), \end{aligned}$$

since $\frac{\partial}{\partial t} G(t, s) = -a(t)G(t, s)$.

Next, if x is a positive T -periodic solution, we have

$$\begin{aligned} (A_\lambda x)(t) &= \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) ds \\ &= \int_t^{t+T} G(t, s) (x'(s) + a(s)x(s)) ds \\ &= \int_t^{t+T} G(t, s) x'(s) ds + \int_t^{t+T} G(t, s) a(s)x(s) ds \\ &= [G(t, s)x(s)]_t^{t+T} - \int_t^{t+T} \left(\frac{\partial}{\partial s} G(t, s) \right) x(s) ds + \int_t^{t+T} G(t, s) a(s)x(s) ds \end{aligned}$$

$$\begin{aligned}
&= [G(t, t+T) - G(t, t)]x(t) - \int_t^{t+T} G(t, s)a(s)x(s) \, ds \\
&\quad + \int_t^{t+T} G(t, s)a(s)x(s) \, ds \\
&= x(t),
\end{aligned}$$

since $\frac{\partial}{\partial s} G(t, s) = a(s)G(t, s)$. Clearly, $(A_\lambda x)(t) \geq 0$ for $t \in [0, T]$, and this completes the proof of the Lemma. \square

In order to prove the existence of a positive T -periodic solution of (2.1), in view of Lemma 2.1.2, it suffices to show that the operator in (2.16) has a fixed point.

Theorem 2.1.1 *Let $f^\infty < 1$ hold. Assume that there are constants $0 < c_1 < c_2$ such that*

(H₁) $f(t, x) \geq 2\delta c_2$ for $x \in K$, $c_2 \leq x \leq \delta c_2$, and $0 \leq t \leq T$, and

(H₂) $f(t, x) < c_1$ for $x \in K$, $0 \leq x \leq c_1$, and $0 \leq t \leq T$.

Then (2.1) has at least three positive T -periodic solutions for

$$\frac{\delta - 1}{2\delta T} < \lambda < \frac{\delta - 1}{\delta T}.$$

Proof We consider the Banach space X defined in (2.14) and the cone K as in (2.15). Since $f^\infty < 1$, there exist $\epsilon \in (0, 1)$ and $\theta > 0$ such that $f(t, x) \leq \epsilon x$ for $x \geq \theta$. Let

$$\gamma = \max_{0 \leq x \leq \theta, 0 \leq t \leq T} f(t, x).$$

Then $f(t, x) \leq \epsilon x + \gamma$ for $x \geq 0$ and $0 \leq t \leq T$. Choose

$$c_4 > \max \left\{ \frac{\gamma}{1 - \epsilon}, \delta c_2 \right\}.$$

Then for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\leq \frac{\lambda \delta}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda\delta}{\delta-1} \int_0^T (\epsilon x(h(s)) + \gamma) \, ds \\
&\leq \frac{\lambda\delta}{\delta-1} \int_0^T (\epsilon \|x\| + \gamma) \, ds \\
&\leq \frac{\lambda\delta(\epsilon c_4 + \gamma)}{\delta-1} T \\
&< c_4.
\end{aligned}$$

Hence $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

Next, define a nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$. Let $c_3 = \delta c_2$ and $\phi_0(t) = \phi_0$, where ϕ_0 is any given number satisfying $c_2 < \phi_0 < c_3$. Then $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$, by (H_1) we have

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\geq \frac{\lambda}{\delta-1} \int_0^T f(s, x(h(s))) \, ds \\
&\geq \frac{\lambda}{\delta-1} 2\delta c_2 T \\
&> c_2.
\end{aligned}$$

Thus, property (i) of Theorem 1.2.2 is satisfied.

Now for $x \in \overline{K}_{c_1}$, (H_2) gives

$$\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\leq \frac{\lambda\delta}{\delta-1} \int_0^T f(s, x(h(s))) \, ds \\
&\leq \frac{\lambda\delta}{\delta-1} c_1 T \\
&< c_1,
\end{aligned}$$

that is, $A_\lambda x \in \overline{K}_{c_1}$. Hence, property (ii) of Theorem 1.2.2 is satisfied.

Finally, for $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{\lambda\delta}{\delta-1} \int_0^T f(s, x(h(s))) ds,$$

which implies that

$$\begin{aligned} \psi(A_\lambda x) &\geq \frac{\lambda}{\delta-1} \int_0^T f(s, x(h(s))) ds \\ &> \frac{c_3}{\delta} \\ &= c_2. \end{aligned}$$

This proves that property (iii) of Theorem 1.2.2 holds. Therefore, (2.1) has at least three positive T -periodic solutions. This completes the proof of the theorem. \square

Theorem 2.1.2 *Let $f^\infty < \frac{1}{\beta}$ hold and assume that there exist $0 < c_1 < c_2$ such that*

(H₃) $f(t, x) \geq \delta(\delta-1)c_2$ for $x \in K$, $c_2 \leq x \leq \delta c_2$, and $0 \leq t \leq T$,
and

(H₄) $f(t, x) < \frac{1}{\beta}c_1$ for $x \in K$, $0 \leq x \leq c_1$, and $0 \leq t \leq T$.

Then Eq. (2.1) has at least three positive T -periodic solutions for

$$\frac{1}{\delta T} < \lambda < \frac{1}{T}.$$

The proof of this theorem is essentially the same as the proof of Theorem 2.1.1, but we include it here for the sake of completeness.

Proof Consider the Banach space X given in (2.14) and the cone K as in (2.15). Since $f^\infty < \frac{1}{\beta}$, there exist $\epsilon \in (0, \frac{1}{\beta})$ and $\theta > 0$ such that $f(t, x) \leq \epsilon x$ for $x \geq \theta$ and $0 \leq t \leq T$. Suppose that

$$\gamma = \max_{0 \leq x \leq \theta, 0 \leq t \leq T} f(t, x).$$

Then, $f(t, x) \leq \epsilon x + \gamma$ for $x \geq 0$ and $0 \leq t \leq T$. Choosing $c_4 > \max \left\{ \frac{\gamma\beta}{1-\beta\epsilon}, \delta c_2 \right\}$ and proceeding as in the proof of the Theorem 2.1.1, it can be shown that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

Next, we define a nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$. Choosing $c_3 = \delta c_2$ and $\phi_0(t) = \phi_0 \in R$ satisfying $c_2 < \phi_0 < c_3$, we see

that $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$ is nonempty. Then, for $x \in K(\psi, c_2, c_3)$, using (H_3) we have

$$\begin{aligned}\psi(A_\lambda x) &\geq \frac{\lambda}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\ &\geq \frac{\lambda}{\delta - 1} \delta(\delta - 1) c_2 T \\ &> c_2.\end{aligned}$$

Now for $x \in \overline{K}_{c_1}$, from (H_4) ,

$$\begin{aligned}\|A_\lambda x\| &\leq \frac{\lambda \delta}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\ &< \frac{\lambda \delta}{\delta - 1} c_1 \frac{\delta - 1}{\delta} T \\ &< c_1.\end{aligned}$$

Furthermore, for $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{\delta}{\delta - 1} \lambda \int_0^T f(s, x(h(s))) \, ds,$$

which in turn implies

$$\begin{aligned}\psi(A_\lambda x) &\geq \frac{1}{\delta - 1} \lambda \int_0^T f(s, x(h(s))) \, ds \\ &> \frac{c_3}{\delta} \\ &= c_2.\end{aligned}$$

Therefore, by Theorem 1.2.2, Eq. (2.1) has at least three positive T -periodic solutions, and this completes the proof of the theorem. \square

Theorem 2.1.3 *Let $f^\infty < \frac{1}{\beta^2}$ and assume that there are constants $0 < c_1 < c_2$ such that*

$$(H_5) \quad f(t, x) \geq \frac{c_2}{\alpha} \text{ for } x \in K, c_2 \leq x \leq \delta c_2 \text{ and } 0 \leq t \leq T$$

and

(H₆) $f(t, x) < \frac{c_1}{\beta^2}$ for $x \in K$, $0 \leq x \leq c_1$ and $0 \leq t \leq T$.

Then Eq. (2.1) has at least three positive T -periodic solutions for

$$\frac{1}{T} < \lambda < \frac{\beta}{T}.$$

Proof Choose the Banach space X and cone K as in (2.14) and (2.15), respectively. Clearly, $f^\infty < \frac{1}{\beta^2}$ implies that there exist $\epsilon \in (0, \frac{1}{\beta^2})$ and $\theta > 0$ such that $f(t, x) < \epsilon x$ for $x \geq \theta$ and $0 \leq t \leq T$. Choose γ as in the proof of Theorem 2.1.1. It is easy to show that $A_\lambda : \bar{K}_{c_4} \rightarrow \bar{K}_{c_4}$, where

$$c_4 > \max \left\{ \frac{\beta^2 \gamma}{1 - \beta^2 \epsilon}, \delta c_2 \right\}.$$

Now, define a nonnegative concave continuous functional ψ on K as in the proof of Theorem 2.1.1. Then for $x \in K(\psi, c_2, c_3)$, from (H₅) we have $\|A_\lambda x\| > c_2$. In addition, for $x \in \bar{K}_{c_1}$, (H₆) implies

$$\begin{aligned} \|A_\lambda x\| &\leq \beta \lambda \int_0^T f(s, x(h(s))) \, ds \\ &< \beta \lambda \int_0^T \frac{c_1}{\beta^2} \, ds \\ &< c_1. \end{aligned}$$

The rest of the proof is the same as that of Theorem 2.1.1. Therefore, (2.1) has at least three positive T -periodic solutions. The theorem is now proved. \square

Theorem 2.1.4 Let $f^\infty < \frac{(\delta-1)^2}{\delta^3}$ and assume there exist constants $0 < c_1 < c_2$ such that (H₅) holds and

(H₇) $f(t, x) < \frac{(\delta-1)^2}{\delta^3} c_1$ for $x \in K$, $0 \leq x \leq c_1$ and $0 \leq t \leq T$.

Then Eq. (2.1) has at least three positive T -periodic solutions for

$$\frac{1}{T} < \lambda < \frac{\delta^2}{(\delta-1)T}.$$

Proof Since $f^\infty < \frac{(\delta-1)^2}{\delta^3}$, there exist $\epsilon \in (0, \frac{(\delta-1)^2}{\delta^3})$ and $\theta > 0$ such that $f(t, x) \leq \epsilon x$ for $x \geq \theta$. Suppose that $\gamma = \max_{0 \leq x \leq \theta, 0 \leq t \leq T} f(t, x)$. Set

$$c_4 > \max \left\{ \frac{\delta^3 T}{(\delta - 1)^2 - \delta^3 \epsilon}, \delta c_2 \right\}.$$

Proceeding along the lines of the proof of Theorem 2.1.1, we can show that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

With the nonnegative concave continuous functional ψ on K defined as in Theorem 2.1.1 and using (H_5) , we have

$$\begin{aligned} \psi(A_\lambda x) &\geq \frac{\lambda}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\ &\geq \frac{\lambda}{\delta - 1} (\delta - 1) c_2 T \\ &> c_2. \end{aligned}$$

Next, using (H_7) , for $x \in \overline{K}_{c_1}$,

$$\begin{aligned} \|A_\lambda x\| &\leq \lambda \frac{\delta}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\ &< \lambda \frac{\delta}{\delta - 1} \frac{(\delta - 1)^2}{\delta^3} c_1 T \\ &< c_1. \end{aligned}$$

The remainder of the proof is similar to the proof of Theorem 2.1.1. Consequently, (2.1) has at least three positive T -periodic solutions and this completes the proof of the theorem. \square

Theorem 2.1.5 Assume that $f^\infty < 1$, $f^0 < 1$, and there is a constant $c_2 > 0$ such that (H_1) holds. Then (2.1) has at least three positive periodic solutions for

$$\frac{\delta - 1}{2\delta T} < \lambda < \frac{\delta - 1}{\delta T}.$$

Proof Since $f^\infty < 1$, there exist $0 < \delta_1 < 1$ and $\xi_1 > 0$ such that

$$f(t, x) \leq \delta_1 x \quad \text{for } x \geq \xi_1.$$

Let $\beta_1 = \max_{0 \leq x \leq \xi_1, 0 \leq t \leq T} f(t, x)$. Then,

$$f(t, x) \leq \delta_1 x + \beta_1 \quad \text{for } 0 \leq x < \infty.$$

Choose

$$c_4 > \max \left\{ \frac{\beta_1}{1 - \delta_1}, \delta c_2 \right\}.$$

For $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda \delta}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda \delta}{\delta - 1} \int_0^T (\delta_1 x(h(s)) + \beta_1) \, ds \\ &\leq \frac{\lambda \delta}{\delta - 1} \int_0^T (\delta_1 \|x\| + \beta_1) \, ds \\ &\leq \frac{\lambda \delta (\delta_1 c_4 + \beta_1)}{\delta - 1} T \\ &< c_4. \end{aligned}$$

This shows $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$. Now setting $c_3 = \delta c_2$, defining the nonnegative concave continuous functional $\psi(x) = \min_{t \in [0, T]} x(t)$, and using (H_1) , we can show that condition (i) of Theorem 1.2.2 holds.

Since $f^0 < 1$, there exist $0 < \delta_2 < 1$ and $0 < \xi_2 < \frac{c_2}{2}$ such that

$$f(t, x) \leq \delta_2 x \quad \text{for } 0 \leq x \leq \xi_2.$$

Set $0 < c_1 = \xi_2$; then for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_0^T G(t, s) f(s, x(h(s))) \, ds \\ &\leq \lambda \delta_2 \frac{\delta}{\delta - 1} c_1 T \\ &\leq \delta_2 c_1 \\ &< c_1. \end{aligned}$$

Thus, condition (ii) of Theorem 1.2.2 is satisfied. The proof of (iii) in Theorem 1.2.2 is easy. Consequently, (2.1) has at least three positive T -periodic solutions, which proves the theorem. \square

Example 2.1.1 Consider the equation

$$x'(t) = -\frac{\log 3}{2} |\sin t| x(t) + \frac{12}{35\pi} e^9 x^3(t) e^{-x(t)}, \quad t \geq 0. \quad (2.19)$$

Here $a(t) = \frac{\log 3}{2} |\sin t|$ and $T = \pi$. Setting $\lambda = \frac{3}{7\pi}$, we see that $f(t, x) = \frac{4}{5} e^9 x^3 e^{-x}$. Clearly, $f^\infty < 1$ and $f^0 < 1$. Furthermore, $\int_0^\pi a(t) dt = \log 3$ implies that $\delta = 3$. Hence, $\alpha = \frac{1}{\delta-1} = \frac{1}{2}$ and $\beta = \frac{\delta}{\delta-1} = \frac{3}{2}$. Choosing $c_2 = 3$, we have $c_3 = \delta c_2 = 9$. Then it is easy to see that

$$f(t, x) = \frac{4}{5} e^9 x^3 e^{-x} > \frac{108}{5} > 18 = 2\delta c_2 \quad \text{for } c_2 \leq x \leq c_3,$$

that is, (H_1) holds. In addition, $\lambda = \frac{3}{7\pi} \in \left(\frac{1}{3\pi}, \frac{2}{3\pi}\right) = \left(\frac{\delta-1}{2\delta T}, \frac{\delta-1}{\delta T}\right)$. Thus, by Theorem 2.1.5, Eq. (2.19) has at least three positive T -periodic solutions.

Remark 2.1.1 In general, it is difficult to obtain a function $f(t, x(t))$ satisfying (H_1) and (H_2) , or (H_3) and (H_4) , or (H_5) and (H_6) , or (H_5) and (H_7) , simultaneously. From Theorem 2.1.5, it is easy to verify that the conditions (H_2) , (H_4) , (H_6) , and (H_7) can be replaced by the conditions $f^0 < 1$, $f^0 < \frac{1}{\beta}$, $f^0 < \frac{1}{\beta^2}$ and $f^0 < \frac{(\delta-1)^2}{\delta^3}$, respectively. One may proceed along the lines of the proof of Theorem 2.1.5 to finish the proof.

Remark 2.1.2 It is not difficult to check that

$$\int_t^{t+T} a(s) G(t, s) ds \equiv 1.$$

This leads us to obtain the following new sufficient conditions for the existence of at least three positive T -periodic solutions of (2.1) using the symbol \tilde{f}^h defined earlier.

Theorem 2.1.6 *Let $\tilde{f}^\infty < T$ hold and assume that there are constants $0 < c_1 < c_2$ such that (H_1) and (H_4) hold. Then (2.1) has at least three positive T -periodic solutions for*

$$\frac{\delta-1}{2\delta T} < \lambda < \frac{1}{T}.$$

Proof From $\tilde{f}^\infty < T$, it follows that there exist $\epsilon \in (0, T)$ and $\theta > 0$ such that $f(t, x) \leq \epsilon a(t)x$ for $x \geq \theta$ and $0 \leq t \leq T$. Let

$$\gamma = \max_{0 \leq x \leq \theta, 0 \leq t \leq T} f(t, x).$$

Then $f(t, x) \leq \epsilon a(t)x + \gamma$ for $x \geq 0$ and $0 \leq t \leq T$. Choose

$$c_4 > \max \left\{ \frac{\delta \gamma T}{(\delta - 1)(T - \epsilon)}, \delta c_2 \right\}.$$

Then, for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\leq \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) (a(s)x(h(s)) \epsilon + \gamma) \, ds \\ &\leq \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) (a(s)\|x\| \epsilon + \gamma) \, ds \\ &\leq \lambda \left[\epsilon c_4 \sup_{0 \leq t \leq T} \int_t^{t+T} a(s) G(t, s) \, ds + \sup_{0 \leq t \leq T} \gamma \int_t^{t+T} G(t, s) \, ds \right] \\ &\leq \lambda \left[\epsilon c_4 + \frac{\gamma \delta}{\delta - 1} T \right] \\ &< \frac{1}{T} \left[\epsilon c_4 + \frac{\gamma \delta}{\delta - 1} T \right] \\ &< c_4. \end{aligned}$$

Hence, $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

Next, we define a nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$. Then $\psi(x) \leq \|x\|$. Let $c_3 = \delta c_2$ and $\phi_0(t) = \phi_0$, where ϕ_0 is any given number satisfying $c_2 < \phi_0(t) < c_3$. Then $\phi_0 \in \{x : x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. Furthermore, for $x \in K(\psi, c_2, c_3)$, from (H_1) we have

$$\begin{aligned} \psi(A_\lambda x) &= \min_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\geq \frac{1}{\delta - 1} \lambda \int_0^T f(s, x(h(s))) \, ds \\ &\geq \frac{\lambda}{\delta - 1} 2 \delta c_2 T \\ &> c_2. \end{aligned}$$

Now, let $x \in \overline{K}_{c_1}$; then, using (H_4) ,

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda \delta}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda \delta}{\delta - 1} c_1 \frac{\delta - 1}{\delta} T \\ &< c_1, \end{aligned}$$

that is, $A_\lambda x \in \overline{K}_{c_1}$.

Finally, for $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{\delta}{\delta - 1} \lambda \int_0^T f(s, x(h(s))) \, ds,$$

which in turn implies that

$$\begin{aligned} \psi(A_\lambda x) &\geq \frac{1}{\delta - 1} \lambda \int_0^T f(s, x(h(s))) \, ds \\ &> \frac{c_3}{\delta} \\ &= c_2. \end{aligned}$$

Hence, all the conditions of Theorem 1.2.2 are satisfied and so Eq. (2.1) has at least three positive T -periodic solutions. This completes the proof of the theorem. \square

Theorem 2.1.7 *Let $\tilde{f}^\infty < T$. Assume that there exist constants $0 < c_1 < c_2$ such that (H_4) and*

$$(H_8) \quad f(t, x) \geq 2(\delta - 1)c_2 \quad \text{for } x \in K, \quad c_2 \leq x \leq \delta c_2 \text{ and } 0 \leq t \leq T.$$

Then (2.1) has at least three positive T -periodic solutions for

$$\frac{1}{2T} < \lambda < \frac{1}{T}.$$

Proof The proof of the theorem is quite similar to proof of Theorem 2.1.6. Here, we use (H_8) in place of (H_1) in the following way to show condition (i) of Theorem 1.2.2. Define ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$ and set $c_3 = \delta c_2$. Then,

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\geq \frac{1}{\delta - 1} \lambda \int_0^T f(s, x(h(s))) \, ds \\
&\geq \frac{\lambda}{\delta - 1} 2(\delta - 1) c_2 T \\
&> c_2.
\end{aligned}$$

Thus, by Theorem 1.2.2, (2.1) has at least three positive T -periodic solutions. \square

Theorem 2.1.8 *Let $\tilde{f}^\infty < T$ and $\tilde{f}^0 < T$. In addition, assume that there exists $c_2 > 0$ such that (H_1) holds. Then there exist at least three positive T -periodic solutions of (2.1) for*

$$\frac{\delta - 1}{2\delta T} < \lambda < \frac{1}{T}.$$

Proof Since $\tilde{f}^\infty < T$, there exist $0 < \delta_1 < T$ and $\xi_1 > 0$ such that

$$f(t, x) \leq \delta_1 a(t)x \quad \text{for } x \geq \xi_1 \text{ and } 0 \leq t \leq T.$$

Let

$$\gamma = \max_{0 \leq x \leq \xi_1, 0 \leq t \leq T} f(t, x).$$

Then $f(t, x) \leq \delta_1 a(t)x + \gamma$ for $x \geq 0$ and $0 \leq t \leq T$, so choosing c_4 as in the proof of Theorem 2.1.6, we can show that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$. Defining a concave continuous functional ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$ and using (H_1) , we can prove that the condition (i) of Theorem 1.2.2 holds.

Now $\tilde{f}^0 < T$ implies there exist $\delta_2, 0 < \delta_2 < T$, and $\frac{c_2}{2} > \xi_2 > 0$ such that

$$f(t, x) \leq \delta_2 a(t)x \quad \text{for } 0 \leq x \leq \xi_2 \text{ and } 0 \leq t \leq T.$$

Set $0 < c_1 = \xi_2$; then $0 < c_1 < c_2$. For $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\leq \lambda \delta_2 \sup_{0 \leq t \leq T} \int_0^T G(t, s) a(s) \|x\| \, ds
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda c_1 \delta_2 \sup_{0 \leq t \leq T} \int_0^T a(s) G(t, s) \, ds \\
&\leq \lambda c_1 \delta_2 \\
&< c_1,
\end{aligned}$$

that is, the condition (ii) of Theorem 1.2.2 is satisfied. In a similar way to what was done in the proof of Theorem 2.1.6, we can show that condition (iii) of Theorem 1.2.2 is satisfied. Hence, there exist at least three positive T -periodic solutions of (2.1) proving the theorem. \square

Theorem 2.1.9 *Let $\tilde{f}^\infty < T$, $\tilde{f}^0 < T$ and (H_8) hold. Then there exist at least three positive T -periodic solutions of (2.1) for*

$$\frac{1}{2T} < \lambda < \frac{1}{T}.$$

Proof Since $\tilde{f}^\infty < T$ and $\tilde{f}^0 < T$, we can proceed as in the proof of Theorem 2.1.8 to prove that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$, and conditions (ii) and (iii) of Theorem 1.2.2 hold. To complete the proof of the theorem, it remains to show the condition (i) of Theorem 1.2.2 is satisfied. We consider the nonnegative concave continuous functional ψ as before. Then, for $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}
\psi(A_\lambda x) &\geq \frac{\lambda}{\delta - 1} \int_0^T f(s, x(h(s))) \, ds \\
&\geq \frac{\lambda}{\delta - 1} 2(\delta - 1)c_2 T \\
&> c_2
\end{aligned}$$

by (H_8) . Hence, condition (i) of Theorem 1.2.2 is satisfied, and this completes the proof of the theorem. \square

In [10], Zhang et al. proved a theorem for the existence of at least two positive T -periodic solutions of (2.8) (see [10, Theorem 3.2]). Applying this theorem to (2.5), we obtain the following result.

Theorem 2.1.10 *Let $\lambda = 1$, $\tilde{f}^0 < 1$, and $\tilde{f}^\infty < 1$. In addition, assume that there exists $\rho > 0$ such that $f(t, x) > a(t)|x|$ for $\mu\rho < |x| < \rho$, where $\mu = \exp\{-\int_0^T a(s) \, ds\}$. Then (2.5) has at least two positive T -periodic solutions x_1 and x_2 such that*

$$0 < \|x_1\| < \rho < \|x_2\|.$$

The following Corollary 2.1.1 follows from Theorems 2.1.8 and 2.1.9.

Corollary 2.1.1 *Let $\tilde{f}^0 < T$ and $\tilde{f}^\infty < T$, and assume that there exists a constant $c_2 > 0$ such that either (H_1) or (H_8) holds. Then (2.5) has at least three positive T -periodic solutions for*

$$\frac{\delta - 1}{2\delta T} < \lambda < \frac{1}{T}.$$

Corollary 2.1.1 is different from Theorem 2.1.10. Indeed, the upper bound on \tilde{f}^0 and \tilde{f}^∞ considered in Corollary 2.1.1 is the general period T , whereas in Theorem 2.1.10 it is 1. However, a range on λ has been given in Corollary 2.1.1.

Theorem 2.1.11 *Let $\tilde{f}^\infty < T$. Assume that there are constants $0 < c_1 < c_2$ such that (H_1) holds and*

$$(H_9) \quad f(t, x) < x \quad \text{for } 0 \leq x \leq c_1 \text{ and } 0 \leq t \leq T.$$

Then there exist at least three positive T -periodic solution of (2.1) for

$$\frac{\delta - 1}{2\delta T} < \lambda < \frac{\delta - 1}{\delta T}.$$

Proof Since $\tilde{f}^\infty < T$, there exist $\epsilon \in (0, T)$ and $\theta > 0$ such that $f(t, x) \leq \epsilon a(t)x$ for $x \geq \theta$ and $0 \leq t \leq T$. Let $\gamma = \max_{0 \leq x \leq \theta, 0 \leq t \leq T} f(t, x)$. Then $f(t, x) \leq \epsilon a(t)x + \gamma$ for $x \geq 0$ and $0 \leq t \leq T$. Now choosing

$$c_4 > \max \left\{ \frac{\delta \gamma T}{\delta(T - \epsilon) + \epsilon}, \delta c_2 \right\},$$

we can prove that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$. However, we need the following argument to prove that condition (ii) of Theorem 1.2.2 holds. From (H_9) we have, for $x \in \overline{K}_{c_1}$

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\leq \lambda \frac{\delta}{\delta - 1} c_1 T \\ &< c_1. \end{aligned}$$

The proof of the condition (iii) of Theorem 1.2.2 is easy and hence is omitted. Thus, (2.1) has at least three positive T -periodic solutions proving the theorem. \square

2.2 Positive Periodic Solutions of the Equation

$$x'(t) = a(t)x(t) - \lambda f(t, x(h(t)))$$

This section is concerned with the existence of at least three positive periodic solutions of Eq. (2.3). In this case, Eq. (2.3) is equivalent to the integral equation (2.13) with a different kernel, namely,

$$G(t, s) = \frac{e^{-\int_t^s a(\theta) d\theta}}{1 - e^{-\int_0^T a(\theta) d\theta}}, \quad (2.20)$$

which has the property

$$0 < \frac{\delta}{1 - \delta} \leq G(t, s) \leq \frac{1}{1 - \delta} \quad \text{for } s \in [t, t + T], \quad (2.21)$$

where $\delta = e^{-\int_0^T a(\theta) d\theta} < 1$.

We consider the Banach space X as in (2.14) and a cone K in X given by

$$K = \{x \in X : x(t) \geq \delta \|x\|, 0 \leq t \leq T\}. \quad (2.22)$$

Defining the operator A_λ by (2.16), we see that $A_\lambda(K) \subset K$. As in Sect. 2.1, it can be proved that the existence of a positive periodic solution of (2.3) is equivalent to the existence of a fixed point of A_λ in the above cone and that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$ is completely continuous.

Theorem 2.2.1 *Assume that $\tilde{f}^\infty < T$ and there exist $0 < c_1 < c_2$ such that*

$$(H_{10}) \quad f(t, x) \geq \frac{c_2}{\delta} \quad \text{for } x \in K, \quad c_2 \leq x \leq \frac{c_2}{\delta} \quad \text{and } 0 \leq t \leq T$$

and

$$(H_{11}) \quad f(t, x) < (1 - \delta)c_1 \quad \text{for } x \in K, \quad 0 \leq x \leq c_1 \quad \text{and } 0 \leq t \leq T.$$

Then Eq. (2.3) has at least three positive T -periodic solutions for

$$\frac{1 - \delta}{T} < \lambda < \frac{1}{T}.$$

Proof Since $\tilde{f}^\infty < T$, there exist $\delta_1 \in (0, T)$ and $\mu > 0$ such that $f(t, x) \leq \delta_1 a(t)x$ for $x \geq \mu$ and $0 \leq t \leq T$. Let $M = \max_{0 \leq x \leq \mu, 0 \leq t \leq T} f(t, x)$. Then $f(t, x) \leq \delta_1 a(t)x + M$ for $x \geq 0$ and $0 \leq t \leq T$. Choose

$$c_4 > \max \left\{ \frac{M T}{(1 - \delta)(T - \delta_1)}, \frac{c_2}{\delta} \right\}.$$

Now for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\leq \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) (a(s)x(h(s)) \delta_1 + M) \, ds \\ &\leq \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) (a(s)\|x\| \delta_1 + M) \, ds \\ &\leq \lambda \left[\delta_1 c_4 \sup_{0 \leq t \leq T} \int_t^{t+T} a(s) G(t, s) \, ds + \sup_{0 \leq t \leq T} M \int_t^{t+T} G(t, s) \, ds \right] \\ &\leq \lambda \left[\delta_1 c_4 + \frac{M}{1 - \delta} T \right] \\ &< c_4. \end{aligned}$$

Hence, $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

Next, we define a nonnegative concave continuous functional ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$. Let $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0$ satisfy $c_2 < \phi_0 < c_3$. This shows $\phi_0 \in \{x : x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. Then, for $x \in K(\psi, c_2, c_3)$, from (H_{10}) , we have

$$\begin{aligned} \psi(A_\lambda x) &= \min_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\ &\geq \frac{\lambda \delta}{1 - \delta} \int_0^T \frac{c_2}{\delta} \, ds \\ &\geq \frac{\delta}{1 - \delta} \frac{c_2 T}{\delta} \lambda \\ &> c_2. \end{aligned}$$

Next, for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\leq \frac{\lambda}{1-\delta} \int_0^T (1-\delta) c_1 \, ds \\
&\leq \frac{\lambda}{1-\delta} (1-\delta) c_1 T \\
&< c_1
\end{aligned}$$

from (H_{11}) . Also, for $x \in K(\psi, c_2, c_4)$ and $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{\lambda}{1-\delta} \int_0^T f(s, x(h(s))) \, ds,$$

which gives

$$\begin{aligned}
\psi(A_\lambda x) &\geq \frac{\lambda \delta}{1-\delta} \int_0^T f(s, x(h(s))) \, ds \\
&> \delta c_3 \\
&= c_2.
\end{aligned}$$

Hence, all the conditions of Theorem 1.2.2 are satisfied, so Eq. (2.3) has at least three positive T -periodic solutions. This proves the theorem. \square

The proofs of Theorems 2.2.2–2.2.4 below are similar to that of Theorem 2.2.1 and hence are omitted.

Theorem 2.2.2 *Let $\tilde{f}^\infty < T$ and assume there are constants $0 < c_1 < c_2$ such that*

$$(H_{12}) \quad f(t, x) \geq \frac{2(1-\delta)}{\delta} c_2 \quad \text{for } x \in K, \quad c_2 \leq x \leq \frac{c_2}{\delta} \quad \text{and } 0 \leq t \leq T$$

and

$$(H_{13}) \quad f(t, x) < \delta(1-\delta)c_1 \quad \text{for } x \in K, \quad 0 \leq x \leq c_1 \quad \text{and } 0 \leq t \leq T.$$

Then (2.3) has at least three positive T -periodic solutions for

$$\frac{1}{2T} < \lambda < \frac{1}{T}.$$

Theorem 2.2.3 *Let $\tilde{f}^\infty < T$, $\tilde{f}^0 < T$, and assume that there exists a constant $c_2 > 0$ such that (H_{10}) holds. Then there exist at least three positive T -periodic solutions of Eq. (2.3) for*

$$\frac{1 - \delta}{T} < \lambda < \frac{1}{T}.$$

Theorem 2.2.4 *Let $\tilde{f}^\infty < T$, $\tilde{f}^0 < T$, and (H_{12}) hold. Then there exist at least three positive T -periodic solutions of (2.3) for*

$$\frac{1}{2T} < \lambda < \frac{1}{T}.$$

Remark 2.2.1 We need to choose

$$c_4 > \max \left\{ \frac{MT}{(1 - \delta)(T - \delta_1)}, \frac{c_2}{\delta} \right\}$$

in the proofs of Theorems 2.2.2–2.2.4 in order to show that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$, where M is given in the proof of Theorem 2.2.1.

Remark 2.2.2 It follows from the range on λ in Theorems 2.2.1–2.2.4 that λ and T depend on each other, that is, $\lambda T < 1$. Hence, if we consider the particular case $\lambda = 1$, then $T < 1$. This observation leads to the following theorem.

Theorem 2.2.5 *Let $\lambda \equiv 1$, $\tilde{f}^\infty < 1$, and $\tilde{f}^0 < 1$. Assume that there exists a constant $c_2 > 0$ such that*

$$(H_{14}) \quad f(t, x) \geq a(t)c_2 \quad \text{for } x \in K, \quad c_2 \leq x \leq \frac{c_2}{\delta} \quad \text{and} \quad 0 \leq t \leq T.$$

Then (2.3) has at least three positive T -periodic solutions.

Proof From the fact that $\tilde{f}^\infty < 1$, we can find $\sigma_1 \in (0, 1)$ and $\xi > 0$ such that $f(t, x) < \sigma_1 a(t)x$ for $x \geq \xi$. Let $\max_{0 \leq x \leq \xi, 0 \leq t \leq T} f(t, x) = \eta$. Then $f(t, x) < \sigma_1 a(t)x + \eta$ for $x \geq 0$ and $0 \leq t \leq T$. Set

$$c_4 > \max \left\{ \frac{\eta T}{(1 - \delta)(1 - \sigma_1)}, \frac{c_2}{\delta} \right\}.$$

Then, using the property that $\int_t^{t+T} G(t, s)a(s) ds \equiv 1$, it can easily be shown that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$. Defining a nonnegative concave continuous functional by $\psi(x) = \min_{t \in [0, T]} x(t)$ and a constant $c_3 = \frac{c_2}{\delta}$, (H_{14}) yields

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0 \leq t \leq T} \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&\geq \min_{0 \leq t \leq T} \int_t^{t+T} G(t, s) a(s) c_2 \, ds \\
&\geq c_2.
\end{aligned}$$

Next, since $\tilde{f}^0 < 1$, there exists a ξ_1 , $0 < \xi_1 < c_2$ such that

$$f(t, x) < a(t)x \quad \text{for } 0 < x < \xi_1.$$

Set $\xi_1 = c_1 < c_2$; then for $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|A_\lambda x\| &= \sup_{0 \leq t \leq T} \int_t^{t+T} G(t, s) f(s, x(h(s))) \, ds \\
&< \sup_{0 \leq t \leq T} \int_t^{t+T} G(t, s) a(s) x \, ds \\
&< c_1.
\end{aligned}$$

Property (iii) of Theorem 1.2.2 is easy to verify. This shows that (2.3) has at least three positive T -periodic solutions and proves the theorem. \square

The next corollary follows from Theorem 2.2.5.

Corollary 2.2.1 *Let $\lambda \equiv 1$, $\tilde{f}^0 = 0$, $\tilde{f}^\infty = 0$, and there exists a constant $c_2 > 0$ such that (H_{14}) holds. Then (2.3) has at least three positive T -periodic solutions.*

Remark 2.2.3 Results derived in this section can be extended to Eq. (2.9) with $\lambda = 1$, that is, the equation

$$x'(t) = a(t)x(t) - f(t, x(t - \tau_1(t)), \dots, x(t - \tau_m(t))), \quad (2.23)$$

where a, τ_i , $1 \leq i \leq m$, and f are as defined in (2.9). Zhang et al. [10] obtained several sufficient conditions for the existence of at least two positive periodic solutions of (2.23) using Krasnosel'skii's fixed point theorem [1, 4]. On the other hand, the above results explain the existence of at least three positive periodic solutions under the same sufficient conditions.

2.3 Positive Periodic Solutions of the Equation

$$x'(t) = a(t)x(t) - \lambda b(t)f(t, x(h(t)))$$

In this section, sufficient conditions are obtained for the existence of at least three positive T -periodic solutions of Eq. (2.4). Observe that (2.4) is equivalent to the integral equation

$$x(t) = \lambda \int_t^{t+T} G(t, s)b(s)f(s, x(h(s))) ds,$$

where $G(t, s)$ given in (2.20) satisfies (2.21). Consider the Banach space X as in (2.14) and a cone K as in (2.22) and define an operator A_λ on X by

$$(A_\lambda x)(t) = \lambda \int_t^{t+T} G(t, s)b(s)f(s, x(h(s))) ds.$$

It is easy to show that $A_\lambda : K \rightarrow K$ is completely continuous and the existence of a positive periodic solution of Eq. (2.4) is equivalent to the existence of a fixed point of the operator A_λ in K .

As above, the Leggett-Williams multiple fixed point Theorem 1.2.2 can be used to prove our results. Moreover, the results hold true if $b(t) \equiv 1$. In this case, the ranges on λ in the results obtained in this section are different from the ones given in Sect. 2.2.

Theorem 2.3.1 *Let $f^\infty < 1$ hold and assume that there are constants $0 < c_1 < c_2$ such that*

$$(H_{15}) \quad f(t, x) \geq \frac{2c_2}{\delta} \quad \text{for } x \in K, \quad c_2 \leq x \leq \frac{c_2}{\delta} \quad \text{and } 0 \leq t \leq T$$

and

$$(H_{16}) \quad f(t, x) < c_1 \quad \text{for } x \in K, \quad 0 \leq x \leq c_1 \quad \text{and } 0 \leq t \leq T.$$

Then (2.4) has at least three positive T -periodic solutions for

$$\frac{1 - \delta}{2 \int_0^T b(t) dt} < \lambda < \frac{1 - \delta}{\int_0^T b(t) dt}.$$

Proof Now $f^\infty < 1$, so there exist $\sigma \in (0, 1)$ and $\xi > 0$ such that $f(t, x) \leq \sigma x$ for $x \geq \xi$ and $0 \leq t \leq T$. If

$$M = \max_{0 \leq x \leq \xi, 0 \leq t \leq T} f(t, x),$$

then $f(t, x) \leq \sigma x + M$ for $x \geq 0$ and $0 \leq t \leq T$. Choose $c_4 > 0$ such that

$$c_4 > \max \left\{ \frac{M}{1 - \sigma}, \frac{c_2}{\delta} \right\} > 0.$$

If $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) b(s) f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda}{1 - \delta} \int_0^T b(s) f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda}{1 - \delta} \int_0^T b(s) (\sigma x(h(s)) + M) \, ds \\ &\leq \frac{\lambda}{1 - \delta} \int_0^T b(s) (\sigma \|x\| + M) \, ds \\ &\leq \frac{\lambda(\sigma c_4 + M)}{1 - \delta} \int_0^T b(s) \, ds \\ &\leq (\sigma c_4 + M) \\ &< c_4. \end{aligned}$$

This shows that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

Now, define a nonnegative continuous concave functional ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$. Let $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0$ be any given number satisfying $c_2 < \phi_0 < c_3$. Then $\phi_0 \in \{x; x \in K(\psi, c_2, c_3), \psi(x) > c_2\}$. For $x \in K(\psi, c_2, c_3)$, (H_{15}) gives

$$\begin{aligned} \psi(A_\lambda x) &\geq \frac{\lambda \delta}{1 - \delta} \int_0^T b(s) f(s, x(h(s))) \, ds \\ &\geq \frac{\lambda \delta}{1 - \delta} \frac{2c_2}{\delta} \int_0^T b(s) \, ds \\ &> c_2. \end{aligned}$$

Next, for $x \in \overline{K}_{c_1}$, (H_{16}) implies

$$\begin{aligned}\|A_\lambda x\| &\leq \frac{\lambda}{1-\delta} \int_0^T c_1 b(s) \, ds \\ &< c_1.\end{aligned}$$

For $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{\lambda}{1-\delta} \int_0^T b(s) f(s, x(h(s))) \, ds,$$

which, in turn implies that

$$\begin{aligned}\psi(A_\lambda x) &\geq \frac{\lambda\delta}{1-\delta} \int_0^T b(s) f(s, x(h(s))) \, ds \\ &> \delta c_3 \\ &= c_2.\end{aligned}$$

Hence, by Theorem 1.2.2, Eq. (2.4) has at least three positive T -periodic solutions. This completes the proof of the theorem. \square

Theorem 2.3.2 *Let $f^\infty < 1$ and $f^0 < 1$. Assume that there exists a constant $c_2 > 0$ such that (H_{15}) holds. Then Eq. (2.4) has at least three positive T -periodic solutions for*

$$\frac{1-\delta}{2 \int_0^T b(t) \, dt} < \lambda < \frac{1-\delta}{\int_0^T b(t) \, dt}.$$

Proof We may proceed along the lines of the proof of Theorem 2.3.1 to prove this result. However, the following argument is needed to show that condition (ii) of Theorem 1.2.2 holds.

Since $f^0 < 1$, there exists a $\xi_1 \in (0, \frac{c_2}{2})$ such that $f(t, x) < x$ for $0 \leq x \leq \xi_1$. Choosing $c_1 = \xi_1$, we observe that $f(t, x) < x$ for $0 \leq x \leq c_1$ and $c_1 < c_2$. Then, for $x \in \bar{K}_{c_1}$,

$$\begin{aligned}\|A_\lambda x\| &\leq \frac{\lambda}{1-\delta} \int_0^T b(s) x(h(s)) \, ds \\ &\leq \frac{\lambda}{1-\delta} \int_0^T b(s) \|x\| \, ds\end{aligned}$$

$$\begin{aligned} &\leq \frac{\lambda}{1-\delta} c_1 \int_0^T b(s) \, ds \\ &< c_1. \end{aligned}$$

□

Theorem 2.3.3 *Let $f^\infty < T$ and assume there are constants $0 < c_1 < c_2$ such that*

$$(H_{17}) \quad f(t, x) \geq \frac{2Tc_2}{\delta} \text{ for } x \in K, \, c_2 \leq x \leq \frac{c_2}{\delta} \text{ and } 0 \leq t \leq T$$

and

$$(H_{18}) \quad f(t, x) < c_1 T \text{ for } x \in K, \, 0 \leq x \leq c_1 \text{ and } 0 \leq t \leq T.$$

Then (2.4) has at least three positive T -periodic solutions for

$$\frac{1-\delta}{2T \int_0^T b(t) \, dt} < \lambda < \frac{1-\delta}{T \int_0^T b(t) \, dt}.$$

Proof Since $f^\infty < T$, there exist $\epsilon \in (0, T)$ and $\xi > 0$ such that $f(t, x) < \epsilon x$ for $x \geq \xi$. Let

$$M = \max_{0 \leq x \leq \xi, 0 \leq t \leq T} f(t, x).$$

Then $f(t, x) < \epsilon x + M$ for $x \geq 0$. Choose $c_4 > 0$ such that

$$c_4 > \max \left\{ \frac{M}{T-\epsilon}, \frac{c_2}{\delta} \right\}.$$

Now for $x \in \overline{K}_{c_4}$, we have

$$\begin{aligned} \|A_\lambda x\| &= \sup_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) b(s) f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda}{1-\delta} \int_0^T b(s) f(s, x(h(s))) \, ds \\ &\leq \frac{\lambda}{1-\delta} \int_0^T b(s) (\epsilon x(h(s)) + M) \, ds \\ &\leq \frac{\lambda}{1-\delta} \int_0^T b(s) (\epsilon \|x\| + M) \, ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\lambda(\epsilon c_4 + M)}{1 - \delta} \int_0^T b(s) \, ds \\
&\leq \frac{(\epsilon c_4 + M)}{T} \\
&< c_4.
\end{aligned}$$

This shows that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$.

Define ψ on K by $\psi(x) = \min_{t \in [0, T]} x(t)$ and let $c_3 = \frac{c_2}{\delta}$ and $\phi_0(t) = \phi_0$ be any given number satisfying $c_2 < \phi_0 < c_3$. Then $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\}$. From (H_{17}) , for $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned}
\psi(A_\lambda x) &\geq \frac{\lambda \delta}{1 - \delta} \int_0^T b(s) f(s, x(h(s))) \, ds \\
&\geq \frac{\lambda \delta}{1 - \delta} \frac{2 T c_2}{\delta} \int_0^T b(s) \, ds \\
&> c_2.
\end{aligned}$$

Next for $x \in \overline{K}_{c_1}$, using (H_{18}) we obtain

$$\begin{aligned}
\|A_\lambda x\| &< \frac{\lambda}{1 - \delta} c_1 T \int_0^T b(s) \, ds \\
&< c_1.
\end{aligned}$$

For $x \in K(\psi, c_2, c_4)$ with $\|A_\lambda x\| > c_3$, we have

$$c_3 < \|A_\lambda x\| \leq \frac{\lambda}{1 - \delta} \int_0^T b(s) f(s, x(h(s))) \, ds,$$

which, in turn implies that

$$\begin{aligned}
\psi(A_\lambda x) &\geq \frac{\lambda \delta}{1 - \delta} \int_0^T b(s) f(s, x(h(s))) \, ds \\
&> \delta c_3 \\
&= c_2.
\end{aligned}$$

Hence, by Theorem 1.2.2, Eq. (2.4) has at least three positive T -periodic solutions and this completes the proof of the theorem. \square

Theorem 2.3.4 *Let $f^\infty < T$, and $f^0 < T$, and assume that there exists a constant $c_2 > 0$ such that (H_{17}) holds. Then (2.4) has at least three positive T -periodic solutions for*

$$\frac{\frac{1-\delta}{T}}{2T \int_0^T b(t) dt} < \lambda < \frac{\frac{1-\delta}{T}}{T \int_0^T b(t) dt}.$$

Corollary 2.3.1 *Let $f^\infty = 0$ and $f^0 = 0$. Assume that there exists a constant $c_2 > 0$ such that (H_{15}) holds. Then (2.4) has at least three positive T -periodic solutions for*

$$\frac{\frac{1-\delta}{T}}{2 \int_0^T b(t) dt} < \lambda < \frac{\frac{1-\delta}{T}}{\int_0^T b(t) dt}.$$

Corollary 2.3.2 *Let $f^\infty = 0$ and $f^0 = 0$. Suppose that there exists a constant $c_2 > 0$ such that (H_{17}) holds. Then there exist at least three positive T -periodic solutions of (2.4) for*

$$\frac{\frac{1-\delta}{T}}{2T \int_0^T b(t) dt} < \lambda < \frac{\frac{1-\delta}{T}}{T \int_0^T b(t) dt}.$$

Remark 2.3.1 (Han and Wang [3]) obtained the following sufficient condition for the existence of two positive periodic solutions for the state-dependent delay differential equation

$$x'(t) = a(t, x(t))x(t) - f(t, x(t - \tau_1(t, x(t))), \dots, x(t - \tau_m(t, x(t)))) \quad (2.24)$$

using fixed point theorem in cones [1]. They assumed that $a \in C(R \times R_+, R)$, $a(t + T, x) = a(t, x)$ for any $(t, x) \in R \times R_+$, $f \in C(R \times [R]^m, R_+)$, $f(t + T, x_1, \dots, x_m) = f(t, x_1, \dots, x_m)$, $\tau_i(t + T, x) = \tau_i(t, x)$ for any $x \in R_+$, $t \in R$, $i = 1, \dots, m$, and $T > 0$ is a constant.

Theorem 2.3.5 Han and Wang [3] *Assume that $a_1(t) \leq a(t, x) \leq a_2(t)$ for any $(t, x) \in R \times R_+$, where a_1 and a_2 are nonnegative T -periodic continuous functions on R and $\int_0^T a_1(s) ds > 0$. Let*

$$\limsup_{|u| \rightarrow +0} \frac{f(t, u_1, u_2, \dots, u_m)}{|u|} < \frac{a_2(t)}{\gamma}$$

and

$$\limsup_{|u| \rightarrow +\infty} \frac{f(t, u_1, u_2, \dots, u_m)}{|u|} < \frac{a_2(t)}{\gamma}$$

uniformly for $t \in R$, where $\gamma = \frac{\exp(\int_0^T a_2(t)dt) - 1}{\exp(\int_0^T a_1(t)dt) - 1}$. Next, suppose that there exists a $\rho > 0$ such that the inequality $\sigma\rho \leq |u| \leq \rho$ yields $f(t, u_1, u_2, \dots, u_m) > a_1(t)\rho\gamma$ for $t \in [0, T]$, where $|u| = \max_i \{u_1, u_2, \dots, u_m\}$ and

$$\sigma = \frac{\left(\inf_{0 \leq t \leq s \leq T} \exp \left(\int_t^s a_1(\theta) d\theta \right) \right) \left(\exp \left(\int_0^T a_1(\theta) d\theta \right) - 1 \right)}{\left(\sup_{0 \leq t \leq s \leq T} \exp \left(\int_t^s a_2(\theta) d\theta \right) \right) \left(\exp \left(\int_0^T a_2(\theta) d\theta \right) - 1 \right)}.$$

Then (2.24) has at least two positive periodic solutions x_1 and x_2 such that $0 < \|x_1\| < \rho < \|x_2\|$.

With $\lambda \equiv 1$, Eq. (2.4) becomes

$$x'(t) = a(t)x(t) - F(t, x(h(t))), \quad (2.25)$$

where $F(t, x) = b(t)f(t, x)$. Applying Theorem 2.3.5 to Eq. (2.25) gives the following result.

Theorem 2.3.6 Let $\limsup_{x \rightarrow 0} \frac{F(t, x)}{x} < a(t)$ and $\limsup_{x \rightarrow \infty} \frac{F(t, x)}{x} < a(t)$ for $0 \leq t \leq T$ and assume there exists $\rho > 0$ such that

(H₁₉) $F(t, x) > \rho a(t)$ for $0 \leq t \leq T$ and $\delta\rho \leq x \leq \rho$,

where $\delta = \exp(-\int_0^T a(\theta) d\theta)$. Then (2.25) has at least two positive T -periodic solutions x_1 and x_2 such that $0 < \|x_1\| < \rho < \|x_2\|$.

Corollary 2.3.3 Let $\limsup_{x \rightarrow 0} \frac{F(t, x)}{x} = 0$, $\limsup_{x \rightarrow \infty} \frac{F(t, x)}{x} = 0$, and assume there exists $\rho > 0$ such that (H₁₉) holds. Then (2.25) has at least two positive periodic solutions.

If we ask that $\frac{1}{2} < \frac{\int_0^T b(t) dt}{1 - \delta} < 1$, then from Corollary 2.3.1 we have the following result.

Corollary 2.3.4 Let $f^\infty = 0$ and $f^0 = 0$. Assume there exists a constant $c_2 > 0$ such that (H₁₅) holds. Then Eq. (2.25) has at least three positive T -periodic solutions.

Similarly, if we assume $\frac{1}{2} < \frac{\int_0^T b(t) dt}{1 - \delta} < 1$, then the following corollary follows from Corollary 2.3.2.

Corollary 2.3.5 Let $f^\infty = 0$ and $f^0 = 0$. Assume there exists a constant $c_2 > 0$ such that (H₁₇) holds. Then Eq. (2.25) has at least three positive T -periodic solutions.

Corollaries 2.3.1–2.3.2 and 2.3.4–2.3.5 extend and improve Corollary 2.3.3. In fact, under very similar condition, Corollary 2.3.3 and Corollary 2.3.4 yield that (2.25) has at least three positive periodic solutions.

Example 2.3.1 Consider

$$x'(t) = \frac{1}{4\pi} \left(\frac{3}{2} + \sin^2 t \right) x(t) - \frac{1}{20\pi} (1 + \cos^2 t) e^5 x^2(t - \tau) e^{-x(t-\tau)}, \quad t \geq 0, \quad (2.26)$$

where $\tau > 0$ is a constant. Here $a(t) = \frac{1}{4\pi}(\frac{3}{2} + \sin^2 t)$, $b(t) = 1 + \cos^2 t$, $T = \pi$, $\delta = e^{-\int_0^\pi a(s) ds} = e^{-1/2}$ and $\int_0^\pi b(t) dt = \frac{3\pi}{2}$. Set $f(t, x) = \frac{1}{\pi} e^5 x^2 e^{-x}$ and $\lambda = \frac{1}{20} = 0.05$. Then $f^\infty = 0 < 1$ and $0.04 = \frac{1-\delta}{2 \int_0^T b(t) dt} < 0.05 = \lambda < 0.08 = \frac{1-\delta}{\int_0^T b(t) dt}$. Set $c_2 = 2$ and $c_1 = 0.02$. Clearly, $f(t, x) = \frac{1}{\pi} e^5 x^2 e^{-x} > \frac{1}{\pi} e^5 c_2^2 e^{-c_2 e^{1/2}}$.

Now, for $c_2 = 2$, we observe that $\frac{1}{\pi} e^5 c_2^2 e^{-c_2 e^{1/2}} > 2c_2 e^{1/2}$. This in turn implies that (H_{15}) holds. Since $f(t, x) = \frac{1}{\pi} e^5 x^2 e^{-x} < \frac{1}{\pi} e^5 c_1^2$ for $0 \leq x \leq c_1$, condition (H_{16}) is satisfied for $c_1 = 0.02$. Also, $0 < c_1 < c_2$. Thus, by Theorem 2.3.1, Eq. (2.26) has at least three positive π -periodic solutions.

Example 2.3.2 By Theorem 2.3.3, the equation

$$x'(t) = \frac{1}{2\pi} (1 + \cos^2 t) x(t) - \frac{1}{50\pi} (1 + \sin^2 t) e^6 x^2(t - \tau) e^{-x(t-\tau)}, \quad t \geq 0$$

has at least three positive π -periodic solutions, where $\tau > 0$ is a constant. Here, we need to choose $c_2 = 1$ and $c_1 = 0.024$.

2.4 Periodic Solutions of State-Dependent Differential Equations

Consider the state-dependent delay differential equation

$$x'(t) = -a(t, x(t))x(t) + \lambda f(t, x(t - \tau_1(t, x(t))), \dots, x(t - \tau_m(t, x(t)))) \quad (2.27)$$

where $\lambda > 0$ is a parameter, $T > 0$ is a constant, $a \in C(R \times R_+, R)$, $a(t + T, x) = a(t, x)$ for any $(t, x) \in R \times R_+$, $f \in C(R \times R_+, R)$, $f(t + T, x_1, x_2, \dots, x_m) = f(t, x_1, x_2, \dots, x_m)$, and $\tau_i(t + T, x) = \tau_i(t, x)$ for any $x \in R_+$, $t \in R$, and $i = 1, 2, \dots, m$. We assume that there exist two nonnegative T -periodic functions $b(t)$ and $c(t)$ such that $b(t) \leq a(t, x) \leq c(t)$ for any $(t, x) \in R \times R_+$ and $\int_0^T b(t) dt > 0$.

We wish to point out that the method applied in this section can also be used to obtain similar results for state-dependent delay differential equations of the form (2.24).

If $a(t, x) = a(t)g(x(t))$ and $\tau_i(t, x(t)) = \tau_i(t)$, $i = 1, 2, \dots, n$, where $g \in C([0, \infty), [0, \infty))$, then Eqs. (2.27) and (2.24) take the forms

$$x'(t) = -a(t)g(x(t))x(t) + \lambda f(t, x(t - \tau_1(t)), x(t - \tau_2(t)), \dots, x(t - \tau_m(t))) \quad (2.28)$$

and

$$x'(t) = a(t)g(x(t))x(t) - \lambda f(t, x(t - \tau_1(t)), x(t - \tau_2(t)), \dots, x(t - \tau_m(t))), \quad (2.29)$$

respectively.

Let $X = \{x(t) : x(t + T) = x(t), t \in \mathbb{R}\}$ and $\|x\| = \max_{0 \leq t \leq T} |x(t)|$. Then X is a Banach space endowed with the norm $\|\cdot\|$. Clearly, x is a positive T -periodic solution of (2.27) if and only if x is a T -periodic solution of the integral equation

$$x(t) = \lambda \int_t^{t+T} G(t, s) f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) ds,$$

where

$$G(t, s) = \frac{\exp\left(\int_t^s a(\theta, x(\theta)) d\theta\right)}{\exp\left(\int_0^T a(\theta, x(\theta)) d\theta\right) - 1}.$$

In view of the above, we define an operator A_λ by

$$A_\lambda x = \lambda \int_t^{t+T} G(t, s) f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) ds \quad (2.30)$$

for every $x \in X$ and $t \in \mathbb{R}$. Clearly $A_\lambda x(t + T) = A_\lambda x(t)$ and $A_\lambda : X \rightarrow X$. The Green's kernel $G(t, s)$ satisfies the inequality

$$\alpha = \frac{1}{\exp\left(\int_0^T c(\theta) d\theta\right) - 1} \leq |G(t, s)| \leq \frac{\exp\left(\int_0^T c(\theta) d\theta\right)}{\exp\left(\int_0^T b(\theta) d\theta\right) - 1} = \beta$$

for every $0 \leq t \leq s \leq t + T$. Let $k_1 = \exp\left(\int_0^T b(\theta) d\theta\right)$ and $k_2 = \exp\left(\int_0^T c(\theta) d\theta\right)$. Then,

$$\alpha = \frac{1}{k_2 - 1}, \quad \beta = \frac{k_2}{k_1 - 1}, \quad k_1 \leq k_2, \quad \text{and} \quad \delta = \frac{\beta}{\alpha} = \frac{k_2(k_2 - 1)}{(k_1 - 1)} > 1.$$

For any $x \in X$, we have

$$\|A_\lambda x\| \leq \frac{\lambda k_2}{k_1 - 1} \int_0^T f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) \, ds$$

and

$$\begin{aligned} (A_\lambda x)(t) &\geq \frac{\lambda}{k_2 - 1} \int_0^T f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) \, ds \\ &\geq \frac{(k_1 - 1)}{k_2(k_2 - 1)} \|A_\lambda x\| = \frac{1}{\delta} \|A_\lambda x\|. \end{aligned}$$

Thus, if we define a cone K on X by

$$K = \left\{ x \in X : x(t) \geq \frac{1}{\delta} \|x\| \right\},$$

then $A_\lambda : K \rightarrow K$. It is easy to show that $A_\lambda : K \rightarrow K$ is completely continuous.

Define

$$f^\theta = \limsup_{|x| \rightarrow \theta} \max_{0 \leq t \leq T} \frac{f(t, x)}{c(t)|x|},$$

where $|x| = \max_{1 \leq i \leq m} \{x_1, x_2, \dots, x_m\}$.

Theorem 2.4.1 *Let $f^0 < T$, $f^\infty < T$, and assume that there exists a constant $c_2 > 0$ such that*

$$f(t, x_1, x_2, \dots, x_m) \geq 2Tk_1 \left(\frac{k_2 - 1}{k_1 - 1} \right)^2 b(t)c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2. \quad (2.31)$$

Then Eq. (2.27) has at least three positive T -periodic solutions for

$$\frac{1}{2T} \frac{k_1 - 1}{k_2 - 1} \leq \lambda \leq \frac{1}{T} \frac{k_1 - 1}{k_2 - 1}.$$

Proof First, suppose that $f^\infty < T$. Then there exist $0 < \epsilon < T$ and $c_3 = \delta c_2 > c_2$ such that

$$f(t, x_1, x_2, \dots, x_m) < c(t)(T - \epsilon)|x| \quad \text{for } |x| > c_3 \text{ and } t \in R.$$

Set $c_4 = \delta c_3$. Clearly, $c_4 \geq \|x\| \geq x \geq \frac{1}{\delta} \|x\|$ for $x \in K \cap \overline{K}_{c_4}$. For $x \in \overline{K}_{c_4}$,

$$\|A_\lambda x\| = \lambda \int_t^{t+T} G(t, s) f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) \, ds$$

$$\begin{aligned}
&\leq \lambda(T - \epsilon) \int_t^{t+T} G(t, s) c(s) \max_{1 \leq i \leq m} |x(s - \tau_i(s, x(s)))| ds \\
&\leq \lambda(T - \epsilon) c_4 \int_t^{t+T} \frac{\exp\left(\int_t^s c(\theta) d\theta\right)}{\exp\left(\int_0^T b(\theta) d\theta\right) - 1} c(s) ds \\
&\leq \lambda T c_4 \frac{k_2 - 1}{k_1 - 1} \leq c_4.
\end{aligned}$$

Since $A_\lambda : K \rightarrow K$, then, in addition to the above, it follows that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$. Define a nonnegative continuous function ψ on K by $\psi(x) = \min_{t \in [0, T]} |x(t)|$. Then $\psi(x) \leq \|x\|$. Let $\phi_0(t) = \phi_0$, where ϕ_0 is any given number satisfying $c_2 < \phi_0 < c_3$. Then, $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$. For $x \in K(\psi, c_2, c_3)$, using (2.31) we obtain

$$\begin{aligned}
\psi(A_\lambda x) &= \min_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) ds \\
&\geq 2Tk_1 \left(\frac{k_2 - 1}{k_1 - 1}\right)^2 c_2 \lambda \int_t^{t+T} \frac{1}{\exp\left(\int_0^T c(\theta) d\theta\right) - 1} b(s) ds \\
&\geq 2Tk_1 \left(\frac{k_2 - 1}{k_1 - 1}\right)^2 c_2 \lambda \int_t^{t+T} \frac{\exp\left(\int_t^s b(\theta) d\theta\right) b(s)}{\exp\left(\int_t^s b(\theta) d\theta\right) \left(\exp\left(\int_0^T c(\theta) d\theta\right) - 1\right)} ds \\
&\geq 2Tk_1 \left(\frac{k_2 - 1}{k_1 - 1}\right)^2 c_2 \lambda \frac{1}{\exp\left(\int_0^T b(\theta) d\theta\right)} \int_t^{t+T} \frac{\exp\left(\int_t^s b(\theta) d\theta\right) b(s)}{\exp\left(\int_0^T c(\theta) d\theta\right) - 1} ds \\
&\geq 2Tk_1 \left(\frac{k_2 - 1}{k_1 - 1}\right)^2 c_2 \lambda \frac{1}{k_1} \cdot \frac{k_1 - 1}{k_2 - 1} \geq c_2.
\end{aligned}$$

Hence, condition (i) of Theorem 1.2.2 is satisfied.

Now $f^0 < T$ implies there are $\epsilon_1 > 0$ and $c_1 < c_2$ such that

$$f(t, x_1, x_2, \dots, x_m) < c_1(T - \epsilon_1)c(t)|x| \quad \text{for } 0 < |x| < c_1.$$

Since $c_1 \geq \|x\| \geq x(t) \geq \frac{1}{\delta} \|x\|$ for any $x \in K \cap \overline{K}_{c_1}$, for any $x \in \overline{K}_{c_1}$, we have

$$\begin{aligned}
\|A_\lambda x\| &\leq \lambda \int_t^{t+T} G(t, s)(T - \epsilon_1)c(s) \max_{0 \leq i \leq m} |x(s - \tau_i(s, x(s)))| \, ds \\
&\leq \lambda(T - \epsilon_1)c_1 \int_t^{t+T} \frac{\exp\left(\int_t^s c(\theta) \, d\theta\right) c(s)}{\exp\left(\int_0^T b(\theta) \, d\theta\right) - 1} \, ds \\
&\leq \lambda T c_1 \left(\frac{k_2 - 1}{k_1 - 1}\right) \leq c_1.
\end{aligned}$$

Thus, condition (ii) of Theorem 1.2.2 holds.

Finally, for any $x \in K(\psi, c_2, c_4)$ and $\|A_\lambda x\| > c_3$, we see that

$$c_3 \leq \|A_\lambda x\| \leq \beta \lambda \int_t^{t+T} f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) \, ds,$$

and it follows that

$$\begin{aligned}
\psi(A_\lambda x) &\geq \alpha \lambda \int_t^{t+T} G(t, s) f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) \, ds \\
&\geq \frac{\alpha}{\beta} c_3 = \frac{c_3}{\delta} = c_2.
\end{aligned}$$

Hence, by Theorem 1.2.2, Eq. (2.27) has at least three positive T -periodic solutions. This completes the proof of the theorem. \square

The following theorem follows from the proof of Theorem 2.4.1.

Theorem 2.4.2 *Let $f^0 < 1$, $f^\infty < 1$, and assume there exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, \dots, x_m) \geq 2k_1 \left(\frac{k_2 - 1}{k_1 - 1}\right)^2 b(t)c_2 \text{ for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.27) has at least three positive T -periodic solutions for

$$\frac{1}{2} \frac{k_1 - 1}{k_2 - 1} \leq \lambda \leq \frac{k_1 - 1}{k_2 - 1}.$$

Theorem 2.4.3 *Let $f^0 < T$, $f^\infty < T$, and assume there exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, \dots, x_m) \geq \frac{\beta}{\alpha} T k_1 \left(\frac{k_2 - 1}{k_1 - 1}\right)^2 b(t)c_2 \text{ for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.27) has at least three positive T -periodic solutions for

$$\frac{\alpha}{\beta T} \frac{k_1 - 1}{k_2 - 1} \leq \lambda \leq \frac{1}{T} \frac{k_1 - 1}{k_2 - 1}.$$

Proof Choose c_3 and c_4 as in the proof of Theorem 2.4.1. Proceeding along the lines of that proof, we can show that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$ and conditions (ii) and (iii) of Theorem 1.2.2 hold. In order to complete the proof of the theorem, we need to verify condition (i) of Theorem 1.2.2. Let $\phi_0(t) = \phi_0$, where ϕ_0 is any number satisfying $c_2 < \phi_0 < c_3$. Then, $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$. For $x \in K(\psi, c_2, c_3)$, we have

$$\begin{aligned} \psi(A_\lambda x) &= \min_{0 \leq t \leq T} \lambda \int_t^{t+T} G(t, s) f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) ds \\ &\geq \lambda T k_1 \frac{\beta}{\alpha} \left(\frac{k_2 - 1}{k_1 - 1} \right)^2 c_2 \int_t^{t+T} \frac{1}{\exp\left(\int_0^T c(\theta) d\theta\right) - 1} b(s) ds \\ &\geq \lambda T k_1 \frac{\beta}{\alpha} \left(\frac{k_2 - 1}{k_1 - 1} \right)^2 c_2 \int_t^{t+T} \frac{\exp\left(\int_t^s b(\theta) d\theta\right) b(s)}{\exp\left(\int_t^s b(\theta) d\theta\right) \left(\exp\left(\int_0^T c(\theta) d\theta\right) - 1\right)} ds \\ &\geq \lambda T k_1 \frac{\beta}{\alpha} \left(\frac{k_2 - 1}{k_1 - 1} \right)^2 c_2 \frac{1}{k_1} \int_t^{t+T} \frac{\exp\left(\int_t^s b(\theta) d\theta\right) b(s)}{\exp\left(\int_0^T c(\theta) d\theta\right) - 1} ds \\ &\geq \lambda T \frac{\beta}{\alpha} \left(\frac{k_2 - 1}{k_1 - 1} \right)^2 c_2 \frac{k_1 - 1}{k_2 - 1} \geq c_2. \end{aligned}$$

Hence, by Theorem 1.2.2, Eq. (2.27) has at least three positive T -periodic solutions, and this proves the theorem. \square

The following theorem follows from the proof of Theorem 2.4.3.

Theorem 2.4.4 Let $f^0 < 1$, $f^\infty < 1$, and assume there exists $c_2 > 0$ such that

$$f(t, x_1, x_2, \dots, x_m) \geq \frac{\beta}{\alpha} k_1 \left(\frac{k_2 - 1}{k_1 - 1} \right)^2 b(t) c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.27) has at least three positive T -periodic solutions for

$$\frac{\alpha}{\beta} \frac{k_1 - 1}{k_2 - 1} \leq \lambda \leq \frac{k_1 - 1}{k_2 - 1}.$$

Theorem 2.4.5 Let $f^0 < T$, $f^\infty < T$, and assume there exists $c_2 > 0$ such that

$$f(t, x_1, x_2, \dots, x_m) \geq T \frac{\beta^2}{\alpha^2} c_2 \int_0^T c(s) \, ds \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.27) has at least three positive T -periodic solutions for

$$\frac{\alpha}{\beta^2 T \int_0^T c(s) \, ds} \leq \lambda \leq \frac{1}{\beta T \int_0^T c(s) \, ds}.$$

Proof Let c_3 and c_4 be as in the proof of Theorem 2.4.1. Following the proof of Theorem 2.4.1 with small modifications, it can be shown that $A_\lambda : \overline{K}_{c_4} \rightarrow \overline{K}_{c_4}$ and conditions (ii) and (iii) of Theorem 1.2.2 hold. Let $\phi_0(t) = \phi_0$, where ϕ_0 is any number satisfying $c_2 < \phi_0 < c_3$. Then $\phi_0 \in \{x \in K(\psi, c_2, c_3) : \psi(x) > c_2\} \neq \emptyset$. In order to apply Theorem 1.2.2, we only need to show that $\psi(A_\lambda x) > c_2$ for all $x \in K(\psi, c_2, c_3)$. Now for $x \in K(\psi, c_2, c_3)$,

$$\begin{aligned} \psi(A_\lambda x) &\geq \lambda \alpha \int_t^{t+T} f(s, x(s - \tau_1(s, x(s))), \dots, x(s - \tau_m(s, x(s)))) \, ds \\ &\geq \lambda \alpha \frac{\beta^2}{\alpha^2} T c_2 \int_0^T c(s) \, ds \\ &\geq \frac{\alpha}{\beta^2 T \int_0^T c(s) \, ds} \alpha \frac{\beta^2}{\alpha^2} T c_2 \int_0^T c(s) \, ds = c_2. \end{aligned}$$

By Theorem 1.2.2, Eq. (2.27) has at least three positive T -periodic solutions. The proof of the theorem is complete. \square

The proof of the following theorem should now be clear and we leave the details to the reader.

Theorem 2.4.6 Let $f^0 < 1$, $f^\infty < 1$, and

$$f(t, x_1, x_2, \dots, x_m) \geq \frac{\beta^2}{\alpha^2} c_2 \int_0^T c(s) \, ds \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.27) has at least three positive T -periodic solutions for

$$\frac{\alpha}{\beta^2 \int_0^T c(s) \, ds} \leq \lambda \leq \frac{1}{\beta \int_0^T c(s) \, ds}.$$

Example 2.4.1 Consider the differential equation

$$x'(t) = -\frac{\log 2}{2\pi}(2 + \sin t) \left(1 + \frac{1}{1+x(t)}\right) x(t) + \frac{1}{11\pi} \left(x(t) + 6e^{23}x^2(t)e^{-x(t)}\right). \quad (2.32)$$

Here $T = 2\pi$ and $a(t, x) = \frac{\log 2}{2\pi}(2 + \sin t) \left(1 + \frac{1}{1+x(t)}\right)$. Setting $b(t) = \frac{\log 2}{2\pi}(2 + \sin t)$ and $c(t) = \frac{\log 2}{\pi}(2 + \sin t)$, it follows that $0 < b(t) \leq |a(t, x)| \leq c(t)$ for $(t, x) \in \mathbb{R} \times \mathbb{R}_+$. Then $k_1 = 4$, $k_2 = 16$, $\left(\frac{k_2-1}{k_1-1}\right)^2 = 25$, and $\delta = 80$. Set $\lambda = \frac{1}{11\pi}$ and $f(t, x) = x + 6e^{23}x^2e^{-x}$. Then $f^0 < 2\pi$ and $f^\infty < 2\pi$. Clearly, $\frac{1}{2T} \frac{k_1-1}{k_2-1} \leq \lambda \leq \frac{1}{T} \frac{k_1-1}{k_2-1}$, that is, $0.0159 \leq 0.02894 \leq 0.03183$. To show that (2.32) has at least three positive T -periodic solutions, we choose $c_2 = \frac{1}{6}$. Then for $c_2 \leq x \leq \delta c_2$, that is, for $\frac{1}{6} \leq x \leq \frac{80}{6}$, we have

$$\begin{aligned} f(t, x) &= x + 6e^{23}x^2e^{-x} \geq c_2 + 6e^{23}c_2^2e^{-\delta c_2} \\ &\geq \frac{1}{6} + \frac{1}{6}e^{23-\frac{80}{6}} \\ &\geq \frac{1}{6} \times 2 \times 2\pi \times 25 \times 4 \times \frac{3 \log 2}{2\pi} \\ &\geq 2Tk_1 \left(\frac{k_2-1}{k_1-1}\right)^2 b(t)c_2. \end{aligned}$$

Hence, by Theorem 2.4.1, Eq. (2.32) has at least three 2π -periodic solutions.

Now, we shall apply the previous theorems in this section to delay differential equations with a parameter of the form (2.28). Similar results can be obtained for (2.29).

We assume that $g \in C([0, \infty), [0, \infty))$ and there are constants $0 < l < L$ such that $l \leq g(x) \leq L$ for $x \geq 0$. Set $\sigma = \exp\left(\int_0^T a(\theta) \, d\theta\right)$, $k_1 = \sigma^l$ and $k_2 = \sigma^L$. We have the following relations:

$$\alpha = \frac{1}{\sigma^L - 1}, \quad \beta = \frac{\sigma^L}{\sigma^l - 1}, \quad \delta = \frac{\beta}{\alpha} = \sigma^L \frac{\sigma^L - 1}{\sigma^l - 1} \quad \text{and} \quad \frac{k_2 - 1}{k_1 - 1} = \frac{\sigma^L - 1}{\sigma^l - 1}.$$

Define

$$\tilde{f}^h = \limsup_{|x| \rightarrow h} \max_{0 \leq t \leq T} \frac{f(t, x)}{a(t)|x|}.$$

Applying Theorems 2.4.1–2.4.6 to Eq. (2.28), we obtain the following results.

Theorem 2.4.7 Let $\tilde{f}^0 < LT$, $\tilde{f}^\infty < LT$, and assume there is a constant $c_2 > 0$ such that

$$f(t, x_1, x_2, \dots, x_m) \geq 2Tl\sigma^l \left(\frac{\sigma^L - 1}{\sigma^l - 1} \right)^2 a(t)c_2 \text{ for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.28) has at least three positive T -periodic solutions for

$$\frac{1}{2T} \frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{1}{T} \frac{\sigma^l - 1}{\sigma^L - 1}.$$

Theorem 2.4.8 Let $\tilde{f}^0 < L$, $\tilde{f}^\infty < L$, and assume there exists $c_2 > 0$ such that

$$f(t, x_1, x_2, \dots, x_m) \geq 2l\sigma^l \left(\frac{\sigma^L - 1}{\sigma^l - 1} \right)^2 a(t)c_2 \text{ for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.28) has at least three positive T -periodic solutions for

$$\frac{1}{2} \frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{\sigma^l - 1}{\sigma^L - 1}.$$

Theorem 2.4.9 Let $\tilde{f}^0 < LT$, $\tilde{f}^\infty < LT$, and assume there exists $c_2 > 0$ such that

$$f(t, x_1, x_2, \dots, x_m) \geq \sigma^{L+l} T \left(\frac{\sigma^L - 1}{\sigma^l - 1} \right)^3 l a(t)c_2 \text{ for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.28) has at least three positive T -periodic solutions for

$$\frac{1}{T\sigma^{L-l}} \frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{1}{T} \frac{\sigma^l - 1}{\sigma^L - 1}.$$

Theorem 2.4.10 Let $\tilde{f}^0 < L$, $\tilde{f}^\infty < L$, and assume there is a constant $c_2 > 0$ such that

$$f(t, x_1, x_2, \dots, x_m) \geq \sigma^{L+l} \left(\frac{\sigma^L - 1}{\sigma^l - 1} \right)^3 l a(t)c_2 \text{ for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.28) has at least three positive T -periodic solutions for

$$\frac{1}{\sigma^{L-l}} \frac{\sigma^l - 1}{\sigma^L - 1} \leq \lambda \leq \frac{\sigma^l - 1}{\sigma^L - 1}.$$

Theorem 2.4.11 *Let $\tilde{f}^0 < LT$, $\tilde{f}^\infty < LT$, and assume there exists $c_2 > 0$ such that*

$$f(t, x_1, x_2, \dots, x_m) \geq \sigma^{2L} c_2 T \left(\frac{\sigma^L - 1}{\sigma^l - 1} \right)^2 L \int_0^T a(s) ds \quad \text{for } x \in K \text{ and}$$

$$c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.28) has at least three positive T -periodic solutions for

$$\frac{(e^l - 1)^2}{T(e^L - 1)e^{2L} L \int_0^T a(\theta) d\theta} \leq \lambda \leq \frac{(e^l - 1)}{Te^L L \int_0^T a(\theta) d\theta}.$$

Theorem 2.4.12 *Let $\tilde{f}^0 < L$, $\tilde{f}^\infty < L$, and assume that there exists a positive constant $c_2 > 0$ such that*

$$f(t, x_1, x_2, \dots, x_m) \geq \sigma^{2L} c_2 \left(\frac{\sigma^L - 1}{\sigma^l - 1} \right)^2 L \int_0^T a(s) ds \quad \text{for } x \in K \text{ and}$$

$$c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.28) has at least three positive T -periodic solutions for

$$\frac{(e^l - 1)^2}{(e^L - 1)e^{2L} L \int_0^T a(\theta) d\theta} \leq \lambda \leq \frac{(e^l - 1)}{e^L L \int_0^T a(\theta) d\theta}.$$

Now we direct our attention to the particular case when $a(t) \equiv a$ is a constant and $g(x) \equiv 1$ with $l = L = 1$. Then $\delta = \sigma = e^{aT}$ and $k_1 = k_2$. Let

$$f^{*\theta} = \frac{1}{a} \limsup_{|x| \rightarrow \theta} \max_{0 \leq t \leq T} \frac{f(t, x)}{|x|}.$$

Applying Theorems 2.4.7–2.4.12 to the equation

$$x'(t) = -ax(t) + \lambda f(t, x(t - \tau)), \quad (2.33)$$

we obtain the following interesting results.

Theorem 2.4.13 *Let $f^{*0} < aT$, $f^{*\infty} < aT$, and assume there exists a constant $c_2 > 0$ such that*

$$f(t, x) \geq 2a\delta c_2 T \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.33) has at least three positive T -periodic solutions for

$$\frac{1}{2T} \leq \lambda \leq \frac{1}{T}.$$

Theorem 2.4.14 Let $f^{*0} < a$ and $f^{*\infty} < a$. If there exists a constant $c_2 > 0$ such that

$$f(t, x) \geq 2a\delta c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2,$$

then Eq. (2.33) has at least three positive T -periodic solutions for

$$\frac{1}{2} \leq \lambda \leq 1.$$

Theorem 2.4.15 Let $f^{*0} < aT$, $f^{*\infty} < aT$, and assume there exists $c_2 > 0$ such that

$$f(t, x) \geq aT\delta^2 c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.33) has at least three positive T -periodic solutions for $\lambda = \frac{1}{T}$.

In particular, for $\lambda = 1$, the next theorem follows from Theorem 2.4.15.

Theorem 2.4.16 Let $f^{*0} < a$, $f^{*\infty} < a$, and assume there exists a constant $c_2 > 0$ such that

$$f(t, x) \geq a\delta^2 c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.33) has at least three positive T -periodic solutions for $\lambda = 1$.

Theorem 2.4.17 Let $f^{*0} < aT$ and $f^{*\infty} < aT$. If there exists $c_2 > 0$ such that

$$f(t, x) \geq aT^2\delta^2 c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2,$$

then Eq. (2.33) has at least three positive T -periodic solutions for

$$\frac{e-1}{e^2 a T^2} \leq \lambda \leq \frac{e-1}{e a T^2}.$$

Theorem 2.4.18 Let $f^{*0} < a$, $f^{*\infty} < a$, and assume there exists $c_2 > 0$ such that

$$f(t, x) \geq aT\delta^2 c_2 \quad \text{for } x \in K \text{ and } c_2 \leq |x| \leq \delta c_2.$$

Then Eq. (2.33) has at least three positive T -periodic solutions for

$$\frac{e-1}{e^2 a T} \leq \lambda \leq \frac{e-1}{e a T}.$$

2.5 Applications to Some Mathematical Models

In this section, we apply some of the results obtained in Sect. 2.1 to models of the form of (2.10)–(2.12). In what follows, all the parameters in models (2.10)–(2.12) are assumed to be positive constants.

Example 2.5.1 If $a(t) \equiv a$, $b(t) \equiv b$, $\tau(t) \equiv \tau$, and $\gamma(t) \equiv \gamma$ are positive constants, then (2.10) reduces to

$$x'(t) = -ax(t) + be^{-\gamma x(t-\tau)}. \quad (2.34)$$

Graef et al. [2] and Zhang et al. [10] proved that (2.34) has at least one positive periodic solution. However, to the best of our knowledge, there is no such result for the existence of at least three positive periodic solutions of (2.34). It would be of interest to obtain such results. The following result follows from Theorem 2.1.7.

Theorem 2.5.1 *Let $\gamma < 2e$, $\delta \leq \frac{2e}{2e-\gamma}$, and $\gamma\delta^2 < \delta - 1$ hold, where $\delta = e^{aT}$. Then (2.34) has at least three positive T -periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$.*

Proof Let $f(t, x) = e^{-\gamma x}$. Then $f(t, x) > e^{-\gamma\delta c_2}$ for $c_2 \leq x \leq \delta c_2$, where $\delta = e^{aT}$. Thus, (H_8) holds if and only if $e^{-\gamma\delta c_2} \geq 2(\delta - 1)c_2$ for $c_2 \leq x \leq \delta c_2$. Now choose $c_2 = \frac{1}{\delta\gamma}$. Then $\delta \leq \frac{2e}{2e-\gamma}$ and $c_2 = \frac{1}{\delta\gamma}$ imply that $e^{-\gamma\delta c_2} \geq 2(\delta - 1)c_2$ for $c_2 \leq x \leq \delta c_2$. Hence (H_8) is satisfied. It is clear that $\tilde{f}^\infty < T$. In order to apply Theorem 2.1.7, we need to show the existence of a constant c_1 such that $0 < c_1 < c_2$ and (H_4) holds. Since $f(t, x) < 1$, (H_4) holds if $c_1 > \frac{\delta}{\delta-1}$. Indeed, (H_4) holds if $1 < \frac{\delta-1}{\delta}c_1$ for $0 \leq x \leq c_1$, that is, $c_1 > \frac{\delta}{\delta-1}$. Now we show the existence of c_1 . Clearly, $\gamma\delta^2 < \delta - 1$ implies that $\frac{\delta}{\delta-1} < \frac{1}{\gamma\delta} = c_2$. Thus, there exists a real $c_1 \in (\frac{\delta}{\delta-1}, \frac{1}{\gamma\delta})$ such that $\frac{\delta}{\delta-1} < c_1 < c_2 = \frac{1}{\gamma\delta}$, so $f(t, x)$ satisfies (H_4) . Hence, by Theorem 2.1.7, Eq. (2.34) has at least three positive T -periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$. This proves the theorem. \square

Example 2.5.2 If $a(t) \equiv a$, $b(t) \equiv b$, $\tau(t) \equiv \tau$, and $\gamma(t) \equiv \gamma$ are positive constants, then (2.11) reduces to

$$x'(t) = -ax(t) + bx^m(t - \tau)e^{-\gamma x^n(t-\tau)}. \quad (2.35)$$

Theorem 2.5.2 *Let $m > 1$ and $2e(\delta - 1)\delta^{(m-1)}\gamma^{\frac{m-1}{n}} \leq 1$. Then Eq. (2.35) has at least three positive T -periodic solutions for*

$$\frac{1}{2T} < b < \frac{1}{T}.$$

Proof Let $f(t, x) = x^m e^{-\gamma x^n}$ and set $c_2 = \frac{1}{\delta\gamma^{1/n}}$. Then it is easy to observe that $c_3 = \frac{1}{\gamma^{1/n}}$, and $2e(\delta - 1)\delta^{(m-1)}\gamma^{\frac{m-1}{n}} \leq 1$ imply that $c_2^m e^{-\gamma\delta^n c_2^n} > 2(\delta - 1)c_2$ for $c_2 \leq x \leq \delta c_2$. Hence, (H_8) is satisfied. Moreover, $\tilde{f}^\infty = 0 < T$ and $\tilde{f}^0 = 0 <$

T hold. Then, by Theorem 2.1.9, Eq. (2.35) has at least three positive T -periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$. \square

Although, the condition in Theorem 2.5.2 looks complicated, it is easy to verify. The following corollary follows from Theorem 2.5.2.

Corollary 2.5.1 *Let $m > 1$ and $\delta < \min \left\{ \frac{1}{\gamma^{1/n}}, \frac{1+2e}{2e} \right\}$. Then (2.35) has at least three positive T -periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$.*

Proof In fact, $\delta < \min \left\{ \frac{1}{\gamma^{1/n}}, \frac{1+2e}{2e} \right\}$ implies that $2e(\delta - 1)\delta^{(m-1)}\gamma^{\frac{m-1}{n}} \leq 1$ and hence by Theorem 2.5.2, (2.35) has at least three positive T -periodic solutions for $\frac{1}{2T} < b < \frac{1}{T}$. This completes the proof. \square

Example 2.5.3 If $a(t) \equiv a$, $b(t) \equiv b$, and $\tau(t) \equiv \tau$, are positive constants, and $m = 1$, then (2.12) reduces to

$$x'(t) = -ax(t) + b \frac{x(t - \tau)}{1 + x^n(t - \tau)}. \quad (2.36)$$

Applying Theorem 2.1.9 to Eq. (2.36) we have the following result.

Theorem 2.5.3 *Let $e^{aT} < \frac{3}{2}$ and $T > 1$. Then (2.36) has at least three positive T -periodic solutions for*

$$\frac{1}{2T} < b < \frac{1}{T}.$$

Proof Let $f(t, x) = \frac{x}{1+x^n}$. Then $\tilde{f}^\infty < T$ and $\tilde{f}^0 = 1 < T$. Choose $c_2 = \frac{1}{\delta} \left[\frac{1}{2(\delta-1)} - 1 \right]^{\frac{1}{n}}$. Since $e^{aT} < \frac{3}{2}$, that is, $\delta < 3/2$, we have $c_2 > 0$. In addition, $c_2 = \frac{1}{\delta} \left[\frac{1}{2(\delta-1)} - 1 \right]^{\frac{1}{n}}$ implies that $\frac{1}{1+\delta^n c_2^n} = 2(\delta - 1)$ and hence (H_8) holds. Then, by Theorem 2.1.9, (2.36) has at least three positive T -periodic solutions for

$$\frac{1}{2T} < b < \frac{1}{T}. \quad \square$$

Similarly, applying Theorem 2.1.9 to the autonomous equation

$$x'(t) = -ax(t) + b \frac{x(t - \tau)}{r + x^n(t - \tau)}, \quad (2.37)$$

we obtain the following theorem.

Theorem 2.5.4 *Let $rT > 1$ and $e^{aT} < \frac{3}{2}$. Then (2.37) has at least three positive T -periodic solutions for*

$$\frac{a}{2T} < b < \frac{a}{T},$$

where $\tau > 0$ is a constant.

Example 2.5.4 If $a(t) \equiv a$, $b(t) \equiv b$ and $\tau(t) \equiv \tau$ are constants, then (2.12) reduces to

$$x'(t) = -ax(t) + b \frac{x^m(t - \tau)}{1 + x^n(t - \tau)}. \quad (2.38)$$

Set

$$\mu = 2(\delta - 1)\delta^{2m-1} \frac{n}{1 + n - m} \left(\frac{1 + n - m}{m - 1} \right)^{\frac{m-1}{n}}. \quad (2.39)$$

Applying Theorem 2.1.9 to Eq. (2.38), we obtain the following result.

Theorem 2.5.5 Let $0 < m - 1 < n$. Eq. (2.38) has at least three positive T -periodic solutions for $\frac{\mu}{2T} < b < \frac{\mu}{T}$, where μ is given in (2.39).

Proof Now $\delta > 1$ and $0 < m - 1 < n$ implies $\mu > 0$. Eq. (2.38) can be written as

$$x'(t) = -ax(t) + \frac{b}{\mu} \mu \frac{x^m(t - \tau)}{1 + x^n(t - \tau)}. \quad (2.40)$$

Let $f(t, x) = \mu \frac{x^m}{1 + x^n}$. Since $m > 1$, $\tilde{f}^0 = 0 < T$ and $\tilde{f}^\infty = 0 < T$. To complete the proof of the theorem, in view of Theorem 2.1.9, we need to find $c_2 > 0$ such that (H_8) holds. Set $c_2 = \frac{1}{\delta} \left(\frac{m-1}{1+n-m} \right)^{\frac{1}{n}}$. Now, for $c_2 \leq \|x\| \leq \delta c_2$, we have

$$\mu \frac{x^m}{1 + x^n} \geq \mu \frac{(\|x\|/\delta)^m}{1 + \delta^n c_2^n} \geq \frac{\mu}{\delta^m} \frac{c_2^m}{1 + \delta^n c_2^n} \quad (2.41)$$

and $1 + \delta^n c_2^n = \frac{n}{1+n-m}$. Then, from (2.41) and (2.39),

$$\begin{aligned} \mu \frac{x^m}{1 + x^n} &\geq \frac{c_2^m}{\delta^m} \frac{n - m + 1}{n} 2(\delta - 1)\delta^{2m-1} \frac{n}{n - m + 1} \left(\frac{1 + n - m}{m - 1} \right)^{\frac{m-1}{n}} \\ &\geq 2(\delta - 1)c_2^m \delta^{m-1} \left(\frac{1 + n - m}{m - 1} \right)^{\frac{m-1}{n}} \\ &\geq 2(\delta - 1)c_2^m \delta^{m-1} \frac{1}{\delta^{m-1} c_2^{m-1}} \\ &\geq 2(\delta - 1)c_2. \end{aligned}$$

This completes the proof of the theorem. \square

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