

Chapter 2

Introduction to Linear Programming

The key takeaways for the reader from this chapter are listed below:

- A good understanding of linear programming (LP) problems
- Formulation of the two-variable LP problem
- Understanding optimization in the contexts of minimization and maximization objective functions
- Representing a two-variable LP model graphically.

2.1 Introduction

LP is an optimization model in which the objective functions and constraints are strictly linear. It is used in a wide range of areas such as agriculture, transportation, economics, and industry. The advent of computers has made it the backbone of solution algorithms for other OR models including integer, stochastic, and non-linear programming.

In this chapter, we discuss a two-variable LP model and present its graphical solution. As we already know, any LP model will contain an objective function, set of constraints, and non-negativity restrictions.

Each of the components may involve one or more of the following:

- Decision variables
- Objective function coefficients
- Technical coefficients
- Resources availability.

2.2 Two-Variable LP Model

In this section, a graphical solution for a two-variable LP model is presented, though in reality there will be more than two variables in any situation. A typical product mix problem is considered as an example.

2.2.1 Example: RM Company

Illustration 2.1

RM company produces two products, P_1 and P_2 , from two raw materials, R_1 and R_2 . Typical examples can be: Product P_1 produced from a milling machine using R_1 and R_2 , and product P_2 produced from a drilling machine using the same raw materials R_1 and R_2 . The basic data of the problem are given below (Table 2.1).

According to a market survey, the daily consumption of P_1 cannot exceed that of P_2 by more than 1 ton. Further, the maximum daily demand of P_2 is 2 tons.

It is required to determine the optimum (best) amounts of P_1 and P_2 that maximize the total daily profits.

Solution

Any OR model, including an LP model as mentioned in Chap. 1 has three basic components.

- Decision variable
- Objective (Goal)
- Constraints that need to be satisfied.

In the present case, the decision variables to be determined are the amounts of P_1 and P_2 . We can define them as

X_1 —Tons produced daily of P_1 .

X_2 —Tons produced daily of P_2 .

(Note A company would always like to maximize the profits and minimize the costs. Hence, the objective function would be *Maximize* in case of profits and *Minimize* in case of costs.)

According to the problem, the company wants to maximize its profits. Hence, the objective function can be formulated as

$$\text{Maximize } Z = 5X_1 + 4X_2$$

The constraints are with respect to raw material and demand. The raw material constraint can be verbally expressed as

$$\left(\begin{array}{c} \text{Usage of raw material} \\ \text{by both } P_1 \text{ and } P_2 \end{array} \right) \leq \left(\begin{array}{c} \text{maximum of raw material} \\ \text{availability} \end{array} \right)$$

For the present problem,

$$\text{Usage of raw material } R_1/\text{day} = 3x_1 + 2x_2 \text{ tons}$$

$$\text{Usage of raw material } R_2/\text{day} = 1x_1 + 2x_2 \text{ tons}$$

Table 2.1 The basic data of the problem

	Tons of raw material per ton of		Max daily availability (tons)
	P ₁	P ₂	
Raw material R ₁	3	2	12
Raw material R ₂	1	2	6
Profit/ton (Rs. 000)	5	4	

Daily availability of R₁ and R₂ is limited to 12 and 6 tons. Hence, the constraints can be expressed as

$$3X_1 + 2X_2 \leq 12(R/M R_1) \quad (2.1)$$

$$1X_1 + 2X_2 \leq 6(R/M R_2) \quad (2.2)$$

The demand constraints of daily demand of P₁ cannot exceed that of P₂ by more than 1 ton, which translates to

$$X_2 - X_1 \leq 1 \quad (2.3)$$

Similarly, the second demand constraint of the maximum daily demand of P₂ is 2 tons, which translates to

$$X_2 \leq 2 \quad (2.4)$$

X₁ and X₂ cannot assume negative values. The nonnegativity constraints, therefore, are $x_1, x_2 \geq 0$ and the complete RM company model can be written as

$$\begin{aligned}
 &\text{Maximize } Z = 5X_1 + 4X_2 \\
 &\text{subject to (s.t)} \quad 3X_1 + 2X_2 \leq 12 \\
 &\quad \quad \quad 1X_1 + 2X_2 \leq 6 \\
 &\quad \quad \quad X_2 - X_1 \leq 1 \\
 &\quad \quad \quad X_2 \leq 2 \\
 &\quad \quad \quad X_1, X_2 \geq 0
 \end{aligned}$$

Any values of X₁ and X₂ that satisfy all the above constraints constitute a feasible solution. For example: X₁ = 2 and X₂ = 1 is a feasible solution, since all the constraints are satisfied, including the nonnegativity constraints.

The optimum feasible solution yields maximum total profit while satisfying all the constraints. There can be a number of feasible solutions to a problem but finding the optimum solution requires following a systematic procedure, which will be explained in [Chap. 3](#). The next solution gives the optimum solution obtained using the graphical method. In the above example, the objective function

and constraints are all linear. Linearity means any LP must satisfy two properties: proportionality and additivity.

Proportionality requires that the contribution of each decision variable in the objective function and its requirements in the constraints to be directly proportional to the variable. In the RM Company model, $5X_1$ and $4X_2$ give the profits for producing X_1 and X_2 tons of P_1 and P_2 , respectively, with unit profits per ton, 5 and 4, providing the constants of proportionality. Suppose RM Company gives quantity discounts when sales exceed certain amounts, then profit will no longer be proportional to X_1 and X_2 .

Additivity requires that the total contribution of all the variables in the objective function and their requirements in the constraints are the direct sum of the individual contributions or requirements of each variable. In the RM Company model, the total profit equals the sum of the individual profit components. Suppose the two products compete for the same market share such that increase in sales of one adversely affects the other, then additivity is not satisfied.

Illustration 2.2

For the RM Company Model, construct each of the following constraints and express them with a constant right-hand side:

- The daily demand for P_2 exceeds that of P_1 by at least 1 ton.
- The daily usage of raw material is at most 6 tons and at least 3 tons.
- The demand for P_1 cannot be less than the demand for P_2 .
- The minimum quantity that should be produced of both the paints P_1 and P_2 is 3 tons.
- The proportion of interior paint to the total production of both the paints P_1 and P_2 must not exceed 5.

Solution

All the formulations are given below:

- Here, the daily demand for P_2 exceeds that of P_1 at least by 1 ton, so we have the formulation as

$$X_2 - X_1 \geq 1 \text{ or } -X_1 + X_2 \geq 1$$

- The daily usage of raw material is at most 6 tons $-X_1 + X_2 \leq 6$. The daily usage of raw material is at least 3 tons $-X_1 + 2X_2 \geq 3$.
- If the demand for P_1 cannot be less than P_2 , we have the formulation as $X_2 - X_1 = 0$ or $X_2 = X_1$.
- Here, it is given that the minimum quantity to be produced is 3 tons. We can formulate this as $X_1 - X_2 \geq 3$.
- The proportion can be formulated as $X_2/X_1 + X_2 = 0.5$ or $0.5X_1 - 0.5X_2 \geq 0$.

Illustration 2.3

Suppose it is asked to determine the best “feasible” solution.

Determine the best feasible solution among the following (feasible and infeasible) solutions of the RM Company model:

- (a) $X_1 = 1, X_2 = 4$
- (b) $X_1 = 2, X_2 = 2$
- (c) $X_1 = 3, X_2 = 1.5$
- (d) $X_1 = 2, X_2 = 1$
- (e) $X_1 = 2, X_2 = -1$.

Solution

In order to find the best feasible solution, we have to substitute all the above values of X_1 and X_2 in all the constraint equations of the RM Company Model. The obtained values will have to be less than the right-hand side values of the constraint equations. Then, the solution will be graded as feasible.

(a)

$$X_1 = 1, X_2 = 4$$

Substituting the above values in the constraint equations, we get

$$6.1 + 4.4 = 22(<24)$$

$$1.1 + 2.4 = 9(>6)$$

We can see above that the rule for feasibility is violated in the second constraint as the obtained value is more than the right-hand side value of the equation. Hence, this is infeasible.

(b)

$$X_1 = 2, X_2 = 2$$

Substituting the above values in the constraint equations, we get

$$6.2 + 4.2 = 20(<24)$$

$$1.2 + 2.2 = 6(=6)$$

$$-1.2 + 1.2 = 0(<1)$$

$$1.2 = 2(=2)$$

Here, we see that all feasibility rules are satisfied for all the constraint equations. Hence, this is a feasible solution.

The objective function value will be $Z = 5.2 + 4.2 = 18$.

(c)

$$X_1 = 3, X_2 = 1.5$$

Substituting the above values in the constraint equations, we get

$$\begin{aligned} 6.3 + 4.1.5 &= 24(=24) \\ 1.3 + 2.1.5 &= 6(=6) \\ -1.3 + 1.1.5 &= -1.5(<1) \\ 1.1.5 &= 1.5(<2) \end{aligned}$$

Here, we again see that all feasibility rules are satisfied for all the constraint equations. Hence, this is also a feasible solution.

The objective function value will be $Z = 5.3 + 4.1.5 = 21$.

(d)

$$X_1 = 2, X_2 = 1$$

Substituting the above values in the constraint equations, we get

$$\begin{aligned} 6.2 + 4.1 &= 16(<24) \\ 1.2 + 2.1 &= 4(<6) \\ -1.2 + 1.1 &= -1(<1) \\ 1.1 &= 1(<2) \end{aligned}$$

Here, again all feasibility rules are satisfied for all the constraint equations. Hence, this is also a feasible solution.

The objective function value will be $Z = 5.2 + 4.1 = 14$.

(e)

$$X_1 = 2, X_2 = -1$$

Here, we see that the value of $X_2 = -1$, i.e., negative. So, we can straightaway say that this is an infeasible solution

From all the above calculations, we conclude that option *c* is the best “feasible” solution as the value of the objective function is maximum, i.e., maximum profits are obtained by having $X_1 = 3$, and $X_2 = 1.5$.

Illustration 2.4

A state government owned farm measuring about 150 acres sells all the ladies fingers, radishes, and onions it can produce. The price obtained is Rs. 1.5/kg for ladies fingers and onions, and Rs. 3/kg for radishes. Wages are paid to the farmers at Rs. 25 per man-day. It is required for the farmers to put in 4 man-days for ladies fingers, 5 man-days for onions, 6 man-days for radishes, and a total of 380 man-days of effort to be put in for the entire production. Fertilizers required for the same are available at Rs. 0.60/kg, and the quantity required per acre is 110 kgs for ladies fingers and onions, and 75 kgs for radishes. It is also required that the average yield per acre should be 2,500 kgs for ladies fingers, 1,500 kgs for onions, and 2,000 kgs for radishes. Formulate the above as an LP so as to give the state government maximum profits.

Solution

The above problem gives many details, but the details to be extracted for the purpose of formulation are average yield per acre, measurement of the land, man-days needed for production, cost of labor, and any other costs, if specified.

$$\text{Profit} = \text{Total Sales Value} - \text{Total Cost}$$

Total cost here accounts for labor cost and fertilizer cost.

$$\text{Hence } [\text{Profit} = \text{Total Sales Value} - \text{Total Cost (Labor + Fertilizer)}]$$

Now, the equations can be arrived at.

$$\text{For ladies fingers, profit} = 2,500 - (25.4 + 110.0.6) = 2,334$$

$$\text{For onions, profit} = 1,500 - (25.5 + 110.0.6) = 1,309$$

$$\text{For radishes, profit} = 2,000 - (25.6 + 75.0.6) = 2,805$$

Hence, the objective function becomes

$$\text{maximize} = 2334X_1 + 1309X_2 + 2805X_3$$

$$\text{subject to } X_1 + X_2 + X_3 \leq 150 \text{ (Land constraint)}$$

$$4X_1 + 5X_2 + 6X_3 \leq 380 \text{ (Labor constraint)}$$

$$X_1, X_2, X_3 \geq 0$$

Illustration 2.5

A soft drink plant has two bottling machines, Y and Z. Y is designed for 8-ounce bottles and Z for 16-ounce bottles. However, each can be used on both types with some loss in efficiency. The following data are made available (Table 2.2).

The machine can be run for 8 h a day, and 5 days a week. The profit is 15 and 25 paise on 8-ounce and 16-ounce bottles, respectively. The plant has to limit the

Table 2.2 Data obtained by the two bottling machines, Y and Z

	8-ounce bottles	16-ounce bottles
Y	100/min	40/min
Z	60/min	75/min

weekly production to 3,00,000 ounces, and the market can absorb 25,000 8-ounce and 7,000 16-ounce bottles per week. Formulate the above problem as an LP so as to maximize the plants profits subject to all production and marketing restrictions.

Solution

Let X_1 units of 8-ounce bottles and X_2 units of 16-ounce bottles be produced. As the profits are known, the objective functions can be directly arrived at. The total profit of the plant is given by $0.15X_1 + 0.25X_2$.

The objective function will be maximize d as $Z = 0.15X_1 + 0.25X_2$. Since machines Y and Z work for 8 h a day and 5 days a week, the total working time for machines Y and Z will become 2,400 min per week. Therefore, the time constraints are

$$X_1/100 + X_2/40 \leq 2,400 \text{ (for machine Y)}$$

$$X_1/60 + X_2/75 \leq 2,400 \text{ (for machine Z)}$$

The other constraints are

$$8X_1 + 16X_2 = 3,00,000 \text{ (Production constraint)}$$

$$\left. \begin{array}{l} 0 \leq X_1 \leq 25,000 \\ 0 \leq X_2 \leq 7,000 \end{array} \right\} \text{ (Marketing constraints)}$$

$$X_1, X_2 \geq 0$$

Illustration 2.6

The quality control department of an organization has recruited 8 and 10 inspectors in grades 1 and 2, respectively, for performing quality inspections on products. It is imperative for the inspectors to maintain 95 % accuracy in their work. In order to achieve this, the inspectors can check not more than 20 and 15 pieces, respectively, in an hour, and not more than 1,450 pieces in a day. Labor costs account to Rs. 4/h for grade 1 inspectors, and Rs. 5/h for grade 2 inspectors. If overloaded, there are possibilities for inspection errors and this costs Rs. 3.

It is required to find the optimal assignment of inspectors that minimizes the daily costs for the organization.

Solution

The objective of this problem is to minimize the costs. Hence, the objective function has to be of minimization type. There is 5 % chance of error in the quality control inspections. Hence, the costs incurred by the organization will be:

For grade 1 inspectors: $4 + (0.05 \cdot 3.18) = \text{Rs. } 6.7/\text{h} = 6.7 \cdot 8 = 52.6/\text{day}$

For grade 2 inspectors: $5 + (0.05 \cdot 3.14) = \text{Rs. } 7.1/\text{h} = 7.1 \cdot 8 = 56.8/\text{day}$

Now, we can formulate the objective function.

$$\text{Minimize } Z = 52.6X_1 + 56.8X_2$$

Subject to $X_1 \leq 8, X_2 \leq 10$ (Recruitment constraint)

$$144X_1(18.8) + 112X_2(14.8) \geq 1,450 \text{ (Capability constraint)}$$

$$X_1, X_2 \geq 0$$

2.3 Graphical Solution to an LP Model

The graphical approach to solve an LP problem obtains a feasible solution to the problem, provided the number of variables involved is two. This section demonstrates the graphical approach under possible conditions.

2.3.1 Graphical Solution of a Maximization Model**Illustration 2.7**

Solve the following LP problem by the graphical method.

$$\text{Maximize } Z = 5X_1 + 12X_2$$

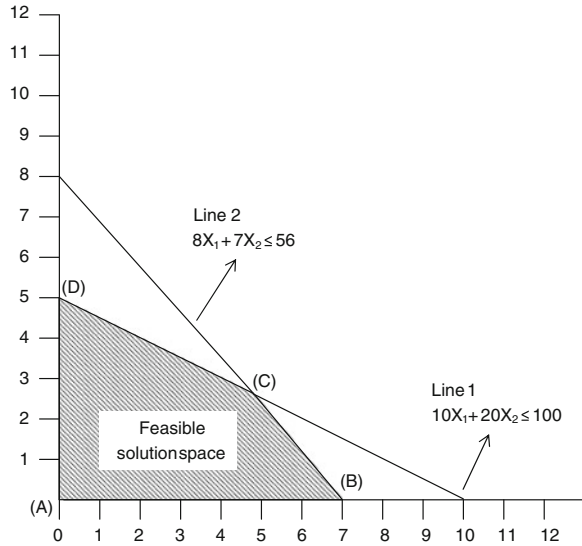
$$\text{Subject to } 10X_1 + 20X_2 \leq 100$$

$$8X_1 + 7X_2 \leq 56$$

$$X_1, X_2 \geq 0$$

Step 1: We have to first account for nonnegativity constraints $X_1, X_2 = 0$. In Fig. 2.1, X_1 and X_2 on the horizontal and vertical axes, respectively, represent P_1 and P_2 . It follows that the nonnegativity constraints restrict the solution space to the first quadrant.

Fig. 2.1 Nonnegativity constraints



To account for the remaining constraints, each inequality is replaced with equations.

Locating two distinct points on it draws a graph of the resulting straight line. In the example, if we replace $10X_1 + 20X_2 = 100$ with the straight line $10X_1 + 20X_2 = 100$, then two points can be determined by setting $X_1 = 0$ and $X_2 = 0$. By setting $X_1 = 0$, we get $20X_2 = 100$, i.e., $X_2 = 5$. Similarly by setting $X_2 = 0$, we get $10X_1 = 100$, i.e., $X_1 = 10$. Thus, the line passes through (10, 0) and (0, 5), which is shown as line 1. Line 1 represents the first constraint. Similarly for the second constraint, we get $X_1 = 7$ by setting $X_2 = 0$, and $X_2 = 8$ by setting $X_1 = 0$. This line passes through (7, 0) and (0, 8) as shown in Line 2. Line 2 represents the second constraint. X_1 and $X_2 = 0$ eliminate the second, third, and fourth quadrants of the X_1 and X_2 plane.

Step 2: Optimum Solution The closed space $A-B-C-D$ represents the feasible solution space. The objective function value at each of the corner points is computed by substituting its coordinates in the objective function as:

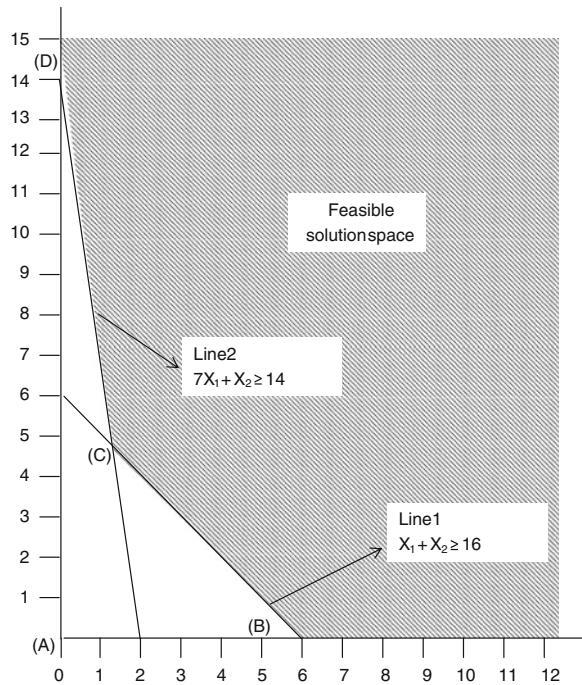
$$Z(A) = 5(0) + 12(0) = 0$$

$$Z(B) = 5(10) + 12(0) = 50$$

$$Z(C) = 5(5) + 12(3) = 61$$

$$Z(D) = 5(2) + 12(5) = 60$$

Since the objective function here is maximization, the solution corresponding to the maximum Z value is to be selected as the optimum solution. The Z value is maximum for the corner point C . The corresponding solution is $x_1 = 5$ and $x_2 = 3$,

Fig. 2.2 Feasible region

and Z (Optimum) = 61. This calls for a daily mix of 5 tons of P_1 and 3 tons of P_2 , giving an associated daily profit of Rs. 61,000.

2.3.2 Solution for a Minimization Problem

Illustration 2.8

Solve the following problem by graphical method:

$$\begin{aligned}
 &\text{Minimize } Z = 2X_1 + 3X_2 \\
 &\text{subject to } X_1 + X_2 \geq 6 \\
 &\quad \quad \quad 7X_1 + X_2 \geq 14 \\
 &\quad \quad \quad X_1, X_2 \geq 0
 \end{aligned}$$

Solution

Let us compute the coordinates on the XY plane relating to constraint equations.

First constraint gives us $X_1 = 6$, when $X_2 = 0$, and $X_2 = 6$, when $X_1 = 0$, represented by

Line 1 in Fig. 2.2.

Second constraint gives us $X_1 = 2$, when $X_2 = 0$, and $X_2 = 14$, when $X_1 = 0$, represented by

Line 2 in Fig. 2.2.

The feasible region is represented by $F-D-C-B-G$. Now, it is required to calculate the value of the objective function at these corner points to obtain the optimum solution.

$$Z(D) = 2.0 + 3.14 = 42$$

$$Z(C) = 2.4/3 + 3.14/3 = 50/3$$

$$Z(D) = 2.6 + 3.0 = 12$$

As our objective function is of minimization type, it is required to choose the minimum value from the above values, which correspond to corner point D of value 12. Therefore, the optimum solution is:

$$X_1 = 6, X_2 = 0, \text{ and } Z (\text{optimum}) = 12.$$

2.4 Review Questions

1. Explain the concepts of an LP model?
2. What are the assumptions of an LP problem?
3. Elaborate the steps involved in solving an LP problem graphically?
4. A textile company can use any or all of the three different processes for weaving its standard white polyester fabric. Each of these production processes has a weaving machine setup cost and per-square-meter processing cost. The costs and capacities of the three production processes are as given below.

Process number	Weaving machine setup cost (Rs.)	Processing cost/m ² (Rs.)	Maximum daily capacity (m ²)
1	75	15	1,000
2	120	10	1,500
3	150	8	1,750

The daily demand forecast for its white polyester fabric is 4,000 m². The company's production manager wants to make a decision concerning which combination of production process is to be utilized to meet the daily demand

forecast and at what production level of each selected production process to be operated to minimize total production costs.

5. Consider the cargo-loading problem, where five items are to be loaded on a vessel. The weight (w_i) and the volume (v_i) of each unit of the different items as well as their corresponding returns per unit (r_i) are as shown below.

The maximum cargo weight (W) and volume (V) are given as 112 and 109, respectively. It is required to determine the optimal cargo load in discrete units of each item such that the total return is maximized. Formulate the problem as an integer programming model. (*Hint*: Similar to LP formulation)

Item i	w_i	v_i	r_i
1	4	1	5
2	7	8	8
3	2	6	7
4	1	5	6
5	6	4	5

6. A firm manufactures two headache pills A and B . Size A contains 2 grains of aspirin and 5 grains of codeine. Size B contains 1 grain of aspirin, 9 grains of bicarbonate, and 6 grains of codeine. It is found by customers that it requires at least 12 grains of aspirin, 74 grains of bicarbonate, and 24 grains of codeine for providing immediate relief. Formulate the problem as an LPP. (*Hint* Minimization problem $-\min X_1 + X_2$, s.t. $2X_1 + X_2 = 12$, $X_1 + 6X_2 = 24$, $5X_1 + 8X_2 = 24$; $X_1, X_2 = 0$).
7. A manufacturer has three machines A , B , and C which produce three different articles P , Q , and R . The different machine times required per article, the amount of time available in any week on each machine, and the estimated profits per article are tabulated below.

Article	Machine time (in hrs)			Profit per article (in Rs.)
	A	B	C	
P	8	4	2	20
Q	2	3	0	6
R	3	0	1	8
Available machine (h)	250	150	50	

Formulate the problem as an LPP. (*Hint*: Similar to illustration).



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