Chapter 2
Topological Horseshoes and Coin-Tossing Dynamics

2.1 Chaos in the "Coin-Tossing" Sense

Definition 2.1 Let X be a metric space, $\psi : X \supseteq D_\psi \to X$ be a map and let $D \subseteq D_\psi$: We say that ψ induces chaotic dynamics on two symbols on the set $\mathscr{D}$ if there exist two nonempty disjoint compact sets

$$\mathscr{K}_0, \mathscr{K}_1 \subseteq D,$$

such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathscr{D}^{\mathbb{Z}}$ such that

$$w_i \in K_{s_i} \quad \text{and} \quad w_{i+1} = \psi(w_i), \quad \forall\, i \in \mathbb{Z} \tag{2.1}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k periodic sequence (that is, $s_{i+k} = s_i, \forall\, i \in \mathbb{Z}$) for some $k \geq 1$, there exists a k-periodic sequence $(wi)_{i\in\mathbb{Z}} \in \mathscr{D}^{\mathbb{Z}}$ satisfying 2.1. To put the emphasis on the sets $\mathscr{K}_j^{\prime}s$, we may also say that ψ induces chaotic dynamics on two symbols on the set $\mathscr{D}$ with respect to $\mathscr{K}_0$ and $\mathscr{K}_1$.

This definition corresponds to the concept of chaos in the coin-tossing sense stated in [1]. However, the definition in [1] is enhanced here with the condition on periodic sequences. To get a simpler feel of Definition 2.1, following Smale in his expository paper [2], we focus our attention on the case $m = 2$ and associate to the set $\mathscr{K}_0$ the name "head" and to the set $\mathscr{K}_1$ the name "tail". If we consider any sequence of symbols $(s_i)_{i\in\mathbb{Z}}$ where, for each i, s_i is either "head" or "tail", then there exists the same itinerary of heads and tails realized by the map ψ. Namely we have a sequence $(w_i)_{i\in\mathbb{Z}}$ of points of the metric space X, with $w_{i+1} = \psi(w_i), \forall\, i \in \mathbb{Z}$, such that $w_i \in \mathscr{K}_0$ or $w_i \in \mathscr{K}_1$ according as the ith term of the sequence $(s_i)_i$ is "head" or "tail". Our definition as commented earlier, extends that in [1] in the sense that any periodic sequence of heads and tails can be realized by suitable points which are periodic points for ψ. We explain this with an example: For instance, there exists a fixed point of ψ in the set $\mathscr{K}_0$ corresponding to the constant sequence of symbols

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$s_i =$ "head", $\forall\, i \in \mathbb{Z}$. There is also a point of period three $w \in \mathscr{K}_1$ with $\psi(w) \in \mathscr{K}_1$ and $\psi^2(w) \in \mathscr{K}_0$, corresponding to the periodic sequence $\ldots TTH\ TTH\ TTH \ldots$, and so on.

Within our approach, we can also obtain periodic points for ψ having a given minimal period. For instance, the sequence $\ldots HHHT\ HHHT\ HHHT \ldots$, is realized by some point $w \in \mathscr{K}_0$ with $\psi(w) \in \mathscr{K}_0$, $\psi^2(w) \in \mathscr{K}_0$, $\psi^3(w) \in \mathscr{K}_1$, $\psi^4(w) = w \in \mathscr{K}_0$, which is a point of minimal period four.

Definition 2.1 agrees with other ones considered in the literature about chaotic dynamics for ODEs with periodic coefficients (see [3–5]).

Definition 2.2 Let Z be a metric space, $\psi : Z \supseteq D_\psi \to Z$ be a map and let $\mathscr{D} \subseteq D_\psi$. Assume also that $m \geq 2$ is an integer. We say that ψ *induces chaotic dynamics on m symbols in the set* $\mathscr{D}$ if there exist m nonempty pairwise disjoint compact sets

$$\mathscr{K}_0,\ \mathscr{K}_1, \ldots, \mathscr{K}_{m-1} \subseteq \mathscr{D},$$

such that, for each two-sided sequence $(s_i)_{i\in\mathbb{Z}} \in \{0,\ldots,m-1\}^{\mathbb{Z}}$, there exists a corresponding sequence $(w_i)_{i\in\mathbb{Z}} \in \mathscr{D}^{\mathbb{Z}}$ such that

$$w_i \in \mathscr{K}_{s_i} \quad \text{and} \quad w_{i+1} = \psi(w_i), \quad \forall\, i \in \mathbb{Z} \tag{2.2}$$

and, whenever $(s_i)_{i\in\mathbb{Z}}$ is a k-periodic sequence (that is, $s_{i+k} = s_i$, $\forall\, i \in \mathbb{Z}$) for some $k \geq 1$, there exists a k-periodic sequence $(w_i)_{i\in\mathbb{Z}} \in \mathscr{D}^{\mathbb{Z}}$ satisfying 2.2. When we want to stress the role of the $\mathscr{K}_j$'s, we also say that ψ *induces chaotic dynamics on m symbols in the set* $\mathscr{D}$ *relatively to* $(\mathscr{K}_0,\ldots,\mathscr{K}_{m-1})$.

Our definition is derived from the characterization of chaos used by Kirchgraber and Stofer [1] in the coin-tossing sense. The fact that the iterates of a map mimic the sequences of coin tossing has long since been regarded as a key feature of complex dynamics of deterministic systems (see, for instance, Smale's remarks [2]). This characterization of chaos is similar to that of "topological horseshoes", for example [6, 7]. A specific advantage of the version adopted here is the possibility of obtaining periodic sequences of two symbols by the periodic itineraries generated by periodic points of ψ. We further observe that a connection to the Bernoulli shift can be derived. Recall the distance function defined on the space of two symbols Σ_2^+

$$\hat{d}(\mathbf{s}', \mathbf{s}'') := \sum_{i\in\mathbb{N}} \frac{d(s_i', s_i'')}{2^{i+1}} \quad \text{for } \mathbf{s}' = (s_i')_{i\in\mathbb{N}},\ \mathbf{s}'' = (s_i'')_{i\in\mathbb{N}} \in \Sigma_2^+ \tag{2.3}$$

(where $d(\cdot\,,\cdot)$ is the discrete distance on $\{0, 1\}$: $d(s_i', s_i'') = 0$ for $s_i' = s_i''$ and $d(s_i', s_i'') = 1$ for $s_i' \neq s_i''$), Also recall that the Bernoulli shift σ is a homeomorphism defined by

$$\sigma((s_i)_i) := (s_{i+1})_i,$$

as shown in [8], σ has positive topological entropy, given by

$$h_{\text{top}}(\sigma) = \log(m).$$

The following lemma relates the dynamical properties of a map satisfying Definition 2.1 to the ones of the Bernoulli shift. Let Λ be a compact metric space and let $\psi : \Lambda \to \Lambda$ be a continuous map. We say that ψ is *semiconjugate to the two-sided m-shift* if there exists a continuous surjective mapping $g : \Lambda \to \Sigma_m$ such that

$$g \circ \psi = \sigma \circ g. \tag{2.4}$$

In a similar manner, if we denote by

$$\Sigma_m^+ = \{0, \dots, m-1\}^{\mathbb{N}}$$

the set of the one-sided sequences of m symbols, endowed with a distance analogous to the one defined in (1.8) we say that ψ is *semiconjugate to the one-sided m-shift* if there exists a continuous surjective mapping $g : \Lambda \to \Sigma_m^+$ such that (2.4) holds.

The following result (which is substantially a standard fact) connects the concept of semiconjugation with the Bernoulli shift to the one of chaotic dynamics expressed in Definition 2.2. Its proof could be easily adapted from similar arguments previously appeared in the literature (see, for instance [6, 7] for semidynamical systems induced by continuous maps of metric spaces), for sake of completeness, we provide here all the details.

Lemma 2.1 [9] *Let Z be a metric space, $\psi : Z \supseteq D_\psi \to Z$ be a map which is continuous and one-to-one on a set $\mathscr{D} \subseteq D_\psi$ and induces therein chaotic dynamics on $m \geq 2$ symbols (relatively to $(\mathscr{K}_0, \dots, \mathscr{K}_{m-1})$). Then, there exists a nonempty compact set*

$$\Lambda \subseteq \bigcup_{j=0}^{m-1} \mathscr{K}_j\,,$$

which is invariant for ψ and such that $\psi|_\Lambda$ is semiconjugate to the two-sided m-shift, so that the topological entropy $h_{\text{top}}(\psi)$ satisfies

$$h_{\text{top}}(\psi) \geq \log(m).$$

Moreover, the subset $\mathscr{P}$ of Λ made by the periodic points of ψ is dense in Λ and if we denote by $g : \Lambda \to \Sigma_m$ the continuous surjection in Eq. (2.4), it holds also that the counterimage through g of any k-periodic sequence in Σ_m contains at least one k-periodic point.

Proof Setting $\mathscr{K} := \bigcup_{j=0}^{m-1} \mathscr{K}_j$, we define

$$\Lambda_0 := \{w \in \mathscr{K} : \psi^i(w) \in \mathscr{K},\ \forall\, i \in \mathbb{Z}\} = \bigcap_{i=-\infty}^{+\infty} \psi^{-i}(\mathscr{K})$$

and

$$\mathscr{P} := \{x \in \Lambda_0 : \exists k \geq 1 \text{ with } \psi^k(x) = x\}.$$

Since $\mathscr{K}$ is compact and ψ is continuous on $\mathscr{K}$, it follows immediately that also Λ_0 is compact and that $\psi(\Lambda_0) \subseteq \Lambda_0$ (that is, Λ_0 is invariant for ψ). Let us now define $g_0 : \Lambda_0 \to \Sigma_m$, as

$$g_0(w) := (s_i)_{i \in \mathbb{N}} \Leftrightarrow \psi^i(w) \in \mathscr{K}_{s_i}, \quad \forall\, i \in \mathbb{N}.$$

By Definition 2.2, the map g_0 is surjective and the counterimage through g_0 of any k-periodic sequence in Σ_m contains at least one k-periodic point (belonging to $\mathscr{P}$). The continuity of g_0 comes from the continuity of ψ on Λ_0, the choice of the distance d in 1.8 and the fact that the sets $\mathscr{K}_j$ are compact and pairwise disjoint. Actually, g_0 turns out to be uniformly continuous as it is defined on a compact metric space. A direct inspection shows that the relation in (2.4) is satisfied and therefore the map g_0 induces a semiconjugation between $\psi|_{\Lambda_0}$ and the two-sided m-shift.

Let

$$\Sigma_m^{\text{per}} \subseteq \Sigma_m$$

be the set of the periodic two-sided sequences of m symbols.

Now,

$$g_0 : \mathscr{P} \to \Sigma_m^{\text{per}},$$

by setting, for each $w \in \mathscr{P}$:

$$g_0(w) := (s_i)_{i \in \mathbb{Z}} \in \Sigma_m^{\text{per}} \Leftrightarrow \psi^i(w) \in \mathscr{K}_{s_i}, \quad \forall\, i \in \mathbb{Z}. \tag{2.5}$$

Notice that

$$g_0 \circ \psi(w) = \sigma \circ g_0(w), \quad \forall \quad w \in \mathscr{P},$$

where σ is the two-sided Bernoulli shift on m symbols.

Now, setting

$$\Lambda := \overline{\mathscr{P}} \subseteq \Lambda_0,$$

it holds that $\psi(\Lambda) \subseteq \Lambda$, so that Λ is compact and invariant for ψ. At last, we extend the uniformly continuous surjective mapping

$$g_0 : \mathscr{P} \to \Sigma_m^{\text{per}} \subseteq \Sigma_m$$

to a continuous surjective function

$$g : \Lambda \to \Sigma_m,$$

such that

$$g \circ \psi(x) = \sigma \circ g(x), \quad \forall\, x \in \Lambda.$$

From the above proved semiconjugacy condition and by [10, Theorem 7.2] it follows that

$$h_{\text{top}}(\psi) \geq h_{\text{top}}(\sigma) = \log(m).$$

Hence we see that all the properties listed in the statement of the lemma are satisfied. The proof is complete.

2.2 Topological Lemmas and Definitions

We now state and prove another result which is more general than the previous Lemma 2.1. In Lemma 2.1 a one-to-one condition is assumed on the map ψ. In the following theorem no such one-to-one condition is assumed. This result is an important and interesting result for the concept of chaotic dynamics as we define it.

Theorem 2.1 [11] *Let ψ be a map inducing chaotic dynamics on two symbols on a set $\mathscr{D}$ and which is continuous on*

$$\mathscr{K} := \mathscr{K}_0 \cup \mathscr{K}_1 \subseteq \mathscr{D},$$

where $\mathscr{K}_0$,$\mathscr{K}_1$ and $\mathscr{D}$ are as in Definition 2.1. Defining the nonempty compact set

$$\mathscr{I}_\infty := \bigcap_{n=0}^{\infty} \psi^{-n}(\mathscr{K}) \tag{2.6}$$

then there exists a nonempty compact set

$$\mathscr{I} \subseteq \mathscr{I}_\infty \subseteq \mathscr{K},$$

on which the following are fulfilled:

(i) $\mathscr{J}$ *is invariant for* ψ *(i.e.* $\psi(\mathscr{J}) = \mathscr{J}$ *).*

(ii) $\psi(\mathscr{J})$ *is semi-conjugate to the Bernoulli shift on two symbols, that is, there exists a continuous map* π *of* $\mathscr{J}$ *onto* $\Sigma_2^+ := 0, 1^{\mathbb{N}}$ *endowed with the distance*

$$d(\boldsymbol{s}', \boldsymbol{s}'') := \Sigma \frac{d(s_i', s_i'')}{2^{i+1}}, \quad \textit{for } \boldsymbol{s}' = (s_i')_{i \in \mathbb{N}}, \ \boldsymbol{s}'' = (s_i'')_{i \in \mathbb{N}} \in \ \Sigma_2^+$$

(where $d(.,.)$ *is the discrete distance on* $\{0, 1\}$: $d(s_i', s_i'') = 0$ *for* $s_i' = s_i''$ *and* $d(s_i', s_i'') = 1$ *for* $s_i' \neq s_i''$*), such that the diagram*

$$\begin{tikzcd} \mathscr{J} \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathscr{J} \arrow[d, "\pi"] \\ \Sigma_2^+ \arrow[r, "\sigma"'] & \Sigma_2^+ \end{tikzcd} \tag{2.7}$$

commutes, where $\sigma : \Sigma_2^+ \to \Sigma_2^+$ *is the Bernoulli shift defined by* $\sigma((s_i)_i) := (s_{i+1})_i, \forall\, i \in \mathbb{N}$.

(iii) *The set P of the periodic points of* $\psi \mid_{\mathscr{J}_\infty}$ *is dense in* $\mathscr{J}$ *and the pre-image* $\pi^{-1}(\boldsymbol{s}) \subseteq \mathscr{J}$ *of every k-periodic sequence* $\boldsymbol{s} = (s_i)_{i \in \mathbb{N}} \in \Sigma_2^+$ *contains at least one k-periodic point.*
Furthermore, from property (ii) it follows that:

(iv) $h_{\text{top}}(\psi) \geq h_{\text{top}}(\psi \mid_{\mathscr{J}}) \geq h_{\text{top}}(\sigma) = \log(2)$, *where* h_{top} *is the topological entropy.*

(v) *There exists a compact positively invariant set* $\Lambda \subseteq \mathscr{J}$ *such that* $\psi \mid_\Lambda$ *is semi-conjugate to the Bernoulli shift on two symbols, topologically transitive and has sensitive dependence on initial conditions.*

Proof Let us start checking that the set $\mathscr{J}_\infty$ in (2.6) is compact and nonempty. By the continuity of the map ψ on $\mathscr{K}$, it follows that $\mathscr{J}_\infty$ is closed and, being contained in the compact set $\mathscr{K}$, it is compact, too. The fact that $\mathscr{J}_\infty$ is nonempty follows from Definition 2.1, by observing that$z \in \mathscr{J}_\infty \Leftrightarrow \psi^n(z) \in \mathscr{K}, \forall\, n \geq 0$. This remark also implies that $\psi(\mathscr{J}_\infty) \subseteq \mathscr{J}_\infty$: indeed, it is straightforward to see that if $z \in \mathscr{J}_\infty$, then also $\psi(z) \in \mathscr{J}_\infty$ Calling $\mathscr{P}$ the subset of $\mathscr{J}_\infty$ made of the periodic points of $\psi \mid_{\mathscr{J}_\infty}$, that is,

$$\mathscr{P} := \{w \in \mathscr{J}_\infty : \exists k \in \mathbb{N} \backslash \{0\}, \quad \psi^k(w) = w\},$$

we claim that$\psi(\mathscr{P}) = \mathscr{P}$. Indeed, if $z \in \mathscr{P}$, then there exists $l \in \mathbb{N} \backslash \{0\}$ such that $\psi^l(z) = z$. Hence, on the one hand, $\psi(z) = \psi(\psi^l(z)) = \psi^{l+1}(z) = \psi^l(\psi(z))$ and thus $\psi(z) \in \mathscr{P}$, too. This shows that $\psi(\mathscr{P}) \subseteq \mathscr{P}$. Note that, repeating the same argument, it is possible to prove that if $z \in \mathscr{P}$, then $\psi^h(z) \in \mathscr{P}$, for any $h \geq 1$. On the other hand, if $\psi^l(z) = z$, for some $l \in \mathbb{N} \backslash \{0\}$, then two possibilities can occur for l, that is, $l = 1$ or $l \geq 2$. In the former case we get $\psi(z) = z$ and so $z \in \psi(\mathscr{P})$, while in the latter we obtain $z = \psi^l(z) = \psi(\psi^{l-1}(z))$. Hence, since $\psi^{l-1}(z) \in \mathscr{P}$

whenever $z \in \mathscr{P}$, we find again $z \in \psi(\mathscr{P})$. In any case we have proved that, if $z \in \mathscr{P}$, then $z \in \psi(\mathscr{P})$, i.e. $\mathscr{P} \subseteq \psi(\mathscr{P})$. The claim is thus checked. At this point we observe that, since $\mathscr{P}$ is contained in the compact set $\mathscr{J}_\infty$, also

$$\mathscr{J} := \overline{\mathscr{P}} \subseteq \mathscr{J}_\infty, \tag{2.8}$$

and moreover $\mathscr{J}$ is compact, as it is closed in a compact set. From $\psi(\mathscr{P}) = \mathscr{P}$, it follows that

$$\psi(\mathscr{J}) = \psi\overline{(\mathscr{P})} \supseteq \psi(\mathscr{P}) = \mathscr{P}.$$

But again, by the compactness of $\psi(\mathscr{J})$, it holds that

$$\psi(\mathscr{J}) \supseteq \overline{\mathscr{P}} = \mathscr{J}.$$

Let us show that also the reverse inclusion is fulfilled for $\mathscr{J}$, that is, $\psi(\mathscr{J}) \subseteq \mathscr{J}$. Indeed, since ψ is continuous, we have

$$\psi(\mathscr{P}) = \psi\overline{(\mathscr{P})} \subseteq \overline{\psi(\mathscr{P})} = \overline{\mathscr{P}} = \mathscr{J}.$$

Hence, the invariance of $\mathscr{J}$ is verified, in agreement with conclusion (i). Let us consider now the diagram

$$\begin{array}{ccc} \mathscr{J}_\infty & \xrightarrow{\psi} & \mathscr{J}_\infty \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \Sigma_2^+ & \xrightarrow[\sigma]{} & \Sigma_2^+ \end{array}$$

and define the map $\pi : \mathscr{J}_\infty \to \Sigma_2^+$ by associating to $w \in \mathscr{J}_\infty$ the sequence $(s_n)_{n\in\mathbb{N}} \in \Sigma_2^+$ such that $s_n = j$ if $\psi^n(w) \in \mathscr{K}_j$, for $j = 0, 1$. More formally, we note that, for any $w \in \mathscr{J}_\infty$, there exists a unique forward itinerary $(w_i)_{i\in\mathbb{N}}$ such that $w_0 = w$ and $\psi(w_i) = w_{i+1} \in \mathscr{K}$, for every $i \in \mathbb{N}$. Hence the function $g_1 : \mathscr{J}_\infty \to, \mathscr{J}_\infty^{\mathbb{N}}$ which maps any $w \in \mathscr{J}_\infty$ into the one-sided sequence of points from the set $\mathscr{J}_\infty$

$$\mathbf{s}_w := (w_i)_{i\in\mathbb{N}} \quad \text{where } w_i := \psi^i(w), \forall\, i \in \mathbb{N},$$

with the usual convention $\psi^0 = Id_{\mathscr{J}_\infty}$ and $\psi^1 = \psi$, is well-defined. Since the sets $\mathscr{K}_0$ and $\mathscr{K}_1$ are disjoint, for every term w_i of $\mathbf{s}_w$ there exists a unique index

$$s_i = s_i(w_i), \quad \text{with } s_i \in \{0, 1\},$$

such that $w_i \in \mathscr{K}_{s_i}$. Therefore the map $g_2 : \mathscr{J}_\infty^{\mathbb{N}} \to \Sigma_2^+$,

$$g_2 : \mathbf{s}_w \mapsto (s_i)_{i\in\mathbb{N}} \in \Sigma_2^+ \tag{2.9}$$

is also well-defined. Thus, by Definition 2.1 the map

$$\pi := g_2 \circ g_1 : \mathscr{I}_\infty \to \Sigma_2^+ \tag{2.10}$$

is a surjection that makes the diagram (2.7) commute and the pre-image through π of any k-periodic sequence in Σ_2^+ contains at least one k-periodic point of $\mathscr{I}_\infty$. To check that π is continuous, we prove the continuity in a generic $\bar{z} \in \mathscr{I}_\infty$ by showing that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall\ z \in \mathscr{I}_\infty$ with $d(z, \bar{z}) < \delta$, $d(\pi(z), \pi(\bar{z})) < \varepsilon$, with $\hat{d}$ as in the statement of the theorem. Let us fix $\varepsilon > 0$ and let $n \in \mathbb{N}$ such that $0 < 1/2^n < \varepsilon$. We notice that it is sufficient to prove that $(\pi(z))_i = (\pi(\bar{z}))_i$, for any $i = 0, \dots, n$. Indeed, if this is the case, by the definition of $\hat{d}$, it follows that $\hat{d}(\pi(z), \pi(\bar{z})) \le 1/2^n < \varepsilon$ Since $\bar{z} \in \mathscr{I}_\infty$, there exists a sequence $(s_0, \dots, s_n) \in \{0, 1\}^{n+1}$ such that

$$\bar{z} \in \mathscr{K}_{s_0}, \psi(\bar{z}) \in \mathscr{K}_{s_1}, \dots, \psi^n(\bar{z}) \in \mathscr{K}_{s_n}.$$

Recalling that $\mathscr{K}_0$ and $\mathscr{K}_1$ are compact and disjoint, it holds that

$$\eta := d(\mathscr{K}_0, \mathscr{K}_1) > 0.$$

Hence, for any $z \in \mathscr{I}_\infty$ with $d(z, \bar{z}) < \eta/2$, it follows that $z2 \in \mathscr{K}_{s_0}$, too. By the continuity of ψ in $\bar{z}$, there exists $\delta_1 > 0$ such that $\forall\ z \in \mathscr{I}_\infty$ with $d(z, \bar{z}) < \delta_1$, then $d(\psi(z), \psi(\bar{z})) < \eta/2$. But this means that $\psi(z) \in \mathscr{K}_{s_1}$. Analogously, by the continuity of ψ^2 in $\bar{z}$, there exists $\delta_2 > 0$ such that $\forall\ z \in \mathscr{I}_\infty$ with $d(z, \bar{z}) < \delta_2, d(\psi^2(z); \psi^2(\bar{z})) < \eta/2$ and thus $\psi^2(z) \in \mathscr{K}_{s_2}$, for any such z. Proceeding in this way until the nth iterate of ψ and setting

$$\delta := \min\{\frac{\eta}{2}, \delta_1, \dots, \delta_n\},$$

we find that, for any $z \in \mathscr{I}_\infty$ with $d(z, \bar{z}) < \delta$, it holds that

$$z \in \mathscr{K}_{s_0}, \psi(z) \in \mathscr{K}_{s_1}, \dots, \psi^n(z) \in \mathscr{K}_{s_n},$$

exactly as for $\bar{z}$. But this means that $(\pi(z))_i = (\pi(\bar{z}))_i$, for any $i = 0, \dots, n$, and hence $\hat{d}(\pi(z), \pi(\bar{z})) \le 1/2^n < \varepsilon$. The continuity of π is thus proved. Considering in diagram (2.7) the restriction of ψ to $\mathscr{P} \subseteq \mathscr{I}_\infty$, we find the commutative diagram

$$\begin{tikzcd} \mathscr{P} \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathscr{P} \arrow[d, "\pi"] \\ \mathscr{P}_2^+ \arrow[r, "\sigma"'] & \mathscr{P}_2^+ \end{tikzcd} \tag{2.11}$$

where $\mathscr{P}_2^+ \subseteq \Sigma_2^+$ is the set of the periodic sequences of two symbols. Note that $\pi(\mathscr{P}) = \mathscr{P}_2^+$ from Definition 2.1. Recalling the fact that $\mathscr{P}_2^+$ is dense in Σ_2^+ by the continuity of π, it follows that

$$\pi(\mathscr{J}) = \pi(\bar{\mathscr{P}}) \subseteq \overline{\mathscr{P}_2^+} = \Sigma_2^+.$$

On the other hand $\pi(\mathscr{J})$ is a compact set containing $\pi(\mathscr{P}) = \mathscr{P}_2^+$ and hence

$$\pi(\mathscr{J}) \supseteq \overline{\mathscr{P}_2^+} = \Sigma_2^+.$$

Therefore, we can conclude that $\pi(\mathscr{J}) = \Sigma_2^+$ and the diagram

$$\begin{array}{ccc} \mathscr{J} & \xrightarrow{\psi} & \mathscr{J} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \Sigma_2^+ & \xrightarrow[\sigma]{} & \Sigma_2^+ \end{array}$$

Moreover, the pre-image through π of any k-periodic sequence in Σ_2^+ contains at least one k-periodic point of $\mathscr{J}$ as $P \subseteq I$. Conclusions (ii) and (iii) are thus proved. Assertion (iv) regarding the positive topological entropy, comes from property (ii) about the semi-conjugacy to the Bernoulli shift. For a proof see [10]. Finally, conclusion (v)pertaining to the existence of a compact set $\Lambda \subseteq \mathscr{J}$ which is positively invariant for ψ and such that $\psi\mid_{\Lambda}$ is semi-conjugate to the Bernoulli shift on two symbols, topologically transitive and has sensitive dependence on initial conditions, follows by applying a result by Auslander and Yorke [12].

2.3 The Concept of “Stretching” as Related to Chaos

In the present section, we provide an essential introduction to our approach for the detection of chaotic dynamics based on the concept of stretching along the paths.

We first need some basic definitions. Let X be a topological space. A path in a metric space X is a continuous map $\gamma : [t_0, t_1] \to X$. Its range will be denoted by $\bar{\gamma}$ that is, $\bar{\gamma} := \gamma[t_0, t_1]$. A *sub-path* ω of γ is the restriction of γ to a closed subinterval of its domain and hence it is defined as $\omega := \gamma|_{[t_0', t_1']}$, for some $[t_0', t_1'] \subseteq [t_0, t_1]$.

Next, if M, N are topological spaces and $\psi : M \supseteq D_\psi \to N$ is a map which is continuous on a set $\mathscr{M} \subseteq D_\psi$, then for any path γ in M with $\bar{\gamma} \subseteq M$, it follows that $\psi \circ \gamma$ is a path in N with range $\psi(\bar{\gamma})$. There is no loss of generality in assuming the paths to be defined on $[0, 1]$. If $\theta_1 : [a_1, b_1] \to M$ and $\theta_2 : [a_2, b_2] \to M$ with $a_i < b_i$, $i = 1, 2$, are two paths in M, we define the equivalence relation $\sim$ between θ_1 and θ_2 by setting $\theta_1 \sim \theta_2$ if there exists a homeomorphism h of $[a_1, b_1]$ onto $[a_2, b_2]$ such that $\theta_2(h(t)) = \theta_1(t)$, $\forall\ t \in [a_1, b_1]$. It can be proved easily that if $\theta_1 \sim \theta_2$, then the ranges of θ_1 and θ_2 coincide. Hence, for any path γ there corresponds an equivalent path defined on [0, 1].

Next we define a concept similar to a path, that of an *arc*. An arc is the homeomorphic image of the compact interval [0, 1], and an open arc, an arc without its end-points.

A continuum of M is a compact connected subset of M.

2.4 Oriented Rectangles

By a *generalized rectangle* we mean a set $\mathscr{R} \subseteq X$ which is homeomorphic to the unit square $[0, 1]^2 \subseteq \mathbb{R}^2$ (Fig. 2.1).

Given a generalized rectangle $\mathscr{R}$ and $\mathscr{S} := [0, 1]^2 \subseteq \mathbb{R}^2$ the associated homeomorphism $h : \mathscr{S} \to h(\mathscr{S}) = \mathscr{R}$, the set $\vartheta R := h(\partial([0, 1]^2))$, where $\partial([0, 1]^2)$ is the usual boundary of the unit square, is named the *contour* of $\mathscr{R}$. the contour ϑR is well-defined as it is independent of the choice of the homeomorphism h. In fact, ϑR is also a homeomorphic image of a simple closed curve, that is, a Jordan curve.

We also call as an oriented rectangle, the pair

$$\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$$

where $\mathscr{R} \subseteq X$ is a generalized rectangle and

$$\mathscr{R}^- := \mathscr{R}_l^- \cup \mathscr{R}_r^-$$

is the union of two disjoint compact arcs $\mathscr{R}_l^-$ and $\mathscr{R}_r^- \subseteq \vartheta\mathscr{R}$ that we call the *left* and *right* sides of $\mathscr{R}^-$. Since $\vartheta\mathscr{R}$ is a Jordan curve it follows that $\vartheta\mathscr{R} \backslash (\mathscr{R}_l^- \cup \mathscr{R}_r^-)$ consists of two open arcs,

$$\mathscr{R}^+ := \overline{\vartheta R \setminus \mathscr{R}^-}$$

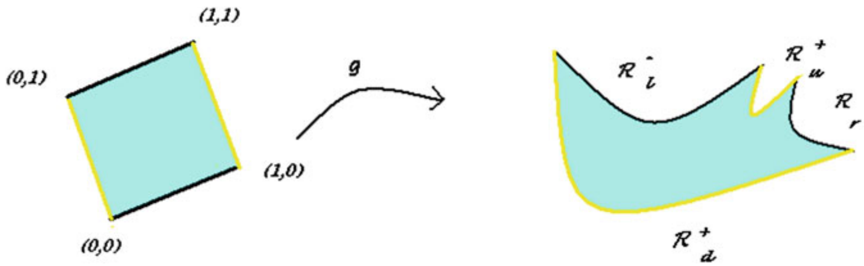

Fig. 2.1 The generalized rectangle with the oriented sides, homeomorphic to the unit square under the homeomorphism g

is also the union of two disjoint arcs. That is, we write

$$\mathscr{R}^+ = \mathscr{R}_b^+ \cup \mathscr{R}_t^+,$$

representing the "base" and "top" components of $\mathscr{R}^+$ as $\mathscr{R}_b^+$ and $\mathscr{R}_t^+$. We would like to mention here that the order in which we label "left" and "right" the two components of $\mathscr{R}^-$ (and, respectively, "base" and "top" the components of $\mathscr{R}^+$) is not important for what follows.

Both the term generalized rectangle for $\mathscr{R}$ and the decomposition of the contour $\vartheta\mathscr{R}$ into $\mathscr{R}^-$ and $\mathscr{R}^+$ are inspired by the construction of rectangular domains around hyperbolic sets arising in the theory of Markov partitions ([13], p. 291), [14], as well as by the Conley–Ważewski theory ([15], p. 133). Broadly speaking, the sets labeled as $[.]^-$, or as $[.]^+$, are made by those points which are moved by the flow outward, or inward, respectively with respect to $\mathscr{R}$. Although in the applications discussed in the book, the space X is simply the interval or the Euclidean plane and the generalized rectangles are compact regions bounded by graphs of some functions, our definitions are general enough to be applied to different situations.

It is important to notice that, given an oriented rectangle in a metric space X, we can always find a homeomorphism $g : [0,1]^2 \to g([0,1]^2) = \mathscr{R}$ (with g possibly different from the map h defined above) such that

$$g(\{0\} \times [0,1]) = \mathscr{R}_l^-, \quad g(\{1\} \times [0,1]) = \mathscr{R}_r^-. \tag{2.12}$$

(see [16]).

In view of this remark, some properties related to $\mathscr{R}$ can be transferred to the unit square $[0,1]^2 \subseteq \mathbb{R}^2$ oriented in its natural (left side–right side) manner.

2.5 Stretching Along Paths

Definition 2.3 Suppose that $\psi : X \supseteq D_\psi \to X$ is a map defined on a set D_ψ and let $\widetilde{\mathscr{A}} := (\mathscr{A}, \mathscr{A}^-)$ and $\widetilde{\mathscr{B}} := (\mathscr{B}, \mathscr{B}^-)$ be oriented rectangles of a metric space X. Let $\mathscr{K} \subseteq \mathscr{A} \cap D_\psi$ be a compact set. We say that $(\mathscr{K}, \psi)$ *stretches* $\widetilde{\mathscr{A}}$ *to* $\widetilde{\mathscr{B}}$ *along the paths* and write

$$(\mathscr{K}, \psi) : \widetilde{\mathscr{A}} \Leftrightarrow\!\!\!\!\to \widetilde{\mathscr{B}}, \tag{2.13}$$

if the following conditions hold:

- ψ is continuous on $\mathscr{K}$;
- for every path $\gamma : [0,1] \to \mathscr{A}$ such that $\gamma(0) \in \mathscr{A}_l^-$ and $\gamma(1) \in \mathscr{A}_r^-$ (or $\gamma(0) \in \mathscr{A}_r^-$ and $\gamma(1) \in \mathscr{A}_l^-$), there exists a sub–interval $[t', t''] \subseteq [0,1]$ such that

$$\gamma(t) \in \mathscr{K}, \quad \psi(\gamma(t)) \in \mathscr{B}, \quad \forall\, t \in [t', t'']$$

and, moreover, $\psi(\gamma(t'))$ and $\psi(\gamma(t''))$ belong to different components of $\mathscr{B}^-$.

In the special case in which $\mathscr{K} = \mathscr{A}$, we simply write

$$\psi : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}.$$

We would like to explain Definition 2.3, of stretching with the aid of the following diagrams:

We point out that both in Definition 2.3 and the next one in Definition 2.4 the generalized rectangles $\mathscr{A}$ and $\mathscr{B}$ could be assumed to be contained in different metric spaces. For the applications under consideration here, we consider the case in which a single metric space is involved. If in particular (2.13) is satisfied by two or more disjoint compact sets $\mathscr{K}_i's$ we get a multiplicity of fixed points. On the other hand, when (2.13) holds for some iterate of ψ the existence of periodic points is ensured. Since the stretching along the paths property is preserved under composition of mappings (see Lemma 2.3), the presence of chaotic dynamics follows.

Definition 2.4 Let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set on a set D_ψ and let $\widetilde{\mathscr{A}} := (\mathscr{A}, \mathscr{A}^-)$ and $\widetilde{\mathscr{B}} := (\mathscr{B}, \mathscr{B}^-)$ be oriented rectangles of a metric space X. Let also $\mathscr{D} \subseteq \mathscr{A} \cap D_\psi$ be a compact set and let $m \geq 2$. We say that $(\mathscr{D}, \psi)$ stretches $\widetilde{\mathscr{A}}$ to $\widetilde{\mathscr{B}}$ along the paths with crossing number m and write

$$(\mathscr{D}, \psi) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow^m \widetilde{\mathscr{B}},$$

if there exist m pairwise disjoint compact sets

$$\mathscr{K}_0, \ldots, \mathscr{K}_{m-1} \subseteq \mathscr{D}$$

such that

$$(\mathscr{K}_i, \psi) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}, \quad i = 0, \ldots, m-1.$$

When $\mathscr{D} = \mathscr{A}$, we simply write

$$\psi : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow^m \widetilde{\mathscr{B}}.$$

A brief explanation of the Figs. 2.2 and 2.3 The rectangles $\mathscr{A}$ and $\mathscr{B}$, embedded in the plane, have been oriented by selecting, respectively, the sets $\mathscr{A}^-$ and $\mathscr{B}^-$ (drawn by thick lines). We describe the action of a map ψ such that $(\mathscr{H}, \psi) : \widetilde{\mathscr{A}} \Bumpeq\!\!\!\!\!\!\longrightarrow \widetilde{\mathscr{B}}$, In the first figure we have darkened the compact subset $\mathscr{H}$ of $\mathscr{A}$. For a path $\gamma : [0, 1] \to \mathscr{A}$ with $\gamma(0) \in \mathscr{A}_r^-$ and $\gamma(1) \in \mathscr{A}_l^-$ belonging to different components of $\mathscr{A}$, there is a sub path $\gamma(t')$ to $\gamma(t'')$. Under the composition with ψ, $\psi(\gamma(t')) \in \mathscr{B}_r^-$ and $\psi(\gamma(t'')) \in \mathscr{B}_l^-$, the different components of $\mathscr{B}^-$ shown with darker lines.

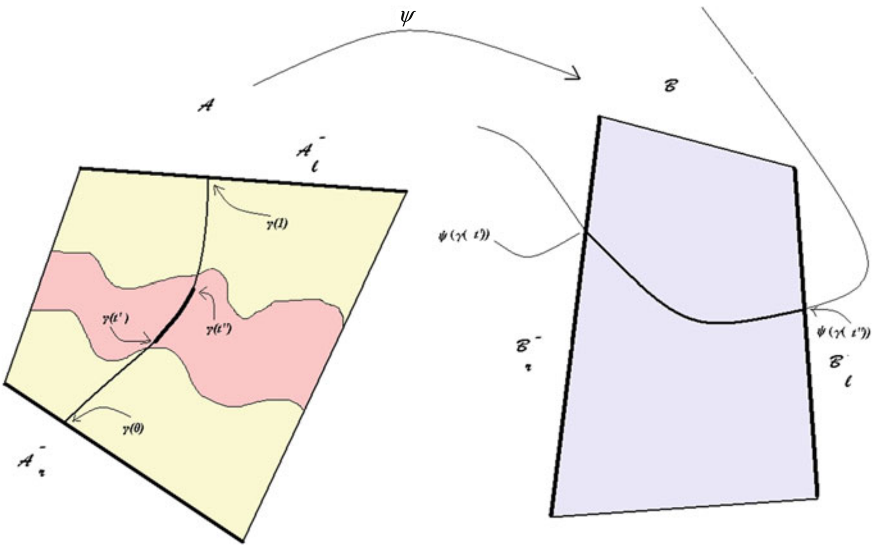

Fig. 2.2 The stretching of curve γ under the map ψ

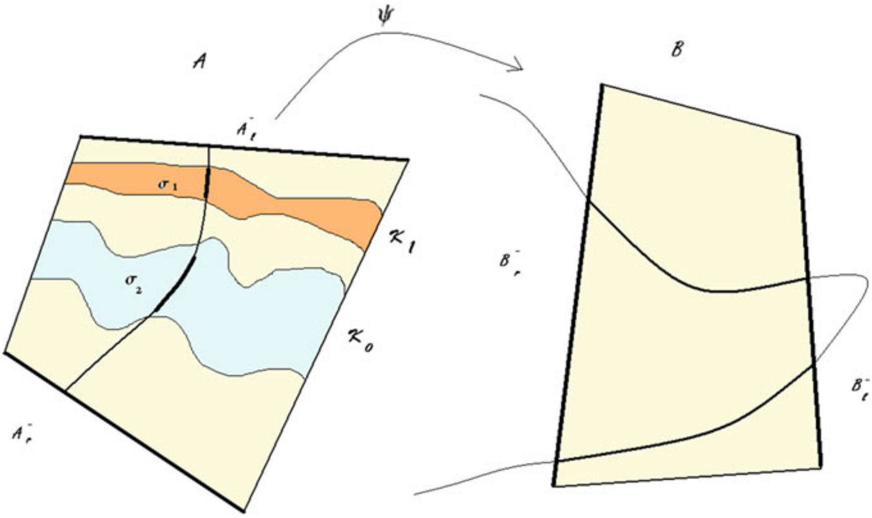

Fig. 2.3 The stretching of curves σ_1 and σ_2 under the map ψ

The term "generalized rectangle" for $\mathscr{R}$ as well as the decomposition of the contour $\vartheta\mathscr{R}$ into $\mathscr{R}^-$ and $\mathscr{R}^+$ are inspired, as commented earlier on by the constructions of rectangular domains around hyperbolic sets arising in the theory of Markov partitions ([13], p. 291), as well as by the Conley–Ważewski theory [15, 17]. Roughly speaking, in such frameworks, the set labeled as $[.]^-$ is made by those points which are moved by the flow outward and inward respectively, with respect to $\mathscr{R}$. One can find other similarities between our approach and the works quoted above, though with a different degree of generality.

One important feature of the "stretching" property is that, when 2.13 is satisfied with $\mathscr{A} = \mathscr{B}$, it is possible to find a fixed point for the map ψ in $\mathscr{K}$. Thus, we get information not only on the existence of such a point in the domain of ψ, but we can also localize it to the compact set $\mathscr{K}$. Hence if the stretching condition is satisfied with respect to different, disjoint $\mathscr{K}'s$, we obtain multiple fixed points.

2.6 The Crossing Number

We would like to point out that our stretching property is more restrictive than the analogous one considered by Kennedy and Yorke [7], concerning connections and preconnections. Indeed, in ([7], horseshoe hypotheses), the authors deal with a locally connected and compact subset Q of the separable metric space X, on which they select two disjoint and compact sets $end_0, end_1 \subseteq Q$ so that any component of Q intersects both of them. On Q a continuous map $f : Q \to X$ is defined in such a way that every compact connected set $\Gamma \subseteq Q$ that joins end_0 and end_1 (i.e. a connection according to [7]) admits at least $m \geq 2$ disjoint compact and connected subsets, whose images under f are again connections. Such sub-continua are named *preconnections* and m is the so-called *crossing number*. Making use of Kennedy and Yorke's language and notation, we can see that the sets $\mathscr{R}_l^-$ and $\mathscr{R}_r^-$ in our definition of the oriented rectangle can be seen to be the same as the special kind of the sets end_0 and end_1. Moreover, any path γ with values in $\mathscr{R}$ and joining $\mathscr{R}_l^-$ with $\mathscr{R}_r^-$ determines a connection via its image $\bar{\gamma}$. Similarly, any sub-path ω of γ with $\omega = \gamma \mid_{[t', t'']}$ as in the "stretching" definition, makes $\bar{\omega}$ a preconnection. As a consequence, when there exist $m \geq 2$ pairwise disjoint compact sets $\mathscr{K}_0, \ldots, \mathscr{K}_{m-1} \subseteq \mathscr{R}$ such that

$$(\mathscr{K}_i, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}}, \quad \forall\, i = 0, \ldots, m-1,$$

we obtain the situation discussed by Kennedy and Yorke with m as a crossing number. Therefore, broadly speaking, our stretching condition may be looked upon as a special case of [7].

2.7 An Example of Oriented Rectangles with Reference to the Nonlinear Pendulum Equation

In this book we explore the presence of chaos in Nonlinear Pendulum type equations. To illustrate this we revert back to the application of the classical pendulum depicted earlier in Eqs. (1.13) and (1.14) by the systems of equations I_b and I_a.

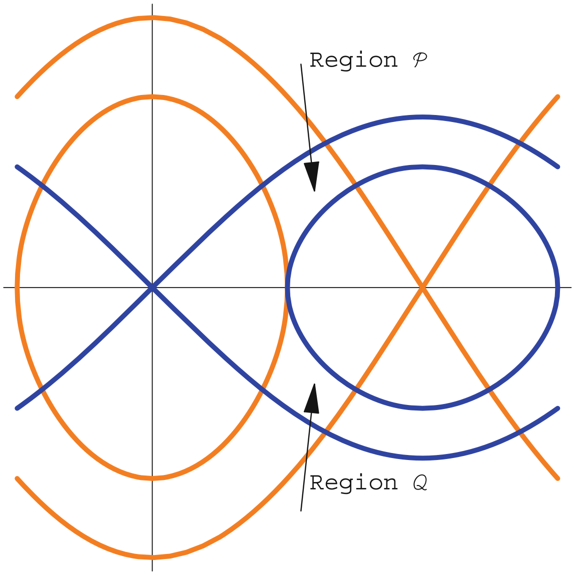

Fig. 2.4 An example of linked annuli determining the two regions $\mathscr{P}$ and $\mathscr{Q}$ for $f(x) = \sin(\pi x)$

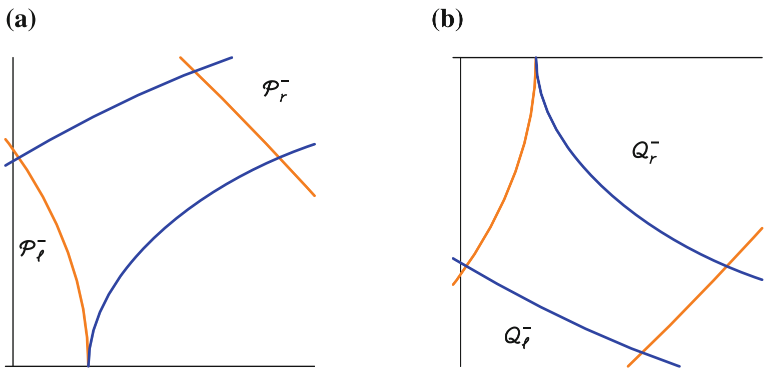

Fig. 2.5 The two regions $\mathscr{P}$ and $\mathscr{Q}$ determined by the two systems of level lines are then oriented, by suitably choosing which are the $[\cdot]^-$-sets. **a** The upper region $\mathscr{P}$. **b** The lower region $\mathscr{Q}$

A superimposition of the phase portraits of the two systems gives rise to linked annuli from which we construct the oriented rectangles. Such oriented rectangles are obtained by intersecting a region included between a pair of level lines of one system with a region between a pair of level lines of the other. The intersection of these two sets gives rise to two regions $\mathscr{P}$ and $\mathscr{Q}$ with no interior points in common as shown in Fig. 2.4.

The two rectangles $\mathscr{P}$ and $\mathscr{Q}$ are then oriented suitably by choosing the $[\cdot]^-$-sets (see Fig. 2.5).

We next establish a few important theorems and lemmas integral to our approach:

Theorem 2.2 *Let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set D_ψ and let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ be an oriented rectangle of a metric space X: If $\mathscr{K} \subseteq \mathscr{R} \cap D_\psi$ is a compact set for which*

$$(\mathscr{K}, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}},$$

then there is at least one point $z \in \mathscr{K}$ such that $\psi(z) = z$.

Proof By definition of the oriented rectangle, one can find a homeomorphism g of the plane onto itself, such that $g(\mathscr{G}) = \mathscr{R}$ and satisfying (2.12), where $\mathscr{G} := [0, 1]^2$.

We investigate the presence of a fixed point by taking the inverse of g, and confining ourselves to the compact set $\mathscr{H} := g^{-1}(\mathscr{K})$ for the continuous planar map $\phi := g^{-1} \circ \psi \circ g : \mathscr{G} \supseteq D_\phi \to \mathscr{G}$, defined on $D_\phi := g^{-1}(D_\psi)$. The stretching assumption $(\mathscr{K}, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}}$ can now be written as

$$(\mathscr{H}, \phi) : \widetilde{\mathscr{G}} \Leftrightarrow\!\!\!\!\!\rightarrow \widetilde{\mathscr{G}}, \tag{2.14}$$

where the unit square $\mathscr{G}$ is oriented in the usual left–right manner. We would like to mention that if w is a fixed point for ϕ in $\mathscr{H}$, then $z := g(w)$ is a fixed point for ψ in $\mathscr{K}$. To obtain a fixed point for $\phi = (\phi_1, \phi_2)$ in $\mathscr{G}$ we solve the system of equations

$$x_1 - \phi_1(x) = 0, \quad x_2 - \phi_2(x) = 0, \quad \text{for } x = (x_1, x_2) \in \mathscr{H}. \tag{2.15}$$

The first equation in (2.15) suggests that we introduce the compact set

$$W := \{x \in \mathscr{H} : 0 \le \phi_2(x) \le 1,\ x_1 - \phi_1(x) = 0\}. \tag{2.16}$$

The next step in the proof is to verify the following claim:

Claim *W contains a continuum (i.e. a compact connected set) $\mathscr{C}$ which joins the lower side $[0, 1] \times \{0\}$ to the upper side $[0, 1] \times \{1\}$ of $\mathscr{G}$.*

Once the existence of such a set $\mathscr{C}$ is proved, the hypothesis easily follows. By the definition of W, it follows that $\phi_2(z) \in [0, 1], \forall\ z \in \mathscr{C}$. Hence, for any $p = (p_1, p_2) \in \mathscr{C} \cap ([0, 1] \times \{0\})$ we have $p_2 - \phi_2(p) = 0 - \phi_2(p) \le 0$ and, similarly, for every $p = (p_1, p_2) \in \mathscr{C} \cap ([0, 1] \times \{1\})$ we have $p_2 - \phi_2(p) = 1 - \phi_2(p) \ge 0$. Since $x_2 - \phi_2(x)$ changes its sign (in a weak sense) along the connected set $\mathscr{C}$, by the Bolzano theorem we are ensured of the existence of at least a point $w = (w_1, w_2) \in \mathscr{C} \subseteq \mathscr{H}$ such that $w_2 - \phi_2(w) = 0$ and thus w is a solution of the second equation in (2.15). On the other hand, $w \in \mathscr{C} \subseteq W$ and therefore it is a solution of the first equation in (2.15) too. As a consequence, w is a solution of (2.15) and is therefore a fixed point of ϕ in $\mathscr{H}$.

Thus, to complete the proof, we have only to verify the above stated claim. For this purpose, we rely on the following known property from plane topology [18] that we call the *Crossing Lemma* as in [19]. It asserts that a compact subset of the unit square

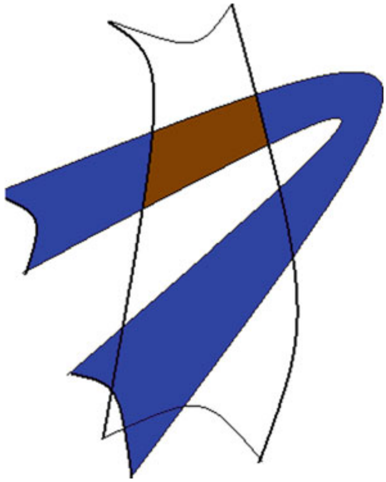

Fig. 2.6 Oriented cells showing crossings after a stretch and bend. Oriented cells crossing into a slice and thus giving a fixed point in for a homeomorphism. The $[.]^{-}$ sets are indicated with a *bold line*. Among the two cells which are the connected components of the intersection only one that is upper cell has the proper crossing

$\mathscr{G}$ which cuts any path in $\mathscr{G}$ connecting two opposite sides of the boundary of $\mathscr{G}$ must contain a continuum joining the other two sides. In other words, we have the following.

Lemma 2.2 *Let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^{-})$ be an oriented rectangle in a metric space X and suppose that $S \subseteq \mathscr{R}$ is a compact set such that*

$$s \cup \bar{\gamma} \neq \emptyset,$$

for each path $\gamma : [0, 1] \to \mathscr{R}$ satisfying $\gamma(0) \in \mathscr{R}_l^{-}$ and $\gamma(1) \in \mathscr{R}_r^{-}$, then there exists a compact connected set $\mathscr{C} \subseteq S$ such that

$$\mathscr{C} \cap \mathscr{R}_d^{+} \neq \emptyset, \quad \mathscr{C} \cap \mathscr{R}_u^{+} \neq \emptyset.$$

We give only a sketch of the proof. The missing details can be found in [20] or in [16, 21] (For a figure of a general oriented rectangle see Fig. 2.6.).

Firstly using the inverse of the homeomorphism $h : \mathscr{G} \to \mathscr{R}$, we confine ourselves to the study of a compact set $S \subseteq \mathscr{G}$ having the property of meeting any path γ contained in $\mathscr{G}$, with γ joining the left and the right sides of $\mathscr{G}$. If, by contradiction, we assume that S does not contain any compact connected set $\mathscr{C}$ intersecting both the lower and the upper sides of $\mathscr{G}$, by the Whyburn Lemma (the details of which we do not give here but which can be seen in [22]) we may find a decomposition of S into two disjoint compact subsets A and B such that

$$A \cup ([0, 1] \times \{0\}) \neq \emptyset, \quad A \cup ([0, 1] \times \{1\}) \neq \emptyset$$

$$B \cup ([0, 1] \times \{1\}) \neq \emptyset, \quad B \cup ([0, 1] \times \{0\}) \neq \emptyset.$$

The contradiction is now achieved by showing that there is a path γ contained in $\mathscr{G}\backslash S$ and joining the left and the right sides of $\mathscr{G}$. The existence of such a special path avoiding $S = \mathscr{A} \cup \mathscr{B}$ may be proved by different techniques of topological or combinatorial nature.

The result just proved differs from the classical Brouwer fixed point theorem in the following sense. It is a well-known fact that the fixed point property for continuous maps is preserved by homeomorphisms. Therefore, any continuous self-map of a set homeomorphic to the closed unit ball of $\mathbb{R}^N$ has at least a fixed point. In particular, a continuous map ψ satisfying

$$\psi : \mathscr{R} \to \mathscr{R}, \tag{2.17}$$

with $\mathscr{R}$ a generalized rectangle of a metric space X, has a fixed point in $\mathscr{R}$. The situation depicted in Theorem 2.2 is quite different. Firstly, the stretching assumption $(\mathscr{K}, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}}$ *does not imply* $\psi(\mathscr{R}) \subseteq \mathscr{R}$ and hence neither Eq. (2.17). Secondly, ψ needs to be continuous only on $\mathscr{K}$ and not necessarily on the whole set $\mathscr{R}$. Finally, as already pointed out, our result also localizes the presence of a fixed point in the subset $\mathscr{K}$. From the point of view of the applications, this means that we are able to obtain a multiplicity of fixed points provided that the stretching condition is satisfied with respect to different compact subsets of $\mathscr{R}$. A development of this perspective is contained in Theorem 2.4 below where we also consider the iterations of a given map, in order to find periodic points. For the sake of simplicity, we discuss only the case of two disjoint compact subsets of $\mathscr{R}$ for which the stretching hypothesis is satisfied. This situation can be easily extended to the framework of $m \geq 2$ compact pairwise disjoint subsets [9].

2.8 Chaotic Dynamics Induced by the Stretching Property

We next state and prove an important theorem:

Theorem 2.3 *Let* $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ *be an oriented rectangle of a metric space* X *and let* $\mathscr{D} \subseteq \mathscr{R} \cap D_\psi$, *with* D_ψ *the domain of map* $\psi : D_\psi \to X$. *If* $\mathscr{K}_0$ *and* $\mathscr{K}_1$ *are two disjoint compact sets with* $\mathscr{K}_0 \cup \mathscr{K}_1 \subseteq \mathscr{D}$ *and*

$$(\mathscr{K}_i, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\!\!\!\rightarrow \widetilde{\mathscr{R}}, \quad \text{for } i = 0, 1,$$

then ψ *induces chaotic dynamics on two symbols on* $\mathscr{D}$ *relatively to* $\mathscr{K}_0$ *and* $\mathscr{K}_1$. *It follows that the map* ψ *possesses the properties (i)–(v) of Theorem 2.1.*

The proof follows from some intermediate results which we prove. As a first step, we prove a simple lemma which shows that the "stretching" property is preserved by the composition of maps.

Lemma 2.3 *Let* $\phi : X \supseteq D_\phi \to X$ *and* $\psi : X \supseteq D_\psi \to X$ *be maps defined on the sets* D_ϕ *and* D_ψ *respectively, and let* $\widetilde{\mathscr{A}} := (\mathscr{A}, \mathscr{A}^-)$, $\widetilde{\mathscr{B}} := (\mathscr{B}, \mathscr{B}^-)$ *and* $\widetilde{\mathscr{C}} :=$

$(\mathscr{C}, \mathscr{C}^-)$ be oriented rectangles of a metric space X. Assume that $\mathscr{H} \subseteq \mathscr{A} \cap D_\phi$ and $\mathscr{K} \subseteq \mathscr{B} \cap D_\psi$ are compact sets such that

$$(\mathscr{H}, \phi) : \widetilde{\mathscr{A}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{B}} \quad (\mathscr{K}, \psi) : \widetilde{\mathscr{B}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{C}}$$

Then it follows that

$$(\mathscr{H} \cap \phi^{-1}(\mathscr{K}), \phi \circ \psi) : \widetilde{\mathscr{A}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{C}}.$$

Proof Let $\gamma : [0, 1] \to \mathscr{A}$ be a path such that $\gamma(0)$ and $\gamma(1)$ belong to the different sides of $\mathscr{A}^-$. Then, since $(\mathscr{H}, \phi) : \widetilde{\mathscr{A}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{B}}$, there exists a sub-interval $[t', t''] \subseteq [0, 1]$ such that

$$\gamma(t) \in \mathscr{H}, \quad \phi(\gamma(t)) \in \mathscr{B} \quad \forall\, t \in [t', t'']$$

and, moreover, $\phi(\gamma(t'))$ and $\phi(\gamma(t''))$ belong to different components of $\mathscr{B}^-$. Let ω be the the restriction of γ to the sub-interval $[t', t'']$ and define $\nu := \phi \circ \omega : [t', t''] \to \mathscr{B}$. Note that $\nu(t')$ and $\nu(t'')$ belong to the different sides of $\mathscr{B}^-$ and so, by the stretching hypothesis $(\mathscr{K}, \psi) : \widetilde{\mathscr{B}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{C}}$ there exists a sub-interval $[s', s''] \subseteq [t', t'']$ such that

$$\nu(t) \in \mathscr{K}, \quad \psi(\nu(t)) \in \mathscr{C}, \quad \forall\, t \in [s', s''],$$

with $\psi(\nu(s'))$ and $\psi(\nu(s''))$ belonging to different components of $\mathscr{C}^-$. But, rewriting all in terms of γ this means that we have found a sub-interval $[s', s''] \subseteq [0, 1]$ such that

$$\gamma(t) \in \mathscr{H} \cap \phi^{-1}(\mathscr{K}), \quad \psi(\phi(\gamma(t))) \in \mathscr{C}, \quad \forall\, t \in [s', s'']$$

and $\psi(\phi(\gamma(s')))$ and $\psi(\phi(\gamma(s'')))$ belong to the different sides of $\mathscr{C}^-$. By the arbitrariness of the path γ, the stretching property

$$(\mathscr{H} \cap \phi^{-1}(\mathscr{K}), \phi \circ \psi) : \widetilde{\mathscr{A}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{C}}.$$

is thus proved. We just point out that the continuity of the composite mapping $\psi \circ \phi$ on the compact set $\mathscr{H} \cap \phi^{-1}(\mathscr{K})$ follows from the continuity of ϕ on $\mathscr{H}$ and of ψ on $\mathscr{K}$ respectively.

Lemma 2.4 *Let $\psi : X \supseteq D_\psi \to X$ be a map defined on a set $D - \psi$ and let $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ be an oriented rectangle of a metric space X. If $\mathscr{K}_0$ and $\mathscr{K}_1$ are two disjoint compact sets with $\mathscr{K}_0 \cup \mathscr{K}_1 \subseteq \mathscr{R} \cap D_\psi$ and*

$$(\mathscr{K}_i, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\rightarrow \widetilde{\mathscr{R}}, \quad \textit{for } i = 0, 1,$$

then the following conclusions hold:

- *The map* ψ *has at least a fixed point in* $\mathscr{K}_i,\ i = 0,\ 1$;
- *Given an integer* $j \geq 2$ *and a* $j+1-$*uple* $(s_0, \ldots, s_j)$ *with each* $s_i\ \in \{0, 1\}$, $i = 0, \ldots, j$ *and* $s_0 = s_j$ *then there exists a point* $w \in K_{s_0}$ *such that*

$$\psi^i(w)\ \in \mathscr{K}_{s_i},\quad \forall\, i = 0, \ldots, j \text{ and } \psi^j(w) = w.$$

Proof The first conclusion is an immediate consequence of Theorem 2.1. As regards the second assertion, it also follows from Theorem 2.1 applied this time to the composite mapping $\phi := \psi^j$. Indeed let us consider the compact set $\mathscr{H} := \{x \in \mathscr{K}_{s_0} : \psi^I(x) \in \mathscr{K}_{s_i},\ \forall\, i = 1, \ldots, j\} \subseteq \mathscr{K}_{s_0}$ Then, by Lemma 2.3 it can be shown that the stretching relation

$$(\mathscr{H}, \phi) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\!\!\!- \widetilde{\mathscr{R}}$$

is fulfilled and now the thesis follows from Theorem 2.1 and by the definition of the compact set $\mathscr{H}$.

In order to prove Theorem 2.3 we need another lemma stated now:

Lemma 2.5 *Let* $\psi : X \supseteq D_\psi \to X$ *be a map defined on a set* D_ψ *and let* $\widetilde{\mathscr{R}} := (\mathscr{R}, \mathscr{R}^-)$ *be an oriented rectangle in a metric space* X. *Assume that* $(\mathscr{K}_j)_{j\in\mathbb{Z}}$ *is a sequence of compact sets with* $\mathscr{K}_j \subseteq \mathscr{R} \cap D_\psi$ *and*

$$(\mathscr{K}_j, \psi) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\!\!\!- \widetilde{\mathscr{R}},\quad \forall\, j \in \mathbb{Z}.$$

Then the following conclusions hold:

- *There is a sequence of points* $(w_j)_{j\in\mathbb{Z}}$ *such that* $w_j\ \in \mathscr{K}_j$ *and* $\psi(w_j) = w_{j+1}$, *for all* $j \in \mathbb{Z}$;
- *For any* $l < m$ *couple of integers such that* $\mathscr{K}_l = \mathscr{K}_m$, *there exists a finite sequence of points* $(z_i)_{l\leq i\leq m}$, *with* $z_i\ \in \mathscr{K}_i$ *and* $\psi(z_i) = z_{i+1}$ *for each* $i = l, \ldots, m-1$, *such that* $z_m = z_l$ *that is,* z_l *is a fixed point of* ψ^{m-l}.

We prove the second point first. By 2.3 it holds that

$$(\mathscr{H}, \psi^{m-l}) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\!\!\!- \widetilde{\mathscr{R}}, \tag{2.18}$$

where

$$\mathscr{H} := \{x\ \in \mathscr{K}_l : \psi^{i-l}(x)\ \in \mathscr{K}_i,\ \forall\, i = l+1, \ldots, m\}.$$

Setting $\phi := \psi^{m-l}$, we can now write condition (2.18) as $(\mathscr{H}, \phi) : \widetilde{\mathscr{R}} \Longleftrightarrow\!\!\!\!\!\!\!- \widetilde{\mathscr{R}}$ and therefore the thesis follows by Theorem 2.1 and by the definition of the compact set $\mathscr{H}$. Note that ϕ is continuous on $\mathscr{H} \subseteq \mathscr{K}_l$ by the continuity of the map ψ on each $\mathscr{K}_i,\ i = l, \ldots, m-1$. Let's turn now to the first assertion. In analogy to what was done before, let us define the closed set

$$\mathscr{S} := \{x \in \mathscr{K}_0 : \psi^i(x) \in \mathscr{K}_i, \ \ \forall\, i \geq 1\} \tag{2.19}$$

and fix a path $\gamma : [0,\ 1] \to \mathscr{R}$ with $\gamma(0)$ and $\gamma(1)$ belonging to different components of $\mathscr{R}^-$. Then, since $(\mathscr{K}_0, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\to \widetilde{\mathscr{R}}$, there exists a sub-path $\gamma_0 : [t_0',\ t_0''] \to \mathscr{K}_0$ of γ such that $\psi(\gamma_0([t_0',\ t_0''])) \subseteq \mathscr{R}$, with $\psi(\gamma_0(t_0'))$ and $\psi(\gamma_0(t_0''))$ belonging to different sides of $\mathscr{R}^-$. Similarly, since $(\mathscr{K}_1, \psi) : \widetilde{\mathscr{R}} \Leftrightarrow\!\!\!\to \widetilde{\mathscr{R}}$ we can find a sub-path $\omega_1 : [t_1',\ t_1''] \to \mathscr{K}_1$ of $\omega_0 := \psi(\gamma_0)$ such that $\psi(\omega_1([t_1',\ t_1''])) \subseteq \mathscr{R}$, with $\psi(\omega_1(t_1'))$ and $\psi(\omega_1(t_1''))$ belonging to different components of $\mathscr{R}^-$. Setting

$$\Gamma_1 := \{x \in \gamma_0([t_0',\ t_0'']) : \psi(x) \ \in \omega_1([t_1',\ t_1''])\} \subseteq \{x \in \mathscr{K}_0 : \psi(x) \ \in \mathscr{K}_1\}$$

and proceeding by induction, we obtain a decreasing sequence of nonempty compact sets

$$\Gamma := \gamma([0,\ 1]) \supseteq \Gamma_0 := \gamma_0([t_0',\ t_0'']) \supseteq \cdots \supseteq \Gamma_n \supseteq \Gamma_{n+1} \supseteq \cdots$$

such that $\psi^{j+1}(\Gamma_j)$ joins the two components of $\mathscr{R}^-$, for $j \geq 0$. Moreover, for every $i \geq 1$, it holds that

$$\Gamma_i \subseteq \{x \in \mathscr{K}_0 : \psi^j(x) \in \mathscr{K}_j, \ \ \forall\, j \ : 1 \leq j \leq i\}.$$

By the Cantor Lemma we have that $\bigcap_{j=0}^{+\infty} \Gamma_j \neq \emptyset$ and for any $w \ \in \bigcap_{j=0}^{+\infty} \Gamma_j$ it holds that $\psi^n(w) \ \in \mathscr{K}_n, \ \ \forall\, n \in \mathbb{N}$. Thus the set $\mathscr{S}$ is nonempty. The thesis follows by a standard diagonal argument which allows to extend the result to bi-infinite sequences once it has been proved for one-sided sequences.

We now state and prove the following theorem which is central to our proof of the presence of chaotic dynamics.

Theorem 2.4 ([23]) *Let $\psi_r \ : X \supseteq D_{\psi_r} \to X$ and $\psi_s \ : X \supseteq D_{\psi_s} \to X$ be given maps. Let $\widetilde{\mathscr{M}} := (\mathscr{M}, \mathscr{M}^-)$ and $\widetilde{\mathscr{N}} := (\mathscr{N}, \mathscr{N}^-)$ be oriented rectangles in X with $\mathscr{M} \subseteq D_{\psi_r}$ and $\mathscr{N} \subseteq D_{\psi_s}$. Suppose that the following conditions are satisfied:*

($\mathbf{H_r}$) *There exists $m \geq 2$ pairwise disjoint compact sets $\mathscr{K}_1, \ldots, \mathscr{K}_m \subseteq \mathscr{M}$ such that*

$$(\mathscr{K}_i, \psi_r) : \widetilde{\mathscr{M}} \Leftrightarrow\!\!\!\to \widetilde{\mathscr{N}}, \quad \forall\, i = 1, \ldots, m;$$

($\mathbf{H_s}$) $\psi_s \ : \widetilde{\mathscr{N}} \Leftrightarrow\!\!\!\to \widetilde{\mathscr{M}}$. *Then the map $\psi := \psi_s \circ \psi_r$ induces chaotic dynamics on m symbols in the set*

$$\mathscr{K} := \bigcup_{i=1}^{m} \mathscr{K}_i\,.$$

Moreover, for each sequence of m symbols $\mathbf{s} = (s_n)_n \in \{1, \ldots, m\}^{\mathbb{N}}$, there exists a compact connected set $\mathscr{C}_{\mathbf{s}} \subseteq \mathscr{K}_{s_0}$ with

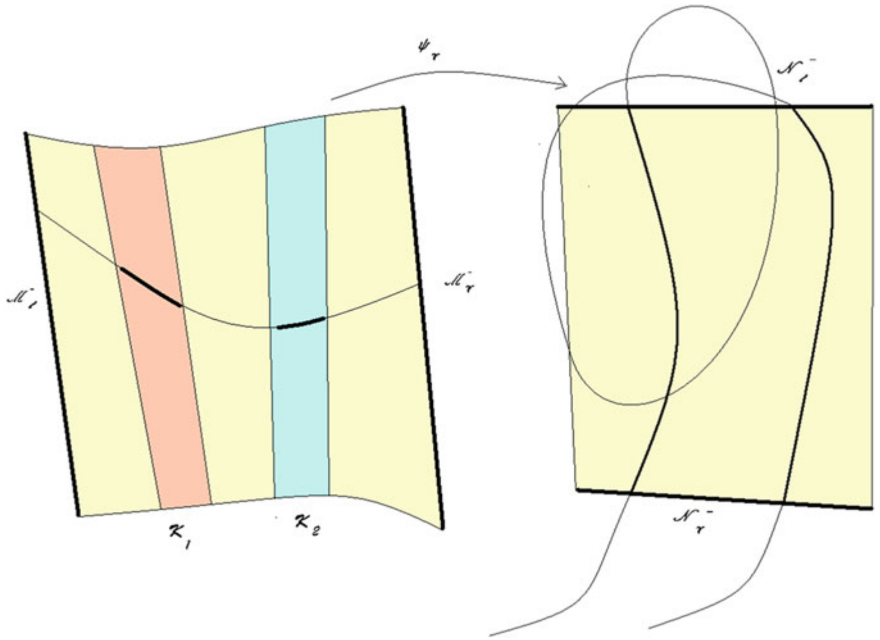

Fig. 2.7 The condition $(\mathbf{H_r})$

$$\mathscr{C}_s \cap \mathscr{M}_b^+ \neq \emptyset, \quad \mathscr{C}_s \cap \mathscr{M}_t^+ \neq \emptyset$$

and such that, for every $w \in \mathscr{C}_s$ *there exists a sequence* $(y_n)_n$ *with* $y_0 = w$ *and* $y_n \in \mathscr{K}_{s_n}, \quad \psi(y_n) = y_{n+1}, \ \forall\, n \geq 0.$

An explanation of the concepts $(\mathbf{H_r})$ and $(\mathbf{H_s})$ can be visualized with the help of the Figs. 2.7, and 2.8.

We would like to explain the figure giving the condition $(\mathbf{H_r})$. In the Fig. 2.7 there is a path across two disjoint compact sets $\mathscr{K}_1$ and $\mathscr{K}_2, \subseteq \widetilde{\mathscr{M}}$ with two sub paths shown by darkened lines one in each compact set. Following the Definition 2.3 of *stretching along paths* we see that the map ψ_r stretches these two sub paths across $\widetilde{\mathscr{N}}$, shown as two darkened vertical paths intersecting $\mathscr{N}_l^-$ and $\mathscr{N}_r^-$.

In the Fig. 2.8, for $(\mathbf{H_s})$, a sub path shown as a darkened part of a vertical path cutting $\mathscr{N}_l^-$ and $\mathscr{N}_r^-$, is stretched across $\widetilde{\mathscr{M}}$ in the correct direction, under the stretching action of the map ψ_s, which is across the ends $\mathscr{M}_l^-$ and $\mathscr{M}_r^-$. See [23] for a proof and [19] for some remarks and extensions. In the applications of Theorem 2.4 to the ODE models considered in the present book, we take $X \equiv \mathbb{R}^2$ and the maps ψ_r , ψ_s to be the Poincaré maps associated with some planar systems for different time intervals.

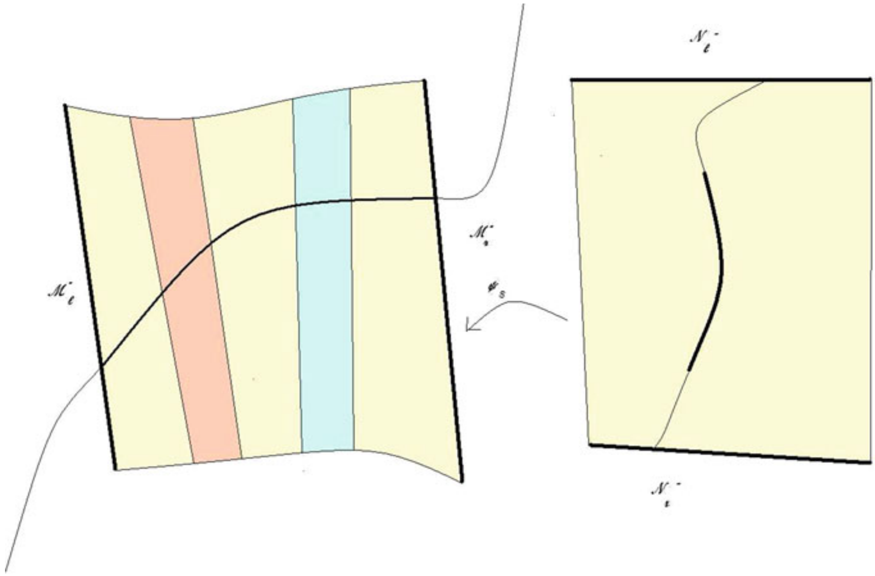

Fig. 2.8 The condition ($\mathbf{H_s}$)

2.9 Applications of the Concept of “Stretching Along Paths” to Some Pendulum Type Equations

2.9.1 The Vertically Driven Planar Pendulum

In this book we apply the concept of “stretching” to show the presence of Chaotic Dynamics in two different cases. The first of these is the case of a vertically driven planar pendulum (For a visualization of this pendulum see Fig. 2.9.). We investigate the presence of chaotic-like dynamics for a class of second order scalar ODEs of the form

$$\ddot{x} + q(t)f(x) = 0, \qquad \left(\dot{x}(t) = \frac{d}{dt}x(t)\right) \tag{2.20}$$

where $f = f(x) : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and $q = q(t) : \mathbb{R} \to \mathbb{R}$ is a T-periodic weight function which belongs to $L^1([0, T])$. The equation of a vertically driven planar pendulum equation with variable length could be given generally choosing $f(x) = \sin(x)$ as

$$\ddot{x} + q(t)\sin(x) = 0. \tag{2.21}$$

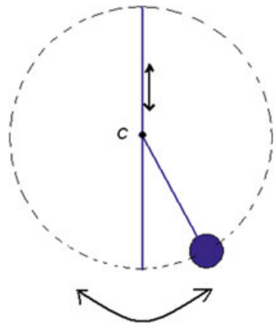

Fig. 2.9 The vertically driven pendulum, the point of suspension C moves up and down along with the too and fro motion

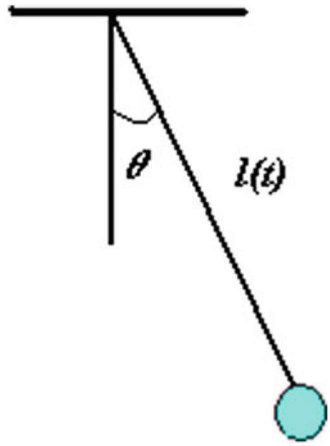

Fig. 2.10 The pendulum, whose length l is a function of time t

2.9.2 A Pendulum with Variable Length

We next apply the concept of "stretching" to show the presence of Chaotic Dynamics in the case of a pendulum with variable length. The figure, (Fig. 2.10) shows a frictionless pendulum, where the length $l(t)$ is a function of t.

The details of the presence of chaotic dynamics in these two cases are elaborated upon in detail in the next two chapters.

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