Chapter 2
Harmonic Univalent Mappings and Minimal Graphs

Zach Boyd and Michael Dorff

2.1 Introduction

In this chapter we will discuss some topics about planar harmonic mappings. These functions can be thought of as a generalization of analytic maps, and so we will first present a brief background of analytic univalent mappings. Then we will discuss harmonic mappings with an emphasis on three topics: the shearing technique, inner mapping radius, and convolutions. Finally, we will discuss the connection between planar harmonic mappings and minimal surfaces.

2.1.1 Analytic Univalent Maps

Harmonic maps naturally generalize analytic functions by relaxing the requirement of analyticity while still retaining some important features. We begin with an overview of the relevant properties of analytic functions to make clear the analogy with harmonic maps. In both cases, we focus entirely on functions which are *univalent*, or one-to-one, although much interesting work has been done on multivalent functions as well.

Definition 1.1 Let $F : D \subset \mathbb{C} \to \mathbb{C}$. The function $F(x, y) = u(x, y) + iv(x, y)$ is *analytic* if:

- F is continuous;
- u and v are real harmonic in D; and
- u and v are harmonic conjugates (that is, $u_x = v_y$ and $u_y = -v_x$).

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In this context, a function $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ is called *real harmonic* if $u_{xx} + u_{yy} = 0$.

While analytic functions may map from any open, connected set in general, the following theorem allows us to restrict attention to the unit disk in many cases.

Theorem 1.2 (Riemann Mapping Theorem) *Let $G \neq \mathbb{C}$ be a simply-connected domain with $a \in G$. Then there exists a unique univalent, onto analytic function $F : G \to \mathbb{D}$ such that $F(a) = 0$ and $F'(a) > 0$.*

Thus, if D is a simply-connected, proper subset of the complex plane, we may replace the function $f : D \to \mathbb{C}$ by the function $f \circ \phi : \mathbb{D} \to \mathbb{C}$, where the existence of $\phi : \mathbb{D} \to D$ is guaranteed. Therefore, in the study of univalent (one-to-one) analytic functions, we may restrict our attention to the following class of functions.

Definition 1.3 The family of analytic, normalized, and univalent functions denoted by S is

$$S = \{F : \mathbb{D} \to \mathbb{C} \mid F \text{ is analytic, univalent with } F(0) = 0, F'(0) = 1\}.$$

This family of functions is also known as *schlicht* functions. Note that $F \in S$ implies $F(z) = z + a_2 z^2 + a_3 z^3 + \cdots$. The following are two essential examples that will be used throughout the chapter.

Example 1.4 (The Analytic Right Half-Plane Mapping)

$$F_h(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \cdots \in S.$$

Example 1.5 (The Koebe Function)

$$F_k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n = z + 2z^2 + 3z^3 + \cdots \in S.$$

Observe that F_k maps to the entire complex plane minus a slit from $-1/4$ to ∞ (Fig. 2.1).

Some important properties of the family S include

- The uniqueness condition in the Riemann Mapping Theorem.
- (de Branges' Theorem) For $F \in S$, $|a_n| \leq n$, for all n.
- (Koebe $\frac{1}{4}$-Theorem) If $F \in S$, then $F(\mathbb{D})$ contains the disk $G = \{w : |w| < \frac{1}{4}\}$.

See [14] for more background in univalent analytic functions.

2.1.2 *Harmonic Univalent Maps*

Complex-valued harmonic functions are a generalization of the analytic functions in which one of the requirements is relaxed.

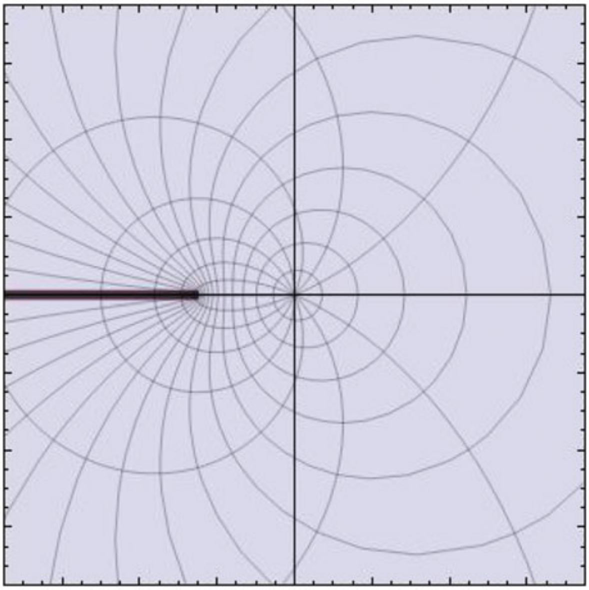

Fig. 2.1 The image of $\mathbb{D}$ under $F_k(z) = \frac{z}{(1-z)^2} \in S$

Definition 1.6 Let $f : D \subset \mathbb{C} \to \mathbb{C}$. The function $f(x, y) = u(x, y) + iv(x, y)$ is a (complex-valued) *harmonic function* if:

- f is continuous; and
- u and v are real harmonic in D.

This definition views harmonic functions as being composed of real and imaginary parts. If D is simply-connected, we have a more useful characterization ([3]).

Theorem 1.7 *If $f = u + iv$ is harmonic in a simply-connected domain G, then $f = h + \overline{g}$, where h and g are analytic.*

Note that $f = h + \overline{g}$ is equivalent to $f = \mathrm{Re}\{h + g\} + i\mathrm{Im}\{h - g\}$. Also, one consequence of this theorem is that a harmonic function f is represented by a power series of the form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n.$$

In particular, every harmonic function with domain $\mathbb{D}$ is just the sum of analytic and coanalytic parts, represented by h and g, respectively. To see the geometric effect of including $\overline{g}$, we recall that an analytic map is called *conformal* if its derivative never vanishes. The conformal property means that intersecting curves in the domain are mapped to intersecting curves in the image, and the angle of intersection is preserved. A harmonic map is the sum of two maps, one which preserves angles, and another which reverses them. After some reflection, it should be clear that if $|h'(z_0)| > |g'(z_0)|$, then the map is *sense-preserving* at z_0, meaning that positive angles

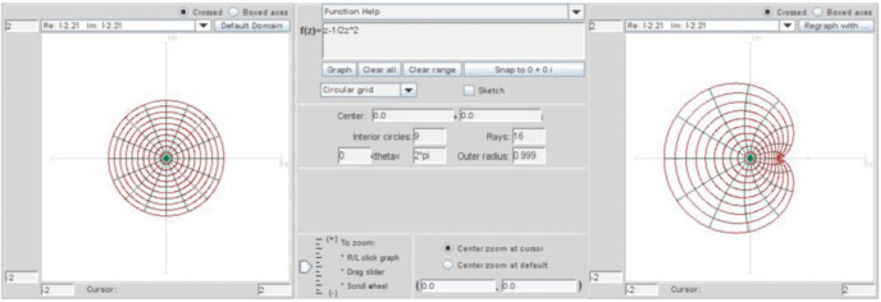

Fig. 2.2 The image of $\mathbb{D}$ under F_p

remain positive, and negative angles remain negative under the map f. Equivalently, we say that a function is sense-preserving if the left-hand side of a curve is mapped to the left-hand side of its image. The following theorem formalizes this intuition.

Theorem 1.8 **(Lewy [22])** *$f(z) = h(z) + \overline{g(z)}$ is locally univalent and sense-preserving if and only if $|\omega(z)| = |g'(z)/h'(z)| < 1$, for all $z \in \mathbb{D}$.*

The function $\omega = g'/h'$ is known as the *dilatation* of $f = h + \overline{g}$.

Observe that in the harmonic case, terms involving $\overline{z}$ are permissible, but terms involving $z\overline{z}$ are not. Also, the graphics highlight the fact that the images of radial and circular lines intersect at right angles in the conformal case, but not in the harmonic case.

The boundary of $f_p(\mathbb{D})$ in Fig. 2.3 consists of concave arcs and the boundary of $f_h(\mathbb{D})$ in Fig. 2.5 gets mapped to just two points, $w = -\frac{1}{2}$ and $w = \infty$. These examples illustrate a difference between analytic and harmonic maps and an important fact about the boundary behavior of certain harmonic functions.

Theorem 1.9 *Let $f = h + \overline{g}$ be a sense-preserving harmonic map with dilatation $\omega = g'/h'$. If $|\omega(z)| = 1$ for almost all z in an arc γ of $\partial\mathbb{D}$, then the image of γ under f is either a concave arc or a stationary point.*

Example 1.10 In the following pages, graphs of functions are usually the image of the unit disk under the function in question. Also, many of these images have been created by the online applet *ComplexTool* [9] (Figs. 2.2–2.5)

Example 1.11 The uniqueness part of the Riemann mapping theorem fails in the harmonic case, since both maps, F_h and f_h, send the disk to the same right half-plane.

Open Problem 1 *What is the analogue of the Riemann mapping theorem for harmonic mappings?*

As a final point in this section, we note that, in analogy to S, we define the classes S_H and S_H^O as follows.

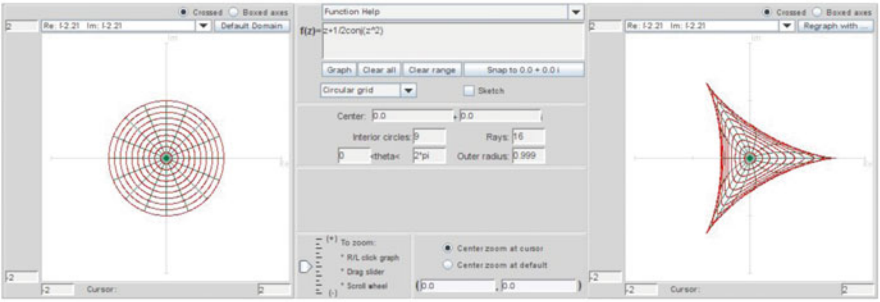

Fig. 2.3 The image of $\mathbb{D}$ under f_p

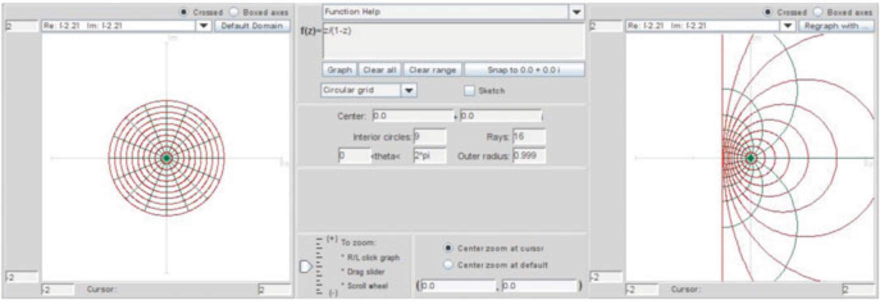

Fig. 2.4 The image of $\mathbb{D}$ under F_h

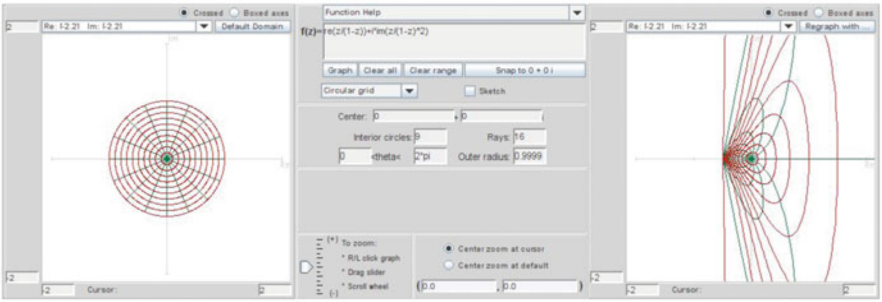

Fig. 2.5 The image of $\mathbb{D}$ under f_h

- **Analytic polynomial map:** $F_p(z) = z - \frac{1}{2}z^2$
- **Harmonic polynomial map:** $f_p(z) = z + \frac{1}{2}\bar{z}^2$
- **Analytic right half-plane map:** $F_h(z) = \frac{z}{1-z}$
- **Harmonic right half-plane map:** $f_h(z) = \mathrm{Re}(\frac{z}{1-z}) + i\,\mathrm{Im}(\frac{z}{(1-z)^2})$

Definition 1.12 Let S_H be the family of complex-valued harmonic, univalent mappings that are normalized on the unit disk, that is,

$$S_H = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is harmonic, univalent with}$$
$$f(0) = a_0 = 0, f_z(0) = a_1 = 1\}.$$
$$S_H^O = \{f \in S_H \mid f_{\bar{z}}(0) = b_1 = 0\}.$$

Thus, $S \subset S_H^O \subset S_H$. Other important classes include K, K_H, and K_H^O, which are the subclasses of S, S_H, and S_H^O containing only the *convex* functions, which are exactly those whose image is a convex domain in $\mathbb{C}$.

We now introduce some major unsolved problems in the field that have obvious analogues in the theory of analytic functions. For years, the biggest problem in the theory of univalent analytic functions was the *Bieberbach Conjecture*, now known as DeBrange's Theorem. Solving this problem allows us to know the sharp bounds on growth and distortion of harmonic maps, among other things. In the nonanalytic case, we have the following.

Conjecture 1 (Harmonic Bieberbach Conjecture) Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \bar{z}^n \in S_H^O.$$

Then

- $|a_n| \leq \frac{1}{6}(n+1)(2n+1)$,
- $|b_n| \leq \frac{1}{6}(n-1)(2n-1)$,
- $||a_n| - |b_n|| \leq n$.

Currently, the best bound is that for all functions $f \in S_H^O$, $|a_2| < 49$ ([15]). The conjecture is that $|a_2| \leq \frac{5}{2}$.

Open Problem 2 *Prove a bound on $|a_2|$ that is lower than 49.*

Recall that for analytic functions we have the Koebe $\frac{1}{4}$-Theorem, which expresses bounds on the distortion of the unit disk under normalized analytic maps. In the harmonic case, we have

Conjecture 2 If $f \in S_H^O$, then $f(\mathbb{D})$ contains the disk $G = \{w : |w| < \frac{1}{6}\}$.

Currently, the best result is that the range of $f \in S_H^O$ contains the disk $\{w : |w| < \frac{1}{16}\}$.

Open Problem 3 *Prove that the radius can be increased to K where $\frac{1}{16} < K \leq \frac{1}{6}$.*

2.2 Shearing

In their paper, Clunie and Sheil-Small introduced the shearing technique that provides a procedure for constructing harmonic maps $f = h + \overline{g}$ that are univalent. Before describing the shearing technique, we need the following definition.

Definition 2.1 A domain Ω is convex in the horizontal direction (CHD) if every line parallel to the real axis has a connected intersection with Ω.

We can now state the shearing theorem.

Theorem 2.2 (Clunie and Sheil-Small, [3]) *Let $f = h + \overline{g}$ be a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$). The function $F = h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain if and only if $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain.*

Summary of the Shearing Technique: To use the shearing technique we start with

- an analytic function F that is CHD, and
- an analytic function ω such that $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

Then we

- write F as $F = h - g$ and ω as $\omega = g'/h'$, and
- explicitly solve for h and g.

The resulting harmonic function $f = h + \overline{g}$ is guaranteed to be univalent.

Notice that it is easy to reformulate Clunie and Sheil-Small's shearing theorem for functions which are convex in other directions. In particular, consider the case of convex in the vertical direction (CVD) which we will use in this chapter.

Definition 2.3 A domain Ω is CVD if every line parallel to the imaginary axis has a connected intersection with Ω.

Theorem 2.4 *Let $f = h + \overline{g}$ be a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$). The function $F = h + g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CVD domain if and only if $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a CVD domain.*

Example 2.5 Consider the analytic function

$$F_p(z) = z - \tfrac{1}{2}z^2.$$

This is the analytic polynomial map F_p given in Example 2.10. It is CHD. Now choose a dilatation. We will choose

$$\omega(z) = g'(z)/h'(z) = z.$$

Note that $|\omega(z)| < 1\ \forall z \in \mathbb{D}$. Next, set $h(z) - g(z) = F_p(z) = z - \frac{1}{2}z^2$. Taking the derivative of both sides, yields $h'(z) - g'(z) = 1 - z$. Since $g'(z) = zh'(z)$, we substitute $g'(z)$ into the previous equation to get $h'(z) = 1$. Integrating this and normalizing it so that $h(0) = 0$, yields $h(z) = z$. Because $g'(z) = zh'(z)$, we can solve for g to get $g(z) = \frac{1}{2}z^2$. Hence, by the Shearing Theorem

$$f_p(z) = h(z) + \overline{g(z)} = z + \tfrac{1}{2}\,\overline{z}^2 \in S_H^O.$$

Thus, we have constructed a harmonic function f_p that is univalent and CHD. Note that this is the harmonic polynomial function f_p in Example 2.10.

Example 2.6 Consider

$$F_k(z) = h(z) - g(z) = \frac{z}{(1-z)^2} \qquad \text{with} \qquad \omega(z) = z.$$

Using the same approach as above, we get

$$f_k(z) = h(z) + \overline{g(z)} = \mathrm{Re}\Big(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\Big) + i\mathrm{Im}\Big(\frac{z}{(1-z)^2}\Big) \in S_H^O.$$

The harmonic function f_k is a slit mapping which maps $\mathbb{D}$ onto $\mathbb{C}$ minus a slit on the negative real axis with the tip of the slit at $-\frac{1}{6}$. There is considerable evidence that f_k can fill a role in harmonic function theory similar to that of the Koebe function in analytic function theory, and for this reason, f_k is called the *harmonic Koebe function*.

To help explore how shearing affects the geometry between analytic and harmonic mappings, one can use the online applet *ShearTool* [9]. The image below demonstrates the functionality of this applet, which simultaneously plots both $h - g$ and $h + \overline{g}$ (Fig. 2.6).

Almost all examples of shearing have used dilatations that are finite Blaschke products. One important type of mappings that are not finite Blaschke products is a singular inner function. We give a brief description of this topic. For more details, see [21].

Definition 2.7 A bounded analytic function f is called an *inner function* if $|\lim_{r\to 1^-} f(re^{i\theta})| = 1$ almost everywhere with respect to Lebesgue measure on $\partial\mathbb{D}$. If f has no zeros on $\mathbb{D}$, then f is called a *singular inner function*.

Every inner function can be expressed in the form

$$f(z) = e^{i\alpha} B(z) \exp\left(-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(e^{i\theta})\right),$$

where $\alpha, \theta \in R$, μ is a positive measure on $\partial\mathbb{D}$, and $B(z)$ is a Blaschke product, i.e., $B(z) = e^{i\theta} \prod_{j=1}^{\infty} (\frac{z-a_j}{1-\overline{a_j}z})^{m_j}$, for some series of constants $|a_j| < 1$ satisfying $\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$.

The function $f(z) = e^{\frac{z+1}{z-1}}$ is an example of a singular inner function. Weitsman [29] provided the following example.

Example 2.8 Shear

$$h(z) - g(z) = \frac{z}{1-z} + \frac{1}{2} e^{\frac{z+1}{z-1}} \qquad \text{with} \qquad \omega(z) = e^{\frac{z+1}{z-1}}.$$

By a result by Pommenke [27], it can be shown that $h - g$ is convex in the direction of the real axis. Shearing $h - g$ with $\omega(z) = e^{\frac{z+1}{z-1}}$ and normalizing yields

$$h(z) = \int \frac{1}{(1-z)^2}\, dz = \frac{z}{1-z}.$$

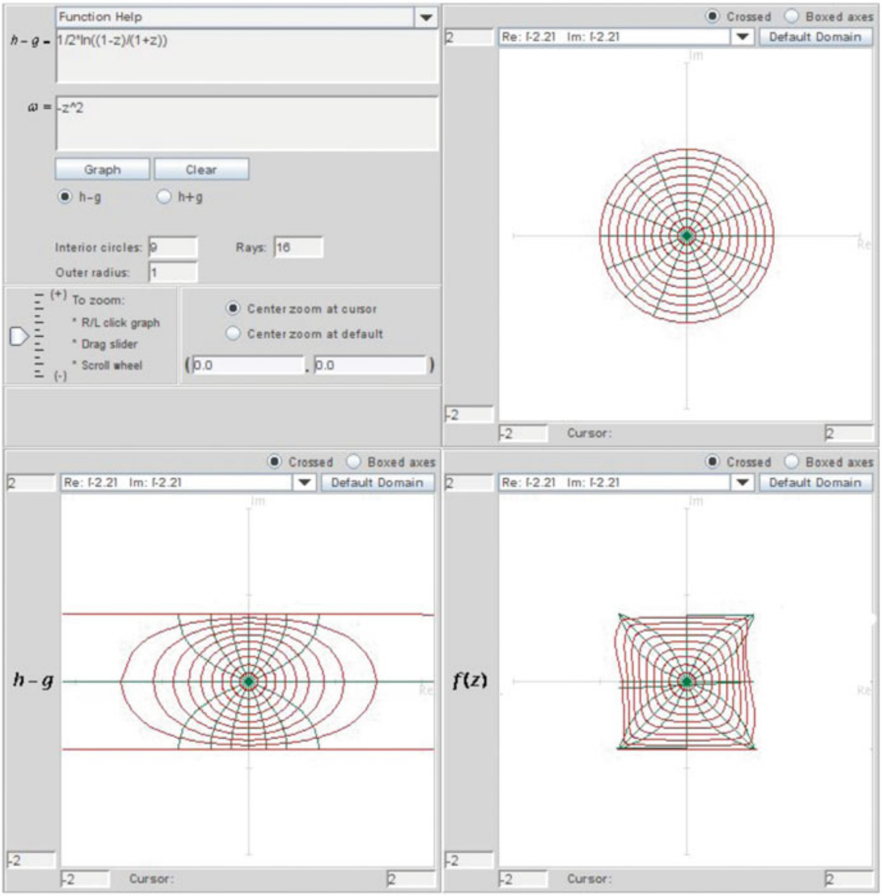

Fig. 2.6 The image of $\mathbb{D}$ under the $f = h + \overline{g}$ is shown in the *bottom right*, where f is constructed from shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = -z^2$

Solving for g we get

$$g(z) = -\frac{1}{2}e^{\frac{z+1}{z-1}}.$$

The image given by the map is similar to the image given by the right half-plane map $\frac{z}{1-z}$ except that there are an infinite number of cusps (Fig. 2.7).

A technique to find harmonic mappings whose dilatations are singular inner functions involves using a theorem by Clunie and Sheil-Small [3].

Theorem 2.9 *Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that $h + \epsilon g$ is convex for some $|\epsilon| \leq 1$. Then f is univalent.*

To develop the technique, we let $\epsilon = 0$ in Theorem 9. This means that if h is analytic convex and if ω is analytic with $|\omega(z)| < 1$, then $f = h + \overline{g}$ is a harmonic

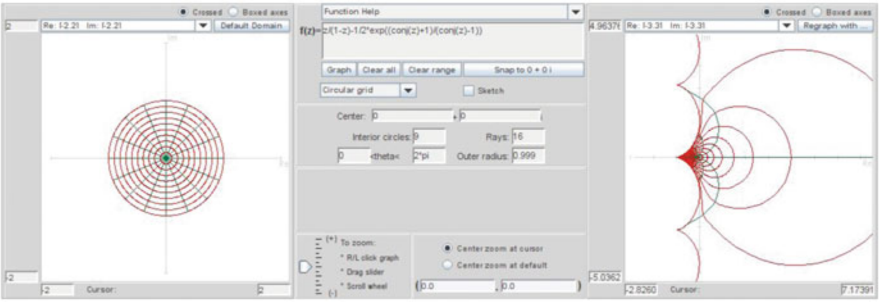

Fig. 2.7 Image of $\mathbb{D}$ under $f(z) = \frac{z}{1-z} - \frac{1}{2}e^{\frac{\bar{z}+1}{\bar{z}-1}}$

univalent mapping. To establish that a function f is convex, we will use the following theorem ([14]).

Theorem 2.10 *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then f is univalent and maps onto a convex domain if and only if*

$$\mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geq 0, \textit{ for all } z \in \mathbb{D}.$$

Example 2.11 Let

$$h(z) = z + 2\log(z+1) \quad \text{with} \quad \omega(z) = g'(z)/h'(z) = e^{\frac{z-1}{z+1}}.$$

Using Theorem 10, we can show that h is convex. Then solving for g we get $g(z) = (z+1)e^{(z-1)/(z+1)}$.

Hence,

$$f(z) = h(z) + \overline{g(z)} = z + 2\log(z+1) + (\bar{z}+1)e^{\frac{\bar{z}-1}{\bar{z}+1}}.$$

By Theorem 9, $f = h + \bar{g}$ is univalent. The image of $\mathbb{D}$ under f is shown in Fig. 2.8.

Open Problem 4 *Construct examples of harmonic univalent functions whose dilatation is a singular inner function and determine properties of these functions.*

2.3 Inner Mapping Radius

The analytic Koebe function F_k is an important function. It is extremal (or maximal) in several important senses. It is the function in S that gives equality for the coefficient bounds in deBranges' Theorem. It is the function that maps the unit disk to a domain that contains the largest possible disk centered at the origin as described in the Koebe $\frac{1}{4}$-Theorem. It is the function that exhibits both the largest and smallest possible http://www.springer.com/978-81-322-2112-8

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