

Chapter 2

Limits of Real Sequences

Any real number is made accessible through its rational approximations, for example, cutting off the decimals starting with $(n + 1)$ th one. As n increases, these approximations come closer to the given real number, a process that lies at the heart of the subject of convergence. The study of many algorithms (such as the Babylonian algorithm for extracting the square root) needs some theoretical considerations of convergence and limits, which can be found in this chapter.

2.1 Convergent Sequences

The notion of a sequence of real numbers is motivated by the various algorithms that make available a certain object by its successive approximations in a class of well-behaved objects. From this point of view, the main problem in connection with a sequence is its behavior for large values of indices.

2.1.1 Definition A sequence $(a_n)_{n \geq 0}$ is called *convergent to the number ℓ* (abbreviated, $a_n \rightarrow \ell$) if for each $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$, we have $|a_n - \ell| < \varepsilon$.

In other words, $a_n \rightarrow \ell$ means that for each $\varepsilon > 0$, there is an index N such that for all $n \geq N$, we have

$$\ell - \varepsilon < a_n < \ell + \varepsilon.$$

The real number ℓ , which appears in the previous definition, if it exists, is unique. In fact, if $a_n \rightarrow \ell$ and $a_n \rightarrow \ell'$, then for each $\varepsilon > 0$ there is an index N such that $|a_n - \ell| < \varepsilon/2$ and $|a_n - \ell'| < \varepsilon/2$ for all $n \geq N$. This yields

$$\begin{aligned} |\ell - \ell'| &\leq |\ell - a_n| + |a_n - \ell'| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $n \geq N$. Since $\varepsilon > 0$ was arbitrarily fixed, we get

$$|\ell - \ell'| \leq \inf \{\varepsilon : \varepsilon > 0\} = 0,$$

and thus $\ell - \ell' = 0$.

We call the number ℓ which appears in Definition 2.1.1 the *limit* of the sequence $(a_n)_n$ and denote the convergence of $(a_n)_n$ to ℓ also by

$$\lim_{n \rightarrow \infty} a_n = \ell$$

(read: the limit of a_n as n tends to infinity equals ℓ). The precise meaning of the phrase “ n tends to infinity” is clarified at the end of Sect. 2.5, where we discuss the sequences with infinite limits.

Intuitively, the convergence means that the terms a_n become arbitrarily close to the limit for n sufficiently large.

The *convergent sequences* are the sequences $(a_n)_n$ for which there exists a real number ℓ such that $a_n \rightarrow \ell$.

The sequences that are not convergent are said to be *divergent*.

The theory of convergent sequences can be adapted mutatis mutandis to the case of sequences indexed over sets of the form $\{n \in \mathbb{Z} : n \geq n_0\}$.

Every constant sequence is convergent (precisely, to the common value of its terms).

The sequence $(\frac{1}{n})_{n \geq 1}$ is convergent to 0 since $|\frac{1}{n} - 0| < \varepsilon$ for all $n \geq \lfloor \frac{1}{\varepsilon} \rfloor + 1$. More generally,

$$\frac{1}{n^\alpha} \rightarrow 0 \quad \text{for all } \alpha > 0.$$

The alternating sequence $((-1)^n)_n$ is bounded and divergent.

The next result is an easy consequence of the definition of convergence.

2.1.2 Lemma *If two sequences differ in a finite number of terms, then they have the same status from the point of view of convergence (and the same limit when they are convergent).*

Actually, more is true. If we change, add, or remove a finite number of terms of a sequence, then the resulting sequence has the same status.

2.1.3 Lemma *Every subsequence of a convergent sequence is convergent to the same limit.*

Proof Suppose that $a_n \rightarrow \ell$ and $(a_{k_n})_n$ is a subsequence of $(a_n)_n$. We need to show that $a_{k_n} \rightarrow \ell$.

Let $\varepsilon > 0$. By our hypothesis, there exists an index N such that for all $n \geq N$, we have $|a_n - \ell| < \varepsilon$. Since $k_n \geq n$ for every n , it follows that $|a_{k_n} - \ell| < \varepsilon$ whenever $n \geq N$. Consequently, $a_{k_n} \rightarrow \ell$. \square

2.1.4 Lemma *A sequence $(a_n)_n$ converges to a number ℓ if and only if every subsequence of it contains a subsequence that is convergent to ℓ .*

Proof The necessity follows from Lemma 2.1.3. The sufficiency part can be argued by reductio ad absurdum. If $(a_n)_n$ is not convergent to ℓ , then (by negating in Definition 2.1.1) we infer that there exist $\varepsilon > 0$ and a subsequence $(a_{k_n})_n$ such that

$$|a_{k_n} - \ell| \geq \varepsilon \quad \text{for all } n \in \mathbb{N}.$$

Or, in this case $(a_{k_n})_n$ has no subsequence convergent to ℓ , in contradiction with our hypothesis. Consequently, $a_n \rightarrow \ell$. \square

2.1.5 Lemma *Every convergent sequence is bounded.*

Proof Suppose that $a_n \rightarrow \ell$. According to the definition of convergence, for $\varepsilon = 1$ there exists N such that

$$|a_n - \ell| < 1 \quad \text{for all } n \geq N,$$

equivalently,

$$\ell - 1 < a_n < \ell + 1 \quad \text{for all } n \geq N.$$

Then

$$\inf \{a_0, a_1, \dots, a_N, \ell - 1\} < a_n < \sup \{a_0, a_1, \dots, a_N, \ell + 1\}$$

for all $n \in \mathbb{N}$, and thus the sequence $(a_n)_n$ is bounded. \square

As already noticed, a bounded sequence may not be convergent. However, as we show in Theorem 2.2.4, it always admits convergent subsequences.

2.1.6 Lemma *If $(a_n)_n$ converges to ℓ , then $(|a_n|)_n$ converges to $|\ell|$.*

Proof This is an immediate consequence of the following inequality:

$$||a_n| - |\ell|| \leq |a_n - \ell|.$$

\square

2.1.7 Corollary *A sequence $(a_n)_n$ is convergent to 0 if and only if the sequence $(|a_n|)_n$, of its absolute values converges to 0.*

An important issue of Lemma 2.1.6 is the possibility to reduce the theory of convergent sequences to the case of positive sequences convergent to zero. In fact,

$$a_n \rightarrow \ell \quad \text{if and only if} \quad |a_n - \ell| \rightarrow 0.$$

Moreover, in order to establish the convergence $a_n \rightarrow \ell$, it suffices to find a suitable sequence $(b_n)_n$ such that

$$|a_n - \ell| \leq b_n \rightarrow 0.$$

We illustrate this idea in our discussion on the algebraic operations with convergent sequences.

2.1.8 Theorem (a) Let $(a_n)_n$ and $(b_n)_n$ be two convergent sequences such that $a_n \rightarrow \ell$ and $b_n \rightarrow \ell'$. Then:

(a) $(a_n + b_n)_n$ converges to $\ell + \ell'$.

(b) $(a_n b_n)_n$ converges to $\ell \ell'$.

Proof (a) Let $\varepsilon > 0$. Then, starting with an index N ,

$$|a_n - \ell| < \varepsilon/2 \quad \text{and} \quad |b_n - \ell'| < \varepsilon/2$$

and thus for every $n \geq N$, we get

$$\begin{aligned} |(a_n + b_n) - (\ell + \ell')| &\leq |a_n - \ell| + |b_n - \ell'| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

(b) According to Lemma 2.1.5, there exists a number $A > 0$ such that $|a_n| \leq A$ for all n . Let $\varepsilon > 0$. The convergence of $(a_n)_n$ to ℓ and the convergence of $(b_n)_n$ to ℓ' gives us a number N such that

$$|a_n - \ell| < \frac{\varepsilon}{2|\ell'| + 1} \quad \text{and} \quad |b_n - \ell'| < \frac{\varepsilon}{2A}$$

for every $n \geq N$. Then,

$$\begin{aligned} |a_n b_n - \ell \ell'| &\leq |a_n b_n - a_n \ell'| + |a_n \ell' - \ell \ell'| = |a_n| \cdot |b_n - \ell'| + |\ell'| \cdot |a_n - \ell| \\ &\leq A |b_n - \ell'| + |\ell'| \cdot |a_n - \ell| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for every $n \geq N$, that is, $a_n b_n \rightarrow \ell \ell'$. □

2.1.9 Corollary If $(a_n)_n$ converges to ℓ and α is a real number, then $\alpha a_n \rightarrow \alpha \ell$.

2.1.10 Theorem If $(a_n)_n$ converges to ℓ and $\ell \neq 0$, then $a_n \neq 0$ starting with some integer N and

$$\frac{1}{a_n} \rightarrow \frac{1}{\ell}.$$

Proof By Lemma 2.1.6, $|a_n| \rightarrow |\ell|$. According to the definition of convergence, for $\varepsilon = |\ell|/2$ there must exist N such that $||a_n| - |\ell|| < |\ell|/2$ for every $n \geq N$. Consequently, for $n \geq N$ we get

$$|a_n| \geq |\ell| - \frac{|\ell|}{2} = \frac{|\ell|}{2} > 0$$

and thus

$$\left| \frac{1}{a_n} - \frac{1}{\ell} \right| = \frac{|a_n - \ell|}{|a_n| |\ell|} \leq \frac{2}{|\ell|^2} |a_n - \ell|,$$

which yields $1/a_n \rightarrow 1/\ell$. \square

From Lemma 2.1.6 and Theorem 2.1.8(a) we can infer easily the order properties of convergent sequences.

2.1.11 Theorem (Order properties of convergent sequences)

(a) If $(a_n)_n$ is a sequence of nonnegative numbers, convergent to ℓ , then ℓ is also a nonnegative number.

(b) If $(a_n)_n$ converges to ℓ , $(b_n)_n$ converges to ℓ' and $a_n \leq b_n$ for every $n \in \mathbb{N}$, then $\ell \leq \ell'$.

Notice that even if all terms of a convergent sequence are positive, the limit may be zero.

2.1.12 The Squeeze Theorem Let $(a_n)_n$, $(b_n)_n$, and $(c_n)_n$ be three sequences such that

$$a_n \leq b_n \leq c_n$$

for all n . If $(a_n)_n$ and $(c_n)_n$ are convergent to the same limit ℓ , then $(b_n)_n$ also is convergent to ℓ .

Proof In fact, $|b_n - \ell| \leq \sup \{|a_n - \ell|, |c_n - \ell|\}$ for all $n \in \mathbb{N}$. \square

2.1.13 Corollary The product of a sequence convergent to 0 by a bounded sequence is also a sequence convergent to 0.

We illustrate the Squeeze Theorem by proving that the sequence $(\sqrt[n]{n})_{n \geq 2}$ is convergent to 1.

In fact, $\sqrt[n]{n} = 1 + r_n$, where $r_n > 0$. We show that $r_n \rightarrow 0$. For, note that for all $n \geq 2$ we have

$$\begin{aligned} n &= (1 + r_n)^n = \binom{n}{0} + \binom{n}{1}r_n + \binom{n}{2}r_n^2 + \cdots + \binom{n}{n}r_n^n \\ &> \binom{n}{2}r_n^2 = \frac{n(n-1)}{2} r_n^2, \end{aligned}$$

so that $0 < r_n^2 < 2/(n-1)$, that is, $0 < r_n < \sqrt{2/(n-1)}$. As $(\sqrt{2/(n-1)})_n$ is convergent to 0, we can apply Theorem 2.1.12 to conclude that $r_n \rightarrow 0$ too.

2.1.14 Remark In connection with the notions of boundedness and convergence, we may consider the following linear spaces:

$\mathcal{F}_0(\mathbb{N}, \mathbb{R})$, the space of all sequences of real numbers, having only finitely many nonzero terms;

$c_0(\mathbb{N}, \mathbb{R})$, the space of all sequences of real numbers convergent to 0;

$c(\mathbb{N}, \mathbb{R})$, the space of all convergent sequences of real numbers;

$\ell^\infty(\mathbb{N}, \mathbb{R})$, the space of all bounded sequences of real numbers.

The linear operations are inherited from the space $\mathcal{F}(\mathbb{N}, \mathbb{R})$ of all real-valued sequences, and are defined by the formulas

$$\begin{aligned}(a_n)_n + (b_n)_n &= (a_n + b_n)_n \\ \alpha(a_n)_n &= (\alpha a_n)_n.\end{aligned}$$

According to Remark 1.2.9 and Lemma 2.1.6, each of these spaces is an example of linear lattice of functions (defined on \mathbb{N}) with respect to the ordering

$$(a_n)_n \leq (b_n)_n \text{ if and only if } a_n \leq b_n \text{ for all } n.$$

An important role in real analysis on intervals is played by *positive sequences*, that is, by sequences $(a_n)_n$ such that $a_n \geq 0$ for all indices n .

Exercises

1. Prove that a sequence of integer numbers is convergent if and only if it is eventually constant (that is, constant from some index onward).
2. Suppose that $(a_n)_n$ is a sequence convergent to ℓ . Prove that $(a_{\sigma(n)})_n$ is also convergent to ℓ , whatever is $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ an injective map.
3. Compute $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right)$.
4. Prove that if $(a_n)_n$ converges to ℓ and $(b_n)_n$ converges to ℓ' , then

$$\max \{a_n, b_n\} \rightarrow \max \{\ell, \ell'\} \quad \text{and} \quad \min \{a_n, b_n\} \rightarrow \min \{\ell, \ell'\}.$$

[Hint: See the formulas $\max \{a_n, b_n\} = \frac{1}{2} (a_n + b_n + |a_n - b_n|)$ and $\min \{a_n, b_n\} = \frac{1}{2} (a_n + b_n - |a_n - b_n|)$.]

5. Let $(a_n)_n$ be a strictly increasing sequence of positive numbers, in arithmetic progression.
 - (a) Prove that $\sqrt{a_{2k-1}a_{2k+1}} < a_{2k}$ for all $k \in \mathbb{N}^*$.
 - (b) Infer from the preceding result that

$$x_n = \frac{a_1}{a_2} \frac{a_3}{a_4} \cdots \frac{a_{2n-1}}{a_{2n}} < \sqrt{\frac{a_1}{a_{2n+1}}}$$

for all n and conclude that $\lim_{n \rightarrow \infty} x_n = 0$.

6. Prove that $\lim_{n \rightarrow \infty} \sqrt[n]{c} = 1$ for every $c > 0$.

7. Let $a_1, \dots, a_p, c_1, \dots, c_p$ be positive numbers. Prove that

$$\lim_{n \rightarrow \infty} \left(c_1 a_1^n + \dots + c_p a_p^n \right)^{1/n} = \max \{a_1, \dots, a_p\}.$$

2.2 Monotone Convergence Theorem

One of the best known criteria for convergence of sequences of real numbers is the following:

2.2.1 Monotone Convergence Theorem *Every increasing and bounded above sequence of real numbers is convergent to its least upper bound.*

Analogously, every decreasing and bounded below sequence is convergent to its greatest lower bound.

Proof Let $(a_n)_n$ be a bounded above and increasing sequence, and let ℓ be its least upper bound. We show that $(a_n)_n$ is convergent to ℓ .

Let $\varepsilon > 0$. By the definition of least upper bound we infer the existence of an index N such that $\ell - \varepsilon < a_N \leq \ell$. Since the sequence is increasing, it follows that

$$\ell - \varepsilon < a_N \leq a_n \leq \ell < \ell + \varepsilon$$

for all $n \geq N$, and the proof is done. \square

2.2.2 Examples (a) The convergence to 0 of the sequence $(\frac{1}{n^\alpha})_{n \geq 1}$ (for $\alpha > 0$) can be seen as a consequence of the Monotone Convergence Theorem. In fact, this sequence is decreasing and its infimum is 0.

(b) Let $a \in \mathbb{R}$. The sequence $(a^n)_{n \geq 1}$ is convergent if $a \in (-1, 1]$ and divergent if $a \in (-\infty, -1] \cup (1, \infty)$. Moreover,

$$a^n \rightarrow \begin{cases} 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1. \end{cases}$$

The case $a = 1$ is trivial. If $|a| < 1$, then the sequence of general term $a_n = |a|^n$ is decreasing and bounded below by 0. Therefore, by the Monotone Convergence Theorem it is convergent. Let ℓ be its limit. Since

$$a_{n+1} = |a| \cdot a_n \quad \text{for all } n$$

we get, by taking the limit on both sides, that $\ell = |a| \cdot \ell$, from which it follows that $\ell = 0$. An application of Corollary 2.1.7 concludes the proof in the case where $a \in (-1, 1)$.

When $a \in (-\infty, -1] \cup (1, \infty)$, we have $|a_{n+1} - a_n| = |a|^n |a - 1| \geq |a - 1|$ for all n , and in this case the sequence $(a^n)_n$ is divergent.

(c) The summation of positive sequences is also an illustration of the Monotone Convergence Theorem. If $(a_n)_n$ is such a sequence, we attach to it the sequence of partial sums

$$S_0 = a_0, S_1 = a_0 + a_1, S_2 = a_0 + a_1 + a_2, \dots$$

which is increasing. We say that $(a_n)_n$ is *summable*, with *sum* S , if the sequence of its partial sums converges to S . Usually, this is denoted by

$$a_0 + a_1 + a_2 + \dots = S \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = S.$$

For example, if $0 < q < 1$, then

$$1 + q + q^2 + \dots = \frac{1}{1 - q}.$$

(d) The decimal representation of a real number is another illustration of this topic. Indeed, the fact that

$$\begin{aligned} a &= d_n \dots d_0.d_{-1}d_{-2}d_{-3} \dots \\ &= \sup_{k \geq 0} (d_n \cdot 10^n + \dots + d_{n-k} \cdot 10^{n-k}) \end{aligned}$$

is equivalent to

$$\begin{aligned} a &= \lim_{k \rightarrow \infty} (d_n \cdot 10^n + \dots + d_{n-k} \cdot 10^{n-k}) \\ &= d_n \cdot 10^n + d_{n-1} \cdot 10^{n-1} + d_{n-2} \cdot 10^{n-2} + \dots \end{aligned}$$

We will come back to the problem of summing numerical sequences in Chap. 4.

As already noticed, every convergent sequence is bounded and there exist bounded sequences that are not convergent. However, every bounded sequence has a convergent subsequence.

2.2.3 Lemma *Every sequence has a monotone subsequence.*

Proof Let us consider the set $A = \{i : a_i < a_j \text{ for all } j > i\}$.

There are two possibilities:

Case 1. A is an infinite set of elements $i_0 < i_1 < i_2 < \dots$. Then, according to the definition of A , $a_{i_0} < a_{i_1} < a_{i_2} < \dots$.

Case 2. A is a finite set. Put $i_0 = \sup A + 1$, if A is nonempty and $i_0 = 0$ if A is empty. Since $i_0 \notin A$, there exists $i_1 > i_0$ such that $a_{i_0} \geq a_{i_1}$. Also, $i_1 \notin A$, so that there exists $i_2 > i_1$ such that $a_{i_1} \geq a_{i_2}$ and so on. \square

By the Monotone Convergence Theorem and Lemma 2.2.3 we infer:

2.2.4 Bolzano-Weierstrass Theorem *Every bounded sequence has a convergent subsequence.*

2.2.5 Corollary *If a bounded sequence does not converge to a number ℓ , then a subsequence of it converges to a limit which is different from ℓ .*

Proof This follows from Lemma 2.1.4 and Bolzano-Weierstrass Theorem. \square

Exercises

1. Consider the sequence $(a_n)_{n \geq 1}$ defined by

$$a_n = \underbrace{\sqrt{a + \sqrt{a + \cdots + \sqrt{a}}}}_{n \text{ roots}},$$

where $a > 0$ is a parameter. Prove that the sequence is convergent and indicate its limit.

[Hint: Notice that $a_1 < a_2$ and $a_{n+1} = \sqrt{a + a_n}$ for $n \geq 1$.]

2. (The Babylonian Algorithm for computing square roots).
 (a) Prove that for every positive numbers a_0 and a , the sequence $(a_n)_n$ defined by

$$a_{n+1} = \frac{1}{2} \left(a_n + \frac{a}{a_n} \right) \quad \text{for } n \geq 0$$

is convergent to \sqrt{a} .

- (b) Consider the particular case where $a_0 = 3$ and $a = 8$. Show that $a_{n+1}^2 - 8 < (a_n^2 - 8)^2 / 32$, and thus

$$0 < a_n^2 - 8 < 32^{1-2^n}$$

for all $n \geq 1$. Then infer that $0 < a_n - \sqrt{8} < 10 \cdot 10^{-3 \cdot 2^{n-2}}$, for all $n \geq 1$. Since $10 \cdot 10^{-3 \cdot 2^{n-2}} < \frac{1}{2} 10^{-(3 \cdot 2^{n-2} - 2)}$, this shows that a_n provides an approximation of $\sqrt{8}$ with at least $3 \cdot 2^{n-2} - 2$ exact decimals, whenever $n \geq 2$.

(c) Using a computer, one can easily see that the upper bound of $a_n - \sqrt{8}$ indicated at point (b) is pretty rough. Find a better upper bound.

3. (The continued fraction expansion of the golden ratio). Consider the sequence

$$a_0 = 1, a_1 = 1 + \frac{1}{1}, a_2 = 1 + \frac{1}{1 + \frac{1}{1}}, \dots,$$

whose terms are related by the formula $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geq 0$.

(a) Prove that both subsequences $(a_{2n})_n$ and $(a_{2n+1})_n$ are monotone and bounded.

(b) Infer that $(a_n)_n$ is convergent to the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$.

In connection with Exercise 3, note that $\varphi = \lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n}$, where $(F_n)_n$ is the sequence of Fibonacci numbers.

The sequences associated to a function $F : X \rightarrow X$ via the formula

$$x_0 = a, \quad x_{n+1} = F(x_n) \quad \text{for } n \geq 0$$

are usually called *recurrent sequences*. If X is an interval and F is an increasing function, then the associated recurrent sequences are monotone, while if F is decreasing, the subsequences $(x_{2n})_n$ and $(x_{2n+1})_n$ are monotone (of opposite monotonicity). See Appendix A for additional information on recurrent sequences.

The next exercise is an example of recurrent sequence in \mathbb{R}^2 , presented with the means of one real variable analysis:

4. (Gauss' Arithmetic-Geometric Mean). Let a and b be two numbers such that $a \geq b > 0$. Prove that the sequences $(x_n)_n$ and $(y_n)_n$ defined by

$$\begin{aligned} x_0 &= a, \quad y_0 = b, \\ x_{n+1} &= \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_n y_n} \quad \text{for } n \geq 0 \end{aligned}$$

are convergent and they have a common limit $M(a, b)$. Follow the next steps:

(a) $x_n \geq y_n$ for all n .

(b) The sequence $(x_n)_n$ is decreasing, while the sequence $(y_n)_n$ is increasing. Conclude from here that both are convergent.

(c) Take the limit of the recursive relation $x_{n+1} = (x_n + y_n)/2$ to obtain that the two sequences $(x_n)_n$ and $(y_n)_n$ have the same limit.

An application to integral calculus of Gauss' Arithmetic-Geometric Mean makes the objective of Exercise 13, Sect. 9.2.

5. Prove that every real number is the limit of a strictly increasing sequence of rational numbers and of a strictly decreasing sequence of rational numbers; a similar statement is true when rationals are replaced by irrationals.
6. Infer the Bolzano-Weierstrass Theorem from the Nested Intervals Lemma. [Hint: If $(a_n)_n$ is included in $[a, b]$ and $c = (a + b)/2$, then at least one of the subintervals $[a, c]$ and $[c, b]$ contains a subsequence $(a_n^{(1)})_n$ of $(a_n)_n$. Then iterate the argument and conclude that $(a_n^{(n)})_n$ is convergent.]

2.3 The Number e

Consider the sequences

$$a_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad b_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$$

indexed over $n \geq 1$. We show that the two sequences above have a common limit, usually denoted e .

2.3.1 Lemma *The sequence $(a_n)_{n \geq 1}$ is strictly increasing.*

Proof According to the AM-GM inequality,

$$\sqrt[n+1]{\left(1 + \frac{1}{n}\right)^n} = \sqrt[n+1]{1 \left(1 + \frac{1}{n}\right) \cdots \left(1 + \frac{1}{n}\right)} < \frac{1+n \left(1 + \frac{1}{n}\right)}{n+1} = 1 + \frac{1}{n+1},$$

whence $a_n < a_{n+1}$. □

2.3.2 Lemma *We have $a_n < b_n < 3$ for every $n \geq 2$.*

Proof In fact,

$$\begin{aligned} a_n &= \left(1 + \frac{1}{n}\right)^n = 1 + \binom{n}{1} \cdot \frac{1}{n} + \binom{n}{2} \cdot \frac{1}{n^2} + \cdots + \binom{n}{n} \cdot \frac{1}{n^n} \\ &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\ &< 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} = b_n. \end{aligned}$$

On the other hand,

$$\begin{aligned} b_n &= 1 + \frac{1}{1!} + \cdots + \frac{1}{n!} = 2 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{2 \cdot 3 \cdots n} \\ &< 2 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} < 3. \end{aligned} \quad \square$$

Lemmas 2.3.1 and 2.3.2 show that the sequence $\left(\left(1 + \frac{1}{n}\right)^n\right)_n$ verifies the Monotone Convergence Theorem and thus it is convergent.

2.3.3 Definition The number e is defined as the limit of the sequence $\left(\left(1 + \frac{1}{n}\right)^n\right)_n$.

The number e was introduced by Jacob Bernoulli in connection with a question on compound interest. The notation e is due to Leonhard Euler, who actually revealed its importance in mathematics.

The number e is used as the base of natural logarithms.

We show that e is also the limit of the sequence $(b_n)_n$ above.

In fact, this sequence is strictly increasing and bounded above, so it also obeys the Monotone Convergence Theorem. Let ℓ be its limit. By Lemma 2.3.2, we infer that $\ell \geq e$. For $N \in \mathbb{N}$ arbitrary fixed and $n \geq N$, we have

$$\begin{aligned} a_n &= 1 + \frac{1}{1!} + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right) \\ &\geq 1 + \frac{1}{1!} + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{N!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{N-1}{n}\right) \end{aligned}$$

so that, passing to the limit as $n \rightarrow \infty$, we get that $e \geq b_N$. Since N was arbitrarily fixed, this yields also $e \geq \ell$. Consequently, $\ell = e$.

2.3.4 Lemma *The inequality*

$$\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}.$$

holds for every $n \geq 1$.

Proof By the AM-GM inequality we infer easily that the inverse of the sequence $\left(\left(1 + \frac{1}{n}\right)^{n+1}\right)_n$ is strictly increasing (and thus the sequence itself is strictly decreasing). Therefore,

$$\left(1 + \frac{1}{n}\right)^{n+1} > \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+1} = e. \quad \square$$

It was noticed by I. Schur that the sequence $a_\alpha(n) = \left(1 + \frac{1}{n}\right)^{n+\alpha}$ is decreasing if $\alpha \in [\frac{1}{2}, \infty)$, and increasing for $n \geq N(\alpha)$ if $\alpha \in (-\infty, 1/2)$. See Exercise 5, Sect. 8.2.

2.3.5 Remark The sequence $(b_n)_n$ approaches e quite fast. In fact,

$$0 < e - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{n!}\right) < \frac{1}{n! \cdot n} \quad (2.1)$$

for every $n \geq 1$. This can be argued as follows:

$$\begin{aligned} e - b_n &= \lim_{k \rightarrow \infty} \left(\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{(n+k)!} \right) \\ &= \lim_{k \rightarrow \infty} \frac{1}{(n+1)!} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \cdots + \frac{1}{(n+2) \cdots (n+k)} \right) \\ &\leq \frac{1}{(n+1)!} \lim_{k \rightarrow \infty} \left(1 + \frac{1}{n+2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{(n+2)^{k-1}} \right) \\ &= \frac{1}{(n+1)!} \cdot \frac{1}{1 - \frac{1}{n+2}} = \frac{n+2}{(n+1)! (n+1)} \\ &= \frac{1}{n! \cdot n} \cdot \frac{n(n+2)}{(n+1)^2} < \frac{1}{n! \cdot n}. \end{aligned}$$

From the inequality (2.1) it follows that e is an irrational number. In fact, assuming that $e = p/q$, with $p, q \in \mathbb{N}^*$, we are led to

$$0 < \frac{p}{q} - \left(1 + \frac{1}{1!} + \cdots + \frac{1}{q!}\right) < \frac{1}{q!q}.$$

By multiplying both sides by $q!$, we infer that between 0 and $1/q$ would exist an integer number, which is false. The number e is transcendental. Its value with ten exact decimals is $e = 2.7182818285 \dots$.

Exercises

1. An immediate consequence of Lemma 2.3.4 is that

$$\frac{1}{n+1} < \log(n+1) - \log n < \frac{1}{n}.$$

Use this fact to prove that the sequence $c_n = 1 + \frac{1}{2} + \dots + \frac{1}{n} - \log n$ is decreasing and bounded (and thus it is convergent).

Remark The limit of this sequence, usually denoted γ , is known as *Euler's constant*; we have $\gamma = 0.57721 \dots$. The problem whether γ is rational or irrational is still open!

2. (a) Infer from the previous exercise that

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \dots + \frac{1}{2n} \right) = \log 2.$$

(b) Use induction (or a summation trick) to prove the *Botez–Catalan identity*,

$$1 - \frac{1}{2} + \dots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \dots + \frac{1}{2n},$$

and conclude that

$$\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \dots + \frac{(-1)^{n-1}}{n} \right) = \log 2.$$

3. It is easy to see that the sequence

$$x_n = \left(1 + \frac{1}{2n} \right) \left(1 + \frac{1}{n} \right)^n$$

is convergent to e . Use some values of n to see that this sequence gives by far better approximations of e than both the sequences

$$a_n = \left(1 + \frac{1}{n} \right)^n \quad \text{and} \quad b_n = \left(1 + \frac{1}{n} \right)^{n+1}.$$

What is the reason behind this fact?

2.4 The Cauchy Completeness of \mathbb{R}

The Bolzano-Weierstrass Theorem yields a very important theoretical criterion of convergence, asserting the equivalence between the property of the terms of being close to a certain point and that of being close to each other.

2.4.1 Definition A sequence $(a_n)_n$ of real numbers is called a *Cauchy sequence* (or a *fundamental sequence*) if for every $\varepsilon > 0$, there exists an index N such that for all $m, n > N$, we have $|a_m - a_n| < \varepsilon$.

The following three results are immediate:

2.4.2 Lemma *Every convergent sequence is a Cauchy sequence.*

2.4.3 Lemma *Every Cauchy sequence is bounded.*

2.4.4 Lemma *A Cauchy sequence that contains a convergent subsequence is convergent.*

Proof Let $(a_n)_n$ be a Cauchy sequence with $a_{k_n} \rightarrow \ell$. We show that $a_n \rightarrow \ell$.

Let $\varepsilon > 0$. Since $(a_n)_n$ is a Cauchy sequence, there exists an index N_1 such that for all $m, n \geq N_1$, we have $|a_m - a_n| < \varepsilon/2$. Since $a_{k_n} \rightarrow \ell$, there exists an index N_2 such that for all $n \geq N_2$, we have $|a_{k_n} - \ell| < \varepsilon/2$. Then, for all $n \geq N = \max\{N_1, N_2\}$, we get

$$|a_n - \ell| \leq |a_n - a_{k_n}| + |a_{k_n} - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which ends the proof. □

The three lemmas above can be combined to yield an important theoretical criterion of convergence.

2.4.5 Theorem (Cauchy's Criterion) *A sequence $(a_n)_n$ of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof If $(a_n)_n$ is a Cauchy sequence, then it is bounded (see Lemma 2.4.3) and so, by Bolzano-Weierstrass Theorem, it has a convergent subsequence $(a_{k_n})_n$. By Lemma 2.4.4, the whole sequence $(a_n)_n$ must be convergent.

The converse is obvious and was stated as Lemma 2.4.2 above. □

Exercises

1. Prove the convergence of the sequence given by the formula

$$a_n = N_0 + \frac{N_1}{10} + \cdots + \frac{N_n}{10^n},$$

where N_1, N_2, N_3, \dots are arbitrary numbers in $\{0, 1, \dots, 9\}$. Notice the connection of this fact with the decimal representation of real numbers.

2. Prove that the sequence of general term

$$a_n = \frac{\{\sqrt{3}\}}{1 \cdot 2} + \frac{\{\sqrt{4}\}}{2 \cdot 3} + \cdots + \frac{\{\sqrt{n+2}\}}{n(n+1)}$$

is a Cauchy sequence.

2.5 The Extended Real Number System

We add to \mathbb{R} two symbols, $-\infty$ (minus infinity) and ∞ (plus infinity), and extend the original order in \mathbb{R} as follows:

$$\begin{aligned} -\infty &< x && \text{for all } x \in \mathbb{R} \\ x &< \infty && \text{for all } x \in \mathbb{R} \\ -\infty &< \infty. \end{aligned}$$

The set $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ together with the above order relation is called the *extended real number system*. The main feature of $\overline{\mathbb{R}}$ is that every nonempty subset A of $\overline{\mathbb{R}}$ admits simultaneously a least upper bound and a largest lower bound (in $\overline{\mathbb{R}}$). If A is a subset of \mathbb{R} bounded above, then the existence of $\sup A$ follows from Completeness Axiom, while for those subsets that are not bounded above, necessarily $\sup A = \infty$. Thus, $\sup A = \infty$ represents an alternative way to outline that a subset A of \mathbb{R} is not bounded above in \mathbb{R} . In a similar manner, one can discuss the case of subsets bounded below.

It is useful to extend the algebraic structure of \mathbb{R} , by defining some operations with infinite elements. More precisely, the addition is supplemented by

$$\begin{aligned} x + (-\infty) &= (-\infty) + x = -\infty && \text{for all } x \in \mathbb{R} \\ x + \infty &= \infty + x = \infty && \text{for all } x \in \mathbb{R} \\ (-\infty) + (-\infty) &= -\infty && \text{and } \infty + \infty = \infty, \end{aligned}$$

while multiplication is supplemented by

$$\begin{aligned} x \cdot (-\infty) &= (-\infty) \cdot x = \begin{cases} \infty & \text{if } x \in \overline{\mathbb{R}}, x < 0 \\ -\infty & \text{if } x \in \overline{\mathbb{R}}, x > 0 \end{cases} \\ x \cdot \infty &= \infty \cdot x = \begin{cases} -\infty & \text{if } x \in \overline{\mathbb{R}}, x < 0 \\ \infty & \text{if } x \in \overline{\mathbb{R}}, x > 0. \end{cases} \end{aligned}$$

All these operations are motivated by their companions in terms of limits. See Exercise 2.

We do not define

$$\infty - \infty, \quad (-\infty) + \infty, \quad 0 \cdot (-\infty), \quad (-\infty) \cdot 0, \quad 0 \cdot \infty, \quad \infty \cdot 0$$

nor

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 1^\infty, \quad \infty^0 \quad \text{and} \quad 0^0.$$

These expressions are known as “indeterminates”. See Exercise 7 for an explanation. In Chap. 11, devoted to Lebesgue integral, the operations $0 \cdot \infty$, $\infty \cdot 0$, $0 \cdot (-\infty)$ and $(-\infty) \cdot 0$ will be considered legitimate (and equal to 0), under certain special circumstances.

Following the model of bounded intervals (described in Sect. 1.2), one can consider intervals of the form

$$[a, b], (a, b], [a, b) \text{ and } (a, b),$$

with $a, b \in \overline{\mathbb{R}}$, $a \leq b$. For example,

$$[-\infty, b) = \{x \in \overline{\mathbb{R}} : -\infty \leq x < b\}, \quad (-\infty, \infty) = \mathbb{R} \text{ and } [-\infty, \infty] = \overline{\mathbb{R}}.$$

An interval is said to be *nondegenerate* if it is nonempty and does not reduce to a single point. The intervals included in \mathbb{R} are called *real intervals*.

A subset I of \mathbb{R} is called *convex* if

$$x, y \in I \text{ and } t \in [0, 1] \text{ implies } (1-t)x + ty \in I.$$

A combination of the form $(1-t)x + ty$ with $t \in [0, 1]$ is called a *convex combination* of x and y . A convex set contains, together with any pair of points, all their convex combinations.

2.5.1 Proposition *The real intervals and the convex subsets of \mathbb{R} are the same.*

Proof Clearly, every real interval is a convex set. Conversely, let I be a nonempty convex subset of \mathbb{R} and put $a = \inf I$ and $b = \sup I$. If $a = b$, then $I = \{a\} = [a, a]$ and the proof is done. If $-\infty < a < b < \infty$, then for every $\varepsilon \in (0, \frac{b-a}{2})$, there are points $a_\varepsilon \in [a, a + \varepsilon) \cap I$ and $b_\varepsilon \in (b - \varepsilon, b] \cap I$, which implies

$$[a_\varepsilon, b_\varepsilon] \subset I,$$

due to the property of convexity of I . As a consequence, I must contain

$$\bigcup_{\varepsilon \in (0, \frac{b-a}{2})} [a_\varepsilon, b_\varepsilon] = (a, b)$$

and thus I is one of the intervals (a, b) , $(a, b]$, $(a, b]$ and $[a, b]$. The other cases, when a and/or b belong to $\{-\infty, \infty\}$ can be treated similarly. \square

In Sect. 2.1 we introduced the notion of convergent sequence by using the absolute value function to measure how close are the terms a_n to the limit ℓ . However, this function can be avoided by noticing that the concentration of the terms near the limit admits an equivalent formulation based on some special intervals around the limit, the so-called ε -neighborhoods of ℓ :

$$I_\varepsilon(\ell) = (\ell - \varepsilon, \ell + \varepsilon).$$

2.5.2 Proposition $a_n \rightarrow \ell$ in \mathbb{R} if and only if for every $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$,

$$a_n \in I_\varepsilon(\ell).$$

In the case of infinite elements, it is natural to define the ε -neighborhoods as

$$I_\varepsilon(-\infty) = [-\infty, -\varepsilon) \quad \text{and} \quad I_\varepsilon(\infty) = (\varepsilon, \infty]$$

and to take the condition in Proposition 2.5.2 as the definition for sequences with infinite limits. This leads to the following definitions.

A sequence $(a_n)_n$ of real numbers has the limit $-\infty$ (equivalently, $a_n \rightarrow -\infty$ or $\lim_{n \rightarrow \infty} a_n = -\infty$) if for every $\varepsilon > 0$, there is an index N such that $a_n < -\varepsilon$ for all $n \geq N$.

A sequence $(a_n)_n$ of real numbers has the limit ∞ (equivalently, $a_n \rightarrow \infty$ or $\lim_{n \rightarrow \infty} a_n = \infty$) if for every $\varepsilon > 0$, there is an index N such that $a_n > \varepsilon$ for all $n \geq N$.

It is worth noticing that unlike the case of convergent sequences, here of interest are the big values of ε .

In this way, every decreasing sequence of real numbers, that is not bounded below, has the limit $-\infty$, and every increasing sequence that is not bounded above, has the limit ∞ . In particular, the sequence of natural numbers has the limit ∞ .

Taking into account the Monotone Convergence Theorem, we infer that *all monotone sequences of real numbers have limit (finite or infinite)*. Thus, Bolzano-Weierstrass Theorem can be extended in the following form: *Every sequence of real numbers contains a subsequence with limit (finite or infinite)*.

Exercises

1. Prove that:

$$(a) \lim_{n \rightarrow \infty} \log n = \infty; \text{ and } (b) \lim_{n \rightarrow \infty} a^n = \infty \text{ for all } a > 1.$$

2. (The extension of algebraic operations with sequences). Let $(a_n)_n$ and $(b_n)_n$ be two sequences of real numbers such that $a_n \rightarrow \ell$ and $b_n \rightarrow \ell'$ in \mathbb{R} . Prove that $a_n + b_n \rightarrow \ell + \ell'$ and $a_n b_n \rightarrow \ell \ell'$ as long as the operations with ℓ and ℓ' make sense.

3. Suppose that $(a_n)_n$ is a sequence of real numbers bounded below and $(b_n)_n$ is a sequence of real numbers with limit ∞ . Prove that $(a_n + b_n)_n$ also has the limit ∞ .
4. Let $(a_n)_n$ be a sequence with infinite limit. Prove that:
 - (a) there exists a natural number N such that $a_n \neq 0$ for all $n \geq N$.
 - (b) $\lim_{n \rightarrow \infty} \frac{1}{a_n} = 0$.
5. Suppose that $(a_n)_n$ is a sequence of positive numbers convergent to 0. Prove that $\lim_{n \rightarrow \infty} \frac{1}{a_n} = \infty$. What is happening if the hypothesis on positivity is dropped?
6. Let $P(x) = a_0x^N + a_1x^{N-1} + \cdots + a_N$ be a polynomial function with real coefficients, of degree $N \geq 1$. Prove that

$$\lim_{n \rightarrow \infty} P(n) = (\operatorname{sgn} a_0) \cdot \infty.$$

7. (a) (An explanation for not defining $\frac{\infty}{\infty}$). Prove, with examples, that for every $\ell \in [0, \infty]$, there exist pairs of sequences $(a_n)_n$ and $(b_n)_n$ with limit ∞ such that $\frac{a_n}{b_n} \rightarrow \ell$.
 (b) Explain why not defining $\frac{0}{0}$, $\infty - \infty$ and 1^∞ .
8. Let $(a_n)_n$ be a sequence of positive numbers such that $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \ell$ exists. Prove that:
 - (a) if $\ell < 1$, then $\lim_{n \rightarrow \infty} a_n = 0$;
 - (b) if $\ell > 1$, then $\lim_{n \rightarrow \infty} a_n = \infty$;
 - (c) the case $\ell = 1$ is not conclusive, due to the existence of sequences without limit. Give an example.
9. (Comparing the order of convergence). Prove that

$$\lim_{n \rightarrow \infty} \frac{P(n)}{a^n} = 0; \quad \lim_{n \rightarrow \infty} \frac{a^n}{n!} = 0; \quad \lim_{n \rightarrow \infty} \frac{n!}{n^n} = 0$$

for all polynomial functions P with real coefficients and all number $a > 1$.

[Hint: Use the above exercise.]

10. Suppose that a sequence of real numbers is divergent. Prove that either it has a limit belonging to \mathbb{R} or it contains two subsequences with different limits in \mathbb{R} .
 [Hint: Negate the property of being a Cauchy sequence and then apply the extension of Bolzano-Weierstrass Theorem to \mathbb{R} .]

2.6 Limit Inferior and Limit Superior of a Sequence

The *limit inferior* and *limit superior* of a sequence $(a_n)_n$ of real numbers are defined (as elements of $\overline{\mathbb{R}}$), respectively, by the formulas

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} a_k \right)$$

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\sup_{k \geq n} a_k \right).$$

Since the sequence $(\inf_{k \geq n} a_k)_n$ is increasing and the sequence $(\sup_{k \geq n} a_k)_n$ is decreasing,

$$\liminf_{n \rightarrow \infty} a_n = \sup_{n \geq 0} \left(\inf_{k \geq n} a_k \right) \quad \text{and} \quad \limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 0} \left(\sup_{k \geq n} a_k \right).$$

Clearly,

$$-\infty \leq \inf_{n \geq 0} a_n \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq \sup_{n \geq 0} a_n \leq \infty.$$

For the sequence of general term $a_n = \frac{1}{n}$, we have

$$\liminf_{n \rightarrow \infty} a_n = \sup_{n \geq 1} \left(\inf_{k \geq n} \frac{1}{k} \right) = \sup_{n \geq 1} 0 = 0$$

and

$$\limsup_{n \rightarrow \infty} a_n = \inf_{n \geq 1} \left(\sup_{k \geq n} \frac{1}{k} \right) = \inf_{n \geq 1} \frac{1}{n} = 0.$$

In general, whenever the ordinary limit exists, the limit inferior and limit superior are both equal to it. Therefore, each can be considered a generalization of the ordinary limit.

2.6.1 Theorem *Let $(a_n)_n$ be a sequence of real numbers.*

(a) *If the sequence $(a_n)_n$ is convergent to a number ℓ , then*

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \ell.$$

(b) *If there is a real number ℓ such that the equality above holds, then the sequence $(a_n)_n$ is convergent to ℓ .*

Proof (a) Suppose that $a_n \rightarrow \ell$ in \mathbb{R} . Then, for any $\varepsilon > 0$, there is a natural number N such that for all $n \geq N$, we have $\ell - \varepsilon < a_n < \ell + \varepsilon$. Thus,

$$\ell - \varepsilon \leq \inf_{k \geq n} a_k \leq \sup_{k \geq n} a_k \leq \ell + \varepsilon$$

for all $n \geq N$, from where, by taking the limit, we obtain that

$$\ell - \varepsilon \leq \liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n \leq \ell + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily fixed, this implies

$$\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \ell.$$

(b) Suppose that $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = \ell \in \mathbb{R}$. Clearly,

$$\inf_{k \geq n} a_k \leq a_n \leq \sup_{k \geq n} a_k \quad \text{for all } n \in \mathbb{N}.$$

The convergence of $(a_n)_n$ to ℓ is now a consequence of the Squeeze Theorem. \square

For the sequence $a_n = (1 + (-1)^n)/2$, we have

$$\liminf_{n \rightarrow \infty} a_n = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} a_n = 1$$

and its subsequences may converge to 0 or to 1. This example illustrates a general phenomenon.

2.6.2 Theorem *Let $(a_n)_n$ be a sequence of real numbers. Then:*

$$\liminf_{n \rightarrow \infty} a_n = \inf \left\{ \ell \in \overline{\mathbb{R}} : \text{there is } (k_n)_n \text{ such that } a_{k_n} \rightarrow \ell \right\}$$

and

$$\limsup_{n \rightarrow \infty} a_n = \sup \left\{ \ell \in \overline{\mathbb{R}} : \text{there is } (k_n)_n \text{ such that } a_{k_n} \rightarrow \ell \right\}.$$

The proof of Theorem 2.6.2 makes the objective of Exercise 3.

Exercises

1. Find the limits inferior and superior to the sequence of general term

$$a_n = \frac{(1 + (-1)^{\lfloor n/2 \rfloor}) \log n}{\log 2n}.$$

2. Prove (by contradiction) that:

(a) if $\liminf_{n \rightarrow \infty} a_n > \ell$, then at most finitely many terms a_n are less than ℓ .

(b) if $\limsup_{n \rightarrow \infty} a_n < \ell$, then at most finitely many terms a_n are bigger than the number ℓ .

3. Prove Theorem 2.6.2 by using the previous exercise.

4. Suppose that $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers such that $\lim_{n \rightarrow \infty} a_n = \ell$ exists in \mathbb{R} . Prove that:

$$(a) \quad \limsup_{n \rightarrow \infty} (a_n + b_n) = \ell + \limsup_{n \rightarrow \infty} b_n;$$

- (b) $\limsup_{n \rightarrow \infty} (a_n b_n) = \ell \limsup_{n \rightarrow \infty} b_n$ provided that $\ell > 0$.
 (c) The assertions (a) and (b) also work when limit superior is replaced by limit inferior.
5. Let $(a_n)_n$ be a sequence of positive numbers. Prove that

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{a_n} \leq \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} \leq \limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$$

and conclude that if the limit $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \ell$ exists, then the limit $\lim_{n \rightarrow \infty} \sqrt[n]{a_n}$ also exists and equals ℓ .

6. Let $(a_n)_n$ be a sequence of real numbers such that $a_{m+n} \leq a_m + a_n$ for all indices m and n . Prove that

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} = \inf_{n \geq 1} \frac{a_n}{n}.$$

7. Let $(a_n)_n$ be a sequence of positive numbers having limit in $\overline{\mathbb{R}}$ and let p be a natural number. Prove that

$$\limsup_{n \rightarrow \infty} \left(\frac{1 + a_{n+p}}{a_n} \right)^n \geq e^p.$$

Find a sequence for which equality occurs.

2.7 The Stolz–Cesàro Theorem

In connection with the operations with sequences having limits (finite or infinite) we ran into the problem of “eliminating the indeterminates”. We know that if $a_n \rightarrow \ell$ and $b_n \rightarrow \ell'$ (with $\ell' \neq 0$), we have

$$\frac{a_n}{b_n} \rightarrow \frac{\ell}{\ell'}.$$

It may happen that $a_n \rightarrow 0$ and $b_n \rightarrow 0$, but the sequence $(\frac{a_n}{b_n})_n$ is still convergent! For example, this is the case when $a_n = b_n = \frac{1}{n}$.

The general cases when “the elimination of indeterminates” is possible make the object of the so-called Stolz–Cesàro theorems (that are nothing but the discrete analogous of Bernoulli–L’Hôpital Rules in Sect. 8.3).

2.7.1 Theorem (Stolz–Cesàro Theorem, the case $\frac{0}{0}$) *Let $(a_n)_n$ and $(b_n)_n$ be two sequences of real numbers convergent to 0, such that $(b_n)_n$ is strictly monotone and*

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell \quad (\text{possibly in } \overline{\mathbb{R}}).$$

Then the limit $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ also exists and equals ℓ .

Proof Clearly, it suffices to consider only the case when $(b_n)_n$ is strictly decreasing.

Case 1: $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell \in \mathbb{R}$. Then for any $\varepsilon > 0$, there is an index N such that

$$\ell - \varepsilon < \frac{a_{n+1} - a_n}{b_{n+1} - b_n} < \ell + \varepsilon$$

for all $n \geq N$. By the hypothesis, $b_{n+1} - b_n < 0$ for all n , so that

$$(\ell - \varepsilon)(b_{n+1} - b_n) > a_{n+1} - a_n > (\ell + \varepsilon)(b_{n+1} - b_n)$$

for all $n \geq N$. For a fixed such number n , we write down the inequalities corresponding to $n, n+1, \dots, n+p$, and we add them side by side. We get

$$(\ell - \varepsilon)(b_{n+p} - b_n) > a_{n+p} - a_n > (\ell + \varepsilon)(b_{n+p} - b_n).$$

Taking the limit as $p \rightarrow \infty$, we obtain

$$(\ell - \varepsilon)(-b_n) \geq -a_n \geq (\ell + \varepsilon)(-b_n)$$

from which we conclude that $\ell - \varepsilon \leq \frac{a_n}{b_n} \leq \ell + \varepsilon$ for all $n \geq N$.

Case 2: $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \infty$. Then for any $\varepsilon > 0$, there is an index N such that

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} > \varepsilon$$

for all $n \geq N$. As a consequence, we get

$$\begin{aligned} a_n - a_m &= \sum_{k=n}^{m-1} (a_k - a_{k+1}) > \varepsilon \sum_{k=n}^m (b_k - b_{k+1}) \\ &= \varepsilon(b_n - b_m) \end{aligned}$$

for all $m > n \geq N$, and thus

$$\frac{a_n}{b_n} > \varepsilon \left(1 - \frac{b_m}{b_n}\right) + \frac{a_m}{b_n}.$$

Keeping n fixed and taking the limit as $m \rightarrow \infty$, we obtain that $\frac{a_n}{b_n} \geq \varepsilon$ (for all $n \geq N$). Consequently, $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \infty$.

Case 3: $\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = -\infty$. This case is similar to the second. \square

In a similar way, one can prove the following result.

2.7.2 Theorem (Stolz–Cesàro Theorem, the case of sequences with infinite limit) *Let $(a_n)_n$ and $(b_n)_n$ be two sequences of real numbers such that $(b_n)_n$ is strictly increasing to ∞ and*

$$\lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell \quad (\text{possibly in } \overline{\mathbb{R}}).$$

Then the limit $\lim_{n \rightarrow \infty} \frac{a_n}{b_n}$ also exists and equals ℓ .

The Stolz–Cesàro Theorem (in its different variants) has numerous applications. We indicate here two examples. The first example shows that the arithmetic (as well as the geometric) mean of terms of a convergent sequence, also converges (and to the same limit).

2.7.3 Corollary (A.-L. Cauchy) (a) *If $a_n \rightarrow \ell$ in $\overline{\mathbb{R}}$, then*

$$\frac{a_1 + \cdots + a_n}{n} \rightarrow \ell.$$

(b) *If $a_n \rightarrow \ell$ in $\overline{\mathbb{R}}$ and all terms a_n are positive, then*

$$(a_1 \cdots a_n)^{1/n} \rightarrow \ell.$$

In connection with Corollary 2.7.3, let us call a numerical sequence $(a_n)_n$ *Cesàro convergent* to ℓ if $\frac{a_1 + \cdots + a_n}{n} \rightarrow \ell$. While convergence implies Cesàro convergence, the converse is not true. See the case of the sequence $((-1)^n)_n$. However, we can relate the Cesàro convergence with the convergence of some subsequences. This makes the objective of Theorem 2.8.1.

A second application of the Stolz–Cesàro Theorem refers to the rate of convergence of a sequence. We know that

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \rightarrow \gamma,$$

from where it follows that $\frac{1}{n+1} + \cdots + \frac{1}{2n} \rightarrow \log 2$. Theorem 2.7.1 allows us to precise this conclusion as follows:

$$\lim_{n \rightarrow \infty} n \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} - \log 2 \right) = -1/4.$$

Exercises

1. Prove that if $a_n \rightarrow \ell$ and $\ell \neq 0$ then $\frac{n}{\frac{1}{a_1} + \dots + \frac{1}{a_n}} \rightarrow \ell$.
2. For $a \in (0, 1)$, we define the recurrent sequence $(x_n)_n$ as follows:

$$x_0 = a \quad \text{and} \quad x_{n+1} = x_n(1 - x_n) \quad \text{for } n \geq 0.$$

- (a) Prove that the sequence $(x_n)_n$ converges to 0.
 - (b) Prove that $nx_n \rightarrow 1$.
3. Infer from Corollary 2.7.3 that every sequence $(a_n)_n$ of positive numbers for which $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \ell$, verifies also $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \ell$. As an application, show that

$$\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

4. Show by example that the converse of Theorem 2.7.2 may fail. Then prove that if $(a_n)_n$ and $(b_n)_n$ are two sequences of real numbers such that $(b_n)_n$ is strictly increasing to ∞ and $\lim_{n \rightarrow \infty} \frac{b_n}{b_{n+1}} = b \in \mathbb{R} \setminus \{1\}$, then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \ell \quad \text{implies} \quad \lim_{n \rightarrow \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \ell.$$

2.8 Notes and Remarks

The rigorous definition of limit was given by Bernard Bolzano (*Der binomische Lehrsatz*, Prague, 1816, work little noticed at the time), and by Weierstrass in his lectures at Berlin University.

A nice account on the algorithms provided by iterative sequences (including the Babylonian algorithm) is given by Bailey [1]. A method for finding an approximation to a square root equivalent to two iterations of the Babylonian algorithm at each step is described in an ancient Indian mathematical manuscript called the *Bakhshali manuscript*. Given a number $a > 0$, one considers N^2 , the largest perfect square less than a and \sqrt{a} is approximated by the iterative sequence

$$x_0 = N \quad \text{and} \quad x_{n+1} = x_n + \frac{a - x_n^2}{2x_n} - \frac{\left(\frac{a - x_n^2}{2x_n}\right)^2}{2\left(x_n + \frac{a - x_n^2}{2x_n}\right)} \quad \text{for } n \geq 0.$$

In the case of $a = 336009$, this sequence starts with $x_0 = 579$. The first iterate $x_1 = 579.662\,833\,033\,259 \dots$ represents an approximation of

$$\sqrt{336009} = 579.662\,833\,033\,134 \dots$$

with 12 significant digits! See the paper by Bailey and Borwein [2] for details.

In Exercise 3, Sect. 2.2, we touched the subject of continued fractions. The *simple continued fractions* algorithm allows us to represent each real number x by a (finite or infinite) string of the form

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}} = [a_0; a_1, a_2, \dots], \quad (2.2)$$

where $a_0 \in \mathbb{Z}$ and all other coefficients a_n are strictly positive integers. This is done inductively as follows. In the first step, we choose $a_0 = \lfloor x \rfloor$. If $\{x\} = 0$, the algorithm stops. If $\{x\} > 0$, then we pass to the next step. We have $x = a_0 + \{x\}$ and $x_1 = \frac{1}{\{x\}} > 1$. We choose $a_1 = \lfloor x_1 \rfloor$. This yields $x_1 = a_1 + \{x_1\}$ and

$$x = a_0 + \frac{1}{a_1 + \{x_1\}}.$$

If $\{x_1\} = 0$, the algorithm stops. If $\{x_1\} > 0$, the algorithm continues with the decomposition of $x_2 = \frac{1}{\{x_1\}} > 1$ and so on. Clearly, this algorithm produces an infinite continued fraction if and only if x is an irrational number.

A good introduction to the theory of continued fractions can be found in the books of Khinchin [3] and Lang [4]. It is proved that every irrational number x is the limit of the sequence of fractions

$$\frac{p_n}{q_n} = [a_0; a_1, a_2, \dots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots a_{n-1} + \frac{1}{a_n}}}},$$

called *convergents* and

$$\frac{1}{q_n (q_n + q_{n+1})} < \left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}}$$

for all $n \in \mathbb{N}$. Moreover, the fractions $\frac{p_n}{q_n}$ provide the best approximation of x by rationals in the sense that for any other fraction $\frac{r}{s}$ with $0 < s < q_n$, we have

$$|sx - r| > |q_n x - p_n|.$$

The golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ has the continued fraction expansion $[1; 1, 1, 1, \dots]$. See Exercise 3, Sect. 2.2. Since none of its coefficients is greater than 1, φ is one of the most “difficult” real numbers to approximate with rational numbers. Indeed, a result due to Adolf Hurwitz asserts that any real number x can be approximated by infinitely many fractions p/q such that $\left|x - \frac{p}{q}\right| \leq \frac{1}{q^2\sqrt{5}}$. The case of φ shows that one cannot change $\sqrt{5}$ by any other greater constant.

The recent handbook by Cuyt et al. [5] offers a nice account on the present state of art of continued fractions and their applications.

The result of Corollary 2.7.3, concerning the arithmetic mean of terms of a convergent sequence can be considerably improved using the notion of convergence in density.

The main idea is to eliminate a negligible part of the indices and to take into consideration the rest of the terms. Precisely, if $J \subset \mathbb{N}$ is a subset such that $\mathbb{N} \setminus J$ is infinite, one may define in \mathbb{R} limits of the form

$$\lim_{\substack{n \rightarrow \infty \\ n \notin J}} a_n = \ell,$$

having the meaning that for any $\varepsilon > 0$, we can choose an index N for which $|a_n - \ell| < \varepsilon$, whenever $n \geq N$, $n \notin J$.

Call a subset $J \subset \mathbb{N}$ a set of *zero density* if $\lim_{n \rightarrow \infty} \frac{|\{j \in J : 0 \leq j \leq n\}|}{n} = 0$. Clearly, the complementary set of a set of zero density is infinite.

A numerical sequence $(a_n)_n$ *converges to ℓ in density* if there is a set of zero density $J \subset \mathbb{N}$ such that $\lim_{n \rightarrow \infty, n \notin J} a_n = \ell$. The main feature of this type of convergence is its connection with the convergence in the sense of Cesàro:

2.8.1 Theorem (Bernard O. Koopman and John von Neumann) *Suppose that $(a_n)_n$ is a bounded sequence of positive numbers. Then the following two conditions are equivalent:*

- (a) $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} a_k = 0$;
- (b) $a_n \rightarrow 0$ in density.

Proof Assuming $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a_k = 0$, we associate to each $\varepsilon > 0$ the set $A_\varepsilon = \{n \in \mathbb{N} : a_n \geq \varepsilon\}$. Since

$$\frac{|\{1, \dots, n\} \cap A_\varepsilon|}{n} \leq \frac{1}{n} \sum_{k=1}^n \frac{a_k}{\varepsilon} \leq \frac{1}{\varepsilon n} \sum_{k=1}^n a_k \rightarrow 0$$

as $n \rightarrow \infty$, we infer that each of the sets A_ε has zero density. Therefore, $a_n \rightarrow 0$ in density.

Suppose now that $(a_n)_n$ is bounded and converges to 0 in density. Then for every $\varepsilon > 0$, there is a set J of zero density outside which $a_n < \varepsilon$. Since

$$\frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} \sum_{k \in \{1, \dots, n\} \cap J} a_k + \frac{1}{n} \sum_{k \in \{1, \dots, n\} \setminus J} a_k \leq \frac{|\{1, \dots, n\} \cap J|}{n} \cdot \sup_{k \in \mathbb{N}} a_k + \varepsilon$$

and $\lim_{n \rightarrow \infty} \frac{|\{1, \dots, n\} \cap J|}{n} = 0$, we conclude that $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n a_k = 0$. \square

An application of Theorem 2.8.1 to the convergence of series makes the objective of Exercise 4, Sect. 4.2. Theorem 2.8.1 was recently extended by Niculescu and Popovici [6, 7] to higher order densities, in particular to the *harmonic density*,

$$d_h(A) = \lim_{n \rightarrow \infty} \frac{1}{\log n} \sum_{k=1}^n \frac{\chi_A(k)}{k},$$

where χ_A represents the characteristic function of A .

See also the Notes and Remarks at the end of Chap. 4.

Cesàro convergence extends the concept of limit beyond the class of convergent sequences, but not to all bounded sequences. This was accomplished by Stefan Banach. Using Axiom of Choice, he proved the existence of *generalized limits*, that is, of the mappings $\text{LIM} : \ell_{\mathbb{R}}^{\infty}(\mathbb{N}) \rightarrow \mathbb{R}$ having the following four properties:

- (LIM1) If $(x_n)_n$ is a convergent sequence, then $\text{LIM}((x_n)_n) = \lim_{n \rightarrow \infty} x_n$;
- (LIM2) (Linearity): $\text{LIM}(a(x_n)_n + b(y_n)_n) = a \text{LIM}((x_n)_n) + b \text{LIM}((y_n)_n)$ for all numbers $a, b \in \mathbb{R}$ and all bounded sequences $(x_n)_n$ and $(y_n)_n$;
- (LIM3) (Positivity): If $(x_n)_n \geq 0$, then $\text{LIM}((x_n)_n) \geq 0$;
- (LIM4) (Invariance): The limit of a sequence $(x_n)_n$ is the same as the limit of its translate to the left, $(x_{n+1})_n$.

The generalized limits are not unique. Necessarily, they verify the double inequality

$$\liminf_{n \rightarrow \infty} x_n \leq \text{LIM}((x_n)_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

Details can be found in the book of Bhatia [8], pp. 34–35.

References

1. Bailey, D.F.: A historical survey of solution by iteration. *Math. Mag.* **62**, 155–166 (1989)
2. Bailey, D.H., Borwein, J.M.: Ancient Indian square roots: an exercise in forensic Paleo-mathematics. *Am. Math. Mon.* **119**, 646–657 (2012)
3. Khinchin, A.Ya.: *Continued Fractions*. The University of Chicago Press, Chicago (1965)
4. Lang, S.: *Introduction to Diophantine Approximations*. Addison-Wesley, Reading (1966)

5. Cuyt, A., Petersen, V.B., Verdonk, B., Waadeland, H., Jones, W.B.: Handbook of Continued Fractions for Special Functions. Springer, Berlin (2008)
6. Niculescu, C.P., Popovici, F.: The behavior at infinity of an integrable function. *Expo. Math.* **30**, 277–282 (2012)
7. Niculescu, C.P., Popovici, F.: The asymptotic behavior of integrable functions. *Real Anal. Exch.* **38**(1), 157–168 (2012/2013)
8. Bhatia, R.: Notes on Functional Analysis. Hindustan Book Agency, New Delhi (2009)

Real Analysis on Intervals

Choudary, A.D.R.; Niculescu, C.

2014, XI, 525 p. 36 illus., Hardcover

ISBN: 978-81-322-2147-0