Chapter 2
Maxwell and Eddy Current Equations

2.1 Introduction

Maxwell equations stand for the set of partial differential equations that describe electric and magnetic phenomena. Maxwell equations were derived in several steps successively by Coulomb, Faraday, Ampère and Maxwell. Their treatment contains numerous difficulties either from the mathematical or numerical point of view. In particular, the presence of a large number of unknown fields, conditions at the infinity, high frequency require specific techniques to handle them. The literature in Mathematics and Physics is rather plentiful in this field and the reader is referred to most popular textbooks in electromagnetic theory (e.g. Feynman [74], Jackson [107], Landau and Lifshitz [116], Robinson [155]) and to Nédélec [138], Monk [131] and many others for the numerical solution of these equations.

Our purpose throughout this textbook is to study Maxwell equations in the particular situation where the source current in an electromagnetic setup has a low frequency. The term "low" means here that the characteristic length of the considered conducting bodies is small when compared to the wavelength of the inflowing current. Dimensional analysis considerations show that, in this case, propagation (hyperbolic) terms can be neglected beside all other terms. In other words, wave propagation phenomena are neglected and we have the creation of the so-called eddy currents inside the conductors. Such configurations are present in some specific industrial setups when induction properties of electromagnetic phenomena have to be exploited. For example, electric conduction generates heat by dissipation (Joule effect) and this feature can be used to raise conductor temperature for many purposes (e.g. forging, welding, surface processing). Another typical situation is the one where Lorentz forces can be used to stir liquid metals (e.g. cold crucibles, solidification). Many other examples can be found in metallurgy and other fields of application. In all these situations, the use of low frequency currents helps creating eddy currents with a negligible effect of displacement currents.

In this chapter, we start by presenting the general setting of Maxwell equations. Through these equations, we shall show the existence of a *potential vector* which

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will play a central role in the analysis and modeling of electromagnetic phenomena. We shall then restrict ourselves to the main object of the present monograph: Study of low frequency regimes. We show the validity of such an approximation and consider static (time independent) cases for which we derive models in electrostatics and magnetostatics. Then, we consider time harmonic regimes that are useful to study time periodic currents.

2.2 Maxwell Equations

In all the sequel, we shall denote as usual by $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{J}$ respectively magnetic induction field, magnetic field, electric displacement current field, electric field and electric current density field.

2.2.1 General Setting

Maxwell–Ampère and Faraday equations are respectively given by:

$$\frac{\partial \boldsymbol{D}}{\partial t} - \mathbf{curl}\,\boldsymbol{H} + \boldsymbol{J} = 0, \tag{2.1}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\,\boldsymbol{E} = 0. \tag{2.2}$$

Sometimes, (2.1) is replaced by an equation to take into account a *source current* $\boldsymbol{J}_S$, that is

$$\frac{\partial \boldsymbol{D}}{\partial t} - \mathbf{curl}\,\boldsymbol{H} + \boldsymbol{J} + \boldsymbol{J}_S = 0. \tag{2.3}$$

We have in addition the magnetic flux conservation equation

$$\operatorname{div} \boldsymbol{B} = 0. \tag{2.4}$$

Remark that taking the divergence of (2.2) yields

$$\frac{\partial}{\partial t} \operatorname{div} \boldsymbol{B} = 0.$$

This means that (2.4) can be interpreted as an initial condition to (2.2) since its validity for the initial time $t = 0$ guarantees it for all times thanks to (2.2).

Equations (2.1)–(2.4) are valid in the whole space $\mathbb{R}^3$ and for all times $t > 0$.

The related constitutive equations for this system are:

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{2.5}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}, \tag{2.6}$$

in $\mathbb{R}^3$. The functions μ and ε stand for *magnetic permeability* and *electric permittivity* respectively. Relation (2.5) is called *Magnetic induction law* and (2.6) is the *Electric induction law*. In the sequel, we shall assume, for obvious physical reasons that the functions ε and μ fulfill the following conditions:

$$0 < \mu_m \le \mu \le \mu_M, \tag{2.7}$$

$$0 < \varepsilon_m \le \varepsilon \le \varepsilon_M, \tag{2.8}$$

where $\mu_m, \mu_M, \varepsilon_m, \varepsilon_M$ are defined lower and upper bounds for μ and ε. In addition, μ and ε are constant equal to μ_0 and ε_0, called respectively *Magnetic permeability* and *electric permittivity of the vacuum*.

Remark 2.2.1. The charge density ϱ_q can be deduced by a charge conservation equation that is

$$\operatorname{div} \boldsymbol{D} = \varrho_q. \tag{2.9}$$

2.2.2 Presence of Conductors

In the presence of conductors Ω moving with velocity $\boldsymbol{v}$, we adopt the *Ohm's law*:

$$\boldsymbol{J} = \sigma\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \qquad \text{in } \mathbb{R}^3, \tag{2.10}$$

where σ is the *electric conductivity* of the given conductor occupying Ω, and $\sigma = 0$ outside the conductors. The function σ is assumed to satisfy the hypothesis:

$$0 < \sigma_m \le \sigma \le \sigma_M. \tag{2.11}$$

Outside the conductors we have $\boldsymbol{J} = 0$ and (2.10) can be considered with $\sigma = 0$. In most situations, we will consider eddy currents in non moving conductors for which we have $\boldsymbol{J} = \sigma \boldsymbol{E}$.

2.2.3 Wave Propagation

Equations (2.1)–(2.6) are of hyperbolic type. To see this, let us consider, for instance, the simple case of a non moving homogeneous isotropic medium, i.e. the case where ε, μ and σ are constant and $\boldsymbol{v} = 0$. We have from (2.1)–(2.6),

$$\varepsilon \frac{\partial \boldsymbol{E}}{\partial t} - \mathbf{curl}\, \boldsymbol{H} + \boldsymbol{J} = 0,$$
$$\mu \frac{\partial \boldsymbol{H}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} = 0.$$

Taking the **curl** of the first equation and the time derivative of the second one, we obtain by subtracting, the equation

$$\varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \mathbf{curl}\,\mathbf{curl}\, \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{J}.$$

Since $\operatorname{div} \boldsymbol{H} = \mu \operatorname{div} \boldsymbol{B} = 0$, we deduce

$$\varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \Delta \boldsymbol{H} = \mathbf{curl}\, \boldsymbol{J}.$$

If the current density $\boldsymbol{J} = \boldsymbol{J}_S$ is given, we obtain a hyperbolic problem that describes propagation of electromagnetic waves in the space. If the Ohm's law (2.10) is assumed in Ω, we obtain the equations:

$$\varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} + \mu\sigma \frac{\partial \boldsymbol{H}}{\partial t} - \Delta \boldsymbol{H} = 0 \qquad \text{in } \Omega,$$
$$\varepsilon\mu \frac{\partial^2 \boldsymbol{H}}{\partial t^2} - \Delta \boldsymbol{H} = 0 \qquad \text{in } \Omega_{\text{ext}},$$

with appropriate interface conditions and condition at the infinity.

Here also we have wave propagation but, waves are damped in the conductors, the damping being proportional to the electric conductivity σ. We have the same conclusion if the coefficients ε, μ and σ are not constant but the equations are slightly more complex.

2.2.4 The Vector Potential

One of the principal ingredients in electromagnetism is the use of a vector potential. This one is defined in the following way: Using (2.4), we deduce from Theorem 1.3.4 the existence of a vector valued function $\boldsymbol{A} : \mathbb{R}^3 \to \mathbb{C}^3$, called *vector potential*, such that

$$\boldsymbol{B} = \mathbf{curl}\, \boldsymbol{A} \quad \text{in } \mathbb{R}^3. \tag{2.12}$$

Such a vector field is in addition unique if we impose the gauge condition

$$\operatorname{div} \boldsymbol{A} = 0. \tag{2.13}$$

As far as the regularity of the vector potential $\boldsymbol{A}$ is involved, we see that if the unknowns of the Maxwell equations are sought in the space $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)$, Theorem 1.3.4 says that necessarily $\boldsymbol{A} \in \boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)$.

We shall characterize later this vector in more specific situations.

2.3 Low Frequency Approximation

In many situations, like in alternating current configurations, low frequencies enable neglecting the displacement current term $\partial \boldsymbol{D}/\partial t$ in the Maxwell equations. This leads to the set of equations:

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J}, \tag{2.14}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\, \boldsymbol{E} = 0, \tag{2.15}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{2.16}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}. \tag{2.17}$$

Some authors have rigourously proved the validity of such an approximation. We mention here a result of Ammari, Buffa and Nédélec [13] where the authors use a formulation with source currents. They show that the external magnetic and electric fields (outside the conductors) are approximated at the first order with respect to the frequency by the system (2.14)–(2.17).

Our study deals mainly with this set of equations, and especially in the presence of the so-called *Eddy Currents* in the conductors. It is also to be specified that, in this section, we consider the magnetic permeability μ as a known function of the position $\boldsymbol{x}$. This allows later to describe nonlinear problems in which μ depends on $\boldsymbol{H}$. However, when we address the numerical solution of this kind of problems, we use an iterative method in which μ can be taken variable but given.

2.3.1 A Vector Potential Formulation

Let us now see how the vector potential $\boldsymbol{A}$ can be characterized using the system of equations (2.14)–(2.17). Looking for solutions of (2.14)–(2.17) in the space $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)$, we deduce from Theorem 1.3.4 that $\boldsymbol{A} \in \boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)$. From (2.14) and (2.17), we deduce

$$\mathbf{curl}\,(\mu^{-1}\,\mathbf{curl}\, \boldsymbol{A}) = \boldsymbol{J} \qquad \text{in } \mathbb{R}^3. \tag{2.18}$$

Let us now assume that the current density $\boldsymbol{J}$ is a function of $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)$ with a compact support contained in a domain Ω. We have the system of equations:

$$\mathbf{curl}\,(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{A}) = \boldsymbol{J} \qquad \text{in } \mathbb{R}^3, \tag{2.19}$$

$$\operatorname{div}\boldsymbol{A} = 0 \qquad \text{in } \mathbb{R}^3, \tag{2.20}$$

$$|\boldsymbol{A}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^{-1}) \qquad \text{for } |\boldsymbol{x}| \to \infty, \tag{2.21}$$

the condition at the infinity being a consequence of $\boldsymbol{A} \in \boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)$.

To prove existence and uniqueness of a solution of (2.19)–(2.21) for given $\boldsymbol{J}$, we derive a variational formulation of it. Let us take a function $\boldsymbol{w} \in \mathscr{D}(\mathbb{R}^3)$. If $\boldsymbol{J}$ is smooth enough, we have by the Green formula,

$$\int_{\mathbb{R}^3} \mathbf{curl}\,(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{A}) \cdot \overline{\boldsymbol{w}}\,d\boldsymbol{x} = \int_{\mathbb{R}^3} \mu^{-1}\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\overline{\boldsymbol{w}}\,d\boldsymbol{x}.$$

This leads to the variational formulation of (2.19)–(2.21):

$$\text{Find } \boldsymbol{A} \in \boldsymbol{\mathcal{V}} \quad \text{such that} \quad \mathscr{B}(\boldsymbol{A},\boldsymbol{w}) = \mathscr{L}(\boldsymbol{w}) \qquad \forall\, \boldsymbol{w} \in \boldsymbol{\mathcal{V}}, \tag{2.22}$$

where $\boldsymbol{\mathcal{V}}$ is the space

$$\boldsymbol{\mathcal{V}} := \{\boldsymbol{w} \in \boldsymbol{\mathcal{W}}^1(\mathbb{R}^3);\ \operatorname{div}\boldsymbol{w} = 0\},$$

equipped with the semi-norm $|\cdot|_{\boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)}$, which is a norm on $\boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)$ (see [62], Vol. 4, p. 118), and

$$\mathscr{B}(\boldsymbol{A},\boldsymbol{w}) := \int_{\mathbb{R}^3} \mu^{-1}\,\mathbf{curl}\,\boldsymbol{A} \cdot \mathbf{curl}\,\overline{\boldsymbol{w}}\,d\boldsymbol{x},$$

$$\mathscr{L}(\boldsymbol{w}) := \int_{\Omega} \boldsymbol{J} \cdot \overline{\boldsymbol{w}}\,d\boldsymbol{x}.$$

We have the following result.

Theorem 2.3.1. *Assume that μ satisfies (2.7). Then (2.22) has a unique solution $\boldsymbol{A} \in \boldsymbol{\mathcal{V}}$. Moreover, there is a constant C such that*

$$\|\boldsymbol{A}\|_{\boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)} \le C\,\|\boldsymbol{J}\|_{\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)}. \tag{2.23}$$

Proof. Using Theorem 1.3.2, we deduce that the quantity

$$|\!|\!|\boldsymbol{w}|\!|\!| := (\|\,\mathbf{curl}\,\boldsymbol{w}\|^2_{\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)} + \|\operatorname{div}\boldsymbol{w}\|^2_{\mathcal{L}^2(\mathbb{R}^3)})^{\frac{1}{2}} = \|\,\mathbf{curl}\,\boldsymbol{w}\|_{\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)}$$

defines a norm on the space $\boldsymbol{\mathcal{V}}$ that is equivalent to the norm of $\boldsymbol{\mathcal{W}}^1(\mathbb{R}^3)$. This implies that the sesquilinear form $\mathscr{B}$ is continuous and coercive on $\boldsymbol{\mathcal{V}} \times \boldsymbol{\mathcal{V}}$. The antilinear form $\mathscr{L}$ is also continuous on $\boldsymbol{\mathcal{V}}$. The Lax–Milgram Theorem 1.2.1 gives then the conclusion. □

In the particular case where the magnetic permeability μ is constant (equal to μ_0), we have an integral formula for the vector potential $\boldsymbol{A}$ and the magnetic induction $\boldsymbol{B}$.

Theorem 2.3.2. *Let $\boldsymbol{J}$ be a given vector field in the space $\boldsymbol{\mathcal{L}}^2(\mathbb{R}^3)$ with a compact support and assume that $\mu = \mu_0$ in $\mathbb{R}^3$. Then the potential $\boldsymbol{A}$ and the magnetic induction $\boldsymbol{B}$ are respectively given by:*

$$\boldsymbol{A}(\boldsymbol{x}) = \mu_0 \int_{\mathbb{R}^3} G(\boldsymbol{x}, \boldsymbol{y}) \, \boldsymbol{J}(\boldsymbol{y}) \, d\boldsymbol{y}, \tag{2.24}$$

$$\boldsymbol{B}(\boldsymbol{x}) = \mu_0 \int_{\mathbb{R}^3} \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \times \boldsymbol{J}(\boldsymbol{y}) \, d\boldsymbol{y}, \tag{2.25}$$

for $\boldsymbol{x} \in \mathbb{R}^3$, where G is the Green kernel in dimension 3, defined by (1.21).

Proof. Let us first note that when $\mu = \mu_0$ is constant, Eqs. (2.19)–(2.21) become:

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} = \mu_0 \boldsymbol{J} \qquad \text{in } \mathbb{R}^3, \tag{2.26}$$

$$\operatorname{div} \boldsymbol{A} = 0 \qquad \text{in } \mathbb{R}^3, \tag{2.27}$$

$$|\boldsymbol{A}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^{-1}) \qquad \text{for } |\boldsymbol{x}| \to \infty. \tag{2.28}$$

Using the vector identity

$$-\Delta \boldsymbol{A} = \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{A} - \nabla \operatorname{div} \boldsymbol{A},$$

we deduce

$$-\Delta \boldsymbol{A} = \mu_0 \boldsymbol{J} \qquad \text{in } \mathbb{R}^3.$$

A vector field that satisfies the above identity and (2.28), can be written, thanks to (1.23),

$$\boldsymbol{A}(\boldsymbol{x}) = \mu_0 \int_{\mathbb{R}^3} \boldsymbol{J}(\boldsymbol{y}) \, G(\boldsymbol{x}, \boldsymbol{y}) \, d\boldsymbol{y}.$$

Remark that this solution is unique and since the support of $\boldsymbol{J}$ is compact we have (2.21).

The proof of (2.25) is simply obtained by applying the **curl** operator to (2.24), the integrand being an integrable function. □

Relation (2.25) enables calculating the magnetic induction generated by a conductor Ω where an electric current of density $\boldsymbol{J}$ flows.

2.3.2 A Scalar Potential Problem

The material developed in the previous subsection shows that when $\mu = \mu_0$, the magnetic induction can be directly calculated from the current density $\boldsymbol{J}$ by the formula (2.25). When this is not the case (2.19)–(2.21) is not well adapted to numerical solution. For this, we can proceed as the following.

Let us consider that the electric current density $\boldsymbol{J}$ is a given vector field and has a compact support Ω with boundary Γ. We assume that $\mu = \mu_0$ outside $\overline{\Omega}$. Let us introduce a magnetic field $\boldsymbol{H}_0$ defined by

$$\boldsymbol{H}_0(\boldsymbol{x}) = \int_{\mathbb{R}^3} \nabla_{\boldsymbol{x}} G(\boldsymbol{x}, \boldsymbol{y}) \times \boldsymbol{J}(\boldsymbol{y})\, ds(\boldsymbol{y}) \qquad \boldsymbol{x} \in \mathbb{R}^3. \tag{2.29}$$

This field is due to $\boldsymbol{J}$ when $\mu = \mu_0$ in Ω. According to (2.25), we have the equations:

$$\mathbf{curl}\, \boldsymbol{H}_0 = \boldsymbol{J}, \tag{2.30}$$

$$\operatorname{div} \boldsymbol{H}_0 = 0 \tag{2.31}$$

in $\mathbb{R}^3$. Subtracting (2.30) from (2.14), we find

$$\mathbf{curl}\, (\boldsymbol{H} - \boldsymbol{H}_0) = 0 \qquad \text{in } \mathbb{R}^3.$$

This implies (see Theorem 1.3.4) the existence of a scalar field $\psi : \mathbb{R}^3 \to \mathbb{C}$ such that

$$\boldsymbol{H} - \boldsymbol{H}_0 = -\nabla\psi \qquad \text{in } \mathbb{R}^3. \tag{2.32}$$

Multiplying this equation by μ and using (2.16), (2.17), we obtain

$$\operatorname{div}(\mu(\nabla\psi - \boldsymbol{H}_0)) = 0 \qquad \text{in } \mathbb{R}^3,$$

and consequently

$$\operatorname{div}\, (\mu\nabla\psi) = \operatorname{div}(\mu\boldsymbol{H}_0) \qquad \text{in } \Omega \cup \Omega_{\text{ext}}, \tag{2.33}$$

where Ω denotes the conductor, $\Omega_{\text{ext}} = \mathbb{R}^3 \setminus \overline{\Omega}$ and in addition

$$\Big[\mu \frac{\partial\psi}{\partial n}\Big]_\Gamma = [\mu\boldsymbol{H}_0 \cdot \boldsymbol{n}]_\Gamma.$$

By using (2.31), (2.16), (2.17), we successively obtain

$$\left[\mu \frac{\partial \psi}{\partial n}\right]_\Gamma = [\mu \boldsymbol{H}_0 \cdot \boldsymbol{n}]_\Gamma = [(\mu - \mu_0)\boldsymbol{H}_0 \cdot \boldsymbol{n}]_\Gamma = (\mu_0 - \mu)\boldsymbol{H}_0 \cdot \boldsymbol{n} \qquad \text{on } \Gamma.$$

We then obtain the problem:

$$- \operatorname{div}(\mu \nabla \psi) = - \operatorname{div}(\mu \boldsymbol{H}_0) \qquad \text{in } \Omega, \tag{2.34}$$

$$\Delta \psi = 0 \qquad \text{in } \Omega_{\text{ext}}, \tag{2.35}$$

$$[\psi]_\Gamma = 0, \tag{2.36}$$

$$\left[\mu \frac{\partial \psi}{\partial n}\right]_\Gamma = (\mu_0 - \mu)\, \boldsymbol{H}_0 \cdot \boldsymbol{n}, \tag{2.37}$$

$$\psi(\boldsymbol{x}) = \mathcal{O}(|\boldsymbol{x}|^{-1}) \qquad \text{for } |\boldsymbol{x}| \to \infty. \tag{2.38}$$

Let us now show that (2.34)–(2.38) is well posed if we seek $\psi \in \mathcal{W}^1(\mathbb{R}^3)$. Using the exterior Steklov–Poincaré operator P defined in Sect. 1.3.5, we can formulate equations (2.35)–(2.38) as

$$- \mu \frac{\partial \psi^-}{\partial n} = \mu_0\, P\psi - (\mu - \mu_0)\, \boldsymbol{H}_0 \cdot \boldsymbol{n}. \tag{2.39}$$

Multiplying (2.34) by a function $\overline{\theta} \in \mathcal{H}^1(\Omega)$ and using the Green formula with $\operatorname{div}(\mu_0 \boldsymbol{H}_0) = 0$, we obtain then

$$\int_\Omega \mu \nabla\psi \cdot \nabla\overline{\theta}\, d\boldsymbol{x} - \int_\Gamma \mu \frac{\partial \psi^-}{\partial n}\, \overline{\theta}\, ds = \int_\Omega (\mu - \mu_0)\boldsymbol{H}_0 \cdot \nabla\overline{\theta}\, d\boldsymbol{x} - \int_\Gamma (\mu - \mu_0)\, \boldsymbol{H}_0 \cdot \boldsymbol{n}\, \overline{\theta}\, ds.$$

By using (2.39), this leads to the variational problem:

$$\begin{cases} \text{Find } \psi \in \mathcal{H}^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \mu\, \nabla\psi \cdot \nabla\overline{\theta}\, d\boldsymbol{x} + \mu_0 \int_\Gamma P(\psi)\, \overline{\theta}\, ds = \int_\Omega (\mu - \mu_0)\, \boldsymbol{H}_0 \cdot \nabla\overline{\theta}\, d\boldsymbol{x} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall\, \theta \in \mathcal{H}^1(\Omega). \end{cases} \tag{2.40}$$

Let us prove the following result.

Theorem 2.3.3. *Assume that μ is given in $\mathcal{L}^\infty(\Omega)$ and satisfies (2.7). Assume in addition that the restriction of $\boldsymbol{H}_0$ to Ω is given by (2.29). Then (2.40) admits a unique solution.*

Proof. We define the sesquilinear and antilinear forms:

$$\mathscr{B}(\psi, \theta) := \int_\Omega \mu \, \nabla\psi \cdot \nabla\overline{\theta} \, d\boldsymbol{x} + \mu_0 \int_\Gamma P(\psi) \, \overline{\theta} \, ds,$$
$$\mathscr{L}(\theta) := \int_\Omega (\mu - \mu_0) \, \boldsymbol{H}_0 \cdot \nabla\overline{\theta} \, d\boldsymbol{x}.$$

Since $\boldsymbol{H}_0 \in \boldsymbol{\mathcal{L}}^2(\Omega)$, then we have by using (2.7),

$$\begin{aligned} |\mathscr{L}(\theta)| &\le (\mu_0 + \mu_M) \, \|\boldsymbol{H}_0\|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \, \|\nabla\theta\|_{\boldsymbol{\mathcal{L}}^2(\Omega)} \\ &\le C \, \|\theta\|_{\mathcal{H}^1(\Omega)}. \end{aligned}$$

The form $\mathscr{L}$ is hence continuous on $\mathcal{H}^1(\Omega)$. The sesquilinear form $\mathscr{B}$ is also continuous since we have from Theorem 1.3.12, the trace theorem [92] and (2.7),

$$|\mathscr{B}(\psi, \theta)| \le C \, \|\psi\|_{\mathcal{H}^1(\Omega)} \, \|\theta\|_{\mathcal{H}^1(\Omega)}.$$

The coercivity of $\mathscr{B}$ is obtained thanks to Theorem 1.3.12, (2.7) and (1.5),

$$\begin{aligned} \mathscr{B}(\theta, \theta) &= \int_\Omega \mu \, |\nabla\theta|^2 \, d\boldsymbol{x} + \mu_0 \int_\Gamma P(\theta) \, \overline{\theta} \, ds \\ &\ge \mu_m \, \|\nabla\theta\|^2_{\boldsymbol{\mathcal{L}}^2(\Omega)} + C_1 \, \|\theta\|^2_{\mathcal{H}^{\frac{1}{2}}(\Gamma)} \\ &\ge C_2 \, \|\theta\|^2_{\mathcal{H}^1(\Omega)}. \end{aligned}$$

Existence and uniqueness of a solution is then a consequence of the Lax–Milgram theorem (Theorem 1.2.1). □

2.4 Static Cases

Static cases stand for configurations where all fields are time independent. We obtain from (2.14)–(2.17) after dropping time derivatives:

$$\mathbf{curl}\, \boldsymbol{H} = \boldsymbol{J}, \tag{2.41}$$
$$\mathbf{curl}\, \boldsymbol{E} = 0, \tag{2.42}$$
$$\operatorname{div}(\mu \boldsymbol{H}) = 0. \tag{2.43}$$

This situation enables decoupling electricity and magnetism in the following way.

2.4.1 Electrostatics

From (2.42) and Theorem 1.3.5, we deduce the existence of a scalar field $\phi : \mathbb{R}^3 \to \mathbb{C}$ such that

$$\boldsymbol{E} = -\nabla\phi \qquad \text{in } \mathbb{R}^3. \tag{2.44}$$

Assuming that Ohm's law is satisfied in the static conductor Ω, i.e.

$$\boldsymbol{J} = \sigma\boldsymbol{E} \qquad \text{in } \Omega, \tag{2.45}$$

where σ is assumed to satisfy (2.11), we obtain from (2.41)–(2.43),

$$\operatorname{div}(\sigma\nabla\phi) = 0 \qquad \text{in } \Omega. \tag{2.46}$$

Equation (2.46) is an elliptic equation that requires appropriate boundary conditions. In many situations, the boundary Γ of Ω is split into parts where Dirichlet or Neumann conditions can be enforced.

– If a part of the boundary is electrically isolated, we prescribe

$$\frac{\partial\phi}{\partial n} = 0$$

on this part (Homogeneous Neumann boundary condition).

– If a part of the boundary is connected to an electricity generator, we prescribe the potential ϕ when we have a voltage generator (Dirichlet condition) or the normal derivative of ϕ by

$$\boldsymbol{J}\cdot\boldsymbol{n} = \sigma\frac{\partial\phi}{\partial n},$$

when we have a current generator (Neumann condition).

Let us assume, for instance, that Γ is divided into three parts Γ_1, Γ_2 and Γ_3 such that

$$\inf_{\boldsymbol{x}\in\Gamma_1,\ \boldsymbol{y}\in\Gamma_3} |\boldsymbol{x}-\boldsymbol{y}| > 0,$$

i.e. Γ_1 and Γ_3 are not connected. Assume furthermore that the potential ϕ satisfies the conditions:

$$\begin{aligned} \phi &= V && \text{on } \Gamma_1, \\ \phi &= 0 && \text{on } \Gamma_3, \end{aligned}$$

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{on } \Gamma_2,$$

where V is given. By defining the space

$$\mathcal{X} := \{\, \psi \in \mathcal{H}^1(\Omega);\ \psi = 0 \text{ on } \Gamma_1 \cup \Gamma_3 \,\},$$

multiplying (2.46) by $\theta \in \mathcal{X}$ and using the Green formula, we obtain

$$\int_\Omega \sigma \nabla\phi \cdot \nabla\theta \, d\boldsymbol{x} = 0. \tag{2.47}$$

Hence, the mathematical problem consists in seeking a function $\phi \in \mathcal{H}^1(\Omega)$ such that $\phi = V$ on Γ_1, $\phi = 0$ on Γ_3 satisfying (2.47) for all $\theta \in \mathcal{X}$. Let ϕ_0 denote a function in $\mathcal{H}^1(\Omega)$ such that $\phi_0 = V$ on Γ_1 and $\phi = 0$ on Γ_3 and let $\psi = \phi - \phi_0$. We easily check that $\psi \in \mathcal{X}$ and

$$\int_\Omega \sigma \nabla\psi \cdot \nabla\overline{\theta} \, d\boldsymbol{x} = \int_\Omega \sigma \nabla\phi_0 \cdot \nabla\overline{\theta} \, d\boldsymbol{x} \qquad \forall\, \theta \in \mathcal{X}.$$

By the Lax–Milgram theorem (Theorem 1.2.1) in $\mathcal{X}$, this problem possesses a unique solution. It follows that (2.47) is well posed and we have $\phi = \psi + \phi_0$.

2.4.2 Magnetostatics

Let us assume we are in presence of a conductor Ω and a given current of density $\boldsymbol{J}_0$ and let us assume, as usual, that $\mu = \mu_0$ in $\Omega_{\text{ext}} = \mathbb{R}^3 \setminus \overline{\Omega}$. We define the vector field

$$\boldsymbol{M} = (\mu - \mu_0)\, \boldsymbol{H},$$

called *Magnetization*. Here above, $\boldsymbol{H}$ is assumed to be the magnetic field generated by $\boldsymbol{J}_0$, i.e. $\textbf{curl}\, \boldsymbol{H} = \boldsymbol{J}_0$.

In general, materials for which the magnetic permeability μ is not constant are called *Ferromagnetic materials*. By definition, $\boldsymbol{M} = 0$ for nonferromagnetic conductors.

In ferromagnetic materials, μ depends generally on the magnetic field $\boldsymbol{H}$. When the function $\mu = \mu(|\boldsymbol{H}|)$ is known, it suffices to compute $\boldsymbol{H}$ in order to deduce $\boldsymbol{M}$. To do this, we define $\boldsymbol{H}_0$ like in (2.30)–(2.31), i.e., $\boldsymbol{H}_0$ is the magnetic field without ferromagnetic conductors. It follows by using Sect. 2.3.2 that

$$\boldsymbol{H} = \boldsymbol{H}_0 - \nabla\psi$$

where ψ satisfies:

$$-\operatorname{div}(\mu\nabla\psi) = -\operatorname{div}(\mu\boldsymbol{H}_0) \qquad \text{in } \Omega, \tag{2.48}$$

$$\Delta\psi = 0 \qquad \text{in } \Omega_{\text{ext}}, \tag{2.49}$$

$$[\psi]_\Gamma = 0 \qquad \text{on } \Gamma, \tag{2.50}$$

$$\left[\mu\frac{\partial\psi}{\partial n}\right]_\Gamma = (\mu_0 - \mu)\,\boldsymbol{H}_0\cdot\boldsymbol{n} \qquad \text{on } \Gamma, \tag{2.51}$$

$$\psi(\boldsymbol{x}) = \mathcal{O}(|\boldsymbol{x}|^{-1}) \qquad \text{for } |\boldsymbol{x}| \to \infty. \tag{2.52}$$

Problem (2.48)–(2.52) is a nonlinear elliptic problem when we replace $\boldsymbol{H}$ in $\mu(|\boldsymbol{H}|)$ by $\boldsymbol{H}_0 - \nabla\psi$. Note that the nonlinearity appears as well in the partial differential equation (2.48) as in the boundary condition (2.51). We shall consider such problems in view of applications (Chap. 11).

2.5 Time–Harmonic Regime

We are frequently faced with the case where data are periodic functions of time. This corresponds to the case where a source alternating (AC) current is given. To handle this situation, a time–harmonic solution can be sought. This one is considered by developing the solution in Fourier series in time. We then seek solutions of (2.1)–(2.6) of the form:

$$\begin{aligned}
\boldsymbol{H}(\boldsymbol{x},t) &= \operatorname{Re}(e^{\mathrm{i}\omega t}\boldsymbol{H}(\boldsymbol{x})),\\
\boldsymbol{D}(\boldsymbol{x},t) &= \operatorname{Re}(e^{\mathrm{i}\omega t}\boldsymbol{D}(\boldsymbol{x})),\\
\boldsymbol{J}(\boldsymbol{x},t) &= \operatorname{Re}(e^{\mathrm{i}\omega t}\boldsymbol{J}(\boldsymbol{x})),\\
\boldsymbol{E}(\boldsymbol{x},t) &= \operatorname{Re}(e^{\mathrm{i}\omega t}\boldsymbol{E}(\boldsymbol{x})),
\end{aligned}$$

for $\boldsymbol{x} \in \mathbb{R}^3$, where $\omega \in \mathbb{R}$ is the angular frequency that we choose positive for convenience. Relations (2.1)–(2.2), (2.5)–(2.6) lead to

$$\mathrm{i}\omega\boldsymbol{D} - \mathbf{curl}\,\boldsymbol{H} + \boldsymbol{J} = 0, \tag{2.53}$$

$$\mathrm{i}\omega\boldsymbol{B} + \mathbf{curl}\,\boldsymbol{E} = 0, \tag{2.54}$$

$$\boldsymbol{B} = \mu\boldsymbol{H}, \tag{2.55}$$

$$\boldsymbol{D} = \varepsilon\boldsymbol{E}. \tag{2.56}$$

Note that we have, for the sake of simplicity, kept the same notations for the involved fields although we are now concerned with time–independent complex functions. Note also that, since div $\mathbf{curl} = 0$, (2.4) is a consequence of (2.54) if $\omega \neq 0$.

Remark 2.5.1. An analog to Sect. 2.2.3 can be made for (2.53)–(2.54). We have, when ε and μ are constant,

$$\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{H} - \omega^2\varepsilon\mu\,\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{J}.$$

Using the relation div $\boldsymbol{H} = 0$, we obtain the Helmholtz equation

$$-\varDelta\boldsymbol{H} - \omega^2\varepsilon\mu\boldsymbol{H} = \mathbf{curl}\,\boldsymbol{J}.$$

2.6 Eddy Current Equations

The remaining chapters are devoted to the derivation and analysis of eddy current models. We consider, in the sequel, a low frequency approximation of the system of equations (2.53)–(2.56) with appropriate behaviour at the infinity. In this case, we can neglect the term $\mathrm{i}\omega\boldsymbol{D}$ in (2.53).

As far as problem data are concerned we are faced with two types of approaches:

1. A first approach consists in assuming that a *source current* $\boldsymbol{J}_0$ is given with a support contained in one (or many) conductor(s). The current density is then written as $\boldsymbol{J} = \tilde{\boldsymbol{J}} + \boldsymbol{J}_0$ where $\tilde{\boldsymbol{J}}$ is the induced current density that is supposed to obey to Ohm's law (2.10) with null velocity ($\boldsymbol{v} = 0$). We obtain then the system of equations:

$$\mathbf{curl}\,\boldsymbol{H} - \tilde{\boldsymbol{J}} = \boldsymbol{J}_0 \quad \text{in } \mathbb{R}^3, \tag{2.57}$$

$$\mathrm{i}\omega\mu\boldsymbol{H} + \mathbf{curl}\,\boldsymbol{E} = 0 \quad \text{in } \mathbb{R}^3, \tag{2.58}$$

$$\tilde{\boldsymbol{J}} = \sigma\boldsymbol{E} \quad \text{in } \mathbb{R}^3, \tag{2.59}$$

$$|\boldsymbol{H}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^{-1}) \quad \text{for } |\boldsymbol{x}| \to \infty, \tag{2.60}$$

$$|\boldsymbol{E}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^{-1}) \quad \text{for } |\boldsymbol{x}| \to \infty, \tag{2.61}$$

   with σ extended by 0 outside $\varOmega$. Note here that the actual current density $\tilde{\boldsymbol{J}}$ satisfies div $\tilde{\boldsymbol{J}} = 0$ only if the source current $\boldsymbol{J}_0$ is divergence free. This condition is furthermore necessary to ensure that the eddy current problem is a good approximation of the Maxwell equations when ω is small enough (See [13]).
2. An alternative method consists in assuming that we are given either voltage or total current intensity that can be directly prescribed by a power generator. The difficulty relies here on the obtention of an adapted formulation that has the

voltage or the current as unique data. A variant consists in supplying current power. This corresponds more to realistic and industrial setups.

The first method is the most used one in the literature. Actually, the inductors are supplied with currents and it is not necessary to prescribe the electric source in the system. This method is simpler to formulate but does not correspond to realistic situations unless the conductors supporting source currents are thin enough so one can approximate a current density with its average.

The second procedure corresponds to an idealization of the real setup in the sense that voltage is given by prescribing a *cut* in the inductor represented by a non simply connected domain. This cut (see Fig. 1.1) stands for a virtual link of the inductor to the power generator and problem data are the constants given in Theorems 1.3.5 and 1.3.6. In this case, we are constrained to assume that (2.54) is valid in Ω and in Ω_{ext} but not in the whole space in order to introduce a source current. As in [34, 36], we have chosen to treat in most applications this second category of formulations.

Time harmonic eddy current equations are given by the set of partial differential equations:

$$\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J} = 0 \qquad \text{in } \mathbb{R}^3, \tag{2.62}$$

$$\mathrm{i}\omega\boldsymbol{B} + \mathbf{curl}\,\boldsymbol{E} = 0 \qquad \text{in } \Omega \cup \Omega_{\text{ext}}, \tag{2.63}$$

$$\operatorname{div}\boldsymbol{B} = 0 \qquad \text{in } \mathbb{R}^3, \tag{2.64}$$

$$\boldsymbol{B} = \mu\boldsymbol{H} \qquad \text{in } \mathbb{R}^3, \tag{2.65}$$

$$\boldsymbol{J} = \sigma\boldsymbol{E} \qquad \text{in } \mathbb{R}^3, \tag{2.66}$$

$$|\boldsymbol{H}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^{-1}) \qquad \text{for } |\boldsymbol{x}| \to \infty, \tag{2.67}$$

$$|\boldsymbol{E}(\boldsymbol{x})| = \mathcal{O}(|\boldsymbol{x}|^{-1}) \qquad \text{for } |\boldsymbol{x}| \to \infty. \tag{2.68}$$

This set of equations has to be supplemented with appropriate boundary and interface conditions on Γ. For this, let us remark that:

1. As said before, in the case where a source current density is prescribed, (2.62) is to be replaced by

$$\mathbf{curl}\,\boldsymbol{H} - \boldsymbol{J} = \boldsymbol{J}_S \qquad \text{in } \mathbb{R}^3. \tag{2.69}$$

2. Equation (2.64) is necessary, since from the previous remark, this one is no more a consequence of (2.63). However, if (2.63) is satisfied with $\omega \neq 0$, then (2.64) is equivalent to assuming that the jump $[\boldsymbol{B}\cdot\boldsymbol{n}]$ is null.
3. Equation (2.62) implies

$$\operatorname{div}\boldsymbol{J} = 0 \quad \text{in } \mathbb{R}^3. \tag{2.70}$$

4. From the set of equations (2.62)–(2.68) we can derive interface conditions (involving continuities and jumps) at boundaries of the conductors. For this, if we formally obtain by using relations

$$\operatorname{div} \boldsymbol{J} = 0 \quad \text{in } \mathbb{R}^3, \qquad \boldsymbol{J} = 0 \quad \text{in } \Omega_{\text{ext}},$$

that

$$\boldsymbol{J} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma. \tag{2.71}$$

In addition, (2.62) implies that

$$[\boldsymbol{H} \times \boldsymbol{n}]_{\Gamma} = 0, \tag{2.72}$$

when $\boldsymbol{J}$ has no Dirac masses on Γ, i.e., no surface currents flow on Γ.
5. Equation (2.63) is assumed to be valid only in the conductors and the free space. This is necessary when specific data are to be prescribed like voltage and total current. Depending on the models this restriction is to be relaxed by a prescription of the equation on $\mathbb{R}^3 \setminus S$, or even $\mathbb{R}^3 \setminus \partial S$, where S is a cut (or union of cuts) in the conductors. This restriction is however not necessary ((2.63) is valid in $\mathbb{R}^3$) if a source current density is specified.

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