Chapter 2
The Adventure as Experienced by the Students

Before taking the reader into the classroom, we need to introduce the children who will be found there. Other chapters introduce the school in which the classroom was located and the teachers who carried out the lessons, but here we are focusing on the students in a particular classroom. Who were they? The first key piece of information is that since the school was an essential element of the COREM (Center for Observation and Research on Mathematics Teaching) admissions were emphatically not selective. The school was the public school for a blue collar neighborhood, and its students were the ones who lived around it. Parents were kept informed about the unusual aspects of the teaching, but there were no special requirements or requests of them. On the other hand, the lessons we visit took place in the fifth grade with students most of whom had been at the school since age three or four, so all of their expectations for what would happen in a mathematics class were built around the kind of activity and responsibility we see in action. They needed no persuasion to involve themselves.

Enjoy joining them!

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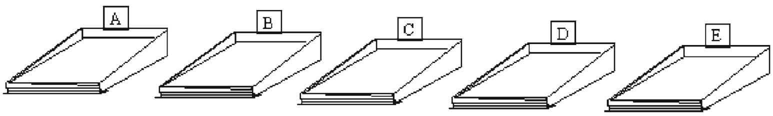
Module 1: Introducing Rational Numbers as Measurements

Lesson 1: Measurement of the Thicknesses of Sheets of Paper by Commensuration

The objective of the first set of lessons is to have the students invent a way to measure something so thin that their previous methods of measurement cannot be applied. The challenge is to find the thickness of a sheet of paper, which they clearly can't do directly with the usual measuring devices. They discover that "repeating the thickness" – that is, stacking the sheets of paper – provides sufficient thickness for their rulers to give a reading.

The Set Up

On a table at the front of the classroom are five stacks (or half-boxes) containing 200 sheets each of paper. All the paper is of the same color and format, but each box contains paper of a different thickness from the others (for example, card stock in one, onionskin in another, etc.) The boxes are set up in a random order and labeled A, B, C, D, E. Some of the differences should be impossible to determine by touch alone. The teacher needn't know the exact measurements, since there is no "good measurement" to be discovered.



- On another table at the back of the classroom are five more stacks or boxes of the same papers, in a different order, which will be used in phase 2.
- Each group of five students has two slide calipers (a device for measuring thickness, standard in French elementary classrooms)
- The ends of the room are screened from each other in some way – a curtain or a screen.

The Search for a Code

(a) The teacher divides the class into teams of four or five students and presents the situation and their assignment:
"Look at these sheets of paper that I have set up in the boxes A,B,C,D,E. Within each box all of the sheets have the same thickness, but from one box to another the thickness may vary. Can you feel the differences?"
Some sheets from each box circulate, so that the students can touch them and compare them.
"How do businesses distinguish between types?" (weight)

"You are going to try to invent another method to designate and recognize these different types of paper, and to distinguish them entirely by their thickness. You are grouped in teams. Each team must try to find a way of designating the thicknesses of the sheets. As soon as you have found a way, you will try it out in a communication game. You may experiment with the paper and these calipers."

The students almost invariably start by trying to measure a single sheet of paper in order to obtain an immediate solution to the assignment. This results in comments to the effect that "It's way too thin, a sheet has no thickness" or "it's much less than a millimeter" or "you can't measure one sheet!"

At this point there is frequently a moment of disarray or even discouragement for the students. Then they ask the teacher if they can take a bunch of sheets. Very quickly then they make trial measurements with five sheets, ten sheets – until they have a thickness sufficient to be measured with the calipers. Then they set up systems of designation such as:

10 sheets 1 mm

60 sheets 7 mm

or 31 = 2 mm[1]

In this phase, the instructor intervenes as little as possible. He makes comments only if he observes that the students are not following – or have simply forgotten – the assignment.

The students are allowed to move around, get more paper, change papers, etc.

When most of the groups have found a system of designation (and the children in each group agree to the system or code) or when time runs out, the teacher proceeds to the next phase: the communication game – going on even if not every group has found a system.

The Communication Game

"To test the code you just found, you are going to play a communication game. In the course of the game you will see whether the system you just invented actually permits you to recognize the type of sheet designated. Students on each team are to separate themselves into two groups: one group of transmitters (two students) and one of receivers (two or three students). All the groups of receivers go to one side of the curtain, and the groups of transmitters to the other. The transmitters are to choose one of the types of paper on the original table, which the receivers can't see because of the curtain. They will send to their receivers a message which should permit them to find the type of paper chosen. The receivers should use the boxes of paper set out on the second table at the back of the classroom to find the type of paper chosen by the transmitters.

[1] This use of the equal sign is incorrect. The teacher will mention it during the discussion time.

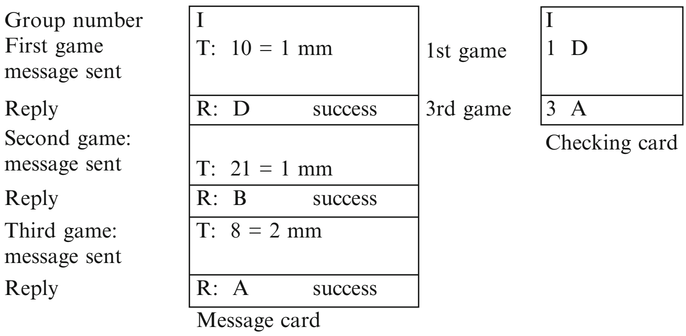


Fig. 2.1 Cards for the communication game

When the receivers have found it and checked it with the transmitters, they become transmitters. Points will be given to the teams whose receivers have correctly found the type of paper chosen by the transmitters."

At the beginning of the game, the teacher puts the curtain in place. Then he

- Passes the messages from the transmitters to the receivers
- Receives the responses of the receivers
- Checks whether this response corresponds to the choice of the transmitters and announces the success or failure to all of the team.

All of the messages are written on the same sheet of paper, which we can call the "message card" (see Fig. 2.1), which the teacher carries back and forth between the transmitters and receivers on the same team, marking whether the receivers have selected the correct paper ("success") or not ("missed"). The team's number is written on the card. In addition, the transmitters write the type of paper that they have chosen on another sheet of paper – the "checking card" – which they keep.

Clearly, the teacher does not introduce superfluous formalism or vocabulary. If certain teams have not arrived at any way of sending effective messages, the teacher could send them back to considering a code together (same assignment as in the first phase). On the other hand, in the first eight identical trials of this material, that never happened. The students always managed to play two or three rounds of the game.

During this game, there are three different strategies commonly observed:

Some choose a particular number of sheets and always measure that number.
Some choose a particular thickness and count how many sheets it takes to make that.
Some look randomly at a thickness and a number of sheets.

The children predictably prefer to choose the sheets of extremes of thickness, either the thinnest or the thickest, to make the job easier for their partners.

Result of the Games and Comparison of the Coding Systems

For this phase, the students go back to their original places in teams of 5, as for the initial phase. The teacher prepares a chart with group names down the side and paper types (A, B, C, D, E) across the top.

Taking turns, each team sends a representative who reads the messages out loud, explains the code chosen and indicates the result of the game. The teacher keeps a record of the groups' messages (and their success) as the reports are made.

The different messages are compared and discussed by the students. Since they are frequently very different, the teacher requests that they choose a common code.

Example: 10; 1 mm
VT (for Very Thin)
60; 7 mm

After discussing these, the class chose: 10; 1 mm and 60; 7 mm.

The children rewrite their messages and present them successively in no particular order on the blackboard. Immediately there are spontaneous remarks like "That can't be!" and "That one's OK", etc.

For example: "Group 2 said 30 sheets of paper C were 2 mm thick, but Group 4 said the same number were 3 mm thick. That can't be!"

The teacher announces that if there are disagreements the groups in question should carry out their measurements again.

The session ends with a request to arrange the chosen messages all on the same chart.

Different Types of Inconsistencies

The students' measurements are collected on a chart such as the following (1977)

Type of paper	Group 1	Group 2	Group 3	Group 4
A	19 s; 3 mm	10 s; 2 mm	20 s; 4 mm	
B	19 s; 3 mm		4 s; 1 mm	15 s; 2 mm
C	19 s; 2 mm	30 s; 2 mm	100 s; 8 mm	30 s; 3 mm 15 s; 1 mm 20 s; 2 mm
D	19 s; 2 mm		100 s; 9 mm	
E			9 s; 4 mm	13 s; 5 mm 7 s; 3 mm

Students look for and discuss the inconsistencies. By the end of the session, they have identified categories of errors among the following:

1st category:

If the sheets are of different types, the same number of sheets should not correspond to the same thickness.

Example:

$$\left.\begin{array}{lll} 19\text{ s; }3\text{ mm} & & \text{Type A} \\ 19\text{ s; }3\text{ mm} & - & \text{Type B} \end{array}\right\} \text{“That can't be!”}$$

2nd category:

If the sheets are of the same type, the same number of sheets should correspond to the same thickness.

Example:

$$\left.\begin{array}{lll} 30\text{ s; }2\text{ mm} & - & \text{Type C} \\ 30\text{ s; }3\text{ mm} & - & \text{Type C} \end{array}\right\} \text{“That can't be!”}$$

3rd category:

If there are twice as many sheets of the same type, it should be twice as thick.

Example:

$$\left.\begin{array}{ll} 30\text{ s; }3\text{ mm -} & \text{Type C} \\ 15\text{ s; }1\text{ mm -} & \text{Type C} \end{array}\right\} \text{“That can't be!”}$$

and the students add: “It should be

$$\begin{array}{c} 30\text{ s; }2\text{ mm} \\ \text{and} \\ 15\text{ s; }1\text{ mm} \end{array} \quad \text{because} \quad \times 2 \downarrow \begin{array}{c} 15\text{ s; }1\text{ mm} \\ \\ 30\text{ s; }2\text{ mm} \end{array} \downarrow \times 2$$

4th category:

A difference in the number of sheets shouldn’t correspond to the same difference in thickness.

Example:

$$\left.\begin{array}{l} 19\text{ s; }3\text{ mm} \\ 20\text{ s; }4\text{ mm} \end{array}\right\} \text{“That doesn't work, because one sheet can't be a millimeter thick!”}$$

The teacher makes no explicit reference to the formal use of the concept of proportionality, and does not ask it of the students either. On the contrary, she favors the explanations given by the students to whatever extent they are understood, but does not at this stage correct the ones that are not understood.

Didactical Results
At the end of this first sequence, all of the students know within this specific set-up

How to measure the thickness of a certain number of sheets of paper
How to write the corresponding ordered pair
And to reject a type of paper that does not correspond to an ordered pair given to them (if the difference is large enough.)

Most of them are thus able to analyze a chart of measurements to point out inconsistencies making *implicit* use of proportionality.
Those who can't do so seem to understand those who do it.

Order: The children know how to find equivalent pairs. They know how to compare the thicknesses of sheets of paper (many by two different methods).

This knowledge is sufficient to undertake (understand the goal and resolve) the situations that follow.

Lesson 2: Comparison of Thicknesses and Equivalent Pairs (Summary of Lesson)

The first step is a review of the chart produced in the previous lesson. Students first study it silently and make individual observations, then discuss these observations as a class. The chart is corrected either by universal agreement, or, where that agreement doesn't occur, by a re-measurement. This process serves to bring out the idea of augmenting the number of sheets counted in order to distinguish between papers of highly similar thicknesses as well as to exercise further the implicit use of proportionality to determine consistency of representations of the same paper.

Working in (non-competitive) groups, students then fill in any empty slots on the chart by counting sheets and then comparing their results with those of other groups. As a confirmation and celebration, they play one more round of the communication game from the previous session, discovering that they are now equipped to handle it even if a couple more types of paper are tossed in. This finishes the second session.

The children must refer with precision to a number of new objects: physical sizes – the thickness of a stack of sheets, the thickness of a single sheet; the *numerical* *expressions* for these thicknesses: a number of sheets and a number of millimeters for the first, the two numbers combined for the second; some *generic terms* for these denominations: “number”, “pair”, “ordered pair”, etc. This vocabulary is not supposed to be taught with formal lessons. Only the accuracy of the thinking counts. The teacher is faced with the difficult task of helping the use and formulation of these concepts move forward without disturbing the expression of the thought processes. This produces a fragile equilibrium to be maintained and developed.

Results The children know how to adapt the number of sheets chosen to meet the needs of discriminating between their thicknesses (increasing the number if the thicknesses are too close). They know how to find, by calculating, which ordered pairs correspond to the same type of paper. All of them now know how to use proportionality to analyze a chart. Some of them are able to use the relationship of proximity between the pairs. Many of the children have been led to make judgments about statements and to make arguments themselves.

Lesson 3: Equivalence Classes – Rational Numbers (Summary of Lessons)

In the following session the completed chart is once more the center of attention, and the central topics are equivalence and comparison. After getting the students to focus on the chart, the teacher presents some other pairs of numbers and asks which kind of paper each pair represents, then has the students invent other representations, listing all of the accepted ones in the same column on the chart. This provides the occasion for introducing the term “**equivalent**”.

“50 s; 4 mm and 100 s; 8 mm are two names, corresponding to different stacks of sheets of the same paper and the thickness of these stacks. We introduce these stacks to identify *the same* object, ***the thickness of one sheet***. Since they designate the thickness of the same sheet, the pairs are ***equivalent***. 50 s; 4 mm is equivalent to 100 s; 8 mm.”

The teacher then produces a new chart with a single name for each kind of paper (the class chooses the name) and the students are told to figure out the order of the papers, from thinnest to thickest. Students work individually, and then discuss their results and their reasoning.

Once an order is agreed on, the teacher introduces another type of paper (fictional this time) and the students figure out where in the ordering it belongs.

As a final step, the teacher returns to the chart with columns containing equivalent ordered pairs for each type of paper and introduces the standard notation a/b to designate the thickness and differentiate it from the varied ways, with a variety of stacks of sheets, they have been using to determine the thickness of one sheet.

The teacher points out that this not only makes it possible to designate the entire class of equivalent pairs, but also gives a designation for the thickness of a single sheet of paper. Thus, a s; b mm designates a stack of sheets and its thickness, b/a mm. is the thickness of each sheet.

The teacher uses the words "ordered pair" and "fraction" without giving a definition for distinguishing the type of notation required. There are many fractions that designate the same thickness.

The lesson finishes with some opportunities for the students to practice the use of this new notation and its connection with types of paper.

Results The children know how to find equivalent pairs. They know how to compare the thickness of sheets (many by two methods). They have a strategy for ordering the pairs, using these comparisons. They know how to use a fraction to designate the thickness of a sheet of paper and how to find equal fractions. They do not know how to check the equality of two fractions in the general case.

They know how to do all these things within a situation. At this particular moment it is not possible to detach a question from the situation and pose it independently. Hence these results cannot yet be built on as knowledge that has been acquired and identified as such by the student.

Module 2: Operations on Rational Numbers as Thicknesses

The next five lessons constitute the second module, which deals with operations in the context of the sheets of paper.

Lesson 1: The Sum of Thicknesses (Summary of Lesson)

By way of motivation for introducing operations, the teacher asks students to consider individually and then discuss with each other the issue of whether the "rational thicknesses" they invented in the previous lessons are numbers. In general the conclusion is that if you have 8/100 the 8 and the 100 are numbers, but 8/100 is two numbers. The teacher points out that we might be able to regard them as numbers if we could do the same things with them that we do with numbers, and asks what those things are. Responses generally include "count objects with them", "put them in order" and "do operations like addition, subtraction, multiplication and so on with them." Quietly tabling the first of these for the moment, the teacher presents the suggestion that to decide whether these are numbers they need to try to do some operations with them.

The first project is to make "cardboard" by sticking together (or rather pretending to do so) a sheet of type A paper (thickness 10/50 mm) and a sheet of type B paper (thickness 40/100 mm.) "How thick do you think the resulting sheet of cardboard will be?" Students agree that that thickness will be 10/50 + 40/100 mm, and most agree that the result will be 50/150 mm, though a few have some doubts about that. After a short discussion, whatever its outcome, they set out to verify the results. The teacher has them count out 50 A sheets and 100 B sheets and begins gluing (that is, pretending to glue) them in pairs, continuing until students realize that a problem is developing and stop the process. Offered an opportunity to correct their proposed solutions, most go immediately to the correct solution. Most are, in fact, so confident that they declare verification unnecessary, but the teacher does it for the sake of the others, counting out 50 more sheets of type A paper and combining the resulting piles. The stack may measure 59 mm or 61 mm, but this they have already learned to deal with.

They then practice by adding some other pairs and triples of fractions, and observe that they are now capable of adding any fractions they want.

Remarks on This Step: The Choice of Values

To offer at this particular moment the sum of two fractions with like denominators would be a didactical error. Certain teachers have tried it with the hope of obtaining an immediate success for everyone. They wanted to avoid having students have the double difficulty of having to decide to reduce to the same number of sheets and doing it in such a way that the sum of the numerators, that is, the thicknesses, would make sense. Doing so gives the children justifications which are easy to formulate and learn, which facilitates the formal learning of the sum of two fractions (we know how to add two fractions whose denominators are the same, so what is left for us to do in the general case is to reduce it to having the same denominators before performing this addition).

But this method gives inferior results. Only the students capable of comprehending simultaneously and immediately both the general case and the reasons for the apparent ease of the particular cases were able to avoid difficulties in developing a correct concept of the sum of two fractions. They were then able to reason directly or make rapid mental calculations. The rest were distracted by the apparent ease of carrying out the action from the pertinent questions (such as why the denominators can't be added) and the efforts necessary to conceive of and validate the concepts. They were invited to learn a method in two stages, with the possibility of some false justifications for the first stage (if I add three hundredths and five hundredths that makes eight hundredths, just the way three chairs and five chairs make eight chairs.) They first learn that it is possible to add fractions which have the same denominator, and how to do it. They also learn that it is not to be done, or can't be done, if the denominators are different (you can't add cabbages and wolves!) Then they learn to solve the other cases by turning them into the first case, not because of the meaning of this transformation, but because it works. The economy of this process is strictly an illusion, because there is no representation to support the memorization. It will furthermore require a large number of formal exercises to make the process stick and to make it possible to distinguish it from other calculations. Some students never do get it figured out.

Using different denominators, on the other hand, all the children are able to come up with the concept and solidify their representations with experimentation and verification in a way that makes any formal teaching unnecessary.

Delaying the introduction of algorithms can, at times, be of considerable benefit to the development of concepts.

Results All the children know how to find the sum of two or more fractions if they represent paper thicknesses and if the conversion to the same number of sheets is "obvious" (one denominator is a simple multiple of the others). Many would be able to work out a strategy in the case of any two fractions, but no method has been formulated, much less learned.

Lesson 2: Practicing the Sum of Thickness. What Should We Know Now?

The next session comes in two sections which look similar but have quite different functions. Each contains a series of problems. Those in the first section are designed to let the children make use of what they have figured out in Lesson 1, both in order to solidify that knowledge and to extend the range of mathematical activities it can be used for. The first problems are strictly review. The teacher writes up several pairs or trios of fractions to add, walks the class through the first one, speaking in terms of thicknesses of the two papers, and turns them loose on the rest. The next problem is to find the thickness of a sheet obtained by gluing together one of thickness 4/25, one of 18/100 and one of 7/50. Following that, they work on 8/45 + 5/30. The last in this set returns to asking the question in terms of the sheets themselves: "A woodworker is making a collage for a piece of furniture. He glues together three pieces of wood of different thicknesses: 40/50 mm, 5/25 mm and 6/10 mm. List these woods in order of thickness, then say how thick the resulting sheet will be."

In each case, the problem or problems are to be solved individually, then to be presented to the class for discussion and validation. Included in the discussion is the possibility of having several correct routes to the same solution.

The object of this phase is to permit the children to make use of the procedures they discovered in the previous session, to generalize them and make them more efficient. That is, to let them evolve.

This session is thus neither a drill nor an assessment. The teacher does not pass judgment on the value of the methods used, nor at any moment say which solution is correct.

For each exercise, she organizes and facilitates the following process:

Individual effort but for collective benefit
Collection of results
Comparison of methods
Discussion and validation by the students

A method is accepted if it gives a correct solution (thus becoming an "acknowledged" and correct method), rejected if not. Among the methods that have been accepted, remarks on length or facility of execution, which the teacher solicits, do not become judgments of value that the child can confuse with judgments of validity. On the contrary, the teacher sees to it that the child takes part in the debate, has a result to offer, is able to discuss his methods and state his position relative to his own knowledge.

The immediate collective correction and rapid discussion of the problems is thus indispensable. It enables the teacher to know each child's stage of assimilation and what she is having difficulty with. The whole class can take part in each student's effort.

The second phase of the session is a set of individual exercises for drill and assessment. It has a classic didactical form: written questions to be answered individually and turned in for correction (outside of class) by the teacher. The problems represent each of the levels of operation with fractions thus far obtained – ordering of fractions with unlike denominators, addition of fractions with denominators which are like, or one of which is a factor of another, or which require a common multiple.

This frequently results in some rather poor papers, especially since part of its function is to accustom students to the as yet unfamiliar task of producing mathematics for which they have no immediate feedback.

Results This lesson gives lots of opportunity for the exercise of mental calculations with two digit numbers (double, half, triple, multiply by 7). All the children know how to organize and formulate their method for finding the sum of several fractions. They start by trying to reduce them to the same denominator (though the term itself has not been introduced.)

The search for a common multiple has been practiced in many ways (despite the rarity until this moment of occasions for doing so.) Many of the students have begun to work out strategies for a systematic search, such as listing the multiples and comparing the lists, or in the case of small numbers even multiplying the denominators.

Not one of these strategies has been identified as stable, much less learned.

Lesson 3: The Difference of Two Thicknesses (As Measure)

The next session proceeds to the subtraction of two thicknesses. It requires more types of paper, with thickness ranging up to that of heavy card stock, but only one sheet of each of them (for demonstration purposes.)

The lesson starts with a rapid discussion of the problems handed in the day before. Only the ones where errors were made need be mentioned, and the teacher needs to restrain herself firmly from letting the discussion of the common denominator in the last problem result in one of the methods taking on the status of Official Method.

The next stage begins with a swift return to the initial situations: what does 8/50 mean? (The thickness of a sheet of paper such that you have to have a stack of 50 of them to measure 8 mm.) And what does 8/50 + 6/100 mean? (The thickness of a sheet made by gluing together an 8/50 thick sheet and a 3/50 thick sheet.)

Remark: It is often useful to insert a reminder like that of preceding situations, for two essential reasons:

In the first place to allow children who have some difficulties or are a little slow to be more thoroughly involved in the present lesson;

Furthermore to allow children who have been absent to understand what happened in the previous lessons and be able to participate in the following one.

The teacher then writes on the board

$$8/50 - 6/100$$

and asks the class what that might mean and how to carry out what it says to do.

This launches a discussion that starts with a predictable set of misinterpretations and arrives fairly swiftly at the realization that it is the card stock that is the very thick one, and it is made up of the thin one glued to one of unknown thickness. With a drawing on the board to represent this combination of sheets and the equation 6/100 + ______ = 8/50 beside it, the students are turned loose to work individually on finding just what that unknown thickness might be, and how to verify their results.

The resulting discussion includes many variations, a number of them correct. Students who have not succeeded give the results they got and say whether they are too large or too small.

Next the class interprets and solves 4/15 – 1/15.

The problem 4/50 – 3/40 is launched by getting the class to state the need for a common denominator, then left for individual work. Then for a final problem, worked individually, they take on 12/8 – 2/5.

> **Results** Once again the reduction to the same denominator has given rise to the search for a common denominator. The children still do not know a systematic way of doing it, but are making progress in the sense that more of them recognize more swiftly at what moment it needs doing, and regularly use speedier methods: mental calculation, products of denominators, intuitive searching for the least common denominator.

Lesson 4: The Thickness of a Piece of Cardboard Composed of Many Identical Sheets: Product of a Rational Number and a Whole Number

This lesson requires ten sheets each of four highly distinguishable types of paper, each with a known thickness. Students are set up in groups and each group is assigned a single type of paper.

They are to determine the thickness of a sheet made up by gluing 3 sheets of their own paper together, then 5 sheets, then 20, 100 and 120. Each group figures out the thickness for each stack of their own sheets, then writes the results on the blackboard. Each group then checks one other group's results and either signifies agreement or supplies an alternate answer. Enough students are solidly in control of the material so that the ensuing discussion produces a general agreement, and the chart of values can be successfully corrected.

Examples of the students' debates.

For the first question, group 1 answered	While group 2 wrote
3/19 × 3 = 9/57	3/19 × 3 = 9/19
Their calculation is based on the following schema: $$\begin{array}{ccc} & 3/19 & \\ \times 3 & \downarrow\;\;\downarrow & \times 3 \\ & 9/57 & \end{array}$$	Discussion: Group 2 answers that the thickness of the cardboard can be found by addition 3/19 + 3/19 + 3/19 = 9/19 They support their result with the following schema which is standard for them at that point:
A student comments that 3/19 and 9/57 are equivalent, so the thickness is the same! It hasn't been multiplied by 3	$3/19 \xrightarrow{\times 3} 9/19$

The final phase of this session is a comparison of the thickness of the various cardboards with 1 mm. The teacher chooses one of the thicknesses in the chart, for instance 57/35, and asks the students whether they have any idea how thick that card really is. Is it thicker or thinner than 1 mm, or equal to it?

In groups of two or three, students set to work. A lot of them take out their rulers to have a more precise idea of a millimeter. Some work out elaborate approximations, many point out that 35 sheets would make up exactly a millimeter, so 57 of them must be thicker than that ("but not 2 mm thick!") and a few are completely bewildered. After a certain amount of discussion of this particular thickness, the assignment becomes: "Look at the chart and see what else you can say about the thicknesses." This gives rise to a lively discussion and a lot of joy in discovery.

Remark: This last part proceeds informally and spontaneously, for the pleasure of exchanging and discussing ideas without any pressure from the teacher. The teacher listens to the remarks and says nothing unless the students ask him to clarify or explain something.

It is essential to emphasize the fact that the teacher has not set out any contract of learning or acquisition. Some children may take the analysis of the situation a huge distance and make subtle, profound remarks. Others have intuitions which they are unable to communicate. These "discoveries" meander a bit, but it doesn't matter – the jubilation of the ones who have found something wins over the ones who listen, approve, look at them in incomprehension or contradict them. Anyone can advance a notion or even say something that proves to have a major glitch. The teacher restricts himself to making sure people take turns, without interfering with the order or the choice of speakers, in order to maintain the group's pleasure in this game. To do that, he has to register his own pleasure, but make sure that his pleasure is not the children's goal.

He takes note of errors and difficulties without trying to correct them right away. If no one notices them, then in general an explanation at that point would do no good. The teacher has to consider it as an obstacle which needs to be taken up later in a prepared didactical activity.

Frequently after a moment a student notices the error and the debate revives.

Obviously, it has to be clear that the teacher's silence doesn't indicate either acceptance or rejection. And it's not enough to **say** it – he has to **do** it.

Results The children have learned to multiply a fraction by a whole number and to distinguish between this operation and the calculation of an equal fraction. The comprehension of this distinction is essential for what follows. When the children begin to make frequent and varied calculations in more complex problems they will tend on their own to automate their procedures. The initial distinction enables them to do so without losing track of what they are doing and hence to correct the errors that are bound to turn up. Many have begun to envisage the comparison of fractions with natural numbers, a question to which they will soon return. Certainly all of this remains connected to the representation of the fractions by thicknesses of paper.

Lesson 5: Calculation of the Thickness of One Sheet: Division of a Rational Number by a Whole Number

First, the students remind themselves how to multiply by figuring the results of gluing together 5 sheets each 3/9 of a millimeter thick. Then they are presented with:

> "I've glued 9 equally thick sheets of paper together and the resulting card is 18/7 mm thick. What could we ask about it? (the thickness of each sheet.) Can you figure out the thickness? If so, write it in your notebook."

Individual work very swiftly produces the correct result and reasoning.

$$18/7 \xrightarrow{\div 9} 2/7 \qquad\qquad 18/7 \xleftarrow{\times 9} 2/7$$

This requires a little delicacy in handling, since they only know for sure that division is defined between whole numbers, but the idea certainly needs confirming, especially after students observe that the operation here can be successfully inverted with a multiplication by 9.

The major point to emphasize is that it is the whole fraction (the thickness) which is to be divided, not just the numerator or denominator. This becomes clearer with the next situation:

"Now I've glued 9 other equally thick sheets together and made a new card. This one is 12/7 mm thick. Can you find the thickness of each of the sheets I glued together?"

Students know how to divide whole numbers. They want to apply the same technique to divide rational numbers by whole numbers. The teacher points out that it might not be the same operation, but accepts it after they compare the properties.

Two out of five groups give the following two-stage response:

$$\begin{array}{ccc} & 12/7 & \\ \times 9 \swarrow & & \searrow \times 9 \\ & 108/63 & \end{array} \qquad\qquad 108/63 \xrightarrow{\div 9} 12/63$$

Students work in groups of 2 or 3, then share their results. Since two of the most accessible solutions are multiplying the numerator and denominator by 3 and multiplying them both by 9, the resulting discussion is likely to include a brief furor until somebody observes the equivalence of 12/63 and 4/21.

The final activity is to work individually on $(13/5) \div 9$, first giving it a meaning, then calculating the result. Students tend to bypass the former and work on the latter, which means the teacher has to lean on them to write the sentence in question. After 5 min or so, the teacher stops the work and sends one or more students to the board to write up their solutions. By and large they multiply by 9/9 and then divide the numerator by 9. Only occasionally does somebody observe that the only thing that has happened is that the denominator has been multiplied by 9, and the level of generality of this observation remains undiscovered.

Results Even though most of the children have carried out the operations brought up in this lesson, and have understood the meaning of their work at the moment and in the particular case, there is no guarantee that they will know afterwards how to divide a fraction by a number. But they will find similar situations often enough to develop their methods of calculation, refine them, become confident with them, and hence learn them.

This lesson will enable them to take on these new situations and to understand them without calling forth a reduction to a procedural technique.

Lesson 6: Assessment

The module finishes with a set of problems for a summative evaluation:

1. Put the following thicknesses in order from thinnest to thickest

   35/100 mm; 3/5 mm; 62/97 mm; 5/25 mm

2. Find the sums of the following thicknesses:

   15/100 + 22/100 + 62/100
   7/25 + 14/50 + 45/100
   3/12 + 1/4 + 2/3
   5/8 + 13/88

3. A piece of cardboard is made by gluing together five identical sheets of paper, each 3/25 mm. thick.

   (a) How thick is the cardboard?
   (b) Is this cardboard thicker or thinner than a millimeter?
   (c) How many sheets would it take to make it thicker than a millimeter?

4. A piece of cardboard is 7/25 mm thick. It is made of eight identical sheets of paper glued together. How thick is a single one of those sheets?
5. Find two fractions equal to 3/18

Note: The fractions are sometimes written with a horizontal fraction bar and sometimes with a slanted one.

Module 3: Measuring Other Quantities: Weight, Volume and Length

The third module (three 1 hour sessions) extends the students' thinking beyond sheets of paper, with the objective of giving them enough similar experiences to make generalization plausible and legitimate. The students use the method of commensuration for three different amounts: volume, mass and length (Fig. 2.2).

Lesson 1: Making Measurements

The first lesson requires a considerable collection of materials:

To measure weight, a balance beam and five different categories of nails;
To measure volume, five small glasses of different sizes, one colored glass to serve as a unit and two (largish) test tubes, one of them with a sticker on it so that they can be distinguished [Note that it is better to do these measurements with fine sand than with water!];
To measure length, strips of construction paper of equal width but different lengths, a single strip of gray cardboard (same width, yet another length) to serve as the unit and a big piece of poster paper to work on.

The glasses, the nails and the strip lengths need to be chosen in such a way that none is an integer multiple or divisor of the unit. For instance, seven nails of one sort might have the same mass (balance on a scale) as eleven of another. If the first serves as a unit, the second weighs 7/11 unit. Similarly, if the content of three "unit" glasses emptied into one tube comes to the same height as the content of five glasses A emptied into the matched tube, then glass A holds 3/5 of a unit. The lengths of the paper strips are between 3 and 30 cm, but they are not any exact whole number of centimeters.

The unit chosen is neither the largest nor the smallest of the available objects. The problem of approximation and precision has to be solved by student agreement with the help of the teacher. An example of how 3/5 of a unit appears for each of these measurements is shown in Fig. 2.3.

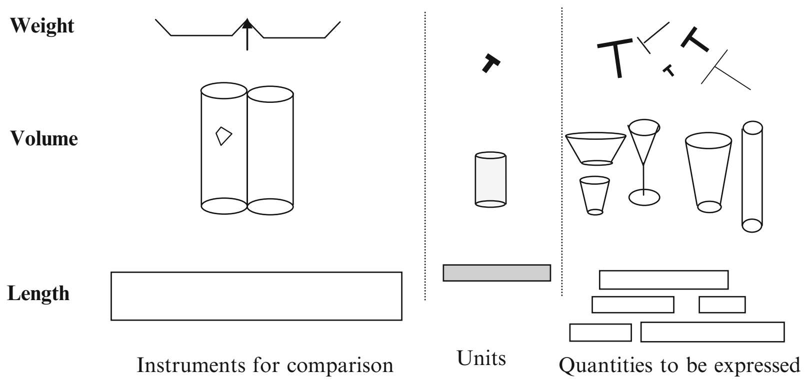
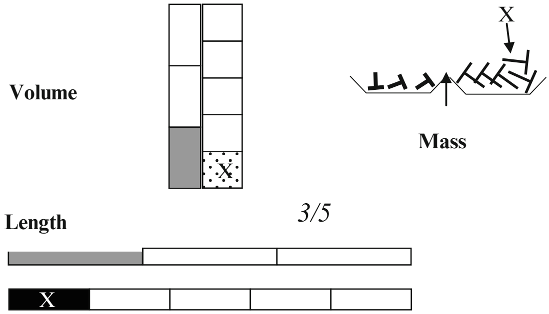


Fig. 2.2 Materials for measuring weight, volume, and length

Fig. 2.3 X is 3/5 of the unit



The Situation

For a class of 24, the teacher sets up 12 stations – for each of the categories of material, two pairs of stations. The class is divided into six teams, one team for each pair of stations. Each team then splits itself between its two stations, which are at some distance from each other. At their stations, they label each size of object, with the unit having the label U. They then work on figuring out a way to designate the measure of each size of object and on writing a message to indicate the measure of one particular object. When both halves of the team have produced a message, they exchange their messages and try to interpret the message they have received. Then they meet to ascertain the success or failure of the communications and to discuss the best form of communication.

At the end of one such cycle, the team moves on to a different category of object. Each team thus needs to carry out three cycles in order to explore all the types of material. Since each cycle has three parts (inventing a message, interpreting a message and discussing the result) the lesson presents a considerable challenge to the teacher. He must adapt to the students, stimulate them without imposing tedious reproductions, get them to work seriously, with a focus on the task at hand and on the understanding needed to accomplish it. Creating and maintaining the enthusiasm and focus require an exceptional pedagogical performance on the part of the teacher: great rigor to keep the activity rapid and efficient and great flexibility to keep from requiring the completion of tasks that are no longer of interest.

In point of fact, there is no need for every experiment to be carried out by every child. The similarity of the methods swiftly leads the students to re-use commensuration with the glasses and the nails. As the lesson progresses, they get more and more interested in what happens with the length measurements, for which the students soon want to proceed in a different way, but don't know how to write the procedure because it is not a commensuration! The students have no mathematical difficulty with the measurement of mass and volume and do not measure all of the quantities available. The rhythm accelerates. The last cycle is abbreviated. All of the students are set to take an interest in the next day's lesson on measurement of lengths and the comparison of commensuration with subdividing the unit.

Conclusions

The class concludes that their codes for commensuration can be used to measure weight, capacity and length. The session finishes with some practice questions, e.g. “What does it mean that this glass has a capacity of 3/4 of the unit? that this paper strip is 17/25 as long as that one? that this nail weighs 20/75 of a unit?”

Lesson 2: Construction of Fractional Lengths: A New Method Appears

In the previous session, the children attached numbers to sizes (they designated a measurement). In this session, they construct objects whose measurement in terms of a unit is given (i.e., they realize a size). The class deals only with lengths. One reason is that it is difficult to construct volumes and masses of a desired size starting with a random unit. But there is another reason: the teacher wants to get the students to discover another way of defining fractions.

The students have already known for a couple of years how to use the usual method for measuring length in the metric system. In this system the method of measurement always consists of comparing the length to be measured with a whole number of smaller units. To increase precision, one switches to a unit that is ten times smaller. And for practicality, rather than re-measuring the whole length, one measures only the piece that sticks out beyond the part that could be measured with the previous unit, as one does with the remainder in division.

The method we want to induce consists of (for instance) realizing 5/4 by first “partitioning” the unit strip by folding it in quarters, then repeating the resulting quarter-length strip five times.

Materials: Strips of construction paper, all the same width (around 2 cm)
12 unit strips (gray) 20 cm.

Four identical sets of six strips (green) whose lengths are respectively:
5 cm (1/4 unit), 10 cm (1/2 or 2/4 unit), 15 cm (3/4 unit), 30 cm (3/2 or 6/4 unit), 35 cm (7/4 unit), and 45 cm (9/4 unit)

Four identical sets of six strips (blue) whose lengths are respectively
4 cm (1/5 unit), 8 cm (2/5 unit), 16 cm (4/5 unit), 24 cm (6/5 unit), 28 cm (7/5 unit), and 36 cm (9/5 unit)

Four identical sets of six strips (yellow) whose lengths are respectively
2.5 cm (1/8 unit), 5 cm (2/8 unit), 12.5 cm (5/8 unit), 17.5 cm (7/8 unit), 22.5 cm (9/8 unit), 27.5 cm (11/8 unit)

Strips of poster paper 50 cm long and 5 cm wide
Long strips of construction paper, all 2 cm wide
Scissors.

The unit strip should be clearly distinct from the strips to be measured, because since measurement by commensuration consists of laying multiple copies of one strip beside multiple copies of the other, the strips are treated identically in that process. The students naturally tend to confuse 4/5 and 5/4 at first. That is one of the inconveniences of commensuration.

Communication Game and Building Lengths Corresponding to a Pair

Assignment

The class is divided into 12 groups of 2 or 3 children. Each group has 1 unit strip and 1 set of 6 strips of the same color.

"Each group is to find fractions representing the lengths of their six colored strips using the (gray) unit strip and write all of them on the same message pad. So each group starts off as a message-sender.

Each group will receive a message from another group. At that point you all become message-receivers. You are to cut strips of white paper in the six lengths indicated on your message.

Next, each receiver-group will meet with the group that sent the message they decoded and verify together (by superposition) that the white paper strips are indeed identical to the ones used to produce the message. If they are identical, the message-senders are winners."

For convenience, it is the teacher who passes the messages. Groups need to receive messages from other groups whose strips are of a different color (and hence a different set of lengths).

Strips of white paper and scissors are given out at the same time as the other strips.

Development

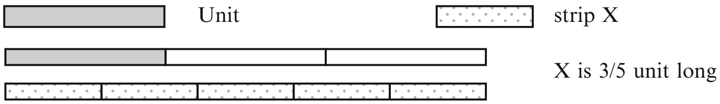
Initially, students use the method that was inaugurated with thicknesses and generalized to masses and volumes. To realize a length of 5/4 of the unit they lay five units end to end and then try to divide the result in four.

For that they make a guess at an approximate length, repeat it four times and compare it to the length of five units. If the result is too long, they snip off a bit and try again. Some of them observe that the strip they are trying should be shortened by a quarter of the extra length.

They verify that their message was well written and well read and that the construction requested was correct, by superimposing the resulting strip on the original. This method calls for a good mastery of the definition and a certain mental flexibility in applying it, but the students have used that a lot in concrete operations.

When the process of using commensuration results in multiples that take too much space, some of the students think of using the method of dividing the unit. They think of it particularly readily when the natural numbers in the ordered pairs

Initial method: commensuration



New method: breaking up the unit to produce an intermediate unit that can be used in the familiar way

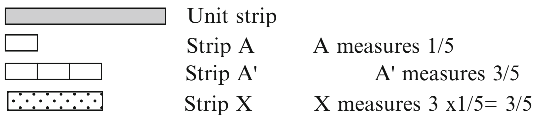


Fig. 2.4 Examples with 3/5

are simple – 2, 3 and 4 – and the denominator is 2 or 4 (a power of 2): for example the lengths of 3/2 or 3/4 inspire them a bit better than 2/3 or 4/3. They fold the unit strip in two or in four. And they can express those measurements orally by halves or quarters using references to everyday life.

Once they have launched the idea with powers of 2, they progress to other denominators, like 5.

But they can’t justify the length directly, with their initial definition. They can only do it by putting five copies of Strip X beside three copies of the unit strip and showing that the lengths are the same, that is demonstrating their equivalence. This will be the subject of the next lesson. Until then, all they can do is write the length of A (1/5) and use multiplication (which they have already encountered): $1/5 \times 3 = 3/5$, trusting to the similarity in writing to carry them through.

Lesson 3: Comparison of Methods, and Demonstration of Equivalence

Summary of the Lesson

This session begins with a follow-up discussion in which by use of the solutions written on the board by the children and a process of observations (by students) and (student-proposed) verifications the teacher guides the class to a conviction that this method of “intermediate units” provides a general solution. For instance, the students can prove step by step that subdividing the unit gives the same result as commensuration because they can write the steps (see Fig. 2.4).

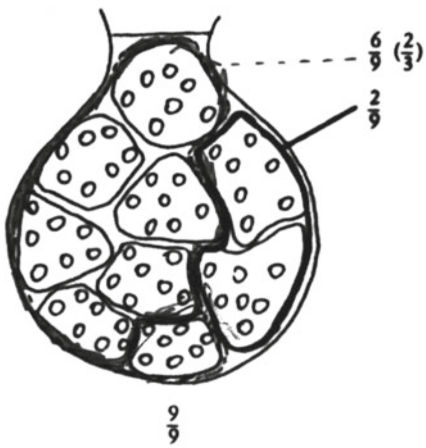
Lesson 4: Fractions of Collections

The follow-up is a pair of problems to be worked on individually and then discussed:

A cloth merchant sells first half of a piece of velvet cloth and then a quarter of the same piece.
What fraction of the piece is left at the end of the day?
The piece was originally 24 m long. What is the length of the remaining piece?

Claude has a bag of marbles. In the course of a game he loses first 2/3 and then 2/9 of his marbles.

What fraction of his marbles has he lost?
What fraction of his marbles does he still have?
At the beginning of the game, he had 63 marbles in his bag. How many does he have at the end of the game?



We expected the problem with the marbles to be a good deal more difficult for the children than the one with the cloth, because it combines a variety of difficulties (it requires adding losses; the number is too large for commensuration to be realized; etc.) We observed moreover that in this situation the children didn't recognize the definitions of fractions that they had previously used with continuous quantities. They especially had difficulty conceiving of the bag of marbles as the unit. But manipulation of discrete quantities eventually enabled them to establish a correspondence with long division that they knew.

This lesson completed the study of fractions as measurements.

Results of This Portion of the Sequence of Lessons

The children can fluently use ordered pairs of numbers to express measurements. In fact, they have solved the practical problems of manipulation, measurement, comparison of sizes, evaluation of sums, equalities, multiplication by a whole number, etc.

The culture at large makes use of an imposing vocabulary to conceive of and express fractions. Three heavy-duty difficulties for the teachers are:

Limiting their vocabulary to terms that have been defined and understood
Limiting their explanations to ones made possible by previous lessons
Avoiding analogies and metaphors.

One question was raised every year: Are there commensurations that will never work? That is, are there pairs of objects for which no amount of repetition of each one will ever result in the exact same measurement for the two?

Module 4: Groundwork for Introducing Decimal Numbers

The first three lessons in module 4 are review lessons, so we present only the fourth and last lesson.

Lesson 4: Whole Number Intervals Around a Fraction

First Phase: Introduction to the Game

(a) Instructions and game
"We are going to learn a game that is to be played by two teams. But to understand the rules well, first we will have two students play it at the board in front of the whole class.

Player A will chose a fraction that is somewhere between 0 and 10 (without saying it out loud). She will write it on a piece of paper which she will put in her pocket.

Player B will try to bracket player A's fraction between two consecutive natural numbers. To do that, he will ask questions. For example: 'Is your fraction between 7 and 9?' A can only reply with 'Yes' or 'No'. B will keep asking questions until he has found two consecutive whole numbers that the fraction is between. At that point, A will show her paper and the whole class will compare her fraction with the interval B found.

After that you're all going to play the game, but this time the players will be two teams each made up of half of the class" (and the teacher swiftly creates two teams).

(b) The playing of the game
Now each team chooses a fraction that all the students on that team write in their notebooks. The students choose a representative to play at the blackboard for them. It is the representatives of each team who take turns posing and answering questions. As they do, they can get help from their team by a discussion between rounds. The first team to bracket the other team's fraction in an interval of length 1 wins.

Remarks

1. The intervals chosen by the two representatives should be written on the board. Note that the class has a previously established convention that intervals are closed on the left and open on the right.

   The board is divided in half, one for each team.

For example, if Team B has chosen 25/30 and the representative of Team A asks “Is your fraction between 0 and 7?”, he writes:

TEAM A
[0,7)

and then after Team B has replied, adds:

TEAM A	
[0,7)	yes

If the student then asks “Is your fraction between 5 and 10?”, then when the opposing team says “No”, he puts a line through the interval:

TEAM A	
[0, 7)	yes
~~[5, 10)~~	no

The convention that intervals are closed on the left and open on the right gives rise to the following:

If the fraction chosen is 25/5, then the interval of length 1 containing it is [5, 6), and if this interval is chosen the opposing team answers that it is “trapped”, because it is the left hand end-point of the interval.
If the fraction chosen is 30/5 and the interval requested is [5, 6), then opposing team says “no”.
If a team brackets the other team’s fraction, it scores one point.
If a team traps the other team’s fraction, it scores two points.

2. There is an appearance of strategies in the choice of the intervals. This first game between the teams gives them a chance to develop interesting strategies in their choices of intervals. By and large on the first round the representatives tend to ask questions randomly, often overlapping intervals in ways that make their teams lose.

   Example: First question: “Is your fraction between 5 and 9?” Second question: “Is your fraction between 3 and 9?”

   This produces some lively discussions within the team. Often already on the second round they make use of the binary nature of the situation. For example: “Is it between 0 and 5?” If the representative of the other team says “No”, they avoid asking “Is it between 5 and 10?”, as they often do in the first round. Often after three or four rounds the students manage to locate the fraction with a minimal number of questions.

Remarks

1. If the instructions, which are long, are not well understood, the team game gives the teacher a chance to explain them better, to check that all students know how to write intervals and that they know how to play the game.

2. The team game needs to be restarted a number of times in order for all of the students to understand the rules (there may well be three or four rounds.)
3. The choice of the fraction at the beginning of a round always produces interesting discussions because students often propose a fraction that is not between 0 and 10. Team mates that disagree, if they want to reject the fraction proposed, have to prove to the rest that it is not between 0 and 10,
4. The students swiftly get to the point of avoiding choosing fractions that can be "trapped", because they don't want their opponents to get two points.

Second Phase: Playing Two Against Two

(a) Presentation
After three or four rounds of the game in large teams, the teacher puts the students in groups of four, so that they play two against two.
(b) Playing the game
Each pair keeps notes on a piece of paper both of the fraction it has chosen and of the intervals they have asked about for locating their adversary's fraction. The teacher does not intervene except to settle conflicts or supply clarifying information requested.

Third Phase: Collective Synthesis

(a) Presentation
During the previous phase the teacher has put the following table on the blackboard

Trapped fractions		Bracketed fractions	
Fraction chosen	Interval requested	Fraction chosen	Interval requested

She interrupts the game played in teams of four after 4–8 min and asks the students:

"Who trapped a fraction?"
"Who bracketed a fraction?"

She writes up the trapped and bracketed fractions along with the intervals in which they were placed. All the students check these results as they are written up, under the guidance of the teacher.

Results At the end of this session all of the students know how to play the game, and almost all are able to locate fractions in intervals of length one.

Module 5: Construction of the Decimal Numbers

Lesson 1: Bracketing a Rational Number with Rational Numbers: Chopping up an Interval

First Phase

(a) Presentation of the problem and review of the game in the previous session.
"During the last class we learned how we could locate a fraction by figuring out which whole numbers it was between. Do you think it could be useful to know which whole numbers a fraction is between? Why?"
Sample answers:

1. Bracketing lets us say whether the number is large or small.
Comparison with whole numbers is useful in measurement and in evaluation.
2. Is bracketing useful for comparing two fractions?

   For example, 156/7 and 149/6.
   First method of comparison: give them the same denominator.
   Second method of comparison: bracket them between two whole numbers.
   Which method is shorter?

3. Bracketing also makes it possible to estimate the sum of several fractions. What interval can one give to the sum when one knows the interval for each fraction?

(b) Instructions.
"We are going to play yesterday's game again. Teams A and B will each choose a fraction and designate a representative who will go to the board and ask questions."
(c) Playing.
The game is played exactly as before until the fractions are bracketed or trapped. But while the fractions are still hidden the teacher interrupts the game.

Second Phase: The Search for a Smaller Interval

(a) Presentation and instructions
"You just bracketed the fractions in an interval of length 1 (for example, [3, 4)). Do you think the fraction you were looking for is the only one in that interval? Find some others!"
(b) Development
The teacher lets the students search individually or in pairs for a minute or two. Then he asks them to come write on the board (or writes himself) the fractions they have found that are in the interval.

The children observe that there are many, and that the interval of length 1 that they found doesn't let them give a precise location for the fraction being searched for. They thus understand – some even say it – that they are going to have to find a smaller interval.

Third Phase: Search for Smaller and Smaller Intervals

(a) Instructions
"We are going to add a new rule to the game: to win, you have to bracket the fraction in the smallest interval you can. So you're going to have to try to find smaller intervals and designate them."

(b) Development
Students work in groups of 2 or 3 (there will be 4 or 5 groups per team). Some of the groups have the idea of writing the end-points of the interval as fractions (for example 6/2 and 8/2 if the interval is [3, 4)) But it also happens at times that a lot of students don't think of it and have difficulties. To avoid discouraging them, the teacher may suggest it to them after a few minutes, which revives their interest.

As soon as they have found and designated a smaller interval, they gather again into their two large teams A and B in which each group proposes the interval it has found. The children on the same team then discuss and agree on which among the 4 or 5 intervals proposed they judge to be the smallest.

Then one of the two representatives of the teams comes back to the board and the game continues:

"Is your fraction between 6/2 and 7/2?" (for example)
To answer the question, the students generally request to get back together as a team.

Remarks

1. To answer the question they often call for help from the teacher, because they can't find the answer or can't agree on it. Some of them think of putting all three fractions (their original fraction and the ones being proposed as end-points of the interval) over the same denominator, others give random answers.

   To sustain the pleasure of the game and the desire to continue, the teacher can aid them by giving a few hints (suggesting a common denominator, for instance, if they haven't thought of that.)
2. It is rare for them to be able in the course of this session to propose more than two intervals. Indeed, the big calculations (which they have not yet really mastered) take a lot of time, because they must:

   - Find smaller intervals and designate them
   - Check to see whether their fraction is in these intervals, which requires common denominator computations that are often complicated
   - Finally, to see which team has won, compare the last two intervals designated.

Few of the students are capable, at the end of this first session, of easily reducing the intervals or of saying whether a fraction is inside of a given interval.

Some strategies observed

1. It rarely happens that in the course of the first game all of the children write the limits of the intervals with denominators 10, 100, 1,000,… That's why the calculations are long and difficult. In fact, one time in a first session a group proposed the interval [6/40, 7/40) to bracket a fraction that was between 0 and 1. And since that fraction was 12/37, it is easy to understand why the children ran into difficulties in calculation!
2. In the course of one first session, one of the teams (A) designated their intervals with factions of denominator 64 because they made binary subdivisions: a group of 2 in this team had initially cut the interval in 2, and then in 4 in designating it. When the team got together the other children said "But we could make the intervals even smaller by continuing to cut them in half!", and they tried successively cutting in 8, then 16, then 32 and stopped at 64, convinced that their interval would be smaller than the other team's.

   At the same time, the other team (B) proposed intervals designated by fractions with denominator 1,000. Why? Because a group of two girls on this team had first marked the interval from one to ten (to look like their rulers, they said!) Then, still working like their rulers, they designated intervals in hundredths, and then in thousandths. Their calculations were done very swiftly!

   When the team got together for discussion, the three other groups, who themselves had made subdivisions of 10, 4 and 2, immediately adopted the subdivision into thousandths.

   When the representatives of the two teams went to propose their intervals, the children in team A were able to respond very quickly. On the other hand, the ones in team B, who were asked questions about intervals in sixty-fourths, had to make long, difficult calculations, which made them say to the others at the end "Next time choose something easier. Ask us easy questions like ours!" The teacher stepped in to ask why it was easier to answer the questions Team B asked than those that Team A asked. Everybody understood that for fractions with denominators 10, 100, 1,000 … the calculations were much easier, and of one accord they requested to play again the next day. During the second session the two teams chose subdivisions of 100, 1,000, 10,000 … but that day one team chose the fraction 14/10, (which was swiftly trapped) and the other 83/9!

Results
At the end of this session, the children understand:

That it is possible to locate a fraction in an interval of length less than 1
That in that interval there are many fractions
That that interval can be reduced.

But depending on the choice of intervals or fractions, more or fewer of the children master the calculations and are able easily to find a smaller interval.

Note

If the game as described is too difficult and too long (which happens in some classes), it is simpler to have the teams play one after the other:

Instead of having to pose questions and simultaneously respond to questions posed by the opposing team, one team chooses a fraction (team A, for instance). The other (team B) asks questions that will allow it to find the fraction chosen by A. Team A answers these questions.

Thus one team has only to find intervals, and the other only to answer questions.

In this case, it is necessary to fix the number of questions for each team (3–5, for example) and to compare the last intervals given. Then the game starts over with B choosing a fraction and A proposing intervals.

Lesson 2: Bracketing a Rational Number Between Rational Numbers, Shrinking the Intervals, and Observing Decimal Filters

First Phase: Return to the Game from the Previous Session

(a) Instructions

The instructions from the previous session are used.

(b) Development

The game proceeds in the same way. (If the two fractions chosen during the previous session have not yet been caught, the children want to continue that same game.)

We need to distinguish between the cases where the students have divided the interval into tenths, hundredths, thousandths, etc. and those where they are still using fractions with a random collection of denominators.

Case 1: Decimal intervals. The game develops more rapidly, and is therefore more engaging because the calculations can be made very quickly and are not an impediment to the development. That makes it possible to play several rounds. There are still two cases:

- The fraction chosen is a decimal fraction. In this case it will swiftly be trapped and the children will want to stop the game and start another round.
- The fraction is not a decimal fraction. The children, who are beginning to master the calculations, ask for smaller and smaller intervals (in general they get as far as 1/10,000 without losing impetus.) But at that moment the team that is looking for the fraction begins to wonder a bit and make some remarks (the other team celebrates.)

This is what happened for the fraction 83/9 mentioned in the previous section, which never got trapped in spite of very small intervals being used. The children said “It must be that it doesn’t have a denominator with zeros, so it should be 7 or 8 or 9!”, and they wanted to stop and see the fraction – which produced a very animated discussion. Some said it couldn’t ever be trapped “because 10, 100, 1,000,… aren’t multiples of 9.” Others held out for the contrary, saying finding shorter and shorter intervals would surely make it all work out.

The problem remained open.

The children’s reactions were exactly the same the previous year when one of the fractions chosen was 22/9.

Case 2: Non-decimal subintervals. If neither team has yet thought of producing subintervals in tenths, hundredths, etc. (which happened one time) the game quickly becomes slow and messy. Before the children lose interest (or become understandably disheartened) it is a good idea for the teacher to stop the game and suggest to the whole class that they think a bit about another strategy. For example, she suggests finding questions (for designating intervals) that make the calculations speedy. After a little time for thinking and collective discussion, if nobody has proposed subdividing in tenths and hundredths the teacher might propose another game in which she herself plays against the whole class:

- Either she is the one who chooses a fraction and writes it behind the board and the students propose intervals (which each one writes in his notebook)
- Or the students choose one together, writing it in their notebooks, and the teacher asks the questions.

In the former case, she asks several children, writes the proposed intervals on the board and only responds to those who have chosen decimal intervals. In the second case, she herself proposes the intervals, and uses only those with denominators of 10, 100, etc.

Remarks

The children’s pleasure in the game is renewed and they notice very quickly that the teacher is specializing in certain intervals. They generally remark on it with a comment to the effect that “All you have to do is add zeros!” They see that the game is faster, and hence more interesting.

To keep up the children’s interest, one can use other variations, such as having them play one against one and then two against two.

It often happens that the fraction proposed is a decimal fraction like 990/100. Children who first bracket it in intervals of tenths find 99/10. The fraction is trapped, but it is not the one that was chosen. So the children say “It’s trapped, but it’s not the one!” The representative, with the help of his team, then proposes equivalent fractions until he finds the required form.

Results

At the end of this session, all the children understand the necessity of choosing intervals in tenths, hundredths, thousandths. They easily manage

– Either to trap the fraction (if it is a decimal)
– Or to bracket it in very small intervals (of the order of ten thousandths or hundred thousandths).

Finally, they have become conscious that there are some fractions that are easy to trap and others that are not. Some of them even spontaneously list them.

Depending on the difficulty of the fractions and of the intervals chosen by the children, it is almost always necessary to carry the game over into a second session (at the request of the children, in fact.)

Lesson 3: Representation on the Rational Number Line

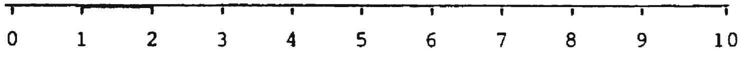
First game:

(a) Instructions
"Today I'm the one who is going to choose a fraction, and I will write it behind the blackboard. You are to catch the fraction by proposing intervals. I will only say "yes" or "no".

(b) Development
The teacher chooses a fraction (145/100, for example), and writes it in a hidden place. The children work in groups of 2 or 3 and write the first intervals in their notebooks. Once the teacher is sure that all the groups have chosen an interval, he asks them one at a time.

The children ask: "Is it between 0 and 5? between 0 and 3?" and so on until they have found an interval of length 1 (in this case, [1, 2)).

The teacher draws a line on the blackboard, represents the different subdivisions and asks a child to come show where the fraction is found:



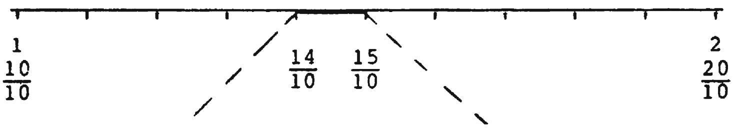
She draws this interval [1, 2) in color. Then she asks the children to find shorter intervals. At each step, the children indicate the length of the interval (at the request of the teacher).

The game continues until the interval [145/100, 146/100) is proposed, at which point the teacher says "Trapped!"

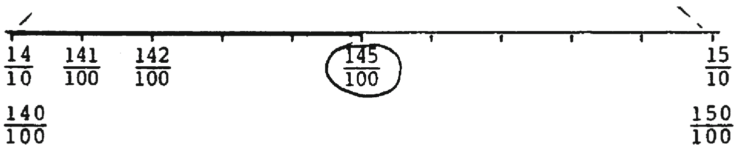
(c) Many strategies emerge
The students propose intervals in hundredths right off, for example [100/100, 150/100), and then progressively [100/100, 125/100) until they get to [145/100, 146/100)
They start with intervals in tenths. For example:

[10/10, 15/10) bracketed
~~[10/10, 13/10)~~ the teacher puts a line through it
~~[13/10, 14/10)~~
[14/10, 15/10) bracketed.
At that point they propose hundredths.

Each time the children propose a new subdivision, the teacher has them come to the board and write the division points as fractions:



At this stage, the students realize that they are going to have a hard time drawing the division of the interval [14/10, 15/10) into ten equal parts. They propose an enlargement of the interval which they will cut into ten equal pieces. At that point a student will come up and mark both the end points and the intermediate points in hundredths:



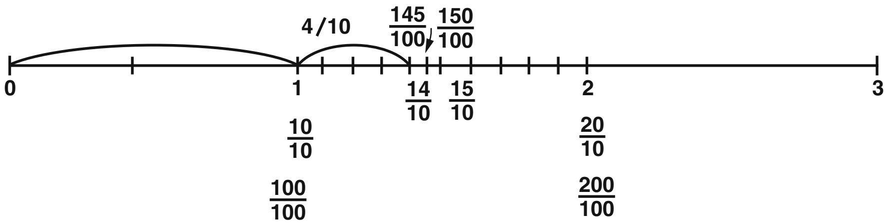
They make new proposals:

~~[140/100, 142/100)~~;
~~[142/100, 144/100)~~;
~~[144/100, 145/100)~~;
[145/100, 146/100) trapped!

Placement on a line

(a) Instructions
"We are going to suppose that this fraction, 145/100, represents the length of a ribbon that we are going to trace in red. So if I put this ribbon along a line marked off from 1 to 10, 145/100 marks the point on the line where the end of the ribbon will be. We are going to put this point on exactly."
(b) Development
This is a collective phase. The activity takes place very quickly in question-and-answer form. On the line drawn on the board, a student comes up and colors the interval [0, 1) in red, then proposes to divide the interval [1, 2) into ten parts, which is also done (either by the teacher or by the student). The endpoints are

marked with fractions as they were in the first phase. He extends the red line to 14/10 and then says “We have to cut it in 10 again to have hundredths.” The teacher asks what has to be cut in ten. The student shows the interval [14/10, 15/10) and marks the fraction 145/100. He finishes by marking in red the interval [140/100, 145/100).



The teacher then asks:

“To measure this ribbon, how many units do we need? How many tenths beyond the 1? How many hundredths?”

And she writes on the board:

Number of units	1
Number of 1/10	4 (so 4/10)
Number of 1/100	5 (so 5/100)

Then she says: “Here is what we measured:”

$$1 + 4/10 + 5/100$$

and asks a student to come to the board and carry out the addition.

The student writes:

$$100/100 + 40/100 + 5/100 = 145/100$$

Remark: The children often say “We took the fraction apart!”

Second game

(a) The teacher proposes that they play another time. For this game the fraction chosen should be a bit different, for example 975/1000.

The game develops exactly as before – the fraction is placed on the line, and then decomposed:

$$9/10 + 7/100 + 5/1000$$

$$900/1000 + 70/1000 + 5/1000 = 975/1000$$

(b) The teacher next asks the children to take apart the fractions that were trapped in the previous activity, for instance 99/10. Everyone writes in their notebook:

$$99/10 = 90/10 + 9/10$$
$$9 + 9/10$$

and puts 99/10 on the line.

Remark: Some of the students notice that the fraction 83/9 that was also chosen in the previous activity can't be placed on this line marked off in 1/10, in 1/100, 1/1000, …

Third game

(a) Instructions
"Do you think now we could guess a fraction very quickly by asking questions about its decomposition? You're going to try to find those questions!"

(b) Development
One student plays against his classmates. He leaves the classroom while the others choose a fraction that they write in their notebooks (243/100, for example).

The child returns and tries to ask his classmates, one at a time, questions that will help him find the fraction very quickly. After a bit of trial and error, he asks (sometimes helped by the teacher) "How many units are there?" "How many tenths?", "How many hundredths?", etc.

His classmates should tell him when he has trapped the fraction. Then he should write it on the board (with the help of the answers he got) and put it on the line.

(c) Remarks
The children note down the information they receive in very different ways. Here are some examples of notations they have used:

Unit rods	2
1/10 rods	4
1/100 rods	3

or

2 units
4 tenths
3 hundredths

or

2
4/10
3/100

The teacher does not intervene in this game. It is the students who protest from their seats if the answers given are not correct or if the student who is looking for the fraction makes a mistake.

The game can be re-played two or three times – the students stay engaged. It's generally the end of the class hour that puts a stop to this game.

The teacher keeps a list of the fractions chosen by the students in this game, because they are going to be needed in the next activity.

Results The students have learned how to put decimal fractions on a number line. Many know how to place them quickly and surely. Some still have difficulties.

They are aware that some fractions can't be put on a line subdivided in powers of ten.

At the end of this activity, they all know how to decompose a decimal fraction and give the number of units, tenths, hundredths, etc.

Lesson 4: From Writing Decimal Rational Numbers as Fractions to Writing Them as Decimal Numerals

Starting a Table

(a) Instructions
The same instructions as before.

(b) Development
A student goes out, her classmates chose a fraction that she is supposed to find by asking the same questions as in the previous activity.

But then the teacher proposes that the information given be marked in the table below (Table A), which will serve for every game.

Table A.

Values of the Intervals	1	1/10	1/100	1/1000	1/10000

For example, if the fraction chosen is 239/1000, the child who is asking the number of units, tenths, hundredths, etc. puts 0 in column 1, 2 in the column 1/10, 3 in the column 1/100, 9 in the column 1/1000 and writes the fraction found in the last column:

Table A.

Values of the intervals	1	1/10	1/100	1/1000	1/10000	
	0	2	3	9	–	239/1000

One or two more children can play and put their information and the resulting fraction in Table A.

Writing Fractions in Table A

(a) Instructions
We're going to put the fractions you chose and guessed in the previous session onto Table A.
(b) Development
The teacher sends several students to the board in turns to write the fractions from the previous game in table A. Then he has them mark other fractions chosen either by the children or by himself (for example 325/100, 1240/10, 85/10000, etc.)

Remark: This phase is collective. All the children participate, either by going to the board, or by making remarks, or by protesting if the one at the board makes a mistake. It should happen quickly like a game.

Other examples are then done individually. The teacher dictates the following fractions which the students put in the copy of Table A they have made in their notebooks:

$$7345\,/\,100,\ 7345\,/\,10,\ 7345\,/\,10{,}000,\ 7345\,/\,100,\ 7345\,/\,1{,}000$$

Passage to Decimal Notation

(a) Information provided
The teacher writes on the board (away from Table A)

7345
7345
7345
7345

and asks the class whether they are all the same number. The students reply that written like that, not in Table A, they are all the same number, even though written in Table A they were different numbers. After discussion with the children about the possible means of distinguishing these numbers, the teacher introduces the decimal point.

73.45, 734.5, 0.7345, 7.345

They immediately note that it is always placed after the units (intervals of length one).
(b) Reading these numbers
The teacher tells the students how these numbers are read: "73 point 45" or "73 units, 45 hundredths" and has them read several.

(c) Individual exercises for drill and verification
The teacher proposes the following exercises which are done individually and corrected at once. That way she can immediately spot any students who are still having difficulties and help them.

1. Write the following fractions as decimal numbers:

$$245/100,\ 48/1000,\ 2/100,\ 7259/10$$

2. Write as fractions:

$$2.5,\ 145.75,\ 13{,}525,\ 3.7425,\ 0.1,\ 0.01$$

Results Almost all of the children understand and can write decimal fractions as decimal numbers and vice versa. When the number is written as a decimal number they can say the number of units, tenths, hundredths, etc. This activity gives them hardly any trouble.

Module 6: Operations with Decimal Numbers (Summary)

In the first four of the lessons above, decimal numbers were always written as decimal fractions. In the last one and its immediate sequel, re-writing them in decimal notation becomes the occasion for various exercises in transcription in both directions, and provides the opportunity for them to make most of the common errors arising from transcription and correct themselves using their knowledge of decimal fractions (Lessons 5.4 and 5.5).

Following that, in Module 6 it is time to "redefine" addition for decimal numbers written as what the students call "numbers with a decimal point". After addition and subtraction, multiplication of a measurement number (a concrete number) by a natural number scalar (an abstract number) is easy to understand and carry out, especially making use of techniques of multiplication and division by 10, 100, 1,000. This cycle of six lessons is a welcome one for the students because it takes them back to a domain that they recognize as a familiar one. The many exercises they are given are much easier to carry out and understand, and the classical errors that normally turn up when the operations are carried out mechanically are easy to flush out with the aid of the knowledge they have developed in the previous activities. Students who spent the previous lessons following along on a route being forged by the class that they could not have forged for themselves find that they finally have some material they can handle on their own. It is a joy to discover all at once that the operations are so easy that they can really handle the reasoning to justify them. They credit the relief to the introduction of decimal notation, in which the operations on the rational numbers can be expressed.

Module 7: Brackets and Approximations (Summary)

Division of decimal numbers by a whole number always stops with the units of the quotient. The students only know how to calculate it exactly in the form of a fraction, so that the result is no longer expressed as a decimal number. The next two lessons therefore deal with systematically extending the bracketing of rational numbers between natural numbers and honing the notion of approximation.

At the end of the two lessons, the students try to bracket as tightly as possible the rational number 4319/29. First they extract the whole numbers by a classical division procedure: the fraction is located between 148 and 149 and there remains 27/29. To bracket this number between two successive tenths, they need to know how many 29/290ths (that is, tenths) there are in 270/290 (that is, 27/29). So they divide 270 by 29. This gives them that the fraction 4319/29 is between 148.9 and 149. And they proceed in the same way. They cut the interval [148.9,149) in ten and check how many 29/2,900ths there are in 90/2,900ths, etc.

When they put together on the board in an organized way the sequence of operations that they had scribbled all over their notebooks the children remark that the sequence of successive divisions looks just like a single division that has been extended. There is a small debate before they accept the idea of giving the name "division" to this new operation that enables them not only to bracket a fraction but to determine the "approaching" decimal number resulting from dividing one whole number by another.

The final session is devoted to a mathematical study of the decimal fractions obtained by approximation (i.e., by division) and comparison with fractions. Are all fractions decimals? Do all divisions come to an end? etc. In the course of this lesson, the students carry out multitudinous divisions, but with an eye to studying their properties, not simply as formal exercises "to learn how". The discovery of periodic sequences produces a passionate interest in these instruments for approaching the infinitesimal.

> We have now traversed the first seven modules of the Manual. Where have we arrived? On a mathematical front, we are at a point that demonstrates with extreme clarity an aspect of teaching on a constructivist model that opens it to criticism by those who either are unfamiliar with it or in disagreement with it. In terms of institutionalized knowledge – knowledge that could be put on a written test with a reasonable expectation that any student who has been paying adequate attention will be able to answer most or all of the questions – the volume is not particularly impressive. Certainly the students can handle basic rational number operations (all the arithmetic operations with the exception of division by a rational number) very comfortably. A notable strength relative to what one commonly observes is that they are equally adept with proper and improper fractions, On the other hand, in terms of making use of

(continued)

(continued)

rational numbers, their knowledge is still limited to the context of measurement. Similarly for decimal numerals, they can dependably carry out all of the basic arithmetic operations with the exception of division by a decimal numeral, and they can convert back and forth between fractions and decimal numerals, provided the decimal numeral in question is a terminating one. They can also make use of decimal numbers, but again only in the context of measurement. If that really represented the whole of their knowledge, then complaints about the paucity of that knowledge would be entirely legitimate.

What an individual, paper-and-pencil test cannot reveal is the depth of their knowledge and their degree of ownership. Also not susceptible to testing, but nonetheless both impressive and valuable is their level of "community understanding" – the body of knowledge that is accessible when they work as a group, as a result of partial understanding by many students and the capacity of all of them to listen to each other and explore each other's thinking. Thanks to those "invisible" forms of knowledge, they will be enabled in the following modules to expand their individual, institutionalized knowledge dramatically and at considerable speed. They will be able to invent and re-invent the concept of a rational number as a linear mapping until they internalize it, to assemble a collection of observations and partial understandings into some very solid knowledge about division and to use both rational numbers and decimals in most of the standard contexts. To a large extent this knowledge will be institutionalized and testable, though there will, of course, be some speculations and queries left to fuel future exploration.

The remaining two sections of the book will summarize and discuss the modules of the manual covering this last stage in the learning of rational and decimal numbers.

Module 8: Similarity

Lesson 1: Enlargement of a Puzzle

The first situation put to students for study of fractions as linear mappings is the following.

Instructions:

Here are some puzzles (Example: Fig. 2.5 below). You are going to make some similar ones, larger than the ones I am giving you, according to the following rules:

The segment that measures 4 cm on the model must measure 7 cm on your reproduction.

When you have finished, you must be able to take any figure made up from pieces of the original puzzle and make the exact same figure with the corresponding pieces of the new puzzle.

I will give a puzzle to each group of four or five students, and every student must either do at least one piece or else join up with a partner and do at least two.

Development:

After a brief planning phase in each group, the students separate to produce their pieces. The teacher puts (or draws) an enlarged representation of the complete puzzle on the chalkboard.

Strategies and Behaviors Observed

Strategy 1: Almost all the students think that the thing to do is to add 3 cm to every dimension. Even if a few doubt this plan, they rarely succeed in explaining themselves to their partners and never succeed in convincing them at this point. The result, obviously, is that the pieces are not compatible. Discussions, diagnostics – the leaders accuse the others of being careless. They don’t blame the plan, they blame

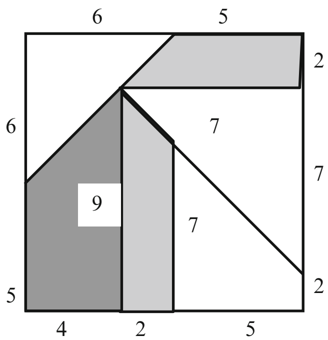


Fig. 2.5 The puzzle

its execution. They attempt verification – some students re-do all the pieces. They need to submit to the evidence, which is not easy to do! The teacher intervenes only to give encouragement and to verify facts, without pushing them in any direction.

Strategy 2: Some of them try a different strategy: they start with the outside square and try adding 3 cm to each of the segments in it. This produces two sides of length 17 cm and two of length 20 cm – not even a square. This is perplexing for the students, who begin to get really skeptical about the plan and often say, “It must be we shouldn’t add 3!”

Strategy 3: Another strategy often tried, either spontaneously or after #1 and #2 have failed, is to multiply each measurement by 2 and subtract 1, since they observe that $4 \times 2 - 1 = 7$. This gives a puzzle that is very similar to the original. Only a few pieces don’t fit well. So occasionally the students work their way out of the situation by a few snips of the scissors here and there. Even if most of them are aware that they are fudging, a few are convinced that they have found the solution. The teacher, invited along with the other groups of students to confirm success, in this case suggests that the competitors use the model to form a figure with some of the original pieces (such as Fig. 2.6) that cannot be reproduced with the pieces they have produced (Fig. 2.7).

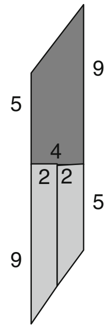


Fig. 2.6 A figure made from pieces of the original puzzle

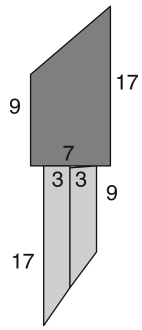


Fig. 2.7 An attempt to produce the same figure after enlarging by ×4-1

To help them see what happened, they can calculate:

$$8 \longrightarrow 14$$

Remarks: (a) For a variety of social and intellectual reasons, there is a general resistance to the idea of reconsidering the initial procedure. Classes often get quite worked up – lively disputes, accusations, threats – but rarely discouraged.

(b) Occasionally a group succeeds in finding the right process and produces the correct puzzle. The whole class and the teacher take note of the success, but the procedure is examined in the following lesson.

Results All the children have tried out at least one strategy, and most have tried two. By the end of the class, they are all convinced that their plan of action was at fault, and they are all ready to change it so they can make the puzzle work. But not one group is bored or discouraged. At the end of the session they are all eager to find "the right way".

Lesson 2: The Image of a Whole Number

Assignment: "The different procedures you tried out yesterday weren't right, because you couldn't make the corresponding models with the results. You found out that adding 2 or multiplying by 4 and subtracting 1 didn't give the right measurements. Today you are going to try to find the right measurements that will let you make the puzzle right.

Development:

To make things easier, the teacher (or sometimes a student who succeeded with the activity the day before) puts the lengths up as a table:

$$\begin{tikzcd} 4 \arrow[r] \arrow[d, bend right, "\div 4"'] & 7 = 70/10 \arrow[d, bend left, "\div 4"] \\ 1 & 70/40 = 35/20 = 175/100 = 1.75 \end{tikzcd}$$

Right off the bat somebody always asks for the image of 8 (which is of no use, but which they nonetheless add to the table)

$$\begin{tikzcd} 4 \arrow[r] \arrow[d, bend right, "\div 2"'] & 7 \arrow[d, bend left, "\div 2"] \\ 2 \arrow[r] & 3.5 \end{tikzcd}$$

This proposition, which is not rejected, may be what leads to the almost instantaneous appearance of another one: "We need the image of 1!"

"Yes, that would let us find all the others"

The teacher then adds 1 to the table and tells the students to find the measurements. The students work in groups of 2 or 3, all of them having copies of the table in their notebooks. As before, the teacher goes from group to group, encouraging them and answering questions, but does not take part.

Some of the procedures observed:

1.

2.

$$\begin{tikzcd} 2 \arrow[r] \arrow[d, bend right, "\div 2"'] & 3.5 = 35/10 \arrow[d, bend left, "\div 2"] \\ 1 \arrow[r] & 35/20 = 175/100 = 1.75 \end{tikzcd}$$

$4 \longrightarrow 7$

Here they are not actually performing a division. They are using cultural knowledge that they have acquired and their explanation is

"Half of 6 is 3
Half of 1 is 1/2 or 0.5
$3+0.5=3.5$"

From there they continue in the same vein:

$$\begin{array}{ccc} 4 & \longrightarrow & 7 \\ 1 & \longrightarrow & ? \end{array}$$

3. An alternative for the last step:

$$2 \longrightarrow 3.5, \text{ which they write as } 3.50.$$

To find the image of 1, they write: half of 3 is 1.50, and to that they add half of 50 hundredths, or 25 hundredths. $1.50+.25=1.75$;

To find the other measurements, they use either of the following procedures:

Either they multiply the image of 1 successively by 5, 6, 7 and 9

Or they add the image of 1 to the image of 5 to get that of 6, the image of 4 and that of 2 to get that of 6, and so on.

Observation:
One of the children, after having correctly found the image of 1, went on to make all of her calculations using 1.7. When the teacher asked her "Why did you multiply by 1.7 after you had found 1.75?" she replied, "Because I can't measure 1.75 with my ruler because it only goes up to millimeters."

The rest of the class broke in to protest: "Yes, you can! If your pencil is good and sharp you can get very close to halfway between two millimeters." This convinced her, so she didn't do the puzzle with the measurements she had found, and therefore never observed the inaccuracy that would result.

Remark: For many children, measuring 12.25 cm or 15.75 cm gives a lot of trouble that teachers often don't register, but that they ought, in fact, to take into consideration.

Comparison of methods and realization of the puzzles:

As soon as all the groups have found the measurements, they compare and discuss their methods.

The teacher then has them make the pieces and reconstitute the puzzle. (The students would ask to do it themselves in any case.)

Remark: This phase is essential, because for the children it is the only proof that is valid and convincing. But above all, it is source of pleasure and enthusiasm for them: their effort is repaid and they have succeeded.

Results All the children know that the image of a whole number can always be found, and almost all of them know how to find it.

Lesson 3: The Image of a Fraction

First phase: review of the two previous activities:

Assignment: "We enlarged a puzzle. To do that, we had a model on which we knew all of the measurements and we had some information about one of the new measurements: we knew that what corresponded to 4 was 7.

$$4 \longrightarrow 7$$

What did you look for?"

Development: The children briefly recall the activity and the teacher provides a quick overview of all the techniques used. What they needed was the image of 1.

$$\begin{array}{ccc} 4 & \longrightarrow & 7 \\ 1 & \longrightarrow & ? \end{array}$$

She runs swiftly through some other examples: "If 9 goes to 11, what does 1 go to?"

The teacher can send one child to the board or ask them all to work it out on scratch paper.

$$\begin{tikzcd} 9 \arrow[r] \arrow[d, bend right, "\div 9"'] & 11 = 11/1 \arrow[d, bend left, "\div 9"] \\ 1 \arrow[r] & 11/9 \end{tikzcd}$$

(They often need to review division here, which they do collectively.)

Remark: It is essential for the teacher to pull the class together on a regular basis to remind them where they are: recall or have them recall what problem was posed and what questions that problem gave rise to. They absolutely must know what it is they are trying to solve. The teacher can even occasionally remind them in the course of an activity. The fact is that many children, in the process of working out the intermediate steps of a problem, forget why it is they are carrying out their calculations.

Second phase: Image of a fraction

Assignment: "Now you know how to find the image of any whole number. You also know that you can designate a measurement by a fraction – what did you do that for? (constructing paper strips). Today you are going to try to find the image of a fraction."

The teacher puts 5/7 in a table of measurements:

$$\begin{array}{ccc} 4 & \longrightarrow & 11 \\ 5/7 & \longrightarrow & ? \end{array}$$

Development:

First she asks the students to think a bit and make sure they all understand the problem posed.

Spontaneously the children suggest adding 1 into the table of measurements, which the teacher does. She then asks them to find the image of 1.

$$\begin{array}{ccc} 4 & \longrightarrow & 11 \\ 1 & \longrightarrow & ? \end{array}$$

One of the students comes to the board and writes this image of 1, which gives one more rapid review. The new table of measurements now reads

$$\begin{array}{ccc} 4 & \longrightarrow & 11 \\ 1 & \longrightarrow & 11/4 \\ 5/7 & \longrightarrow & ? \end{array}$$

Remark: The operator 11/4 is not, and should not be, identified. 11/4 is just a measurement.

At this point, for this particular piece of the problem, the students work in groups of two or three.

Behaviors observed:

1. Many students transform the fraction 11/4 into a decimal numeral: 11/4=275/100=2.75, then stop because they don't know how to multiply 5/7 by 2.75.

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right, "\times 5/7"'] & 11/4 = 2.75 \arrow[d, bend left, "\times 5/7"] \\ 5/7 \arrow[r] & 2.75 \text{ x } 5/7 \end{tikzcd}$$

The majority “trap” 5/7 by doing what they did before[2]

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right, "\times 5/7"'] & 11/4 \arrow[d, bend left, "\times 5/7"] \\ 5/7 \arrow[r] & {} \end{tikzcd}$$

But there again they bump into calculations that they don’t know how to carry out: the multiplication of two fractions: $11/4 \times 5/7$.

We should point out that a certain number of them do actually write out the correct result: $11/4 \times 5/7 = 55/28$ purely by intuition. Obviously, the result can’t be accepted, because they can’t justify it at all.

Another frequent event is that they write 4, 11 and 1 as fractions:

$$4 = 4/1 \quad 11 = 11/1 \quad 1 = 1/1$$

and are stuck there.

The teacher goes from group to group, asking questions, giving encouragement. This activity is difficult for children of their age, and they need to be helped along with questions like “What would you do to trap 5/7?” and “What do you think might be another in-between number?”

Fruitless efforts. They remain stuck. So she organizes a collective discussion.

Third phase: the search for an in-between number

Assignment: The teacher first asks the students to look closely at the table of measurements from before:

$$\begin{array}{lcl} 4 & \longrightarrow & 11 \\ 1 & \longrightarrow & 11/4 \\ 5/7 & \longrightarrow & ? \end{array}$$

She poses the question: “What would make it easy to do the trapping? Think about the calculations you would have to do.”

Development: A phase of collective reflection starts up first. The children think silently, then propose things out loud. The proposals are immediately put to the test while the whole class watches.

A certain number of them lead nowhere: a proposal to put 1 in the table in the form 7/7 or 1/1 which makes no progress on the problem because they don’t know how to get from 7/7 to 2/7, or from 1/1 to 5/7.

Others, for instance 5 or 1/7, do lead to a possible answer to the question the teacher asked.

[2] Some expressions used spontaneously by the children to make themselves understood – either by the teacher or by the other children – are adopted by the whole class and accepted by the teacher. Some terms are thus used that are not necessarily “mathematical” but only serve temporarily for dealing with particular situations. They are not institutionalized, and are therefore later forgotten. Examples: “trap” and “in-between number”

Examples of attempts to check the last two, made at the blackboard by a student or the teacher:

First attempt, using 5:
First 5 is converted to a fraction: 5/1.

Remark: At this point it is once again generally useful to review division by having them quickly carry out small calculations like $25/3 \div 9 = 25/27$; $13/9 \div 5 = 13/45$; $81/13 \div 9 = 9/13$ etc.

(A few students know and recall that they can multiply the denominator by 7 to make the fraction 7 times smaller.)

Second attempt, using 1/7:

$$1/7 \xrightarrow{\times 5} 5/7$$

The children return to their groups of two or three and get back to work on the solution they were working on in the second phase.

Strategies observed:
All the groups make one or the other of the following tables:

Either

$$\begin{array}{lcl} 4 & \longrightarrow & 11 \\ 1 & \longrightarrow & 11/4 \\ 5 & \longrightarrow & \\ 5/7 & \longrightarrow & \end{array}$$

or

$$\begin{array}{lcl} 4 & \longrightarrow & 11 \\ 1 & \longrightarrow & 11/4 \\ 1/7 & \longrightarrow & x \\ 5/7 & \longrightarrow & y \end{array}$$

depending on which of the two proposals they adopt.

Remark: Not all the groups get to the end of the calculations, because the children make mistakes. They have forgotten the techniques they developed during the lessons about operations on fractions. This is perfectly normal, and the teacher should neither worry nor blame the children. On the contrary, this is exactly the moment to re-use the processes they discovered quite a while ago, put them to work and let the children see what the processes are good for.

Collective Synthesis of Methods

First the teacher asks them which groups didn't succeed. She asks them to try to say what messed them up and why they didn't get any result. The children mostly know very well what happened to them: they made mistakes in the multiplication or division of a fraction by a whole number – that's the principal cause of errors.

Remark: Because this is a regular proceeding, the children are perfectly comfortable discussing their mistakes. This is beneficial to the whole class, since exploration of errors can often contribute just as much to understanding as observation of correct procedures.

After that discussion, the teacher sends some students to the board to describe the methods they used:

1)	$4 \longrightarrow 11$		2)	$4 \longrightarrow 11$	
	$1 \longrightarrow 11/4$			$1 \longrightarrow 11/4$	
	$5/1 \longrightarrow 55/4$			$1/7 \longrightarrow 11/28$	
	$5/7 \longrightarrow 55/28$			$5/7 \longrightarrow 55/28$	

Exercises for Practice

The teacher adds two more fractions to the table of measurements and the students calculate their image individually

$$
\begin{aligned}
4 &\longrightarrow 11 \\
1 &\longrightarrow 11/4 \\[1em]
7/9 &\longrightarrow ? \\
6/7 &\longrightarrow ?
\end{aligned}
$$

They discuss their solutions rather than turning them in.

Final step: The teacher inquires: "Does every fraction have an image?"

After some reflection, the children conclude that you can always find the image of a fraction because you can always multiply and divide a fraction by a whole number.

Results At the end of this session, the children understand that you can find the image of any fraction at all provided you know the image of one whole number.

They have also all grasped that you have to figure out the image of 1 and of some "in-between number".

On the other hand, they haven't all mastered the sequence of calculations, and can't all get to the result.

Remark: We emphasize here that this is normal and the teacher shouldn't worry. It would be a serious error to stop and drill the students, because the up-coming activities let them re-use these notions and progressively master them (each child at his own rate.)

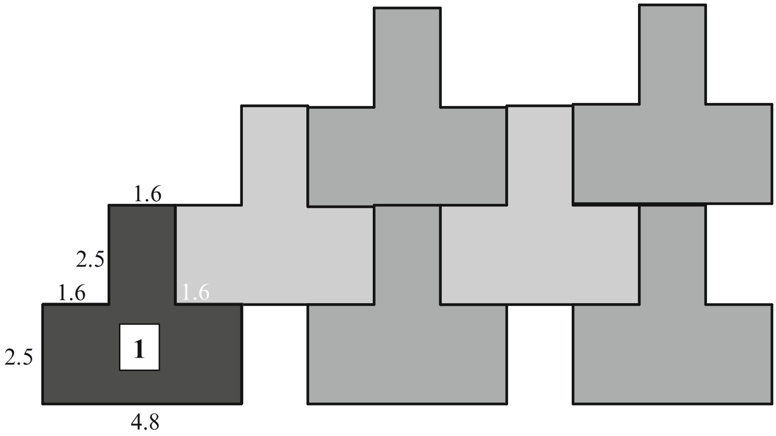


Fig. 2.8 Tesselation to enlarge

Lesson 4: The Image of a Decimal Number

Problem situation: construction of a tessellation

Materials: 3 or 4 cardboard pieces similar to the figure piece marked (1) in Fig. 2.8 or the same figure drawn on the board before the lesson.

Assignment: We are going to make a decorative panel for our classroom. It will be made up of pieces like the one you have, put together like this[3]:

To do that, each of you will make one piece by enlarging the model so that 1 cm on the model corresponds to 3.5 cm on the piece you make.

$$1\text{ cm} \longrightarrow 3.5\text{ cm}$$

Development: The children work in groups of three. They start off looking for ways to find the measurements for the piece

$$\begin{aligned} 1 &\longrightarrow 3.5 \\ 2.5 &\longrightarrow \\ 1.6 &\longrightarrow \\ 4.8 &\longrightarrow \end{aligned}$$

Strategies observed:

1. The most common strategy is to convert the decimal numbers to fractions

$$\begin{aligned} 1 &\longrightarrow 35/10 \\ 25/10 &\longrightarrow \\ 16/10 &\longrightarrow \\ 48/10 &\longrightarrow \end{aligned}$$

[3]The drawing should be prepared before class, either on the board or on paper.

Then by referring to the previous activity they calculate the images and add them to the table, first calculating that the image of 1/10 is 35/100 and then multiplying by 25, 16 and 48 respectively.

2. Another common strategy is to do the calculations by taking apart the decimal numbers as follows:

   For the image of 2.5:

   First find the image of 2 by doubling the 3.5.
   Next find half of 3.5 by finding half of 3 and then half of 0.5
   Then add up all three results.
   For the image of 1.6, add the images of 1, 0.5 and 0.1
   For the image of 4.8, add the images of 4 and 0.5, plus 3 times the image of 0.1

Remark: This last method is generally used by the children who are very good at mental calculations. Many of the calculations described above are invisible – only the results appear. At the request of the teacher (who goes from group to group and keeps on saying "But how did you get that?") the children consent – often with bad grace – to write them (at times in a highly disorderly way.)

Phase 2: Comparison of Methods

The groups that have found the numbers take turns at the board explaining their method. This gives the ones whose numbers didn't work out a chance to find out what went wrong. When children don't succeed with the activity it is always because of errors in calculation.

Conclusion: As at the end of the previous activity, the teacher asks, "Does every decimal number have an image?" Needless to say, the children respond in the affirmative.

Phase 3: Making the Pieces

Each child makes one or more pieces out of colored paper. As happens with the puzzle activity they are again faced with measurements: 5.6 cm, 8.75 cm, 16.8 cm They also have to use a T-square to make their lines, which adds a second interest to this session: construction of geometric figures.

Results This activity gives the children a chance to re-use procedures worked out in the previous session. It enables many of them to master some difficult calculations that they have previously been unable to carry all the way out.

Lesson 5: Division of a Decimal Number by 10, 100, 1,000, … (Summary)

Still using the set-up of 1 → 3.5, the teacher turns the class loose on finding the images of 1/10, 1/100 and 1/1,000. This presents no difficulties, and very soon the teacher is able to put a collectively produced table on the board. She then writes up the problems and results, including some intermediate problems that the students have produced:

$3.5 \div 10 = 0.35$
$0.35 \div 10 = 0.035$
$3.5 \div 100 = 0.035$
$0.035 \div 10 = 0.0035$
$0.35 \div 100 = 0.0035$
$3.5 \div 1{,}000 = 0.0035$

The students contemplate this and make observations, checking against different entries: "It's just the reverse of multiplication"; "You have to move the decimal point backwards", etc.

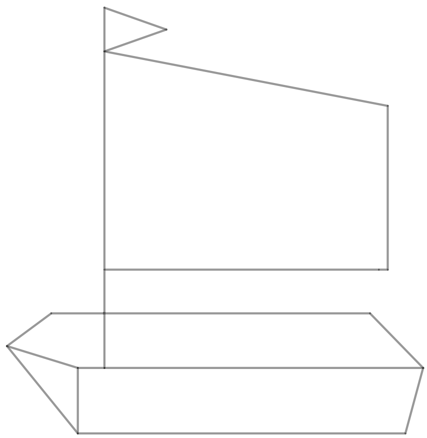
The teacher then leads them to formulate the rule: When you divide by 10 or 100 or …, you have to move the decimal point as many places to the left as there are zeros in the number.

By way of solidifying the rule and pushing the students a little farther, the teacher gives them some exercises, which are done individually and immediately corrected:

1)	$45.87 \div 1000 =$	2)	$135.9 \times \boxed{}$	$= 1359$
	$139.2 \div 10 =$		$4457 \times \boxed{}$	$= 485.7$
	$4750 \div 100 =$		$0.129 \times \boxed{}$	$= 129$
	$25785 \div 10000 =$		$130 \times \boxed{}$	$= 13000$
	$0.08 \div 100 =$		$1675 \times \boxed{}$	$= 16.75$
	$0.08 \div 1000 =$		$5.45 \times \boxed{}$	$= 5450$

Result As was the case for multiplication of a decimal number by 10, 100, etc., even though the children understand, they make a lot of mistakes with the placement of the decimal point. As a result, it is absolutely necessary to keep on regularly giving them exercises (corrected immediately) so that they master these calculations swiftly, because they are going to need them for lots of other activities.

Module 9: Linear Mappings



Optimist

All of the sections of Module 9 revolve around reproductions of the drawing above. It was chosen because not long before the time the lessons were given, the class had had a whole month in which after a morning in school, they spent every afternoon together at a sailing school near-by on a boat called the Optimist. This annual event was a source of great pleasure and of great class bonding as well. The basic drawing is on card stock and has the dimensions listed below. In addition there are 11 reproductions also on card stock, with specified ratios of enlargement or reduction.

Lesson 1: Another Representation of the Optimist (Lesson Summarized)

After introducing the drawing and having the children help her label the parts of it, the teacher puts on the board the list of dimensions:

Height of mast	17.7 cm	Length of boom	14 cm
Height of pennant	1.7 cm	Height of hull	3.4 cm
Side of pennant	4 cm	Length of stem	5.2 cm

She then puts up, beside the original, the reproduction that has a ratio of 1.5 to the original. The children observe and make comments: “That one’s bigger”, “It’s not twice as big – it’s less than that”. They often ask: “Are all the measurements the same?” by which they mean, “Were all the measurements enlarged the same way?” Sometimes they say: “Is it proportional?”

The teacher tells them that they can find that out themselves if they find the measurements of the reproduction. Then she asks: “Would you know how to find the measurements if it were proportional? What information would you need to do it?” The children generally tell her that one is enough.

She sets them up in groups and announces that each group must request in writing the one measurement that it wants. They are then to use that to predict what all the other measurements will be if the enlargement is proportional. Once the calculations are finished, they are to take their rulers up to the reproduction and check the measurements. If all of the actual measurements correspond to their calculated ones they will have the answer to their question.

The groups work together first to find the procedure that will give them the measurements. Then they divide up the measurements – one does the mast, another the boom, etc. As soon as they are done they check their results by measuring. Standard comments: “Yes, it is proportional” or “We blew it! Our measurements aren’t the same”, in which case they go back to their places and start over. (In one sad case a group that had asked for the measurement of the mast proceeded to add 8.55 to all the measurements, because $26.55 - 17.5 = 8.55$.)

When they are done, they have a collective discussion. First the ones who have had trouble describe where the trouble arose, then the groups that succeeded come to the board and present their methods (one presentation per method).

Some of the methods presented by the children:

First strategy – measurement requested was the length of the boom

The students noticed that 14 + 7 (half of 14) = 21. To find each measurement of the reproduction, they added half of the measurement on the original to the measurement itself	14	21
3.4 = 3 + .04	3.4	3.4 + 1.5 + 0.2 = 5.1
5.2 = 5 + 0.2	5.2	5.2 + 2.5 + 0.1 = 7.8
17.7 = 17 + 0.7	17.7	17.7 + 8.5 + 0.35 = 26.55

Second strategy – again starting with the boom

$$\begin{tikzcd} 14 \arrow[r] \arrow[d, bend right, "\div 14"'] & 21 \arrow[d, bend left, "\div 14"] \\ 1 \arrow[r] & 1.5 \end{tikzcd}$$

So the image of 1 is 1.5.

For the side of the pennant, then:

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right, "\times 4"'] & 1.5 \arrow[d, bend left, "\times 4"] \\ 4 \arrow[r] & 6 \end{tikzcd}$$

For the height of the pennant, they write 1.7 as 17/10, then work with the image of 1/10

$$\begin{tikzcd} 1 \arrow[r] \arrow[d, bend right, "\div 10"'] & 1.5 \arrow[d, bend left, "\div 10"] \\ 1/10 \arrow[r] \arrow[d, bend right, "\times 17"'] & 0.15 \arrow[d, bend left, "\times 17"] \\ 17/10 \arrow[r] & 2.55 \end{tikzcd}$$

The same strategy was used by some groups who started with the measurement of the side of the pennant. One such group began by calculating the image of all the integers: 17, 3, 4, 5 and 17 and the image of 0.7, 0.4 and 0.2 and adding appropriately.

A third and fourth strategy were developed by groups who started with the height of the mast. One was to multiply both sides first by 10, so as to have whole numbers to deal with. Another was to convert 17.7 to 177/10 and then divide both sides by 177. Both strategies then match those of the second strategy.

Commentary Like any other lesson that involves making actual measurements and comparing them with the results of computations, this one brings up issues related to approximation. The teacher needs to establish very gradually over the course of all such lessons an understanding within the class of how to treat values arrived at by measuring and those arrived at by calculation. Questions of how large a discrepancy is acceptable should be treated case by case, with student opinion always underlying the decision so that they never think the answer is handed down from on high. Eventually error intervals and the algebra thereof should work their way in, but not as a topic in themselves, always as a means of dealing with a particular situation.

Lesson 2: (Summary of Lesson)

The next lesson is highly similar to the first. The only difference is that the new reproduction is the one with proportionality factor 1.4. As soon as the students see it they notice that the first procedure above won't work. They settle down and swiftly work out the new lengths using one or another of the other procedures. After the solutions have been duly discussed, they discuss which one they found the most effective and institutionalize it as the one to be used in the following activities (generally the one that starts by turning everything into a whole number.)

Lesson 3: Lots of Representations of the Optimist (Summary of Lesson)

This is followed by a very challenging lesson that uses a bunch of the reproductions and poses a new problem.

The teacher holds up five of the reproductions, some larger than the original, some smaller, and some very close in size. First she has the class put them in order by size and labels them A, B, C, D, E, and M for the model. She sets up a table with the letters across the top, starting with M, and the six elements of the boat whose measurements they have been working with down the side. She fills in the column of measurements for the model, then the row of stem measurements for all of the reproductions (which settles whether they got the order right.)

Then she says, “I will tell you that one of these has mast length 13.275. Can you figure out which reproduction I’m talking about?”

This is a real challenge to the students, because it is not obvious to them how to attack it – how to identify the relevant variables. For instance, one tactic would be to calculate the ratio of 13.275 to 17.7 (mast length of the model), then calculate the ratio of each of the stem lengths to 5.2 (stem length of the model) and see which one matched. Another would be to calculate the lengths of all of the masts and see which one comes out to be 13.275. Or then again, one could calculate the stem length of the unknown boat and compare it with the given lengths.

Remark Some of the children simply can’t get their hands on the problem. This is the kind of situation in which the teacher must firmly resist temptation. If she reduces the scope of the problem by pointing out which numbers are relevant and what to calculate, she will take all the interest out of it. The object of the lesson isn’t to accomplish a task, but to determine what it is.

Left to their own devices, the children make remarks like: “The image is smaller than the model, because the mast is 13.275 instead of 17.7” “But not much smaller…” “It can’t be E, because E is bigger.” This strategy of reducing the field of possibilities provides the opportunity for some good work with ratios: “It’s not A, because A is much smaller – the stem is less than half as long as the one on the model.”

This way they whittle the possibilities down to two or three. Then “to be sure”, they decide to do some calculations. But which ones? What’s going to tell them which boat that mast belongs to?

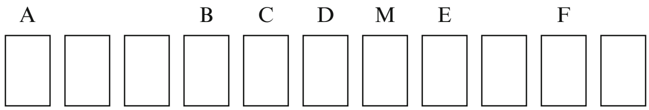
They think it over a while, and after some hesitations and tentative efforts, one of them comes to the board and writes:

Model	Reproduction
17.7 →	13.275
5.2 →	$\boxed{}$

The children work it out in groups of two, and the teacher chooses one to write the correct process on the board. The standard mode of calculation produces a stem length that matches that of boat C.

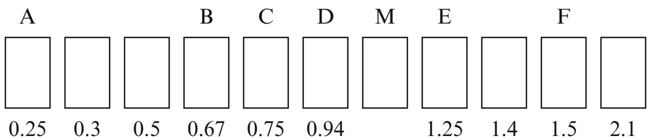
As a follow-up, the class does a series of problems individually, so that each child can figure out whether he actually understands, and whether he knows how to find whatever measurement he needs.

The rest of the lesson is a swift activity aimed chiefly at the motivation and introduction of a new notation. The original drawing and the six reproductions labeled A through F are still posted on the board. [Note: this lesson write-up is based on a report from a different year from the previous one. In the interim a spare reproduction of the Optimist seems to have turned up!] The teacher brings out four more. One at a time she gives a single measurement from each of them, and the class quickly tells her where to put it. In a short, almost playful time, all 11 are lined up on the board.



Question: "What are we going to call these new ones? We need to be able to talk about them." Many of them suggest A, A_1, A_2, B, etc. The teacher says she has another one that goes between A_1 and A_2, after which the class realizes that letters are not going to suffice. They set to work finding an alternative method. Often one of them will suggest using the image of one, since that way they can tell whether it is enlarged by a little bit or a lot. If none of them thinks of that possibility, the teacher suggests it, and asks them to verify that it provides the information needed. It should (a) let them find the image of any of the measurements and (b) let them put any enlargement she gives them in the right place.

She then has students go to the board to show where to put something enlarged by 1.35, by 1.87, by 0.72 (i.e., shrunk), by 0.29, etc. Then she has them reverse the process and find enlargements to go between ones that are already up there. This they do on their own, on scratch paper. Meanwhile the teacher writes under each reproduction the corresponding image of 1:



Enlargements, reductions, 0 or 1?

Next comes a rapid class exchange, launched by the teacher:

"What do you notice about the numbers labeling the reproductions?"
"They're in order from smallest to largest"
"The bigger the enlargement, the bigger the number"
"One of them doesn't have a number – it's the original model."

They decide it ought to have a number, and the class splits between those who propose 0 and those who propose 1.

"All the ones that get smaller have numbers less than 1"
"The numbers bigger than 1 all give enlargements"
"M is in between, so it ought to have a 1."
The teacher steps in with: "If I make a reproduction using 1, what will I get?"
"A reproduction "equal" to the model. It doesn't get smaller and it doesn't get larger."
"That's just it! It does nothing. Enlarging by nothing should mean zero!"

Teacher: "With our convention we have to put 1, but what would a 0 reproduction give?"

$$\begin{array}{l} 1 \longrightarrow 0 \\ 2 \longrightarrow 0 \quad \text{etc.} \end{array}$$

"Nothing!" "A point!" …

With that settled, the teacher goes on to another point: "Do you know how to tell whether the reproduction 0.84 is an enlargement? And 1.10? and 0.01? What would you mean by an enlargement by 2?"

The students answer: $1 \longrightarrow 2$

"How about a reduction by 2? Or an enlargement by 1/2? Is that a contradiction?"

Conclusion: It needs to be called a reproduction $1 \longrightarrow 2$, because the number is all it takes to tell us whether it enlarges or shrinks.

Remarks This lesson is in "Socratic" form – questions and answers. Rather than setting up a Situation of communication, like the ones with which rational numbers as measurements were introduced in Module 1 the teacher contents herself with talking about communication, because here the issue is familiar to the students and nothing new would come of such a Situation.

Students' internalization of the types of Situation that justify the means proposed for managing knowledge is part of the epistemological construction that the teacher is responsible for. This internalization saves time later without losing any of the meaning of the knowledge being created.

Lesson 4: Good Representations, Not So Good Representations

This one requires some special preparation. The teacher needs to make a special transformation of the model that enlarges the model with a horizontal ratio of 1.2 and a vertical ratio of 1.5. The resulting reproduction will be called Z.

As usual, the lesson starts with a review of the preceding lessons, including in this case listing on the board the images of 1 they found for all 11 of the reproductions. This leads up to having the children articulate what constitutes a reproduction that is an enlargement or reduction:

"You take a model.
You measure its dimensions.
You enlarge or shrink all of its dimensions the same way
You get a bigger or smaller picture."

The teacher then divides the class into four groups, with four different tasks. Three of them start with the original model and apply the following three mappings:

1. $1 \xrightarrow{\times 2.2} 2.2$
2. $1 \xrightarrow{+5} 6$
3. $1 \xrightarrow{\times 2 + 3} 5$

Question: "Do these mappings give you enlargements?"

The fourth group gets the new reproduction, Z. Their question is then "Is this an enlargement of the model?"

After the first question is asked, murmurs are audible: "Yes, they're enlargements." There are nevertheless a few who bring up the enlargement of the puzzle: "When we added 3 it didn't work!" From then on the term "enlargement" has some ambiguity for the children, but not all of them can quite say why.

Development: The children in the first three groups decide to calculate all the dimensions using the proposed directions and make the corresponding design with the resulting dimensions. They decide who is to calculate which dimension, then settle down and do it.

Observation: In the course of this phase of the lesson, the children in the groups that were given mappings # 2 and 3 become rather noisy (everybody accuses everybody else of calculating badly): "It's impossible! Our design doesn't look a bit like the model!" "It's not a boat! It's a jam jar!" "The pieces just don't fit!"

They want to quit, and usually call the teacher, who de-dramatizes the situation by smiling and telling them it's OK not to go on with the design.

Class discussion: When they are all done, the teacher has the first three groups put all of their numbers on the board, and the class checks them. Looking at the designs, the children are completely satisfied that #1 gives a proper reproduction. Some of them say "It looks proportional to the model." On the other hand, the designs that correspond to the other two create a lot of laughter. The line segments intersect at weird places, or don't intersect at all. It's impossible to reproduce the shape of the Optimist and use the numbers resulting from the operations in question.

The fourth group has had some problems: their reproduction looks a lot like the Optimist, but they couldn't find a consistent image for 1. Using the mast, they computed the image of 1 to be 1.5, but one of them noticed that the reproduction looked a bit elongated, so they checked the boom. Using 1.5 as the image of 1, the calculated the image that the boom on Z ought to have, and got 21 cm. Then they measured it and got 16.8 cm. They went back and checked all their calculations, but they were all right.

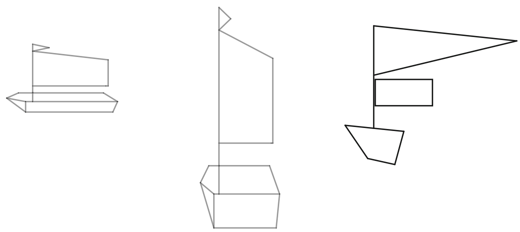
The teacher suggested that they calculate the image of 1 using the boom, so they did that and found 1.2 – not the number they got before! The child who had noticed that the design seemed elongated suggested that they should find the image of the hull with the new image for 1, so they did and it checked out with their measurements. When they presented this to the rest of the class, someone wondered whether all the vertical enlargements might be the same. So the teacher immediately encouraged them to calculate the image of 1 first using the height of the hull and then the height of the pennant. The students noted, “It ought to be the same as for the mast!”

They do a batch of computations and everything works as predicted. Now they know for sure: there is more than one image for 1.

To finish up, the teacher gives them a swift introduction to linearity. She points out that the height of the boat is the height of the mast plus the height of the hull, and writes all three of those measurements on the board. Then she has them compute the images of all three under each of the first three mappings. The first one duly gives images that add up properly, but adding 5, or multiplying by 2 and then adding 3 both give non-matching images. The students remark that it’s just like the puzzle situation – you don’t get a good reproduction unless you just multiply. Addition just messes things up!

Presentation of information: The teacher confirms their conclusion by telling them that if the sum of the images is equal to the image of the sum, then we say that **the mapping is linear**, or that **the numbers are proportional.**

Game: Invent some reproductions that aren’t proportional. Some examples might be:



Lesson 5: Change of Model

The whole flock of reproductions is still on the board. The teacher reminds the class of all they have learned working with the puzzle and the boat reproductions.

Assignment: "Do you think we could start with a different model? For instance, suppose I chose C as my model rather than M. Would I be able to designate the enlargement that gives F by the same number? Could you find a number that designates this enlargement?"

Development: The children and the teacher work together – the teacher poses questions and the children answer them.

"We saw that we could get F from M by the enlargement $1 \longrightarrow 1.5$. If I take C as the model, will $1 \longrightarrow 1.5$ still work?"

To make the comparison, the students propose to calculate the respective lengths of the same piece of the design. They choose the boom measurement, which is easy to work with. Some (re-) calculate the image of 1 in F starting from M, thus confirming that it is 1.5. Others calculate the image of 1 in F starting from C, and find that the mapping that takes C to F is $1 \longrightarrow 2$

Students often comment that "It's reasonable for the enlargement to be bigger, because C is smaller than M, so you have to enlarge it more to get F!"

Conclusion

If you change the model, for each figure you can still figure out the image of 1, but it's different than before.

You can't represent a figure by a number unless you indicate the corresponding model. It can be represented by as many different numbers as there are models to choose from.

Reproduction: the action and the image.

"Since we can have different models for the same figure, we can't just put the number that designates the enlargement or reduction under the reproduction. It has to be put between the model and the reproduction in order to designate the mapping." The resulting example is the familiar list of 11 figures, with a curved arrow going from M to each of the others and the image of 1 on the arrow. They are the numbers of the reproduction-action, not of the reproduction-image.

Calculations with Other Images

Assignment: "If we take as a model the reproduction we used to call 0.5 and reproduce it so as to make $1 \longrightarrow 3$, which figure will we get?"

Since the boom on the reproduction in question measured 7 cm, the new one will measure 21 cm, which is the boom of reproduction F.

Next assignment: "What is the mapping that takes us from A, used as the model, to C?"

The teacher gives the boom lengths for both figures and the children compute as they did before. It turns out to be the same as the mapping that took us from the former 0.5 to F.

Another assignment: "If I use the enlargement 1⟶5, I get F. Which one was the model? What table could we set up to find it?"

The teacher and the children exchange propositions, and then the children calculate individually. They use the ratios in F to find the measurements in the model:

$$
\begin{tikzcd}
\text{Model} & \text{F} \\
1 \arrow[r] \arrow[d, bend right, "\div 5"'] & 5 \arrow[d, bend left, "\div 5"] \\
1/5 \arrow[r] \arrow[d, bend right, "\times 21"'] & 1 \arrow[d, bend left, "\times 21"] \\
21/5 \arrow[r] & 21
\end{tikzcd}
$$

On the model, the boom length is thus 21/5 = 4.2. That's the reproduction formerly known as B.

Images and Reproductions

To determine a proportional representation, how many reproductions do you have to show? Two, the model and its image. For sure, the same proportional reproduction can make each model correspond to a different image. For example, in the first two of the three questions we just worked on we saw the mapping 1 ⟶ 3 first taking 0.5 to F, then A to C.

"Are there any other pairs of designs that share a reproduction-action? How can we find all the enlargements realized in our collection of figures?"

Development: The students just sit down and start calculating random enlargements. The teacher holds out for a system that represents each and every reproduction. She puts a grid on the board with all of the images designated down the side as models and across the top as images, and gets the students to fill it in. This can be a skill exercise, or an effort of a small group armed with a calculator, or a little competition: "Who can find the smallest? The largest? One between this number and that?"

The formulations are not simple, but the children manage to master them, and to laugh at the apparent contradictions that they produce.[4] They finish by putting them all in order, from smallest to largest and checking out the effects of various of them.

[4] "The more you pedal less hard, the less you go forward", as a child once explained to a flabbergasted psychologist.

Remark This lesson can be omitted for fifth graders, but it demonstrates very nicely the need to distinguish between the mapping that produces the reproduction and the image that it produces. Students can get by with thinking of enlargements as operations or the result of operations without being required to make a formal distinction, but the moment the problems start getting complicated, the teacher is left without any way to explain things to the students who are the least competent at constructing their own models. Teachers then have recourse either to teaching algorithms (the traditional solution) or waiting until the questions can be presented formally (current solution). In either case, there is no negotiation and no teaching of the meaning. The difficulty is not resolved, it is just disguised.

Lesson 6: Reciprocal Mappings

Presentation of the Problem

"When we took M as a model, we found that the enlargement 1⟶1.25 produced E as a copy. Now we want to know what would happen if we took E as a model and M as the copy. Every length on E corresponds to a length on M. Is it a good (proportional) reproduction? And if so what is the enlargement factor?"

Protest from the class: "It can't be an enlargement! It's a reduction!"

Assignment: "Since you are sure it's a reduction, let's find it!"

Development: This proceeds via an exchange of remarks, propositions and objections between the teacher and the children (and among the children themselves.)

The teacher writes up the beginnings of a table, with E and M at the top, and the children immediately propose to put in the corresponding measurements, starting with what 1 (in E) maps to in M. This one they calculate very swiftly, and find that the mapping in question is 1⟶0.8. But they still have to verify that this reduction stays the same for all the measurements. This they do individually, though a lot of them think it's unnecessary. Why? "It's just gotta be!" – but they can't articulate a reason. The teacher refrains from making objections.

Information from the teacher:

"The mapping that takes E to M is the mapping reciprocal to the one that takes M to E. (She writes "reciprocal mapping" on the board.) Do you think that every proportional reproduction that we have seen has a reciprocal? If so, would you know how to calculate it? They will also be proportional reproductions."

Exercise: What is the mapping reciprocal to 1⟶5/4?

Some of the students have to re-do the tables and calculations. The result is either that the reciprocal mapping is 1⟶4/5 or 1⟶0.8, depending which tactic the student used.

Challenge: "See if you can find a mapping that is equal to its own reciprocal."

Results: This activity is relatively simple for all the children. It develops rapidly as a game (question and answer.)

Module 10: Multiplication by a Rational Number

Lesson 1: Multiplication by a Rational Number

The process starts with a review of everything the class knows about fractions, bringing back into focus the original construction of fractions as a measurement.

"We constructed fractions, what did we do with them?"
"We put them in order"
"We added them,"
"We did some subtraction problems"
"We converted them into decimal numbers"
"What else do you think we could try to do with them?"
"Multiply them!"

> "We have already calculated the products of two fractions, but we didn't recognize it. We did some calculations that we could have written as one fraction times another fraction. We are going to see if we can find which calculations they were.
>
> We'll need to figure out what it is that lets us put the x sign between two fractions. Why do we have the right to write that when it's a different multiplication from the one we know?"

Remark To justify the use of the + sign on fractions the students contented themselves with verifying that the material operation they carried out, on lengths, for instance, corresponded well with what they were in the habit of associating with addition.

Here the meaning of the product of two fractions is quite different from that of the product of two natural numbers. The only really legitimate way to accept the sign "multiply" would be a detailed examination of the formal properties of the new operation and comparison with the known properties of multiplication. We think that such an exhaustive examination is inappropriate with children of this age, but that it is indispensable to have them inventory a certain number of properties

Either that are conserved (for example distributivity over addition),
Or that change (for example the fact that the product of two whole numbers is equal to or greater than each of the two)

And, of course, to construct a new meaning for multiplication.

Definition of the product of two fractions.

"We know that $3 \times 2/5$ is $2/5 + 2/5 + 2/5$, but is there an addition problem that could replace the operation in $3/7 \times 2/5$? As you might suspect, we need to look at enlargements and reductions and not at additions to construct this new multiplication. We will proceed in three steps.

First step: Let's see if we can find an enlargement in which we might be led to write $3/7 \times 2/5$"

The teacher gives the students a few minutes to think about it and possibly write something on a small piece of paper and put it on the corner of her desk.

At the end of this first short period of reflection (2 or 3 min) the teacher doesn't ask the students for their answer. They will find out whether they were right in the course of the class research.

"Do you know an enlargement that we could call '× 4'? Up to now we haven't put a × sign in front of numbers that designate an enlargement. But people often do put that × sign. See if you can understand why."

The expected response is: "1 on the model corresponds to 4 on the reproduction" As soon as she gets it, she writes it on the board as a conclusion:

Model		Reproduction
1	$\longrightarrow$	4

"Why can we call this enlargement x 4?" And he adds the following measurements:

Model		Reproduction
1	$\longrightarrow$	4
5	$\longrightarrow$	
3.5	$\longrightarrow$	
3/5	$\longrightarrow$	

A student comes to the board and fills in the list of reproductions with 5×4, 3.5×4 and 3/5×4, and often gives the result of the multiplication.

"We can call this enlargement "× 4" because the image of a number is calculated by multiplying the number by 4. Would you know the same way what an enlargement × 5 or × 7 would be?"

Second reflection step: "Now are there some of you who can write in their notebook how they might wind up writing 3/7 x 2/5 in the course of an enlargement?"

The teacher lets them think a few minutes, but doesn't call for the answers. If some of the children, sure that they have found it, put on too much pressure, the teacher can invite them to "deposit" their answer on another little piece of paper on another corner of her desk (so that both they and the teacher can know at the end of the process at what point they knew how to define the product.) At the end of this second period of reflection, the teacher poses a new question:

"On the same principle, what would be an enlargement that we could call 'x 0.25'?" She follows the same procedure as for the fraction.

At the end she introduces a new notation:

"We will write the name of the enlargement on top of the arrow, like this:"

$$\xrightarrow{\times\ 0.25}$$

Third step:
"Now can you find the circumstances in which we could write 3/7 × 2/5?"

And she waits until they give the answer

"1 on the model corresponds to 2/5 on the reproduction
3/7 on the model corresponds to 3/7 x 2/5 on the reproduction."

She can either take answers from the students if she thinks enough of them have it right or give one more piece of information and ask: "If we use the enlargement "× 2/7", what are the images of the following?

$$\begin{array}{lcl} \text{Model} & & \text{Reproduction} \\ 1 & \xrightarrow{\times 2/7} & \\ 2 & \longrightarrow & \\ 5 & \longrightarrow & \end{array}$$

Is it an enlargement or a reduction?" Students write their answers in their notebooks.

Fourth step: **Calculating the product of two fractions.**

Assignment: "Now can you find what we can do to decide what 3/7 × 2/5 means? Do you know how to calculate the result?"

$$\begin{array}{lcl} 1 & \xrightarrow{\times\ 2/5} & 2/5 \\ 3/7 & \xrightarrow{\times\ 2/5} & \boxed{} \end{array}$$

The students work in pairs. They try to reactivate the techniques they discovered in the activity from Module 8, Lesson 3.

Strategies observed:
The students calculate by using either 7 or 3 as an in-between number, and get (in either case) the image 6/35.

Comment: Every year some children write directly

$$3/7 \times 2/5 = 6/35.$$

The teacher, who goes from group to group during the working phase, expresses astonishment and asks how they got it. They tell her they multiplied the two numerators and the two denominators. "Why?" "It's just what you're supposed to do!"

The teacher tells them that she can't accept a step if they can't prove it's right. The children then solve it with one or the other of the strategies above.

Collective correction: The children who found it come to the board to demonstrate their strategies. It's a rapid reminder since they did these calculations many times in the course of the previous module. Many observe and affirm that you can do it by multiplying numerators and then denominators.

The teacher agrees to check out this method, which she baptizes with the name of the student who proposed it.

Study of the method: $\dfrac{\text{Product of numerators}}{\text{Product of denominators}}$

"Do you think this rule always works, no matter what the fractions are?"

The children hesitate and ask the teacher to give them another product. So she gives them $5/7 \times 4/3$. The children set out to calculate it both ways, and the teacher helps the ones who are having a little trouble. In the end they discover with pleasure that the rule works for this one, too, and enthusiastically endorse the "rule" because it worked again.

But the teacher points out that they only tried two examples and if they are going to adopt it ("institutionalize" it) it has to work *all* the time, on any pair of fractions. So she proposes a new form of verification in the form of a game.

Verification of the rule

First game:

1. The teacher asks the children to choose a number corresponding to these letters: a = , b= , c= , d= . Each child writes on scratch paper.
2. Calculate $(a+b)$: "Are you all going to get the same thing? Why?"
3. Calculate $(c+d)$: "Are you all going to get the same thing?"
   "No!"
4. Calculate $(a+b)+(b+c)$
   Then $a+b+c$
   Then $(a+b)+(b+c)-b$

   What do you notice?
   The result is written on the board:

$$(a+b)+(b+c)-b = a+b+c$$

Why?

Remark: The children love this activity. Most of them have seen older children calculating with letters and they say so: "It's just like 6th grade!"

Second game: **"**What does $a/b \times c/d$ mean?

The teacher has a student come to the board while the others look on and comment. The students sets up the usual format

$$1 \xrightarrow{\times\, c/d}$$

$$a/b \xrightarrow{\times\, c/d}$$

With a little help from the teacher, the student works through the whole pattern, getting $(a \times c)/(b \times d)$

Conclusion: The teacher points out that just the way it was in the first game, the letters could represent any numbers at all, so the rule they discovered holds true.

Lesson 2: Multiplying by a Decimal (Summary of Lessons)

The next day's lesson repeats roughly the same process as Lesson 10-1, but with decimal numbers: a search for an example giving rise to 1.25×3.5 and calculations. Clearly it's a question of calculating the length of the image of 1.25 in an enlargement that takes 1 to 3.5. The calculation is carried out initially by expressing the decimal numbers as decimal fractions, then directly, after a rediscovery of the way moving the decimal point represents the denominator. The algorithm is recognized and practiced and given the status of something to be memorized, as it would be in the classical methods.

Results: All the children understand the algorithm and the meaning of this multiplication. But it is interesting to note that it gives them great satisfaction to be able at last to multiply two decimal numbers.

In fact, they have invariably long since been asking the teacher, "Why aren't we learning yet how to multiply two decimal numbers, because we would know how to do it!" The pressure is particularly heavy in the course of the activity about the Optimist (Module 8, lesson 7) because at that point there are always one or two students (either repeating the class or coming in from other schools) who calculate the images of the measurements in the Optimist (whole or decimal numbers) directly by multiplying them by the enlargement or reduction factors.

Since the teacher doesn't take this procedure into account, and doesn't exhibit it when the class does its collective correction these children feel ill treated and ask why their solution hasn't been considered. Often there is one whose response to the teacher's "Because we haven't learned multiplication of two decimals and you don't know what it means" is that he does know – he has learned it.

In that case, the teacher has no choice but to have a collective clearing-up session. She reminds the students of the meaning of the different multiplications that they have already dealt with:

4×125 means $125 + 1{,}125 + 125 + 125$

4×2.5 (where 2.5 may be the length of a stick or the price of an object or the capacity of a container) means $2.5 + 2.5 + 2.5 + 2.5$.

But what does 1.7×0.94 or 4.128×3.67 mean?

Obviously, the children then realize that there exist multiplications whose meaning is different from those that they know, and they all agree that these calculations can't be used at this point in the progression.

It's easy to understand the relief they feel and express during this session, and their desire to do and use this long-awaited calculation!

Lesson 3: Methods of Solving Linear Problems (Summary of Lessons)

Introduction (for the teacher) The examples that follow will permit us to describe the typical progression of the study of a problem and to indicate how the teacher and students draw conclusions from it that are explicit but definitely not learned by heart. We will also indicate as many as possible of the conclusions and remarks that the children may make as they master different methods of solution, different types of questions, different uses of linear functions, etc. In order to avoid presenting a multitude of problems we will concentrate all these conclusions slightly artificially on a few examples. In any case, from the moment that the students start solving the problems the teacher ceases to exercise control over the details of the means of coming to the conclusions, and focuses on keeping the class engaged and with its eye on the goal.

The teacher demonstrates as an example what questions it is worthwhile asking oneself in the course of solving the following problem, while explaining step by step the solution of the problem:

The children collected the cream from 2 l of whole milk and got 32 cl of cream. They also collected the cream from 5 l of low-fat milk and got 40 cl of cream. Can you answer the following questions:

How much cream would you get from 50 l of milk? 125 l of milk? 250 l of milk?
How much milk would you need to get 4 l of cream? 2 l of cream? 10 l of cream?

The students discuss the problem and end up asking the teacher to remove the ambiguity of the questions. This helps prepare them to construct problems themselves.

The teacher makes comments and indicates how to present the givens, how to express the results (in the solution) and how to check the use of numbers and functions. Then he asks the children to recall the different ways they have encountered to solve linear mappings.

Lesson 4: The Search for Linear Situations (Summary of Lessons)

In this lesson the teacher sets up a "tournament of problems ". On a regular basis, students are to come up with problems that involve solving a linear mapping. The problems can be invented or taken from a book, but in any case the student who presents a problem must be able to give a solution if asked.

The tournament will be open until the end of the year. From time to time the class will spend a few minutes "judging the problems" the way pictures are judged at a painting exhibition: which is the most interesting, the most beautiful, the most original, the most trivial – but it is not the students who are being judged, it is the problems. Only the teacher knows which student proposed which problem.

The primary goal of this activity is obviously technical: it develops in the students a knowledge and culture of problems. By trying to classify them: problems about sales, about representations (in the ordinary sense), about relations between physical sizes, about percentages, etc., they observe their similarities and differences and varied characters. They will know them much better than if they were using the teacher's choices. This is not the place for a standardized classification!

It's the activity that matters more than its result. And the traps and counterexamples stand out without it being necessary for a certain number of students to fall victim to them. The discussions, of course, point up ways to look for more examples.

The second goal is in effect psychological. This set up provides a nice safe zoo where they can approach the wild beasts to which they often fall victim.

Module 11: The Study of Linear Situations in “Everyday Life”

Remark to the teachers: In this initial paragraph, we will first familiarize the children with the designation of linear mappings using the vocabulary of fractions.

The problems and the examples should thus be chosen appropriately. For example, in the problem:

“The 25 students in a fourth grade class go to the swimming pool every week. At the end of the year, 4/5 of the students know how to swim. How many students is that?”

The fraction 4/5 is a **ratio** between two sizes: the number of students knowing how to swim and the total number of students. It does not correspond to a **linear mapping**: we are not talking here about a rule for determining how many swimmers some other class should have. In fact, one might expect to compare the ratios. On the other hand, problems that correspond to “rules” – conventions or logical necessities – do furnish examples of linear mappings.

Examples of “rules”:

Composition of a food product (milk, bread, coffee, etc.) and transformations of it: coffee loses 1/7 of its weight in roasting; making fig jam requires a weight of sugar equal to 3/4 of the weight of the fruit, etc.

Lesson 1: Fraction of a Magnitude

(a) *Introduction*: Assignment

Weight of fruit in kilograms		Weight of sugar in kilograms
2	$\longrightarrow$	1.5
12	$\longrightarrow$	9
8	$\longrightarrow$	6
5	$\longrightarrow$	3.75

Is this table produced by a linear mapping?

(b) *Development*:

Students verify it by the methods available to them:

Is the weight of sugar corresponding to the sum of two weights of fruit equal to the sum of the weights of sugar corresponding to the two weights of fruit?

First method: Some students suggest comparing the weight of sugar (15 kg) corresponding to $12+8$ kg of figs with that corresponding to 4×5 kg of figs ($4\times3.75=15$), but they are already assuming that the mapping is linear. Here this method provides no verification.

Second method: Is it true that if we multiply each weight of fruit by some particular number we will have to multiply the corresponding weight of sugar by the same number to find the new amount of sugar?

The children verify that $2\times6=12$, does indeed correspond to $1.5\times6=9$, and $2\times4=8$ to $1.5\times4=6$. Then they realize that they are going to have to do an awful lot of verifications (16, or at least 6). So some suggest working out 1 and using it to verify the rest.

Third method: Do you get the weight of the sugar by multiplying the weight of the fruit by a constant?

To find this number, the children look for the image of 1, then carry out the multiplications 12×0.75; 8×0.75, etc.

Conclusion: the mapping is linear.

Note: The children already know these different methods, and use whichever is the most efficient in a given situation. They will be inventoried and institutionalized a little later.

(c) Summaries of the table

"How can we summarize this table in a short recipe?"
The students propose their standard method using the image of 1:

"You have to multiply by 0.75", swiftly corrected to
"You have to multiply the weight of the fruit by 0.75 to find the corresponding weight of sugar."

The teacher has them convert to fractional notation and simplify the fraction to 3/4.
75/100 = 150/200 = 15/20 = 3/4
Then he reformulates it as

"You have to multiply the weight of the fruit by 3/4 to find the corresponding weight of the sugar."
"You have to apply x 3/4 to the weight of the fruit to find the corresponding weight of the sugar."

He tells the class: "You will often find this said with expressions like
The weight of the sugar is 3/4 the weight of the fruit.
To find the weight of the sugar, you take 3/4 of the weight of the fruit, or you calculate 3/4 …

What you have just done here is to **calculate a fraction of a number.**

Notice that in the table we find opposite the number 4 (in the weights of the fruit) the number 3 (in the weights of the sugar). The weight of the sugar is 3 when the weight of the fruit is 4; the ratio of the weight of sugar to the weight of fruit is 3–4.

Careful! The ratio of the weight of fruit to the weight of sugar isn't the same! What is it?"

Exercises in Formulating Fractions in Terms of Linear Mappings

(a) *Assignment*: Here are some situations formulated in this way.
Translate them into the linear mapping schema.
Then pose some questions, if necessary filling in needed information.

1. A merchant wants her profit to be 2/5 of her purchase price.
2. Wheat gives 4/5 of its weight in white flour.
3. Draw a rectangle whose width is 2/3 of its length.
4. To buy on credit at a store, you have to deposit 3/8 of the selling price at the time of purchase.

(b) *Development*: Recognition of the mapping designated by a fraction and search for a schema.

The children know for example that the first situation has to do with a × 2/5 mapping, and they can represent it by

$$1 \xrightarrow{\times 2/5} 2/5$$

For them, the problem is to know where to put the price and the profit.

Often, if the reference situation is well known, outside information comes in to indicate the solution, for example, in the case where one is taking a part of a whole. Example (not true for this particular case) an image that is smaller than the original quantity… if you take two fifths, then 5 can't correspond to 2, so 2 must correspond to 5.

Here the "semantic" information was intentionally rendered inoperative. The merchant could wish to make a profit of 5/2 of her purchase price, because the profit is not part of the purchase price. Also the situation is not well known to the children. In this case, the formulation itself must be consulted: the expression "2/5 **of** the purchase price" shows that 2/5 **is not** the purchase price – that the purchase price is what you're taking 2/5 **of.**

So we have

Purchase price

$$1 \xrightarrow{\times 2/5} 2/5$$

and it must be that what we get to is the profit.

Here the students go back over the formulations they have already encountered:

For a purchase price of 1, the profit is 2/5
When the purchase is 5 the profit is 2 means that the profit is 2 for [a purchase of] 5
The profit is [2 per 5] (purchase price)

Two fifth is the profit; you have to multiply it by 5 to get 2 times the purchase price:

$$\begin{array}{ccc} 1 & \longrightarrow & 2/5 \\ \times 5 & & \times 5 \\ 5 & \longrightarrow & 2 \end{array}$$

In response, students produce schemas such as

$$\text{Purchase price} \xrightarrow{\times 2/5} \text{profit}$$

$$\text{Weight of wheat} \xrightarrow{\times 4/5} \text{Weight of white flour}$$

$$\text{Sale price} \xrightarrow{\times 3/8} \text{deposit}$$

$$\text{Length of rectangle} \xrightarrow{\times 2/3} \text{Width of rectangle}$$

Remarks for the teacher:

1. The starting number multiplied by the number determining the mapping is equal to the ending number. Since we can get the starting number by dividing the ending number by the fraction, distinguishing the starting and ending numbers is closely linked to the understanding of the product of two fractions or of two numbers.
2. The children should interpret the formulations directly. They must not be formally taught "algorithms" for getting the right answer. Numerous exercises and translations among the different formulations, accompanied by arguments of every sort (most of them particular to a specific example and thus not generalizable) will enable them to make sense of the cultural formulations that they will run into (of which a few are pretty illogical.) In any case, this study will be resumed in Module 14, where, with the composition of mappings, it will be possible to get back the traditional meaning (3/4 means divide by 4 and multiply by 3.)

(c) *Development (continued):* Formulation of questions and problems; search for necessary complementary information.
The students propose, for each of the above situations, several problems obtained by adding questions. For example (for the second situation), they might ask

- The weight of flour that you get
- The weight of grain that was necessary.

But this then requires that one know the weight of the grain in the first case and the weight of the flour in the second.

The children have no difficulty in posing these questions thanks to their familiarity with the tables. Nonetheless, this activity brings up interesting remarks on the relevance of information and questions.

Example: "A car has gone 100 kilometers and its tank is 3/4 empty."

If we add that the tank was full at the start, then we can ask how many more kilometers it can go:

$$\text{3/4 of a tank} \longrightarrow \text{100 kilometers}$$

$$\text{1/4 of a tank} \longrightarrow (100 \div 3)\ \text{kilometers}$$

Since 1/4 of a tank is left, it can go 33 km.

But if instead of the information that the tank was full we say that 20 liters of gas are left, then what we can ask is the capacity of the tank:

$$\text{1/4 of a tank} \longrightarrow \text{20 liters}$$

$$\text{4/4 or a full tank} \longrightarrow 20 \times 4 = 80 \text{ liters}$$

(and in this case the number of kilometers is useless.)

Further remarks for the teacher:

(i) Students often have difficulties in identifying the three elements of the mapping:

(a) The domain set – the thing you are "taking a fraction of", which is at the front of the schema with the arrows, but often named after the fraction is named, as it is in all of the examples above, and at times difficult to identify.

Examples:

"The reservoir is 3/4 empty"
"A worker earns a certain amount and saves 3/40 of it."

(b) The correspondence: the way of finding the image of a given number. Classically, 3/4 describes the operation "multiply by 3 and then divide by four" (or possibly vice versa), which we will describe in module 12. Here the student says that 1 corresponds to 3/4, without making reference to some operation for getting from the 1 to the 3/4. This way we avoid various difficulties linked to

- The impossibility of actually making the division being envisaged
- Too concrete a representation (taking fourths of a sum!)
- Or the complexity of the concrete operations envisaged.

But this makes the a priori identification of the image set all the more vital

(c) The image set of the values that are "a fraction" of another one is at times all the more difficult to distinguish in that the French language[5] permits a constant confusion between an operation and its result, a mapping and its image, an action or fact and the state that is its consequence (the marriage took place on such and such a date and lasted 20 years!)
The language of fractions assumes that it is obvious how to carry out a linear mapping.

(ii) We assume that the schema can be made by the students they before know the question posed and independently of it, using only the language of the representation of the situation of reference. We assume next that the question can be represented before and independently of the solution to be produced.

[5] As well as the English one!

(iii) The students are thus invited to pose questions and to inventory the possible questions:

- The search for the image (a fraction of a quantity)
- The search for the object (the quantity of which we know a fraction)
- The search for the mapping (the fraction taken – or the ratio of the two magnitudes)

Questions provide the justification for the rest of the lesson.

(iv) The students comment that they "can't calculate anything" if they don't know the quantity or the number of which they are "taking a fraction", but they can make the table, just as they could draw a rectangle whose width is two thirds of its length.

Calculations with "fraction-mappings"

(a) *Calculating the image*: The teacher presents the following problems:

1. You buy 6 kg of fruit to make jam. This kind of fruit gives 2/3 of its weight in juice. You need to add a weight of sugar equal to the weight of the juice. How much sugar should you buy?
2. Cotton shrinks when it is washed: it loses 2/9 of its length. If a piece of cotton fabric measures 6.75 m, how long will it be after washing?
3. 4/25 of the volume of milk is cream. How much cream would you get from 3/4 of a liter of milk?

(b) *Mathematization with the children*:
The teacher invites the children to make mathematical remarks about the problems they have just done. Some of them comment that they have calculated a fraction of a whole number, then a fraction of a decimal number, and finally a fraction of a fraction.

The teacher requests more precision: "What operation did you do to take 2/3 of a number?"

"We multiplied the number by 2/3."
"So what operation can we use to describe the mapping 'take 2/3'?"

By similarity to Activity 10.1 (product of two fractions), the children propose:

$1 \longrightarrow 2/3$ is multiplication by 2/3!

$$1 \xrightarrow{\times\, 2/3} 2/3$$

The teacher then puts a frame around

> Taking a fraction of something means multiplying by that fraction.

(c) Calculating a number of which a fraction is known.
For a holiday dinner, you buy a 3.6 kg roast. When it is deboned and cooked, this meat loses 2/9 of its weight.

If you want to have to have 2.1 kg of meat after cooking, how many kilograms do you need to ask the butcher for?

$$\begin{tikzcd}[column sep=huge]
\text{Weight before cooking} & \text{Weight lost} \\
1 \arrow[r, "\times 2/9"] & 2/9
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\text{Weight before cooking} & \text{Weight lost} \\
1 \arrow[r, "\times 2/9"] \arrow[d, bend right, "\times 9"'] & 2/9 \arrow[d, bend left, "\times 9"] \\
9 \arrow[r] & 2
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
\text{Weight before cooking} & \text{Weight after cooking} \\
9 \arrow[r, "\times 7/9"] \arrow[d, bend right, "\times 3"'] & 7 \arrow[d, bend left, "\times 3"] \\
27 \arrow[r] \arrow[d, bend right, "\div 10"'] & 21 \arrow[d, bend left, "\div 10"] \\
2.7 \arrow[r, "\times 7/9"] & 2.1
\end{tikzcd}$$

The teacher asks what operation corresponds to this mapping:

$$\begin{tikzcd}[column sep=huge]
9 \arrow[r] & 7 \\
1 \arrow[r] & 7/9 \\
{} \arrow[r, "\times 7/9"] & {}
\end{tikzcd}$$

Summary of the Remaining Paragraphs and Sections of Module 11

The rest of Lesson 1 takes up the reformulation in terms of fractions of the mappings that they already know: enlargements, etc. Simplification of fractions is recognized as the search for smallest whole number correspondences. The students study different ways of realizing certain fractions of squares or of dividing a segment into any number of parts using a supply of equidistant parallel lines.

Lesson 2 is dedicated to the study of percentages. Lesson 3 presents the correspondences of the same magnitudes (scales). The determination of the length of a segment that is out of their reach leads the students to surround it with a rigid figure and make a reduced model of it. The issue is to show a simple use of scales. By studying the correspondence between different magnitudes, the teacher leads the students to the use of non-scalar linear coefficients like the price per meter or the pounds per unit of volume.

While the second two modules in this section will provide students with occasions for addressing the often hidden difficulty of distinguishing between problems involving ratios and those involving linear mappings, at this stage the original manual simply presents the teacher with a warning of that difficulty.

The Problem-Statement Contest (Commentary 2008)
The curriculum on which we ran our experiments from 1973 to 1980 introduced different aspects of rational and decimal numbers by following new mathematical pathways. It consisted of the modules 1–11 and module 14. Teachers completed it with a classical exploration of the official program – the metric system and application problems – with the standard organization. We were interested, among other research subjects, in evaluating the impact of our new introduction on the performance of the students in these well known domains. Starting in the late 1970s the study of the *didactical contract* led us to study problem solving.

The contest of problem statements, introduced in Module 10.4 of the manual, recounts the first endeavors of this program, which continues to be carried out today.

Developing on the ideas of Polya, who suggested making students experts in the solution of problems by teaching them methods and heuristics, we wanted the students to stop thinking of problems as tests or as individual challenges and instead to develop a culture made up not just of tasks observed, techniques, and known results, but also of knowledge under construction and of emotions. Heuristics can be a useful tool, but if they are taught as skills to learn and apply, they become nothing but bad theorems.[6] Analogy is likewise a helpful but flawed tool: we have shown, among other things, how the abuse of "analogy" raised to the rank of a teaching principle resulted in an augmentation of students' failures.

(continued)

[6]We have shown that if the teaching of "problem-solving methods" follows the classic conceptions relative to knowledge and learning, the teaching will lead to uncontrollable metadidactical slippage and to failure. [Metadidactical slippage is discussed in Chap. 5.]

(continued)

The object of the ***Contest of Problem Statements*** is to improve students' learning by leading them to consider classes of problems and solutions and not just types of problems, to examine what constitutes them, to discuss them. Above all, it is to change the psychological, didactical and social conditions of the students' activity by changing their position in the didactical situation.

Instead of the usual pattern of using the problems to judge the students, here the students judge and determine the value of the problems. Thus, as is done for works of art at a showing, the students themselves award prizes among the problems presented to them: a prize for the longest, the most interesting, the most difficult, the most annoying, the most original, etc.

Ordinarily, problem statements are presented by the teacher. In our project, the students choose some and above all produce some themselves and discuss them with each other and with the teacher.

For teachers and society, problem solutions are regarded as a source of knowledge about the students. Here the solutions become knowledge of the student about mathematics and about the problems.

Thus the search for and making of "original" problems can become a challenge for the students, a motive and instrument for recognition, comparison and classification of problem statements, on condition, clearly, that the focus remain their mathematical solution.

The most important point concerns the status of the *connaissances* that appear and are formulated by the students or the teacher *in the Situation,* and their evolution toward the status of *savoirs.*[7]

In this process, the students learn to analyze these texts, to pose questions, to distinguish the givens. Statement and solution form, in fact, definitions and theorems. The teacher must thus know their grammatical and logical components. It is essential that he not teach them as *savoirs* and especially that he not explain them. The teacher's situation is similar to that of parents of a young child who is learning her language, in that the parents must get her to respect phonetic rules without explaining them. The classifications the children come up with are *connaissances* not *savoirs*. To take this knowledge as a piece of *savoir* constitutes a metadidactical slippage (teaching the meta-object in place of the object) which generally leads to other slippages.

[7] *Connaissances* and *savoir* both translate to "knowledge", but they are used very distinctly. A good working definition is that *connaissance* is general knowledge and *savoir* is reference knowledge. For a more nuanced definition, see Chap. 5.

Module 12: More on the Problem Statement Contest

Lesson 1

Phase 1: Research

(a) *Assignment*: "We will continue our problem statement contest (cf. Module 10, Lesson 4).

Today you are to propose problems that lead to doing a division. You will write the problem statement and simply carry out the operation, but you should prepare the justification for the operation in your head so that you can give it orally to your classmates. You may start with simple examples or ones we have encountered before. Don't get complicated – introduce the dividend and divisor with a sentence and ask for the quotient. What we are interested in is the occasions for doing division.

On the other hand, do try to put as many decimal numbers or improper fractions as you can in the statements and the solutions PROVIDED THE PROBLEM STILL MAKES SENSE.

The session continues. Students present their problem statements and verify under the teacher's guidance that the statement is correct (givens and question) and plausible, and that the solution offered is right. They inventory a certain number of difficulties: confusion between the means of calculating (example: division) and the statement or the means of checking (example: multiplication), classification by a partial calculation, etc.

Phase 4: Production of New Problems and Use of the Criteria

(a) **Examples: creation of a category**

The students propose to put the following two problems into the same category:

"Evelyn divides a 1.50 meter long ribbon into two equal parts. How long is each piece?", and
"Three brothers share a sum of 375 francs equally. How much does each one get?"

T: "Why do you think they are alike?"
S: "Because something is being divided in equal parts."
T: "Still, there are some differences?"
S: "Yes, in this one it's ribbon and in that one it's money."
T: "And can you divide up ribbon the same way as money?"
S: "??? No, but for numbers it's the same."
T: "Find an 'intermediate' problem that shows the similarity: for example, replace the givens from one with the givens from the other one."

The students propose the problems:

"Three brothers share a sum of 1.50 francs equally. How much does each one get?" and
"Evelyn divides a 375 meter long ribbon in 3 equal parts. How long is each piece?"

The teacher accepts the similarity provisionally – this will be a category of "divisions" – but brings the number problem back up.

(b) **Creation of criteria**

T: "So, changing the numbers doesn't change the problem?"

The students are ready to think that changing the numbers does not lead to changing the operation.

T: "Let's take the same problem statement and change the numbers. What's going to happen?"

The remarks that follow can make it possible to clarify the effect of the magnitude of the numbers:

Some numbers are plausible and others are not (value of the givens): a 375 m ribbon is unusual, but a 375 km ribbon is impossible.
If Evelyn divides up a 1.523712 m long ribbon the problem is plausible, but the precision is ridiculous (representation of the givens.)
If the number of brothers were the decimal number 3.2, the problem would make no sense (nature of the numbers.)

The Classifications of Problems (Commentary 2008)
We classified problems about division of rationals using the following criteria:

Classification according to the material or symbolic manipulations carried out: long division of natural numbers, then decimal or rational numbers, measurements, exchanges, successive approximations, equal or unequal shares,…

Classifications according to special vocabularies
Arising from practical or professional activities (scales, percentages, rates)
Arising from applications (speed, physical density, etc.)
Arising from some cultural vestiges (fractional measurements)

Classification according to problematics[8]

Classification according to mathematical concepts,
Either classical (types of operations, fractions, ratios, proportions),
Or more current (order, topology, algebraic laws and structure, measure, scalar, function, …) which are the criteria maintained in the course.

[8] A *problematic* is something that constitutes a problem or an area of difficulty in a particular field of study [Oxford English Dictionary] The French use problematics more specifically to refer the set of questions posed in a science or philosophy with respect to some particular domain.

In Lesson 2, the first problems to classify are those that are familiar to the students. They arise from the conception of long division.

The next paragraph brings up the review of long division based on manipulations: sharing, partitioning, attribution and distribution lead to different strategies according to regular or irregular conditions (leading to the equalization of parts, for instance.) The term “division” unites certain of the conceptions, but not all (such as the search for a remainder.)

Following that, classification according to problematic leads to envisaging the calculation of the unknown term of a product or that of a component of a product measurement. Each conception leads to different manipulations which themselves bring up different modes of calculation. Classical teaching requires that children recognize division “naturally” as the concept common to these varied conditions and at the same time that they support this recognition with concrete arguments!

The extension of long division to division of decimal numbers happens naturally with the method of successive subtractions, which makes it possible to rediscover the reasoning and establish the algorithm for bracketing a number [see Module 5.]

The use and comprehension of division of decimal numbers are facilitated by its similarity to long division in the natural numbers. This is a recognized fact. But it is important to note that this facility hides a difficulty that is easy to observe, which itself hides an epistemological and didactical obstacle that is fundamental for the passage from the use of natural numbers to that of rational and real numbers.

Long division is based on the idea of measuring something using something smaller as the unit. If by considering only the whole number parts of the dividend and the divisor the student can conceive of the long division that solves the problem next door to the problem required, he can simply extend the algorithm by the calculation of decimal parts. For example, $17.4 \div 3.62$ is understood first in the sense of $17 \div 3$, the rest is a matter of the algorithm. This conception collapses if the long division indicated has no meaning. More explicitly, if the divisor is greater than the dividend, or if the dividend is less than two. The operation $0.4 \div 0.62$ is the case that gives rise the most difficulties.

Modules 14 and 15 make it possible to surmount this difficulty in conceptualization by means of a deep comprehension of the structure of rational and decimal numbers.

Module 13: New Division Problems in the Rationals

The first lesson continues the inventory of problems that was undertaken in module 12. The issue is first to have the classification completed by adjoining new problems where the division is defined by a rational linear mapping (i.e., one with a rational coefficient) expressed in any way.

The class should find there the notions studied in the preceding modules, and can make an inventory of them: a measurement divided by a scalar, a measurement divided by a measurement of the same type or of a different type, a scalar divided by a scalar, etc. The class can recall that a division also consists of finding a decimal expression (exact or approximate) for a fraction (module 7). This uses the idea that the result of a division expresses the measurement of the dividend if the divisor is taken as the unit: for example $12 \div 3$ expresses that if we measure 12 with 3 as a unit, the result is 4. Measuring 3 with 4 as a unit gives a result of 3/4, like dividing 3 into 4 parts, … and if the numbers are measurements in meters, then the result is __ meters.

This leads the students to understand that a fraction is the indication of a division that one neither can nor wants to carry out, but about which one can calculate.

(a) But they may also discover that the mappings sometimes pose difficulties. An example of these difficulties: the students know how to find the result of dividing one fraction by another as long as the first is the measurement of the thickness of a piece of cardboard and the second expresses the thickness of a sheet of the paper that makes up the cardboard. But what does $3/4 \div 2/5$ mean in general, in particular when the result is not a whole number? Interpreting this operation with the general idea that division is partitioning does not furnish a practical procedure. The equivalence of commensuration and partitions of unity always presents difficulties.

(b) Since problems often disguise the distinction – well known to our students – between ratios and linear mappings, the teachers propose problems of the nature of the following example – which does not figure in the 1985 Manual:

“A father is 5 times as old as his son. How old is the son if the father is 35 years old? How old will that father be when his son is 10 years old?” For the students, the issue is to recognize that the ratio between the age of the father and that of his son does not determine a linear function between their ages: the father will not be 50, but only 38. There is indeed a function, which fools the children, but it is a translation (+28). Clearly if the question had been the age of the father when the son is 35 years old the error would have been easier to detect.

(c) The study continues with the inventory of the roles of division in the study of a linear mapping: finding the correspondent (the image) of 1 when the coefficient is known, finding the ratio between the two values, calculating the coefficient of the mapping when one original and its image are known, calculating the original when the mapping (the coefficient) and the image are known, etc. The numbers are decimals or fractions.

At this stage, the students conceive of all linear mappings as multiplications (for example, $\times 3/4$).

Lesson 2 has them study linear mappings that are read as "divisions" and that will be understood as the reciprocal of multiplication by a number.

Lesson 2: (Extract) Division as Reciprocal Mapping of Multiplication (The Term Is Not Taught to the Students)

The session proceeds in the form of a sequence of problems that the students carry out rapidly. These problems provide the occasion for posing some mathematical questions. The teacher needs to make clear the distinction that he makes between these mathematical questions and the problems. The mathematical questions are the real object of challenges proposed to the students, the occasion for debates, and the real goal the teaching is aiming for. Problem statements, whether proposed by the teacher or by the students, are there only as means of treating those mathematical questions, or as applications of knowledge newly acquired or discovered.

Division (by a Number), a Linear Mapping

(i) **First problem statement:** "A movie ticket costs 35 francs. The total receipts of a theater are:

3,325 F on Monday	5,250 F on Friday
4,480 F on Wednesday	6,125 F on Saturday
3,675 F on Thursday	6,230 F on Sunday"

First question: How many paying customers were there on each of the days of the week?

The students make a table in which they place the results of their divisions by 35. The teacher asks if it is the result of a linear mapping. Students: "If we add up the receipts and divide that by 35 we ought to get the sum of the numbers of tickets" "If there is twice as much money it is because there are twice as many customers."

Division by a Fraction: Calculation of the Image

The issue is to find out how to divide by 3/8. To support their reasoning, the students must think of a problem. For example:

"I have to divide by 3/8 if I am looking for the number of 3/8 mm sheets it takes to make different given thicknesses, for example 6 mm, 9 mm, etc.

(a) Reasoning by an assumption contrary to fact:
If the sheets were 1/8 mm thick, we would need eight sheets to make a thickness of 1 mm, so we would need 48 sheets to make 6 mm. But the sheets are $3 \times 1/8$ mm, so these are three times thicker, so we need three times less (than 48) to make a 6 mm cardboard. So $6 \div 3/8 = 48/3 = 16$.

(b) Reasoning by equivalence:
6 mm is 48/8 mm, so the reasoning above produces
$48/8 \div 3/8 = 48 \div 3 = 16$.

(c) Using the reciprocal:
The reasoning that leads to a search for the unknown term of a product leads to sentences like: "If I had 3 sheets of 3/8 mm, they would have a thickness of 9/8 mm – there have to be more. 10 sheets → 30/8 – that's a little more than 3 mm – we need still more. It's 16, because $16 \times 3/8 = 48/8 = 6$ mm", which the teacher translates: "So we have to look for the number that you can multiply by 3/8 and get 6. $\_\_\_ \times 3/8 = 6$" and he draws the schema

$$6 \;\underset{\times\, 3/8}{\overset{\div\, 3/8}{\rightleftarrows}}\; \boxed{} \quad ?$$

With, admittedly, some effort, the teacher can then obtain a recollection of module 9.6: to find the object, you have to find the image of 6 by the reciprocal of $\times 3/8$.

The reciprocal of $\times$ 3/8 is $\times$ 8/3, so

$$6 \;\underset{\times\, 3/8}{\overset{\times\, 8/3}{\rightleftarrows}}\; 48/3 = 16$$

From these three methods, one can retain that each time, the student has multiplied the numbers whose image he wanted by 8, then divided by 3.

Division by a fraction is the reciprocal of multiplication by that same fraction (2008 Commentary)

Continuing studies of this nature with other examples was tried, but it is clear that this ambition is not very practicable without a veritable teaching of rules, without intense training with lots of exercises. Even if certain students are able to answer questions of this kind once – and most cannot – the reasoning is uncertain and painful. We explained this difficulty by the complexity and variety of material operations that concretize them and are necessary for verifying them. The solution of this problem is the object of the two modules that follow.

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To illustrate the difficulty of stating general principles, we report the following observation:

The students noticed that a multiplication $a \times b = c$ could give rise to two divisions: $c \div a$ and $c \div b$. But when one of the numbers is a measurement and the other is a scalar (a ratio or a coefficient) one of these divisions may not correspond to a mapping, especially not a familiar mapping.

Example: rate x principal = interest. The mapping principal → interest (rate fixed) and its reciprocal are more frequently envisaged than the mapping rate → interest (principal fixed). At the end of this activity, a lot of the students can answer the question: "One student says 'the reciprocal of (× 7/13) is (÷ 7/13)'. Another one says 'No! The reciprocal of (× 7/13) is (× 13/7)!' Which one is right?" But very few can "prove" it with a calculation.

Lesson 3 brings up various methods of getting the students to consider proportional mappings. A "portrait game" trains the students to recognize and characterize them. A display – the "tapestry of proofs" – helps the students during collective discussions. It enables them to follow and to determine, at any moment of a debate, who claims what and who is supposed to prove what. The aim is to help the teacher to lead the class progressively to distinguish logical argumentation from purely rhetorical exchanges. This design, which was too complex and had insufficient a priori study, never led to any lesson projects that were satisfactory enough to be realized.

The students, with the teacher, pull together and summarize what they now know.

Extracts from the Original Text

Remark: Not every student can achieve a level of comprehension of these questions sufficient to be able to produce individually the proofs sketched below. The proofs should not be required as skills to acquire. Furthermore, the "rules" proposed by the students should not immediately be institutionalized. They should remain in doubt – that is, something to be verified, either by a calculation that uses the representations used in the proofs, which might not be general, or by previously established results.

Conclusions drawn with the students:

1. "We are going to make an inventory of the different ways of writing a linear mapping and of writing its reciprocal.

We have seen that we do the same thing no matter what the numbers are, so let's choose some numbers to work with.

The linear mapping is given by an ordered pair:

$$14 \longrightarrow 27$$

If we want to express it as a multiplication, the mapping is × 27/14
The reciprocal mapping is given by the ordered pair,

$27 \longrightarrow 14$ which is the mapping × 14/27.

If we want to use division to express

$$14 \longrightarrow 27$$

we find the division by looking at the reciprocal expressed as a multiplication. The reciprocal is the mapping × 14/27, so the original mapping is ÷ 14/27.
Let us present these results in a table, with a and b being two random numbers."

Different ways to designate a linear mapping:

Linear mapping	Reciprocal mapping
$a \longrightarrow b$	$b \longrightarrow a$
$1 \longrightarrow b/a$	$1 \longrightarrow a/b$
× b/a	× a/b
÷ a/b	÷ b/a

To summarize this table, all we have to remember is that

$$\boxed{\times\ b/a = \div\ a/b}$$

1. Shrinking and enlarging by multiplying and by dividing:

(a) "We found some mappings that shrank and some that enlarged, all of them expressed as multiplications. Can you give me some?"

The students talk about this apparent paradox, which they encountered in the lessons on the "Optimist" (Module 9), and which surprised them considerably. At that time they had remarked that "Up to now we thought that multiplying always made things bigger, because the only numbers we knew about were numbers bigger than 1!"

They propose mappings (which the teacher writes on the board), at the same time classifying them into two categories: those that enlarge and those that shrink (the teacher might add a few, too.)

× 2/5; × 1/2; × 12/7; × 4/4; × 7/5; × 1; × 1/4; × 1.4; × 0.95; × 2.75; …

The ones that enlarge	The ones that shrink
× 12/7; × 7/5; × 2.75; × 1.4;…	× 2/5; × 1/2; × 1.4; × 0.95;…

That leaves × 1 and × 4/4, which call for a reminder from the "Optimist" section: "We saw that if we made a reproduction using × 1 we got the original back!"

Conclusion: the teacher has them explicitly express the conclusion: "The mappings that shrink things are expressed as multiplication by a number less than 1."

(b) By dividing:

"We just recalled that we can shrink a model using a multiplication. Is it possible to enlarge a model using division?"

To increase their comprehension, the teacher suggests that the students find a translation like the one they are in the habit of using in such cases: looking for the image of 1.

$$1 \longrightarrow 7$$

What is the mapping that is expressed as a division and lets us multiply by 7 (or enlarge 7 times)?

The students suggest writing x 7, but that doesn't answer the question that was asked. To help them, the teacher asks, "What can we divide 1 by to get 7?"

$$1 \xrightarrow{\div\ ?} 7$$

"It's a number less than 1, because when we divide 1 by it we have to get back to 7. So it must be 1/7."

$$1 \xrightarrow{\div 1/7} 7$$

The teacher writes: $1 \xrightarrow{\div\ ?} 5/2$ and then asks, "Can you find some more mappings?" But using the same system, the students only find divisions of the form ÷ 1/n. So the teacher asks them to find a division mapping that takes 1 to 5/2:

As before, they first say "It's × 5/2!" "How can we write it as a division? See if you can find other ways of writing this mapping." The students remember that they had just learned that × 5/2 is the same as ÷ 2/5 (they give the proof: "The reciprocal of × 5/2 is × 2/5, and the reciprocal of × 5/2 is ÷ 5/2. So × 2/5 is the same as ÷ 5/2")

As in the activity before, the teacher has them produce a collection of mappings expressed as divisions and classify them by whether they enlarge or shrink. This they do, observing also that ÷ 4/4 does not change the model.

Conclusion: the teacher has them explicitly express the conclusion: "The mappings that shrink things are expressed as division by a number greater than 1, and those that enlarge are expressed as division by a number less than 1."

Commentary

It might be useful to recall that history has attempted to make of the concept of fraction the universal instrument of measurement and of treatment of proportionality. But in the end this attempt has failed. Today the concept is a mosaic of a plethora of particular expressions in an environment of metaphors that are neither general, nor well adapted to the physical manipulations that they claim to represent, nor well adapted to a general mathematical treatment. An ambition of the reforms of the 1970s was to erase this obstacle a little, but it still holds a major position in our cultures and our practices.

A visit to the field of applications of proportionality traditionally occupied a major part of the program of mathematics. In the 1970s this field was greatly reduced in the curricula of the period. Not wanting to lose any of this essential educational project, we tried to obtain equivalent knowledge with the students, but with fewer lessons specific to different fields and more mathematical reflections, and with a small dose of meta-mathematical and heuristic reflections, on condition that they be formulated by the students and not set up as methods. We would encounter the terms of proportionality on occasion, but we would replace them with the mathematical terms introduced in the lessons.

We knew already that our curriculum (modules 1–11, 14 and 15) brought real improvements to the teachers' and students' possibilities for dealing with applied problems. We also knew already that with the usual conceptions and didactical practices the use of arrows that we had introduced risked provoking a formalist drift and a metadidactical slippage if that use expanded beyond the terrain of the experiment. We then wanted to know what the effect of our mathematical introduction (modules 1–11) would be on problem solving, before the homogenization (the identification of a/b with x a/b.) So we put a first exploration of problems (modules 12 and 13) before the last two modules, which we continued to teach as we had before. Note that this study itself constituted a metadidactical slippage that had to be closely monitored (we expand on this notion in Chap. 5.)

The additional modules were optional. That was a part of the experimental plan. To study the effects of our variations, it was important to maintain the teaching conditions that characterize the whole process under study. It was fundamental to our research plan that rather than the usual practice of evaluating how much

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material it was possible for the students to learn, we set a fixed goal in terms of the content and then compared the efforts and time spent by the teacher and the students to surmount the difficulties within that content. It was thus necessary to let them react to the difficulties, whether by reasonable supplementary effort or possibly by giving up, and these would constitute our indicators.

It was possible for studying and classifying problems to come too early. The study benefited as in the preceding design from the good mathematical knowledge developed in the first 11 modules. But the questions of proportionality had become more difficult to collect and master because of local singularities that appeared there. We observed that augmenting the collection of classical problems presented in different environments resulted in an increase in the volume of the vocabulary and metaphors brought into play and an improvement in the execution of algorithms, but also in a diminution of the students' capacity to verify and explain their calculations.

The phenomenon seemed all the more marked in that we had taken a lot of care to get students to use manipulations, formulations and explanations that were more precise and better based on their actions. Classical knowledge about fractions passes for "concrete". In fact, it consists of metaphors, verbal connections and cultural habits often stripped of real concrete meaning, unlike the knowledge that we developed. The complexity of the concept comes from its roots in the culture, the explosion of the collection of meanings and the absence of a sufficient use of unifying instruments.

Later on the teachers did not always maintain the insertion of the classification of problems before the last two modules, but it did make the last two appear to be a clarification necessary for the teachers and for the students.

Owing to a shortage of researchers and observers, the complementary sessions on the study of problems could not be undertaken with the normal and necessary scientific environment of the COREM. The effects of these modules, when they were carried out, therefore could not be collected and analyzed.

Modules 14 and 15 return to and complete the mathematical study suspended since the end of Module 10. The students unify the conceptions of multiplication and division of fractions and rational numbers that they have learned, by putting them back into the multiplicative group of the rational numbers.

Module 14: Composition of Linear Mappings

Lesson 1: The Pantograph

Materials[9]

One pantograph for every pair of students (the pantographs have different scales)
At least 4 sheets of unlined paper per pair of students
Tape to stick the paper to the desk
One eraser per pair of children

The pantographs are distributed before the lesson begins (Fig. 2.9).

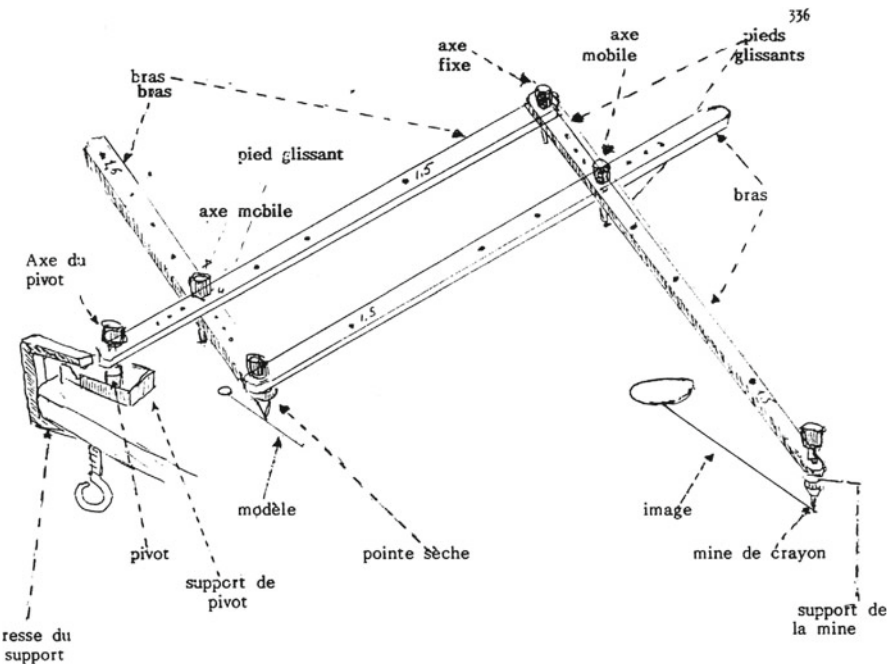


Fig. 2.9 The pantograph

[9]This activity can take place in the context of a Social Studies class – a drawing, for example.

Introduction of the Pantographs

(a) Instructions: "These gadgets are pantographs. Some of you may have seen them or even used them. They let you reproduce designs. So you are going to make a design on one of your sheets of paper and reproduce it on the other one."
(b) Development: The children work in pairs. There is always a moment of hesitation (as is often the case when children are confronted with a new situation or an instrument that they have never used.) But very soon they organize themselves, make a design and figure out how to use the pantograph: the pivot mustn't move, nor the paper (which is taped down), the pointer follows the model, the pencil draws the image.

As soon as they have produced a few designs they ask to modify the form of the pantographs, and the teacher says they may. They modify the scale, start drawing again, are surprised by some of the modifications and amused by others.

This free manipulation of the pantographs can hold their full attention for 30 or 45 min.

Discussion of observations

After this playful phase, the teacher gathers them to make a collection of observations based on the designs and their reproductions.

(a) Instructions: "What did you notice?"
(b) Process: The children make remarks and hypotheses:

- You can enlarge or shrink by exchanging the pencil and the pointer.
- The shape of the image doesn't change no matter how you set up the pantograph.
- The "enlargement" or "shrinkage" varies with the scale of the pantograph.
- The numbers beside the holes indicate the amount of enlargement or shrinkage.

This remark is immediately verified by the children who made it: they display their model and its reproduction, measure a segment on the model and the corresponding segment on the reproduction and write the measurements on the board.

	Design (cm)		Reproduction (cm)
Example:	3.2	$\longrightarrow$	5.4

And calculate 3.2×1.5 (if 1.5 is the number by the hole on the pantograph)

$3.2 \times 1.5 = 5.25$!

General astonishment! "They made a mistake!" The teacher suggests that all the students check the operation – notebooks, calculations … it really is 5.25!

So there is a 1.5 mm error!

The children make comments: "It's not a big error!" "It's bound to happen, because the pantograph isn't very precise." Many of them then want to check whether their own reproduction was better realized, and by pairs they verify using the above procedure. They get quite competitive: each one hopes to have succeeded better than the others!

If the pantograph isn't set up as a parallelogram, the image is deformed.

It always happens that one group sets up its pantograph without paying attention to the numbers by the holes. When the time comes to share their results they are a little embarrassed to present their designs because sometimes they haven't figured out what caused the deformation. The class often gets a good laugh out of the reproductions, which can be bizarre shapes.

Others who figured it out in the phase of free manipulation come to the aid of their comrades by saying that the same thing happened to them, but they noticed that the scales were badly set up and fixed them.

Results All the children know how to use the pantograph. They also all understand the remarks that were made and know thereafter how to get what they want out of a pantograph.

Lesson 2: Composition of Mappings: First Session

Materials

Two pantographs per group of two or three students, one set to enlarge by a factor of 3, the other by a factor of 1.5

Three sheets of paper of different colors per group and for the teacher.

Presentation of the Situation

"On the back of this white piece of paper I made a design. Then, with this pantograph I reproduced the design on the blue sheet of paper. Then finally I reproduced the design on the blue paper on this yellow paper using this pantograph."[10] (The designs are on the backs of the pages, so the students don't see them.)

1. Qualitative predictions
   "What can you guess about these designs? What can you say without seeing them?"

   The teacher can count on the following answers: "They will look like each other" "They will be enlarged or shrunk", "You have to see how the pantograph is set up."

   So the teacher demonstrates where the pointer and the pencil are on the two pantographs. The children then say "The designs are enlarged. The yellow one is the biggest."

[10]The teacher's pantographs are also set to scale factors of 3 and 1.5.

2. Quantitative predictions
   "In a moment you are going to do the same thing: you will make a design on the white paper, reproduce it on the blue paper with the first pantograph, then reproduce the one on blue on the yellow paper using the other pantograph.
   But first, I am going to give you two dimensions of my model:

   4 $\longrightarrow$
   2.5 $\longrightarrow$

   (The teacher writes these measurements on the board.)
   Can you predict the corresponding dimensions on the yellow paper?"
   The students say that they need more information and request either by how much the pantographs enlarge or a corresponding dimension on the yellow paper.

Presentation of a Game: First Try

(a) *Assignment*: "You are going to play a game: in your notebook you are to write the information that you want. I will give it to you. After that it is a matter of making predictions: you choose some numbers that designate measurements on the model and you predict the lengths of the corresponding segments on the yellow sheet.
   You must write these numbers in a table.

   Example:

   4 $\longrightarrow$
   2.5 $\longrightarrow$
   3 $\longrightarrow$

   When you have predicted the corresponding measurements, you can verify if your prediction is right by using the pantographs.

   If you choose a whole number and the prediction is right, you get one point.
   If you choose a decimal number and the prediction is right, you get three points.

(b) Development:
   The children work in groups of two or three. The teacher suggests, if they haven't thought of it, that each one calculate a measurement (because the points can be added up). That way they can have more because in the groups the children have a tendency to calculate the same measurement together, which slows the calculations and at the same time limits their number.
   While the students are making their predictions, the teacher prepares the following table:

	Predictions		Correct predictions	Points
	Whole	decimal		
Group 1				
Group 2				
Group 3				
Group 4				
Group 5				

(c) Verification, done in two parts:

1. First collectively for 4 and 2.5: the teacher has one child from each group come to the board.

   One of the children draws a 4 cm line on a white paper; another uses the pantograph to make the first image on blue paper; another does the second image on yellow paper. Finally still another measures the images and gives the results to the teacher, who puts them on an enlarged representation on the board:

   $$4 \longrightarrow 12 \longrightarrow 18$$

   The teachers asks, "Who got these measurements?" Often there are errors caused by the pantograph, which gives rise to discussions. A consensus is established: predictions that are within three tenths will be accepted: for example, if they predicted 18 and came out with 17.7 or 18.2 their prediction would count as correct.
2. For the other predictions, verification is done in each group with pantographs. A child from a concurrent group comes to check. If the activity takes too long (because the children are still not very adroit at using the pantographs) the teacher can switch to a simple collective verification like the ones for 4 and 2.5.

   The teacher collects the results in the prepared table and scores the points.

Game: Second Try

(a) *Instructions*: "You saw that it was possible to predict the measurements of the last image with calculations, some slow, some fast. You are going to try a second time. You will discuss with your group how to find the speediest way to calculate the results, so that you can make the most predictions possible. Here is a list of numbers. You are to choose the numbers on this list that you want – as many as possible of them. You can even add some if you are very swift and if you want to."

$$4 \longrightarrow \qquad \longrightarrow$$

$$2.5 \longrightarrow \qquad \longrightarrow$$

$$6 \longrightarrow \qquad \longrightarrow$$

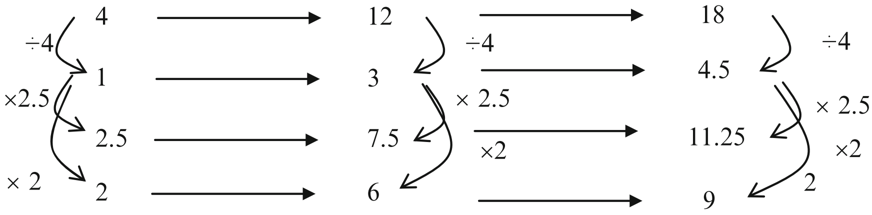
$$2 \longrightarrow \qquad \longrightarrow$$

$$5.1 \longrightarrow \qquad \longrightarrow$$

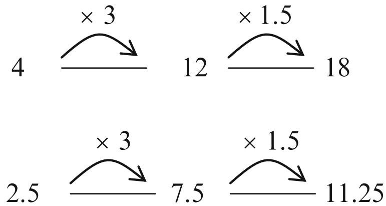
$$14.6 \longrightarrow \qquad \longrightarrow$$

$$2.25 \longrightarrow \qquad \longrightarrow$$

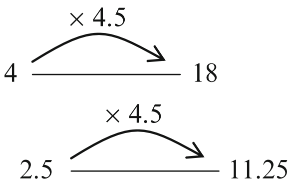
(b) Development: The children continue to work in groups, dividing up the work.
Collective synthesis: correction of results, inventory of methods
(a) Correction of results. The teacher takes the results the children have found and puts them on the board, correcting them in the process, then gives out the points in a way that lets the children know which team won. After that the class makes an inventory of the methods they used:
(b) Inventory of methods: One child from each team comes to the board to explain his method. Clearly not all will come, because after each demonstration, the teacher asks who else used the same method.
First method observed: Calculation of intermediate values by linearity



Second method: Calculation of intermediate values as a product:



Third method: No intermediate calculations:



Obviously, the children who used the third method were able to calculate all the results quickly, while those who used the two others, especially the first, didn't often get to the end of the list. They recognize that the last method is the fastest, but there are always a few who ask: "Why did you multiply by 4.5?", to which some answer "Because $3+1.5=4.5$" and others say "What we did was to multiply 3 by 1.5". The latter is generally not accepted, however, because the children can see immediately without calculating anything that $3+1.5=4.5$, whereas to multiply 3×1.5 they have to carry out a calculation (even if it is a mental one.) The problem therefore stays open.
The teacher writes on the board the following conclusion:
(× 3) F (× 1.5) = (× 4.5), where the F stands for "Followed by"
(c) Open problem: "Can you predict what enlargement you'll get from two pantographs set to 3.5 and to 2? Think about it and give your answer next time."

A very anarchical discussion takes off among the children: some think it is 5.5, others disagree. The teacher doesn't take part, and tells them to think about it for the next day and above all to find a proof that what they are saying is true. The session ends in a state of suspense that excites the interest of the children and sets them up for the next activity.

> **Results** The children have composed two linear mappings. They have anticipated the result, found several methods, and chosen the shortest. They have discovered that they can cut down on calculations by replacing two linear mappings by some linear mapping, but they don't know yet how to calculate it.

Lesson 3: Composition of Linear Mappings: Designation of Composed Mappings

Search for a solution to the open problem and validation

1. Review of preceding activity by the teacher.
   "Last time we enlarged a model with the pantograph set to 3. Then we enlarged the first image with a pantograph set to 1.5. We saw that the enlargement that would let us go straight from the original to the second image was (×4.5) and we wrote that

$$(\times 3)\ \mathrm{F}\ (\times 1.5) = (\times 4.5)$$

2. Open problem
   (a) *Instructions*: "At the end of the last class, I gave you the following assignment: I used the pantograph to make the enlargement (× 3.5) F (× 2) and I asked you what linear mapping could replace these two mappings. If you found a solution, write it in your notebook."
   (b) *Development*: The children write an answer. After two minutes, the teacher asks what answers they wrote in their notebooks, and writes them on the board. There are always two:

$$(\times 3.5)\ \mathrm{F}\ (\times 2) = (\times 5.5)$$

$$(\times 3.5)\ \mathrm{F}\ (\times 2) = (\times 7)$$

   So they proceed to a collective verification.
   (c) *Verification*: "How can we know which one is the right answer?"
   The children propose to verify the mappings with whole numbers or decimals.
   Verification on a whole number measurement, for example 8:

$$\begin{tikzcd} 8 \arrow[r, "\times 3.5"] \arrow[rr, bend right, "\times 7"'] & 28 \arrow[r, "\times 2"] & 56 \end{tikzcd}$$

The mapping that gives 56 as the image of 8 is indeed × 7. The teacher then has the students calculate what the image of 8 would be under the mapping × 5.5. That one gives 44, which doesn't correspond to what they got using intermediate steps. The students thus see clearly that

$$(\times\ 3.5)\ \mathrm{F}\ (\times\ 2) = (\times\ 7)$$

Verification on a decimal number, for example 1.5

The students first calculate the values using the two steps × 3.5 and × 2. Then the teacher has them calculate 1.5×7 and 1.5×5.5, and again the former corresponds to the image found and latter doesn't. This solidifies the conclusion that

$$(\times\ 3.5)\ \mathrm{F}\ (\times\ 2) = (\times\ 7).$$

(d) Rule of composition for two mappings

The teacher has them formulate the rule that they found after these two verifications:

"To find the linear mapping that replaces two linear mappings, you have to multiply the mappings."

Verification on a very simple measurement: 1

The teacher asks the children if they couldn't verify the same thing but avoid messy calculations where they might make mistakes.

"What's the really simple measurement that you could start with and verify that the mapping you found is right?"

A few children suggest 1 (if nobody thinks of it, the teacher suggests it) and they try it right away. One of the students who proposed it comes to the board and writes:

$$\begin{tikzcd} 1 \arrow[r, "\times\ 3.5"] \arrow[rr, bend right, "?\ \times\ 7"'] & 3.5 \arrow[r, "\times\ 2"] & 7 \end{tikzcd}$$

Verification of the rule for any sequence of enlargements or reductions

a) *Instructions*: "Now you know how to find a mapping that lets you replace two successive mappings. But does that work if you have more than two mappings? To know that, you are going to calculate a lot of examples that I am going to write on the board and after that you can say whether the rule is general."

Examples:

(× 1.75) F (× 1) F (× 0.5)
(× 3) F (× 2.5) F (× 1.75)
(× 0.125) F (× 5) F (× 1.5) F (× 2)
(× 4.5) F (× 0.2) F (× 0.7)
(× 2.7) F (× 4.52) F (× 0) F (× 0.425)

Development: The children do the calculations in groups of two or three. The teacher assigns one or two series of mappings to each group, who are to replace the series by a single mapping, proving that it is correct.

c) *Collective correction*:

One student from each group goes to the board to show she has done, under the critical eye of the others, who follow the calculations attentively. They choose 1 as the measurement to start with.

$$
\begin{tikzcd}
1 \arrow[r, "\times 1.75"] \arrow[rrr, bend right, "\times 0.875"'] & 1.75 \arrow[r, "\times 1"] & 1.75 \arrow[r, "\times 0.5"] & 0.875
\end{tikzcd}
$$

First example: (× 1.75) F (× 1) F (× 0.5) = 0.875

The rest of the examples are done the same way. The teacher takes the opportunity offered by this correction process, which gives the students no trouble at all, to get the students to discover the properties of these compositions of mappings: commutativity, associativity, the role of 1, the role of 0, etc. (Many of the children, in fact, start in on long, complicated calculations before noticing that there is a (× 0) in the course of the mappings.)

The teacher points out that no matter what the mappings are (enlargements or reductions), and no matter how many of them there are, they can always be replaced by a single mapping by multiplying them all together.

She points out to the children that these multiplications are different from the ones they already knew (cf. Module 8, activities 2-4 – finding the image of a measurement under a decimal number mapping.)

Individual exercises:

13.4 (× 3.5) F (× 1.5) F (× 4) $\xrightarrow{\quad ? \quad}$

25.86 (× 3.5) F (× 1.5) F (× 4) $\xrightarrow{\quad ? \quad}$

11 (× 3.5) F (× 1.5) F (× 4) $\xrightarrow{\quad ? \quad}$

In the course of correcting these the teacher takes note of the methods. There are still children who carry out the intermediate calculations, thus making mistakes and proceeding much less swiftly than those who go directly to the mapping (× 3.5 × 1.5 × 4). This then provides an occasion for the children to become conscious of the utility of replacing several linear mappings with a single one and of making use of the rules that they learned in the course of the activity.

Results This activity presents no difficulties at all. It not only gives the children a chance to multiply some decimals and rediscover the meaning of this multiplication, but permits them, thanks to the rules discovered, to save some calculations and to design new mappings.

Lesson 4: Different Ways of Writing the Same Mapping

Materials
One pantograph
Review of the rule of composition for mappings

(a) *Instructions*: Here is a sequence of linear mappings:

$$(\times\ 1.5)\ \mathrm{F}\ (\times\ 2)\ \mathrm{F}\ (\times\ 2.5)\ \mathrm{F}\ (\times\ 3)\ \mathrm{F}\ (\times\ 4)$$

Can our pantograph make that enlargement? If so, what linear mapping could one substitute for the sequence of enlargements?

(b) *Development*: The children work on their own in their scratch notebooks, working as fast as possible. The teacher invites them to find the fastest possible calculations.

(c) *Correction*: After 3 min, there is a collective correction. For that, the teacher sends one student to the board to write

$$(\times\ 1.5)\ \mathrm{F}\ (\times\ 2)\ \mathrm{F}\ (\times\ 2.5)\ \mathrm{F}\ (\times\ 3)\ \mathrm{F}\ (\times\ 4) = 1.5 \times 2 \times 2.5 \times 3 \times 4.$$

The child explains how he did it and very often the others propose various solutions from their places:

"What I did was to do everything in my head that I could:

2 times 1.5 makes 3
3 times 3 makes 9
9 times 4 makes 36

and all that's left is to multiply 36 by 2.5, and you can do that in your head, too:

2 times 36 makes 72; half of 36 is 18, 72 plus 18 makes 90!"

Another one says "I started off with 3 times 4, that makes 12, then 2 times 12, that makes 24. 24 multiplied by 1.5 makes $24 + 12 = 36$, and all that's left is 36 times 2.5"

This goes on until all the procedures have been stated. This development lets all the students discover all the possible methods of doing the calculation and find the fastest.

At the end, someone writes on the board

$$(\times\ 1.5)\ \mathrm{F}\ (\times\ 2)\ \mathrm{F}\ (\times\ 2.5)\ \mathrm{F}\ (\times\ 3)\ \mathrm{F}\ (\times\ 4) = (\times\ 90).$$

Different ways to write the same mapping

(a) Presentation of the problem and instructions:

"What can the pantographs that you have been using do? (They can either enlarge a model or shrink a model). You have already used these different possibilities. So you should be able to give the meaning of:

'Shrink by 3'.

Would you know how to do that with this pantograph and write in your notebook what this mapping does?"

(b) Development

The teacher gives the children a moment to think and then asks two of them to come an carry out with the pantograph, in front of all the class, a reduction by 3.

They set the pantograph to 3 and put the pencil between the point and the pivot, helped if necessary by remarks from other children. Then the teacher has them draw a 9 cm line segment on a piece of paper which she then tapes to the board and with the help of the pantograph they have set up, they draw the image of the segment. The class makes comments out loud:

"The image is 3 cm long because it should be 3 times smaller than the model. Maybe it's not quite exact on the design…"

The teacher reminds them of the last question he posed: "Would you know how to write in your notebook what this mapping does?" and lets the children think about it a minute or two. Then one of them comes to the board and writes:

$$9 \xrightarrow{\div 3} 3$$

The teachers adds some measures and asks the students to complete the following table:

$$\begin{array}{ccc} 9 & \xrightarrow{\div 3} & 3 \\ 6 & \longrightarrow & 2 \\ 3 & \longrightarrow & 1 \\ 1 & \longrightarrow & 1/3 \end{array}$$

He proceeds to a rapid collective correction and detaches the last pair on the board:

$$1 \longrightarrow 1/3$$

"How could we designate the mapping that takes 1 to 1/3?"
The children spontaneously answer "(× 1/3)"

$$1 \xrightarrow{\times 1/3} 1/3$$

But they have to check that it really is the same mapping, if it works also with the preceding measurements: 6, 3 and 9, which a child promptly does:

$$6 \xrightarrow{\times\, 1/3} 6/3 = 2$$

$$9 \xrightarrow{\times\, 1/3} 9/3 = 3$$

It gives the same images as the mapping (÷ 3). So we can write (÷ 3) = (x 1/3)

Conclusion

The teacher says: "This mapping is called "dividing by 3" or "multiplying by 1/3". It can be written as "divide by 3" or as "multiply by 1/3". There are lots of ways to write it.

Other names for the mappings ÷4 and ÷2

(a) Instructions: "We are going to try to write some other mappings in a bunch of ways. What reductions can we make with our pantograph using only whole numbers?

– You can shrink by 4 or by 2

"How can we write what those mappings do? Who knows how to find several ways to write what they do?"

(b) Development: The children work on it a moment in their scratch notebooks. Each one makes it a point of honor to find a different name. The teacher proceeds quickly to a collective correction so as to keep the interest lively.

Children take turns coming to the board to write:

$1 \xrightarrow{\times\, 1/4} 1/4$	The mapping is (× 1/4)
$4 \xrightarrow{\div\, 4} 1$	The mapping is (÷4)

"Could we find another one by replacing the fraction 1/4 by a decimal number?"
The children calculate quickly in two different ways:
First way (most often used): 1/4 = 25/100 = 0.25
Second way: by division 1 ÷ 4 = 0.25

(c) Validation of the names

Instructions: "We just found three different names: (÷ 4), (× 1/4) and (× 0.25) Now we need to check whether they really are the same mapping. So we are going to apply them to some other numbers: 2.5, for instance."

Development The teacher suggests that the class divide up the tasks to save time: one row calculates with (÷ 4), one with (× 1/4) and the third with (× 0.25).

Correction: One child from each row comes to the board and writes the calculation for their row's mapping of 2.5. All find 6.25.

So the mappings really are all the same, because they give the same image. So we can write:

$$(\div 4) = (\times 1/4) = (\times 0.25)$$

This equation is written up and left on a corner of the board or on another board.
The teacher goes through all the same steps for reduction by 2, getting (÷ 2), (× 1/2), and (×0.5), and having them check by applying all three to 7.8.
He writes beneath the previous equation:

$$(\div 2) = (\times 1/2) = (\times 0.5)$$

Other examples: generalization

(a) *Instructions*: "If you had a pantograph that shrank things by 5, or 6, or 9 do you think you could find other names for those mappings?"
(b) *Development*: The children work in their scratch notebooks. Some of them, by analogy, immediately write

$$(\div 5) = (\times 1/5) = (\times 0.2)$$

Others still need to calculate the long way:

$$\begin{tikzcd} 5 \arrow[r, bend left, "\div 5"] \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r, bend left, "\div 5"] \arrow[r, bend right, "\times 1/5"'] & 1/5 \end{tikzcd}$$

When it comes to (÷ 6) = (× 1/6) = ?, they hesitate because they observe that "It doesn't come out right!" The teacher decides with them that they will write it as

$$(\times 1/6) = (\times 0.1\underline{6}\ \ldots)$$

Use of the different names

To finish up, the teacher organizes a brief session of mental calculation. He writes on the board the following operation:

$$4 \times 0.25 =$$

and gives them 20 seconds to figure it out without writing it in vertical format.
The students hesitate, start the multiplication mentally and protest when the teacher stops them after 20 seconds. Only one or two have found the answer.

The teacher points out the equalities that are still written on the board: nobody had thought of replacing (x 0.25) by (÷ 4)

$$4 \div 4 = 1.$$

The students catch on and ask for some more calculations to do. The activity finishes up as a real game.

$$18 \times 0.5 = ?$$
$$25 \times 0.2 = ?$$

Results The students know several ways to write the same mapping.

Lesson 5: Rational Linear Mappings

Presentation of the Problem

(a) *Instructions*: "What are the whole number linear mappings you can do with our pantograph?"

The teacher writes on the board, as directed by the students,

$$(\times 2)\ (\times 3)\ (\times 4)$$
$$(\div 2)\ (\div 3)\ (\div 4)$$

"I'm going to take the pantograph that enlarges by 3 and draw an image with it. Then with the pantograph that shrinks by 2 I will shrink the image I got and make a second image. Do you know how to write what I did?"

The class answers: "You multiplied by 3 and then divided by 2." And one of them is invited to write these two successive mappings on the board:

$$\begin{tikzcd} {} \arrow[r, bend left, "\times 3"] & {} \arrow[r, bend left, "\div 2"] & {} \end{tikzcd}$$

(b) *Problem posed*: "Can we combine these two linear mappings to get a fractional or decimal mapping?"

$$\begin{tikzcd} {} \arrow[r, bend left, "\times 3"] \arrow[rr, bend right, "?"'] & {} \arrow[r, bend left, "\div 2"] & {} \end{tikzcd}$$

(c) *Development*: This is done collectively with the teacher. The children suggest taking a number and naturally choose 1 (because that is the number that has always had priority.)

The Teacher Writes

$$1 \xrightarrow{\times 3} \quad \xrightarrow{\div 2}$$

and asks a child to come to the board and complete the diagram, putting in the intermediate numbers:

$$\begin{tikzcd} 1 \arrow[r, "\times 3"] \arrow[rr, bend right, "\times 3/2"'] & 3 \arrow[r, "\div 2", "\times 1/2"'] & 3/2 \end{tikzcd}$$

(÷ 2) is replaced by (× 1/2)

the mapping is (× 3/2)

The teacher writes up the conclusion: (× 3) F (÷ 2) = (× 3/2) = (× 1.5), the last having been rapidly calculated by the children.

It is thus possible to replace two whole number linear mappings with a fractional or decimal linear mapping.

A search for all the rational linear mappings the pantograph can produce

(a) Instructions: "Use the same method to find everything else that we could do with our pantograph by combining all the whole number mappings."

(b) Development: The students work a little while in their scratch notebooks. After 5 min, the teacher asks them to come write on the board what they have found. This gives a sequence of mappings:

$$\xrightarrow{\times 3} \qquad \xrightarrow{\div 3}$$

$$\xrightarrow{\times 3} \qquad \xrightarrow{\div 4}$$

$$\xrightarrow{\times 2} \qquad \xrightarrow{\div 2}$$

$$\xrightarrow{\times 2} \qquad \xrightarrow{\div 3}$$

$$\xrightarrow{\times 2} \qquad \xrightarrow{\div 4}$$

etc.

Making a Table

(a) *Instructions*: The teacher suggests making a double entry table to avoid missing anything or duplicating anything.

	× 2	× 3	× 4	÷ 2	÷ 3	÷4
× 2						
× 3						
× 4						
÷ 2						
÷ 3	x 2/3					
÷ 4						

He tells the children to complete the table, checking their results each time.

Example: (× 2) F (÷ 3)

$$\begin{tikzcd} 1 \arrow[r, bend left, "(\times 2)"] \arrow[rr, bend right, "\times 2/3"'] & 2 \arrow[r, bend left, "(\div 3) = (\times 1/3)"] & 2/3 \end{tikzcd}$$

(b) *Development*: The children work individually. They complete the table they have drawn in their scratch notebooks and don't write in a result until they have checked it as before.

Remarks: While they are filling in their tables they often make comments out loud:

"You don't always get a fraction!"
"You get some things we already learned!"

(They are talking about (× 4) F (÷ 2), for instance, which they saw at the beginning of the year in an activity on functions corresponding to operations on natural numbers.)

(c) *Correction*: After 5–8 min of individual work, the teacher organizes a collective correction. The children take turns coming to the board to fill in the table that the teacher has drawn for them.

In the course of doing it they make more comments like

"If you do (× 3) F (÷ 3) or (× 4) F (÷ 4) it's as if you did (× 1) or (÷ 1)"
"If you do (× 3) F (÷ 2) = (× 3/2) it's the same thing as if you did (÷ 2) F (× 3) = (3/2)"

This way they discover the commutativity of mappings. They verify that it is true for all cases, using the usual format.

After checking all of the entries, the students formulate the rule for composition of mappings:

"Any decimal or fractional linear mapping can be gotten by doing two whole number mappings in a row. The number that is multiplied is always on top, and the number that is divided is on the bottom."

Application exercises, done individually in mathematics notebooks

Instructions:

1. Find the rational and decimal linear mappings when it can be done, to replace two whole number mappings:

   Example: (× 7) F (÷2) = (× 7/2) = (× 3.5)
   (÷ 5) F (× 4) =
   (× 8) F (÷ 5) =
   (× 4) F (÷ 5) =
   (× 12) F (÷ 12) =
   (÷ 5) F (× 5) =

2. Find the mapping that is missing in each of these:

   (× 5) F ( ) = (× 5/3)
   (÷ 4) F ( ) = (× 7/4)
   ( ) F (× 2) = (× 2/9)
   ( ) F (× 3) = (× 1)
   (÷ 5) F ( ) = (× 1)

Results This activity gives the students no trouble at all. They all understand and know how to do the individual exercises. There will, however, be some errors to correct. The activity finishes with a collective correction (which can be done at the beginning of the next session if time runs out.)

Module 15: Decomposition of Rational Mappings. Identification of Rational Numbers and Rational Linear Mappings

Lesson 1: Decomposition of Rational Mappings

In this activity the teacher asks questions that have not been directly addressed, but for which the students can almost instantly find an answer (Socratic *maieutique*)

Decomposition of a Rational Mapping into Natural Number Mappings

Teacher: "I want to carry out a 3/4 enlargement (which is a diminution) but I don't have a pantograph that does × 3/4. With the pantographs we have would we be able to do the enlargement?"

The teacher takes care not to have his request confused with an "enlargement by 3/4", in the sense of adding 3/4, which would in fact be × 7/4.

When the students propose their answers orally the teacher writes them on the board using arrows, as was habitually done since the reproductions of the Optimist (Module 9).

For example:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\div 4"] \arrow[rr, bend right, "\times 3/4"'] & {} \arrow[r, bend left, "\times 3"] & {} \end{tikzcd}$$

"Without arrows we could write (÷4)F(× 3)", which she reads "divide by 4 and then multiply by 3"

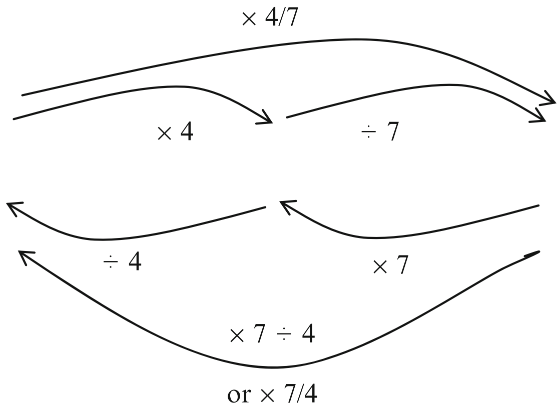
She continues with similar questions: "× 7/12", "take 3/5 of something", then "× 3.67" The children hesitate a moment and them some shout out (instead of writing) "You have to change the 3.67 into a fraction!"

$$3.67 = 367/100 \rightarrow (\times\ 367).(\div 100)$$

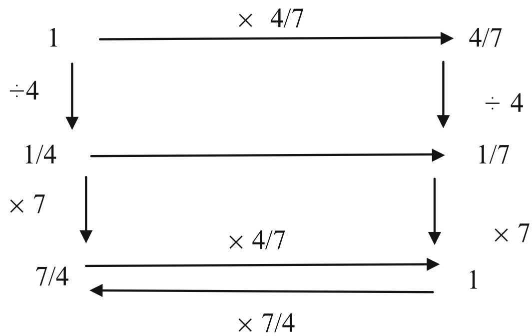
Decomposition of the Reciprocal

"To get from a model to its image I used the pantograph × 4/7. How could I do the reciprocal mapping with whole number pantographs?"

The solution is obvious materially, because pantographs are invertible. The students *explain the action* and invert it by replacing the multiplications by divisions. It's not enough to reverse the arrows!



The students rediscover a result they already knew in another context. The teacher reminds them of this method of representing measures and operations on those measures:



"You can find the reciprocal of a fractional linear mapping by decomposing it, taking the reciprocals and recomposing it."

The teacher then studies in the same way the reciprocal of a ratio:

"Kafor coffee is a mixture: 4/7 of its weight is made up of Arabica coffee. What operation would let us figure out how many pounds of Kafor coffee we could make with various different weights of Arabica?"

Decomposition of the mapping x1; inverse mappings

$$3 \longrightarrow 3$$

$$2.5 \longrightarrow 2.5$$

$$4/7 \longrightarrow 4/7$$

"Here is a mapping. What is it?"
"It's the mapping (×1)."

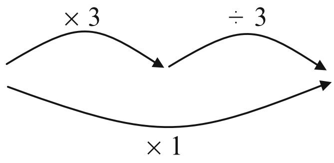
"Can we decompose it?"

First reaction from the student: "Can't do it!"

The teachers pushes them: "There isn't any pair of mappings that can be replaced by $\times 1$?"

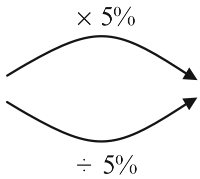
Students: "Oh, yeah! You can enlarge by 3 and shrink by 3…"

The teacher illustrates their suggestion with

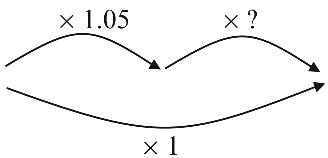


After a few more illustration she poses the question: "There is a report that the number of accidents has increased 5 % over last year. By how much they decrease this year in order for next year's number to be the same as last years?"

Some of the students first try to use the arrows directly:



But when they carry out the calculations they discover that decreasing by 5 % doesn't work. So they pose $1+5/100=1+0.05=1.05$ and represent the calculations to carry out as



Results The students become familiar with new vocabulary and situations. For many of them, handling compositions of mappings seems easy but risks becoming formal, and too quickly escaping from the control that the students need to exercise by verifying the meaning.

Thus for the teacher this is not the moment to institutionalize these new ways of calculating and still less to require mechanical reproduction of them.

Important Remark, 2008

This warning to the teachers was essential and needs to be explained to today's reader to avoid misunderstanding the nature of the teaching and of the practices described here. Up until this moment, the arrows have been used exclusively to designate mathematical objects: either correspondences (not necessarily numerical ones), or natural relationships between numbers in the same set of measures (for example natural differences or later natural ratios, but never both at the same time). Later they were used to indicate rational linear mappings (horizontal arrows for the enlargement x 1.75, etc.) Now they are used in showing that natural ratios like (x3) or (÷3) can be replaced respectively by (÷1/3) or (x1/3). But they have remained a free means of expression and have not been the object of any teaching or any evaluation. In Module 15 the study of compositions of mappings gives them a new status. They become an instrument of analysis, of calculation and even of proofs, and thus also an object of study and discussion. Their disposition may change – but they do not have the properties of a good model. It is therefore essential that the teacher not treat them as a piece of mathematical knowledge, that he not teach them lest he launch a metadidactical slippage that would be difficult to control. They should be used only as a means of expression, a prop for reasoning whose validity the student checks by reference to the actual meaning. It should be well noted that formal operations, their representations by arrows and the reasoning presented in these chapters **are not pieces of formal knowledge to be taught in the classical sense.**

Lesson 2: The Meaning of "Division by a Fraction" (Summary of Lessons)

"Would you know how to give a meaning to the operation 4 ÷ 3/5?", asks the teacher.

What the students need to do is

First interpret the formula by a real situation like the ones they have encountered in solving problems. For example:

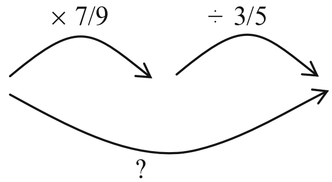
– You divide a 4 m long ribbon into parts that are 3/5 of a meter long
– You buy 3/5 of a meter of ribbon for 4 francs. What is the price of a meter of ribbon?
– A rug with area 4 m^2 has width 3/4 m. How long is it?

Then with the help of their schemas on quantities, relations and mappings they figure out the operations to carry out.

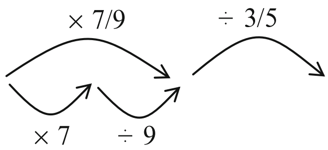
The teacher collects the problem statements they are trying to invent and helps them pull their ideas together, then organizes a discussion among the students about the statements they have come up with. A statement may be interpreted in a variety of ways. The teacher tries to get them to reformulate each of the possible interpretations of the givens as measure, then as linear mapping.

This type of activity is organized by a didactical schema known as a "tournament of problem statements", which is described in Module 12. A few examples of similar questions lead the students to be interested in the interpretation of the value being sought as a ratio of amounts or as a linear mapping.

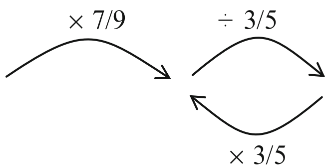
In this lesson the products of fractions are finally conceived as products, that is, compositions of direct or inverse linear mappings that make it possible to ask questions like: What linear mapping does 7/9 ÷ 3/5 represent?



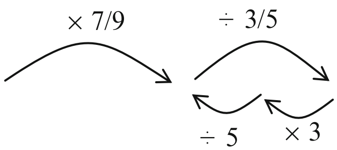
The decomposition of (× 7/9) into (× 7) followed by (÷9) is one they know well



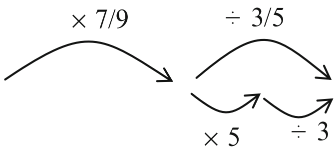
Then (÷ 3/5) is the reciprocal of (× 3/5)



And that in turn can be decomposed into (× 3) followed by (÷5)



Their reciprocals can also be calculated:



Thus the sequence of calculations is established as (× 7)(÷9)(× 5)(÷3), and since the order of linear mappings can be modified, we get × 7×5÷9÷3=35÷27

Commentary
This "proof" calls for some details about the development of the lesson:

1. No, it was not a response given individually by each child in response to a test question or an individual exercise.
2. The teacher offered the exercise for students to reflect on individually for a while, then collected the students' suggestions. Everyone achieved the first step, and the second was routine.
3. The third step gives no difficulties to certain students, but although they respond with "÷ 3 followed by x 5" it is for bad reasons that they can't justify: they broke up the fraction as they would have done with x 3/5, inverting the last sign. The teacher says nothing, but other students express concern. The class is not in the habit of calculating without knowing what they are doing.
4. The teacher guides the discussion: "You don't know how to decompose ÷3/5?" Some of the students recall recent calculations about this kind of linear mapping by taking numbers.
5. The teacher suggests that they know how to decompose the reciprocal. The students then develop the method. Each step is a sort of rapid individual exercise.

So here is a matter of a sequence of "exercises". This would be a problem for students who wanted to solve it by themselves. Certain of them, stimulated by challenges from others, could get there by themselves, but at what price and for what profit (for themselves or the others)? The solutions of the steps are exercises that the teacher rapidly proposes and checks.

This problem is not a lesson and its solution is not a piece of knowledge to be learned. It is simply an occasion for using the knowledge that is in process of being learned and making it more familiar and more easily available. Not every student solves every exercise, but they will see a certain number of them again.

Lesson 3: Division of Decimals

A Mapping For the Calculation of Decimal Numbers

(a) *Assignment*: "Would you know now how to find a meaning for this division: 1.38 ÷ 4.15 ?" (the operation is written on the board.)
(b) *Development*: This phase proceeds like the preceding one: First, time for the students to reflect and try things out in their scratch-notebooks. Then an alternation of individual and collective reflections in the course of which the teacher has them explain the meaning of this division: "We have to find the image of 1.38 under the mapping ÷ 4.15", and has someone write on the board (or writes herself):

$$1.38 \xrightarrow{\div 4.15} ?$$

The children, who by this time are well trained on this kind of exercise, suggest writing 4.15 as a fraction: 415/100

First step:

$$1.38 \xrightarrow{\div 415/100} ?$$

Referring to the activity of 15.2.1, the teacher asks: "What mapping can we replace ÷ 415/100 by?", and writes the mapping (or has a child write)

$$\div 415/100 = \times\, 100/415$$

Second step:

$$1.38 \xrightarrow{\times\, 100/415} ?$$

Third step:

$$\begin{tikzcd} 1.38 \arrow[rr, bend left, "\times\, 100/415"] \arrow[r, bend right, "\times\, 100"'] & {} \arrow[r, bend right, "\div\, 415"'] & ? \end{tikzcd}$$

$$\begin{tikzcd} 1.38 \arrow[rr, bend left, "\times\, 100/415"] \arrow[r, bend right, "\times\, 100"'] & 138 \arrow[r, bend right, "\div\, 415"'] & 138/415 \end{tikzcd}$$

The children calculate 138/415 in their scratch notebooks

$$\begin{tikzcd}
1.38 \arrow[r, bend right, "\times 100"'] \arrow[rr, bend left, "\times 100/415"] & 138 \arrow[r, bend right, "\div 415"'] & 0.332\ldots
\end{tikzcd}$$

Conclusions and Installation of the Algorithm

The teacher asks what calculations they had to make to find $1.38 \div 2.14$. "We had to multiply by 100 first (138) and then divide by 415."

The teacher calls the students' attention to this new method of "division" with a different meaning from the one they knew before (see modules 12 and 13), which they can now calculate rapidly no matter what the numbers (without writing the successive subtractions or finding the intervals by trial and error.)

With a few remarks connecting what they have just learned with the notation and format they used previously, this concludes the curriculum.

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