

Limit Analysis: A Layered Approach for Composite Laminates

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Abstract The present contribution summarizes the results of recent studies carried on by the authors in the last few years concerning the evaluation of the load bearing capacity of single- and multi-pin joints in composite orthotropic plates. The problem, tackled via limit analysis, employs a Tsai-Wu-type yield surface and a non standard treatment of limit analysis approach. Upper and lower bounds to the real peak load value are evaluated by two FE based numerical procedures predicting also the joint failure mode. The whole procedure is implemented at lamina level so taking into account some of the through thickness effects on the joint strength capabilities. A wide number of experimental findings, coming from laboratory tests on real prototypes and available in the relevant literature, is considered to validate, by comparison, the expounded methodology.

1 Premises and Main Assumptions

Limit analysis plays an eminent role among the theoretical and numerical methods aimed at predicting the load bearing capacity of structures or structural elements. It also becomes very effective and attractive for the design of many modern industrial prototypes often manufactured with materials whose constitutive behaviour does not have a well defined mathematical description. In composite laminates, for example, the available constitutive models are often affected by values of material parameters hardly identifiable via experimental tests so that, in such a context, results obtained via a step-by-step post-elastic analysis may be useless for applications of engineering interest. Nevertheless, even in those cases where the constitutive behavior is well

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settled, limit analysis can be used as a preliminary/first design tool: once a good prototype is individuated for a specific problem, sophisticated post elastic FE analysis or experimental laboratory tests can be performed for a deeper and exhaustive understanding of its mechanical behaviour with a considerable saving of efforts either economic or time consuming.

The classical approaches of limit analysis rely upon mathematical programming procedures which have progressed significantly in recent years (see e.g. [15, 16, 19, 20, 35, 36, 40]). A number of contributions in this field (see e.g. [41] and references therein) adopt however a different approach whose basic assumption is that limit state solutions may be developed from sequences of elastic (linear) analyses easy to handle via any commercial finite element code. In this context a numerical FE-based approach has been recently proposed by the authors to a peculiar problem of orthotropic composite laminates, namely the evaluation of the load bearing capacity of pinned-joint composite plates with single- or multi-pin fastenings [24–27]. This is a topical problem in the composite realm as witnessed by the number of recent paper on the subject, more than 700 papers are listed in the work of Mackerle [21] which provides an interesting review of finite elements methods applied for the analysis of joints; remarkable reviews are also the ones given in [6, 10, 31, 37]. For design purposes, Researchers are mainly interested in determining the effects on joints failure strength and failure modes of some peculiar geometrical parameters such as: the ratios between hole-distance from the free plate edge to hole-diameter, or the one between the plate-width to hole-diameter or, also, the relative distances between fastener holes as well as their spatial distribution within the plate [2–4, 9, 12–14, 22].

The results of a research carried on in the last few years and concerning the analysis of mechanically fastened joints in multi layers composite plates are expounded in the following. Details, and deeper explanations are given in the papers [24–27] to which the Reader could refer for more information. In particular, two well known FE procedures have been rephrased: (i) the Linear Matching Method, conceived by Ponter and Carter [28], (see also [8]), has been used to compute an upper bound to the peak load multiplier; (ii) the Elastic Compensation Method, due to Mackenzie and Boyle [18], has been employed to evaluate a lower bound to the peak load multiplier. The former, considering a structure made by a fictitious linear viscous material with elastic parameters spatially varying, allows to construct a collapse mechanism and eventually to evaluate an upper bound to the peak load. The latter, grounding on a stress redistribution procedure pursued by a sequence of elastic analyses in which highly loaded regions are systematically weakened, produces an admissible stress field suitable for a lower bound evaluation.

The assumption of a Tsai-Wu-type yield surface for composite laminates [7, 34, 38, 39] allows one to locate stress states at which the material has exhausted its strength capabilities. The further assumption of a non-associate flow rule makes the whole methodology of general applicability following a nonstandard approach [17, 30]. Limit analysis is used indeed to evaluate the strength capability of the joint by evaluating upper and lower bounds bracketing the real peak load value.

The main novelty of the authors' recent studies hereafter summarized, is related either to the extension of the mentioned FE procedures in the realm of anisotropic

composites and, in general, non associative materials or, also, to their non trivial implementation at layer laminate level. The latter peculiarity reveals undoubtedly more burdensome computations but, as witnessed by the obtained results, allows to take into account the stacking sequence of the laminate. Some of the through-thickness effects depending on the number of the laminae the laminate is made with as well as the effects of fiber orientations within each lamina can indeed be handled with good accuracy. To this aim higher order shell-type multilayered finite elements have been used, performing all the relevant operations at the Gauss points of each element layer (lamina).

In order to verify the reliability of the proposed procedure, checking its ability of bracketing the real (experimentally detected) peak load, a considerable number of experimental tests at rupture, available in the literature [2, 12–14, 22], have been numerically reproduced. The possibility of localizing the collapse zone so predicting the related failure mode has been also investigated highlighting the potentialities of the layer by layer implementation. In conclusion, the proposed procedure seems to guarantee, at least for the studied problem, a great accuracy for different laminate lay-ups and different joint geometries establishing an effective design tool useful to avoid expensive trials on real prototypes.

2 Limit Analysis and Constitutive Assumptions

2.1 The Limit Analysis Problem

Following the kinematic theorem of the limit analysis, considering a body of volume V , for a given distribution of compatible strain rates $\dot{\epsilon}_j$, say $\dot{\epsilon}_j^c$, an upper bound to the (collapse) peak load multiplier is given by:

$$P_{UB} = \frac{\int_V \sigma_j^y \dot{\epsilon}_j^c dV}{\int_{\partial V_i} \bar{p}_i \dot{u}_i^c d(\partial V)} \quad (1)$$

where: $\dot{\epsilon}_j^c = \dot{\lambda} \partial f / \partial \sigma_j$ are the components of the outward normal to the yield surface $f(\sigma_j) = 0$ (with $\dot{\lambda} > 0$ scalar multiplier); P_{UB} denotes the upper bound load multiplier (for simplicity only surface forces, $\bar{p}_i$, acting on the external portion of the body ∂V_i , are considered); σ_j^y are the stresses at yield associated to the compatible strain rates $\dot{\epsilon}_j^c$; $\dot{u}_i^c$ are the related displacement rates. The set $(\dot{\epsilon}_j^c, \dot{u}_i^c)$ indeed defines a collapse mechanism.

On the other hand, the static theorem of limit analysis states that if at every point within V exists a stress field $\tilde{\sigma}_j$ which satisfies the condition $f(\tilde{\sigma}_j) \leq 0$ and in equilibrium with the applied load $P \bar{\mathbf{p}}$ for a certain value of P , say P_{LB} , then P_{LB} is a lower bound to the limit load multiplier.

Two remarks appear necessary to focus what will be expounded in the next sections.

Remark 1 For nonstandard materials, see e.g.: [17, 30, 32, 33], the Radenkovic's first and second theorems state that (after [17]): every value of the limit load for a body made of a nonstandard material is located between two fixed boundaries defined by the values of the limit loads for two corresponding standard materials. The hypothesis of a nonstandard constitutive behavior assumed next will then imply to search for an upper and a lower bound to the (collapse) peak load of the joint.

Remark 2 The two numerical procedures, expounded in Sects. 3 and 4, play two distinct roles as specified in the following. The Linear Matching Method (LMM) is naturally related to the kinematic approach of limit analysis being able to construct a compatible collapse mechanism. It then gives an upper bound, P_{UB} , to the peak load multiplier. The Elastic Compensation Method (ECM) is indeed strictly related to the concept of stress redistribution aimed at producing an admissible stress field. It is then related to the static approach of limit analysis furnishing a lower bound, P_{LB} , to the peak load multiplier.

2.2 Constitutive Assumptions

By hypothesis, the constitutive behaviour of the composite laminate obeys to a second-order tensor polynomial form of the Tsai-Wu failure criterion [39] having the following analytical form:

$$F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + 2F_{12}\sigma_1\sigma_2 + F_1\sigma_1 + F_2\sigma_2 = 1, \quad (2)$$

where:

$$\begin{aligned} F_1 &:= \frac{1}{X_t} + \frac{1}{X_c}; & F_2 &:= \frac{1}{Y_t} + \frac{1}{Y_c}; & F_{11} &:= -\frac{1}{X_t X_c}; \\ F_{22} &:= -\frac{1}{Y_t Y_c}; & F_{66} &:= -\frac{1}{S^2}; & F_{12} &:= -\frac{1}{2}\sqrt{F_{11}F_{22}}; \end{aligned} \quad (3)$$

with X_t , X_c the longitudinal tensile and compressive strengths respectively; Y_t , Y_c the transverse tensile and compressive strengths respectively and S the longitudinal shear strength; moreover, as usual for composite structures, 1 and 2 denote the principal directions of orthotropy in plane stress case while $\sigma_6 \equiv \tau_{12}$. In Eqs. (3) the compressive strengths X_c and Y_c have to be considered intrinsically negative. The quadratic form given by Eq. (2) defines an admissible stress states domain: points within the domain locate stress states pertaining to an orthotropic linear elastic behaviour of the material; points lying on the domain boundary locate stress states at which the material has exhausted its strength capabilities. The Tsai-Wu-type surface, in the quadratic form adopted allows one to apply the standard rules of transformation, invariance and symmetry, and locates an ellipsoid in the stress space which is assumed as yield surface for the orthotropic material considered. Moreover, the orthotropy of the composite laminates infers the general assumption of non associativity so, as noted above, the peak load values of the analyzed specimens will be located by the determination of upper and lower bounds.

Remark 3 On taking into account the previous Remark 1 and the strict convexity of the Tsai-Wu-type yield surface the postulated non associativity can be treated following the Radenkovic approach and, in this case, the assumed yield surface can itself play the double role of inner and outer surface.

3 The Linear Matching Method (LMM)

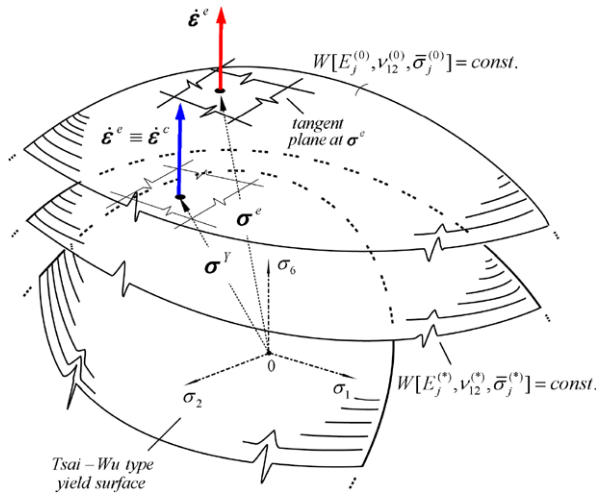
3.1 Fundamentals

An upper bound to the (collapse) peak load multiplier can be evaluated by the LMM which furnishes all the ingredients entering Eq. (1) namely: the kinematic fields $\dot{\epsilon}_j^c, \dot{u}_i^c$, and the associated stresses at yield (with respect to the yield surface (2)), i.e. σ_j^y . To this aim an iterative procedure, involving a sequence of linear FE-based analyses, is performed on the structural element assumed as made of an orthotropic linear viscous *fictitious* material with *spatially varying moduli* and subjected to a distribution of *imposed initial stresses*. At each iteration, the fictitious moduli are then adjusted and the initial stresses are varied so that the computed fictitious linear stresses are brought on the yield surface at a fixed strain rate distribution. This allows one to define a collapse mechanism (strain and displacement rates $(\dot{\epsilon}_j^c, \dot{u}_i^c)$), the related stresses at yield σ_j^y and eventually, by Eq. (1), allows to evaluate a P_{UB} . All the above can be geometrically interpreted as given in next subsection for a quick understanding.

3.2 Linear Matching Method and FE Procedure

The fundamental steps of the LMM are schematically shown in Fig. 1 where a *fictitious linear viscous solution*, computed at a generic Gauss point (GP) inside a generic layer of a generic finite element, is represented in the stress space $(\sigma_1, \sigma_2, \sigma_6)$. In particular, denoting with apex $^{(0)}$ the values of fictitious material parameters and initial stresses at iteration k , and with apex $^{(*)}$ those at iteration $k + 1$, the fictitious solution computed at k is displayed by a stress point, located by the stress vector σ^e , lying on the complementary dissipation rate equipotential surface $W[E_j^{(0)}, \nu_{12}^{(0)}, \bar{\sigma}_j^{(0)}] = \text{const}$ pertaining to the fictitious material. Such solution is also displayed, in terms of linear viscous strain rates, i.e. by the outward normal, $\dot{\epsilon}_j^e$, to $W = \text{const}$ at σ_j^e . The related $\dot{u}_i^e$ being the compatible displacement rates. At iteration $k + 1$, the material moduli and the initial stresses are varied (assigning them the starred values), in such a way that $\dot{\epsilon}_j^e$ can be interpreted as $\dot{\epsilon}_j^c$, the latter being the strain rate at collapse at the current GP. The above operation simply implies to find a (stress) point of assigned normal $(\dot{\epsilon}_j^e = \dot{\epsilon}_j^c)$ belonging either to the yield surface or to a “modified” complementary dissipation rate equipotential surface, say

Fig. 1 Matching procedure at a GP in the FE mesh:
 $(\cdot)^{(0)}$ = values at iteration k ;
 $(\cdot)^{(*)}$ = values at iteration $k + 1$



$W[E_j^{(*)}, \nu_{12}^{(*)}, \bar{\sigma}_j^{(*)}] = \text{const}$, where the starred quantities define indeed the modified $W = \text{const}$ at $k + 1$. The corresponding stresses, σ^Y , are then “stresses at yield”, the related displacement rates the “incipient displacements at collapse”. This operation, which also explains the name of the method, the surface W matches the yield surface at a point of given normal, is performed at each GP of the FE mesh and obviously violates the global equilibrium conditions. The stresses at yield, evaluated by matching, do not satisfy equilibrium with the loads acting at iteration k ; this essential condition is indeed assured at the end of an iterative process. To this concern, a sufficient condition for convergence is given in [29] and is not reported for brevity.

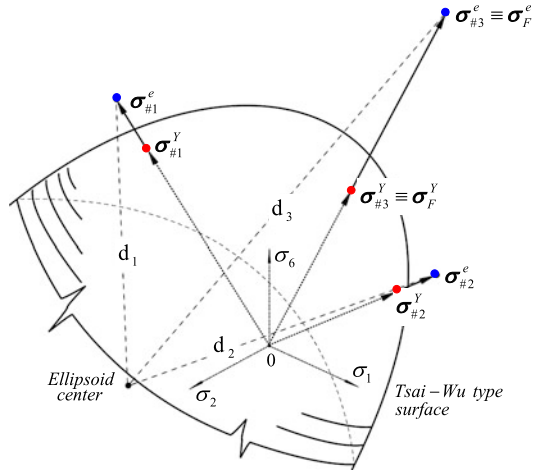
Moreover, grounding on the formal analogy between the linear viscous problem and the linear elastic problem (see e.g. [23]), the fictitious *linear viscous* solution (in rate form) can, in practice, be computed as a fictitious *linear elastic* solution (in finite form), W playing the role of complementary energy potential of the fictitious material. The FE analysis is then achievable as a linear elastic analysis by any commercial FE code.

4 The Elastic Compensation Method (ECM)

4.1 Fundamentals

A lower bound to the peak load multiplier can be evaluated via the ECM aimed at constructing an admissible stress field suitable for the evaluation of a P_{LB} . As the previous procedure, the ECM is also an iterative procedure involving sequences of linear elastic FE-based analyses in which *highly loaded regions* of the structure are *systematically weakened by reduction of the local modulus of elasticity*. Precisely,

Fig. 2 ECM procedure at three generic layers belonging to elements # 1, #2 and #3 in the FE mesh



a sequence of linear elastic FE analyses is carried out in which, for a given load value, the elastic moduli are reduced within “critical regions” of the structure identified by the elements where the stresses attain values greater than the yield one. This allows one to define a *maximum admissible* stress value in the whole structure and for the given load. Increased values of loads are then considered in the subsequent sequences of analyses till further load increase do not allow the maximum stresses to be brought below yield by the reduction (or redistribution) procedure. A P_{LB} load multiplier can be easily evaluated at last admissible stress values attained for a maximum acting load in the spirit of the static approach of limit analysis.

4.2 Elastic Compensation Method and FE Procedure

Also in this case, the fundamental steps of the ECM are schematically shown. In Fig. 2 the solution pertaining to an elastic analysis carried on the structure for a given (arbitrary) fixed load level P_D is displayed in terms of elastic stress vectors evaluated at three generic layers of three generic FEs in the mesh. The latter, for simplicity, are denoted by elements #1, #2 and #3. In particular, with reference to the stress space $(\sigma_1, \sigma_2, \sigma_6)$, the value $\sigma_{\#e}^e$ denotes the stress vector representing the average elastic stress computed within a layer of element $\#e$ (its components are simply the averaged values of the stress components measured at the GPs of the layer); $\sigma_{\#e}^Y$ denotes the corresponding stress at yield (on the Tsai-Wu-type surface) measured on the direction $\sigma_{\#e}^e/|\sigma_{\#e}^e|$.

For the fixed load P_D a sequence of FE elastic analyses is carried out on the structure such that, at the i -th iteration (analysis) of the sequence, within the FEs layers where $|\sigma_{\#e}^e| > |\sigma_{\#e}^Y|$ the elastic moduli of the layer are reduced according to:

$$E_{\#ej}^{(i)} := E_{\#ej}^{(i-1)} \left[\frac{|\sigma_{\#e}^Y|^{(i-1)}}{|\sigma_{\#e}^e|^{(i-1)}} \right]^2. \quad (4)$$

Among all the $\sigma_{\#e}^e$ in the mesh, the “maximum stress”, i.e. the stress point farthest away from the Tsai-Wu type surface, say σ_R (coincident with $\sigma_{\#3}^e$ in Fig. 2), is detected, σ_R^Y being the corresponding stress at yield (measured on the direction $\sigma_R/|\sigma_R^Y|$). The iterations (analyses) are carried out inside the given sequence until this maximum stress in the whole mesh σ_R just reaches (is below) its yield value σ_R^Y , and this by the above reduction or redistribution procedure. Further sequences of elastic analyses are then carried on, each one with an increased value of P_D , and the reduction procedure repeated till further load increases do not allow the σ_R stress to be brought below yield. A lower bound to the collapse load multiplier can then be computed as:

$$P_L := |\sigma_R^Y| \frac{P_D}{|\sigma_R|}. \quad (5)$$

Concerning the evaluation of a lower bound via the ECM, it is worth to mention the remarkable works of Staat and Co-workers (see e.g. [35, 36]), where the effectiveness of the ECM is analyzed through a comparative study with respect to other optimization procedures. In [40] a better performance of the so called primal-dual procedure with respect to the ECM is also shown. The results obtained in the above quoted papers, all concerning von Mises type materials, show, at least for the cases there addressed, that the ECM is too conservative. Such assertion deserves surely further investigations also in the present context even if, for the mechanical problem herein tackled and the material utilized, at least for the analyzed cases, such drawbacks seem do not appear.

5 Layer-by-Layer Formulation

5.1 FE Modeling and Assumptions

From a numerical point of view, both iterative procedures are driven by a Fortran main program which utilizes the results of elastic analyses of the examined structural element carried out using the commercial FE code ADINA [1]. In the discrete model, the number of FEs is chosen taking into account the geometry of the modeled test and with the aim to obtain an enough accurate elastic FE solution. To this end, as general rule, a finer mesh is always employed around the fastener holes and a preliminary sensitive study on the elastic solution is performed for each different considered geometry. Moreover, when the analyzed specimens are made by quasi-isotropic laminates, isoparametric shell elements with 16 nodes and 16 GPs per element are utilized in the mesh, while, when dealing with laminates with a general stacking sequence, higher order isoparametric *multilayered* shell elements with 16 nodes per element, referred as DISP 16 in the quoted code, are employed. In particular, the employed multilayered shell elements are an extension to the case of composite laminates of mixed interpolated tensorial components (MITC) fam-

ily of plates and shell elements (see e.g. [11]). The modelling of the sublamine is so entrusted to the element's layers endowed with 16 (on the top) plus 16 (on the bottom) Gauss points. For each element layer a proper material axes system is fixed and an order 2 Gauss integration is performed in the layer thickness. The number of layers per element is, obviously, chosen equal to the numbers of the laminae the analyzed laminate is made with. In Fig. 3(a) the modeling assumptions, namely the transition from the real stacking sequence of the laminate to the layers' element with the proper definition of the layer material axes system, are sketched. For both procedures, as shown in Fig. 3(b), the updating and/or reduction of the elastic moduli is carried into effect at each GP of the element (or of the layer for the multilayer shells) but, on taking into account that a unique set of E_j —i.e. a unique (orthotropic) material—has to be assigned to each single element layer, the modified moduli are averaged within the single element layer at the beginning of each FE analysis.

Finally, to simulate the presence of the rigid pins, two different schemes have been adopted. The first, used for single-pin fastenings, assumes a cosine load normal distribution to approximate the pressure exerted by the pin on the inner hole surface. The second, used for multi-pin fastenings, assumes boundary conditions of contact type between the pins and the fastener holes. In particular, the constrain function method (see e.g. [5] and references therein), available in the ADINA code, is used. The method requires the definition of a target line (the pin contour) and the definition of a contactor line (the fastener hole contour in the laminate) such that no material overlap between them can occur during the deformation process, while the contactor line can scroll or leave the target line.

5.2 Iterative Schemes for Upper and Lower Bounds Evaluation

5.2.1 Flow-Chart of the LMM Iterative Procedure

- *Initialization*

Knowing the strength values of the orthotropic material ($X_c; X_t; Y_c; Y_t; S$); assign to all FEs layers an initial set of fictitious elastic parameters and initial stresses such that the complementary energy equipotential surface is homothetic to the Tsai-Wu type surface, i.e.:

$$\begin{aligned} E_1^{(0)} &= 1/(2F_{11}); & E_2^{(0)} &= 1/(2F_{22}); & E_6^{(0)} &= 1/(2F_{66}); \\ \nu_{12}^{(0)} &= -f_{12}\sqrt{F_{11}}/\sqrt{F_{22}}; \\ \bar{\sigma}_1^{(0)} &= \alpha_{TW}/\sqrt{F_{11}}; & \bar{\sigma}_2^{(0)} &= \beta_{TW}/\sqrt{F_{22}}; & \bar{\sigma}_6^{(0)} &= 0; \end{aligned}$$

α_{TW} and β_{TW} being the X, Y coordinates of the Tsai-Wu type ellipsoid centre, while F_{11} , F_{22} and F_{66} are functions of the strength values. Set also: $k = 1$,

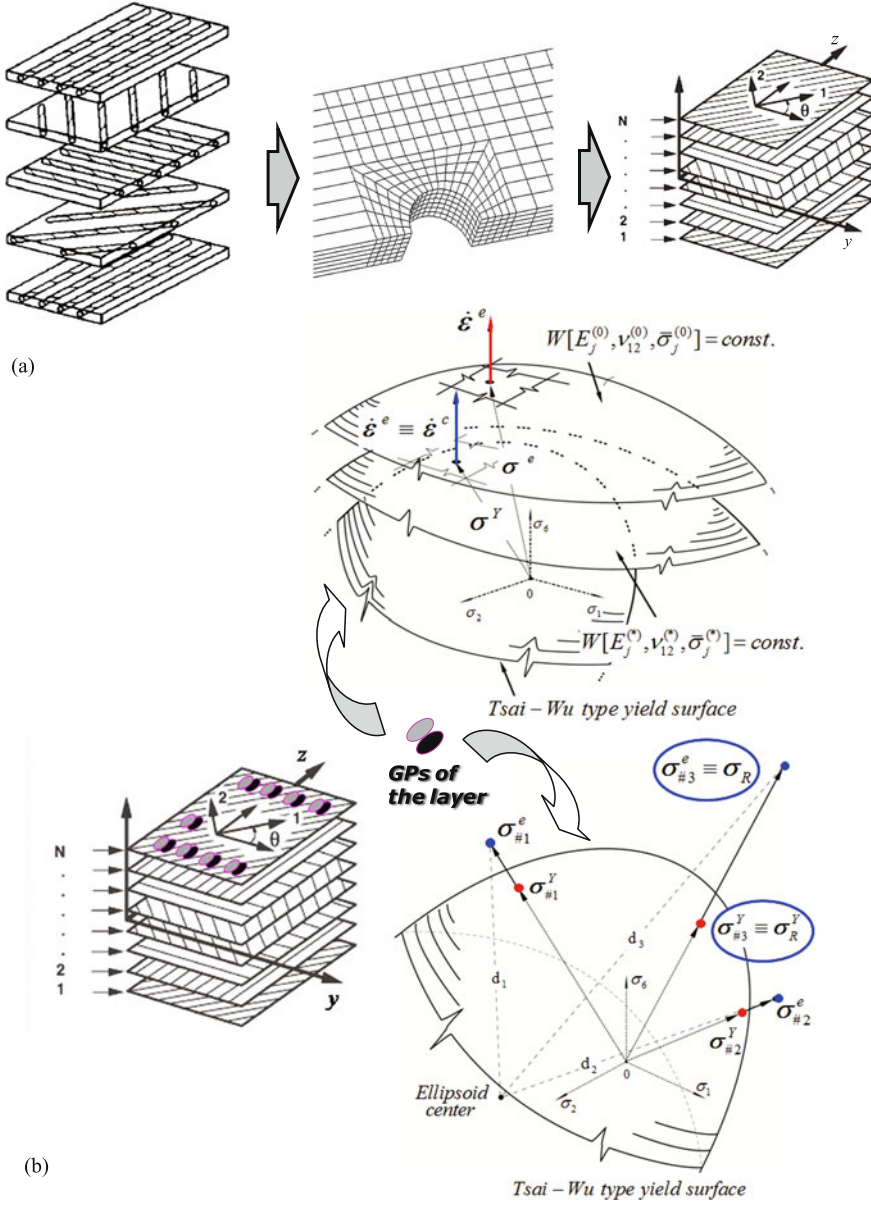


Fig. 3 Modeling assumptions: (a) definition of material axes at element layer following the stacking sequence; (b) sketch of the LMM and the ECM carried into effect at element layer

$P_{UB}^{(k-1)} = P_{UB}^{(0)} = 1$ (for $k = 1$, $P_{UB}^{(0)}$ can be any arbitrary value) and compute the constant $\Omega = 1 + \alpha_{TW}^2 + 2f_{12}\alpha_{TW}\beta_{TW} + \beta_{TW}^2$ for later use.

- *Start iterative procedure*

- *Start elements loop*
- *Start layers loop*

step # 1: perform a fictitious elastic analysis with elastic parameters $E_j^{(k-1)}$, $\nu_{12} = \nu_{12}^{(0)}$, initial stresses $\bar{\sigma}_j = \bar{\sigma}_j^{(0)}$ and with loads $P_{UB}^{(k-1)} \bar{p}_i$ ($\bar{p}_i :=$ reference loads), computing a fictitious elastic solution at Gauss points of each layer, namely: $\dot{\varepsilon}_j^{e(k-1)}$, $\dot{u}_i^{(k-1)}$, $\sigma_j^{e(k-1)}$.

step # 2: compute the constant value of the complementary potential energy:

$$\bar{W}^{(k-1)} = \frac{1}{2} \sigma_j^{e(k-1)} \varepsilon_j^{e(k-1)}$$

step # 3: compute the homothety ratio, namely

$$\Gamma^{(k-1)} = \begin{cases} \sqrt{\Omega / \bar{W}^{(0)}} & \text{for } k = 1 \\ \sqrt{\bar{W}^{(k-2)} / \bar{W}^{(k-1)}} & \text{for } k > 1 \end{cases}$$

step # 4: evaluate stresses at yield:

$$\begin{aligned} \sigma_1^{Y(k-1)} &= [1 - \Gamma^{(k-1)}] \frac{\alpha_{TW}}{\sqrt{F_{11}}} + \Gamma^{(k-1)} \sigma_1^{e(k-1)}; \\ \sigma_2^{Y(k-1)} &= [1 - \Gamma^{(k-1)}] \frac{\beta_{TW}}{\sqrt{F_{22}}} + \Gamma^{(k-1)} \sigma_2^{e(k-1)}; \\ \sigma_6^{Y(k-1)} &= \Gamma^{(k-1)} \sigma_6^{e(k-1)} \end{aligned}$$

step # 5: set $\dot{\varepsilon}_j^{c(k-1)} = \dot{\varepsilon}_j^{e(k-1)}$, $\dot{u}_i^{c(k-1)} = \dot{u}_i^{e(k-1)}$ and evaluate the upper bound multiplier

$$P_{UB}^{(k)} = \frac{\int_V \sigma_j^{Y(k-1)} \dot{\varepsilon}_j^{c(k-1)} dV}{\int_{\partial V_t} \bar{p}_i \dot{u}_i^{c(k-1)} d(\partial V)}$$

- *End layers loop*
- *End elements loop*

step # 6: check for convergence

$$|P_{UB}^{(k)} - P_{UB}^{(k-1)}| \leq \text{TOL} \quad \begin{cases} \text{YES} \Rightarrow \text{EXIT} \\ \text{NOT} \Rightarrow \text{CONTINUE} \end{cases}$$

- *Start elements loop*
- *Start layers loop*

step # 7: compute the $E_j^{(k)}$ distribution accomplishing the matching at each GP of each layer to be utilized at next iteration, namely:

$$E_j^{(k)} = E_j^{(k-1)} [\Gamma^{(k-1)}]^2, \quad j = 1, 2, 6$$

step # 8: average the updated $E_j^{(k)}$ values within each element layer.

- *End layers loop*
- *End elements loop*
- *End iterative procedure*

set $k = k - 1$ and GOTO step #1

5.2.2 Flow-Chart of the ECM Iterative Procedure

- *Initialization*

Assign to all FEs layers the material elastic parameters, i.e.:

$$E_1^{(0)}; \quad E_2^{(0)}; \quad E_6^{(0)}; \quad \nu_{12};$$

set also: $k = 1$, $P_D = 1$ (for $k = 1$, P_D can be a design load value).

- *Start iterative procedure*
- *Start elements loop*
- *Start layers loop*

step # 1: perform an elastic analysis with material parameters $E_j^{(k-1)}$, ν_{12} and loads $P_D \bar{p}_i$ ($\bar{p}_i :=$ reference loads), computing the stresses at Gauss points of each layer, namely: $\sigma_j^{e(k-1)}$.

step # 2: average the stress values inside each element layer so computing the stress vector $\sigma_{\#e}^e$ and the corresponding stress at yield $\sigma_{\#e}^Y$.

step # 3:

IF $P_D = 1$

update the young moduli within the current layer according to:

$$E_{\#ej}^{(k)} := E_{\#ej}^{(k-1)} \left[\frac{|\sigma_{\#e}^Y|^{(k-1)}}{|\sigma_{\#e}^e|^{(k-1)}} \right]^2 \quad (\text{Eq. (4)}),$$

ELSE

update the young moduli within the current layer according to the above equation (Eq. (4)) only when:

$$|\sigma_{\#e}^e|^{(k-1)} > |\sigma_{\#e}^Y|^{(k-1)}$$

ENDIF

step # 4: detect the “maximum stress” σ_R in the whole mesh (i.e. whose corresponding stress point is the one farthest away from the Tsai-Wu type surface) and evaluate the pertinent stress at yield σ_R^Y

- *End layers loop*
- *End elements loop*

step # 5: check if compensated stress σ_R just reaches (is below) the yield value

$$|\sigma_R| > |\sigma_R^Y| \quad \begin{cases} \text{YES} \Rightarrow \text{GOTO step \# 8} \\ \text{NOT} \Rightarrow \text{CONTINUE} \end{cases}$$

step # 6: compute a lower bound as:

$$P_L := |\sigma_R^Y| \frac{P_D}{|\sigma_R|}$$

step # 7: set $P_D > P_L$, set also $k = 1$ and GOTO step # 1 to perform a new sequence of elastic analyses

step # 8:

IF

$$|\sigma_R|^{(k)} \geq |\sigma_R|^{(k-1)}$$

set $P_L = P_D$ of the previous sequence and *EXIT*

ELSE

set $k = k - 1$ and GOTO step #1 to start a new elastic analysis

ENDIF

- *End iterative procedure*

6 Numerical Versus Experimental Findings

6.1 Pinned Joints in FRP Plates: Single- and Multi-hole Fasteners

As already pointed out in the previous sections, the expounded numerical procedures have been validated by reproducing a considerable number of experimental tests for load bearing capacity experimental evaluation available in the literature and concerning single- as well as multi-pin joints composite plates. Four main different configurations have been considered. For all the examined cases, overall 177, the proposed approach has been applied to predict either the upper and lower bounds to the joint peak load value or the joint failure modes. In all cases, due to the symmetry of loading, geometry and material with respect to the longitudinal axis y of the plate, only half of the mechanical problem shown in Figs. 4 and 5 has been considered.

In particular, the first configuration refers to a multilayer composite laminate with a single hole, like the one sketched in Fig. 4 where geometrical, boundary and loading conditions are specified. In this first case, to simulate the action of the pin inside the hole a cosine load normal distribution T_i is assumed with P applied reference load, assumed equal to 1 kN, n_i the unit vector normal to the inner hole surface and ϑ a clockwise angle varying in the range $[-\pi/2, \pi/2]$. The numerical study has involved six different lay-ups (specimen-type), each one with eight different geometries excepted for the sixth specimen-type where seven different geometries have been analyzed, so considering 47 different specimens. For these 47 specimens, the experimental findings are the ones given by Okutan Baba [22] and Aktas and Dirikolu [2]. For benefit of clarity, the 6 layer stacking sequences (*LSS*), identifying 6 different specimen types, are listed in Table 1 together with the thickness of the laminate and the diameter of the fastener hole. Moreover, Table 2 shows the mechanical parameters of the two types of unidirectional laminae forming the examined laminates, i.e. glass-fiber/epoxy and carbon/epoxy. Specimens belonging to this first configuration have been analyzed either following a single layer approach, with 16 nodes single layer shell elements, or using higher order isoparametric multilayered shell elements, referred as DISP 16 in the ADINA code. The number of elements has been varied from 250 to 330 depending on the considered geometry.

Table 1 Layer stacking sequence (*LSS*) and geometrical properties of the single-hole specimens

Specimen type	Material	<i>LSS</i>	<i>t</i> (mm)	<i>D</i> (mm)
A	glass-fiber epoxy	$[0^\circ/\pm 45^\circ]_S$	4.4	5
B	glass-fiber epoxy	$[90^\circ/\pm 45^\circ]_S$	4.4	5
C	glass-fiber epoxy	$[0^\circ/90^\circ/0^\circ]_S$	3.3	5
D	glass-fiber epoxy	$[90^\circ/0^\circ/90^\circ]_S$	3.3	5
E	glass-fiber epoxy	$[\pm 45^\circ]_{2S}$	4.8	5
F	carbon epoxy	$[90^\circ/\pm 45^\circ/0^\circ]_S$	2.64	6.35

Table 2 Mechanical parameters of the unidirectional laminae forming the composite laminates with a single-pin joint

	Glass-fiber epoxy ^a					
	Elastic moduli (GPa) and Poisson ratio	E_1	E_2	G_{12}	ν_{12}	
		44	10.5	3.74	0.36	
	Strength (MPa)	X_t	X_c	Y_t	Y_c	S
		800	350	50	125	120
	Carbon epoxy ^b					
	Elastic moduli (GPa) and Poisson ratio	E_1	E_2	G_{12}	ν_{12}	
		88	8.8	2.8	0.28	
	Strength (MPa)	X_t	X_c	Y_t	Y_c	S
		811	457.7	47.3	109.5	132

^aAfter Okutan Baba [22]^bAfter Aktas and Dirikolu [2]

multi-pin joints fixture test together with the corresponding mechanical model is given in Fig. 5. The latter shows the distributed edge load as well as the shaded rigid pins fixed to the external world. As said in Sect. 5.1 boundary conditions of contact type are assumed between the pins and the fastener holes. With reference to Fig. 5: W and t are the width and thickness plate respectively; the length G indicates the distance between two parallel holes; K their distance from the longer edge; M the distance between two serial holes in the y direction; F the distance between two generic holes in the y direction. Finally, L and E are the distances between the inner and outer holes from the plate edges respectively.

In detail, the second configuration refers to a $[0^\circ/90^\circ/\pm 45^\circ]_S$ laminate with a three pins joint tested in the technical report of Karakuzu et al. [14]. Referring again to Fig. 5, the three holes are located in the positions 1, 2, 3. Each glass-epoxy lamina, forming the composite laminate, is characterized by the mechanical properties reported in Table 3, while the geometrical parameters are: $L = 90$ mm, $t = 1.7$ mm, $D = 5$ mm; finally, the distance K is equal to 10 mm. For this specimen type, 45 different geometries were tested with values of the ratio E/D ranging from 1 to 5, ratio G/D ranging from 3 to 5 and considering also three different ratios for F/D , namely: 2, 4, 6. A number of elements ranging from 544 to 662 has been used for this configuration together with the proposed layered approach.

The third configuration refers to a woven glass/vinylester laminate with two serial pins joint tested in the paper of Karakuzu et al. [12]. The two holes are located in the

Table 3 Mechanical parameters utilized for the numerical simulations of composite laminates with multi-pins joints

Glass/epoxy lamina ^a						
Elastic moduli (GPa) and Poisson ratio	E_1	E_2	G_{12}	ν_{12}		
	38	8	4	0.27		
Strength (MPa)	X_t	X_c	Y_t	Y_c	S	
	687	223	74	109	78	
Glass/vinylester laminate ^b						
Elastic moduli (GPa) and Poisson ratio	E_1	E_2	G_{12}	ν_{12}		
	20.77	20.77	4.13	0.09		
Strength (MPa)	X_t	X_c	Y_t	Y_c	S	
	395	260	395	260	75	

^aAfter Karakuzu et al. [14]

^bAfter Karakuzu et al. [12]

positions 3 and 4 of Fig. 5. The mechanical properties of the laminate are reported in Table 3, while its geometrical parameters are: $L = 85$ mm, $t = 2.8$ mm, $D = 5$ mm. For this specimen type, 40 different geometries were tested considering the ratios $W/D = 2, 4$ and M/D and E/D variable from 2 to 5 and from 1 to 5, respectively. In this case the FE models utilized had a number of 16 nodes single layer shell elements varying from 275 to 530.

Finally, the fourth configuration refers again to a woven glass/vinylester laminate but with two parallel pins joint (see e.g. Karakuzu et al. [13]). With reference to Fig. 5 the two holes are located in the positions 1 and 2. In this case the geometrical and mechanical parameters of the laminate are the same of configuration three, while the 45 different geometries considered are obtained by varying the ratio K/D from 2 to 4, the ratio E/D from 1 to 5, and considering the ratios $M/D = 2, 4, 5$. For this specimen type a FE modelling involving 16 nodes single layer shell elements has been again employed with a number of elements variable from 448 to 725.

6.2 Peak Load Prediction

For sake of synthesis, the obtained numerical results are organized in diagram format. In particular Figs. 6, 7 report, for the 6 specimen types related to the first laminate configuration, the comparison between the experimental findings and the numerical predictions in terms of peak load. Each Figure reports different couples of E/D and W/D ratios, the pertinent experimentally detected values of peak load and the predicted values, in terms of upper and lower bounds, given by the layer-by-layer and equivalent layer formulation of LMM and ECM. Precisely, the plots given in Fig. 6 refer to $E/D = 4$ and W/D variable, while the plots in Fig. 7 refer to $W/D = 4$ and E/D variable. The upper bound values predicted by the proposed layer-by-layer limit analysis are always above, or are practically coincident with the experimental ones. The same is definitely not true for the single layer treatment

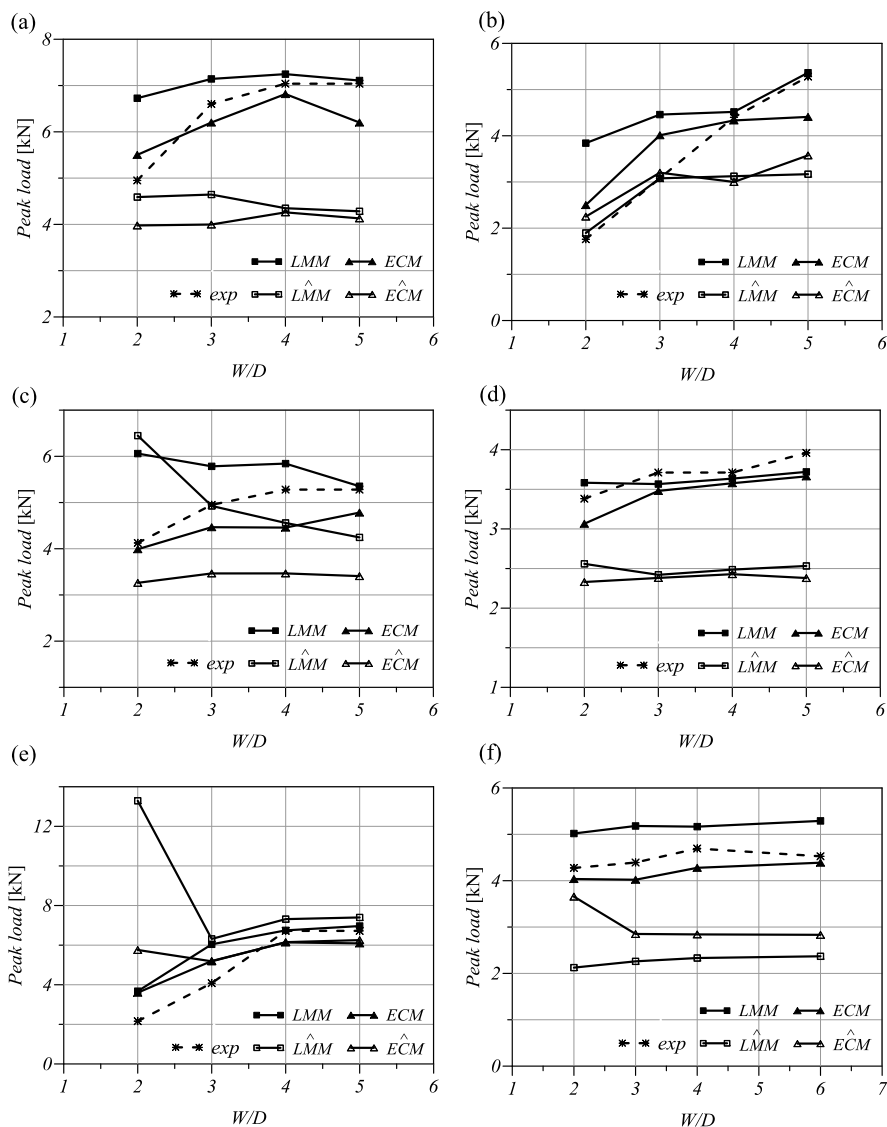


Fig. 6 Peak load values for fixed ratio $E/D = 4$; comparison between experimental data, dashed lines with asterisk marker, and values predicted by LMM and ECM respectively. (a) Specimen type A; (b) Specimen type B; (c) Specimen type C; (d) Specimen type D; (e) Specimen type E; (f) Specimen type F

where the evaluated upper bounds appear far, and even below, from the experimental values. Analogously, the lower bounds evaluated with the layer-by-layer analyses are closer (from below) to the experimental values than the ones given by the single layer treatment, too much conservative. As expected, good accuracy is attained by

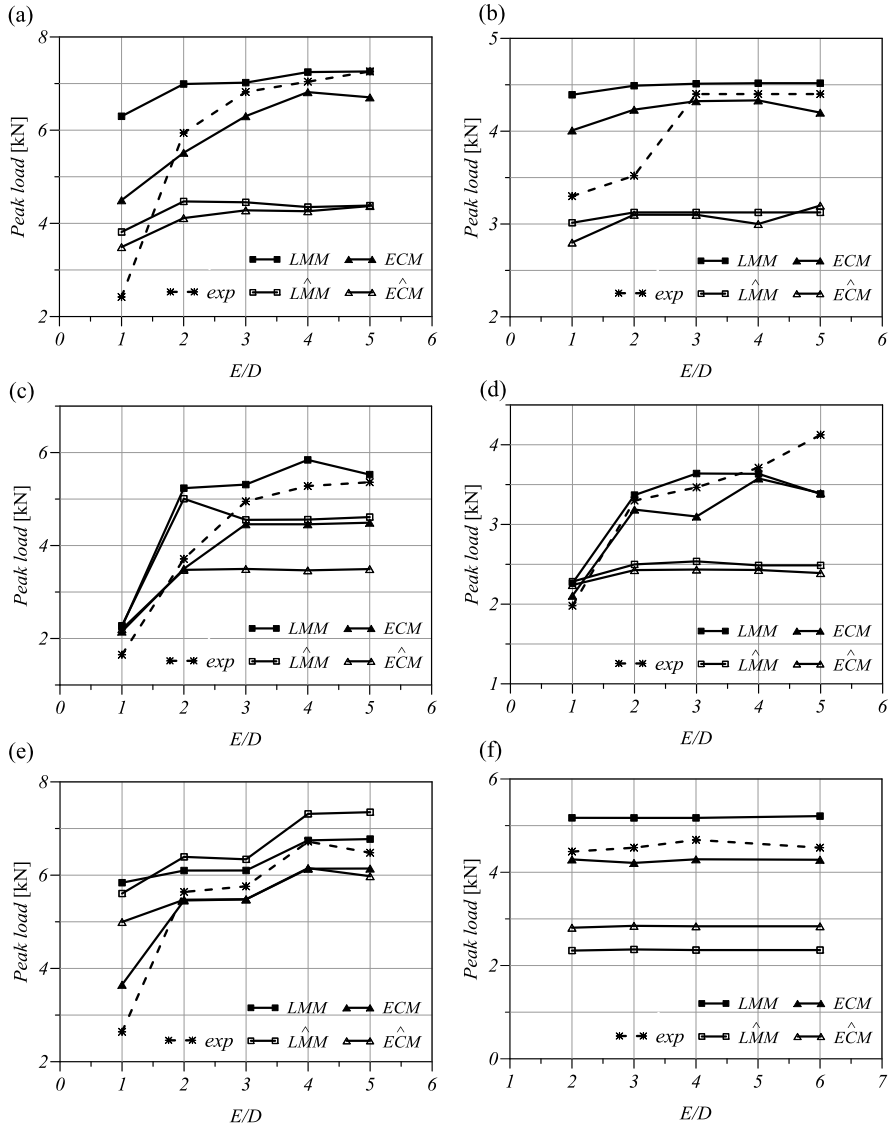
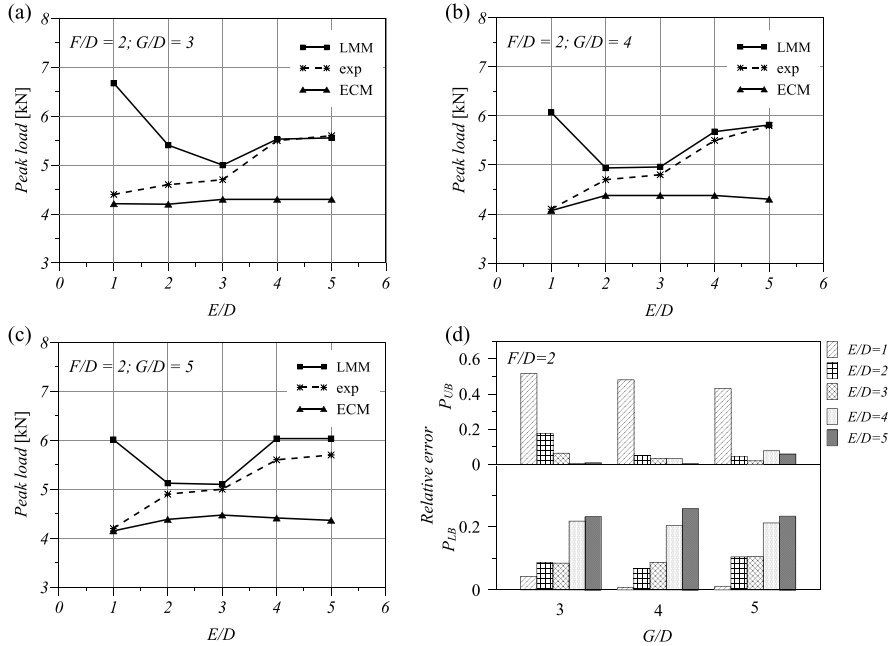


Fig. 7 Peak load values for fixed ratio $W/D = 4$; comparison between experimental data, dashed lines with asterisk marker, and values predicted by LMM and ECM respectively. (a) Specimen type A; (b) Specimen type B; (c) Specimen type C; (d) Specimen type D; (e) Specimen type E; (f) Specimen type F

both single and layer-by-layer procedures when dealing with laminate lay-ups of the type $[\pm 45^\circ]_{ns}$, as the one of specimen E of Figs. 6(e) and 7(e), for which the anisotropy of each layer is mitigated by the sequence itself. The average relative errors, computed as the absolute value of the difference between the experimental

Table 4 Average relative error of the peak load values computed with the layer-by-layer and the single layer approach

Specimen type	LMM	$\hat{LMM}$	ECM	$\hat{ECM}$
A	0.29	0.34	0.17	0.36
B	0.29	0.19	0.17	0.23
C	0.20	0.23	0.12	0.28
D	0.07	0.29	0.08	0.32
E	0.33	0.92	0.20	0.39
F	0.15	0.49	0.06	0.34

**Fig. 8** Peak load values of the multi-layer specimens $[0^\circ/90^\circ/\pm 45^\circ]_S$ for $F/D = 2$. (a) $G/D = 3$; (b) $G/D = 4$; (c) $G/D = 5$; (d) Plot of the relative error given by the two procedures

and the numerical detected peak load values over the experimental ones, are shown in Table 4 for the layered analysis and for the single layer one. As it appears the range of the computed load values bracketing the real one is, by far, more accurate for the layered approach.

With reference to the second configuration (a three holes joint), Figs. 8, 9 and 10 report the comparison between the experimental findings and the numerical predictions in terms of peak load for fixed values of F/D and different values of E/D and G/D . In addition, Figs. 8(d), 9(d) and 10(d) give the relative errors of the numerically predicted P_{UB} and P_{LB} with respect to the corresponding experimentally detected values.

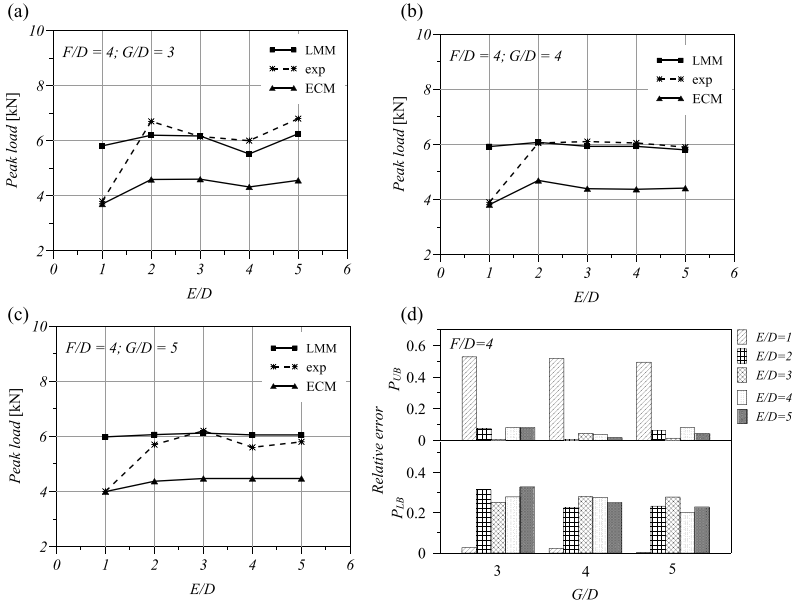


Fig. 9 Peak load values of the multi-layer specimens $[0^\circ/90^\circ/\pm 45^\circ]_S$ for $F/D = 4$. (a) $G/D = 3$; (b) $G/D = 4$; (c) $G/D = 5$; (d) Plot of the relative error given by the two procedures

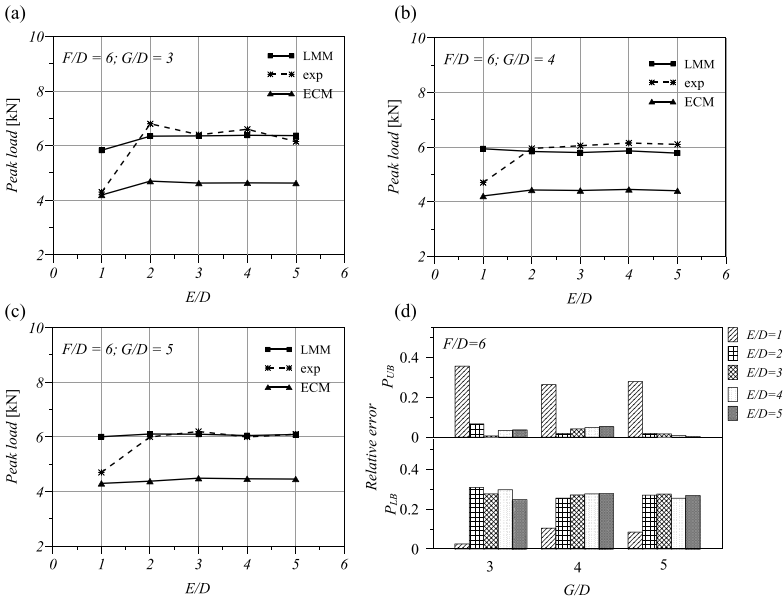


Fig. 10 Peak load values of the multi-layer specimens $[0^\circ/90^\circ/\pm 45^\circ]_S$ for $F/D = 6$. (a) $G/D = 3$; (b) $G/D = 4$; (c) $G/D = 5$; (d) Plot of the relative error given by the two procedures

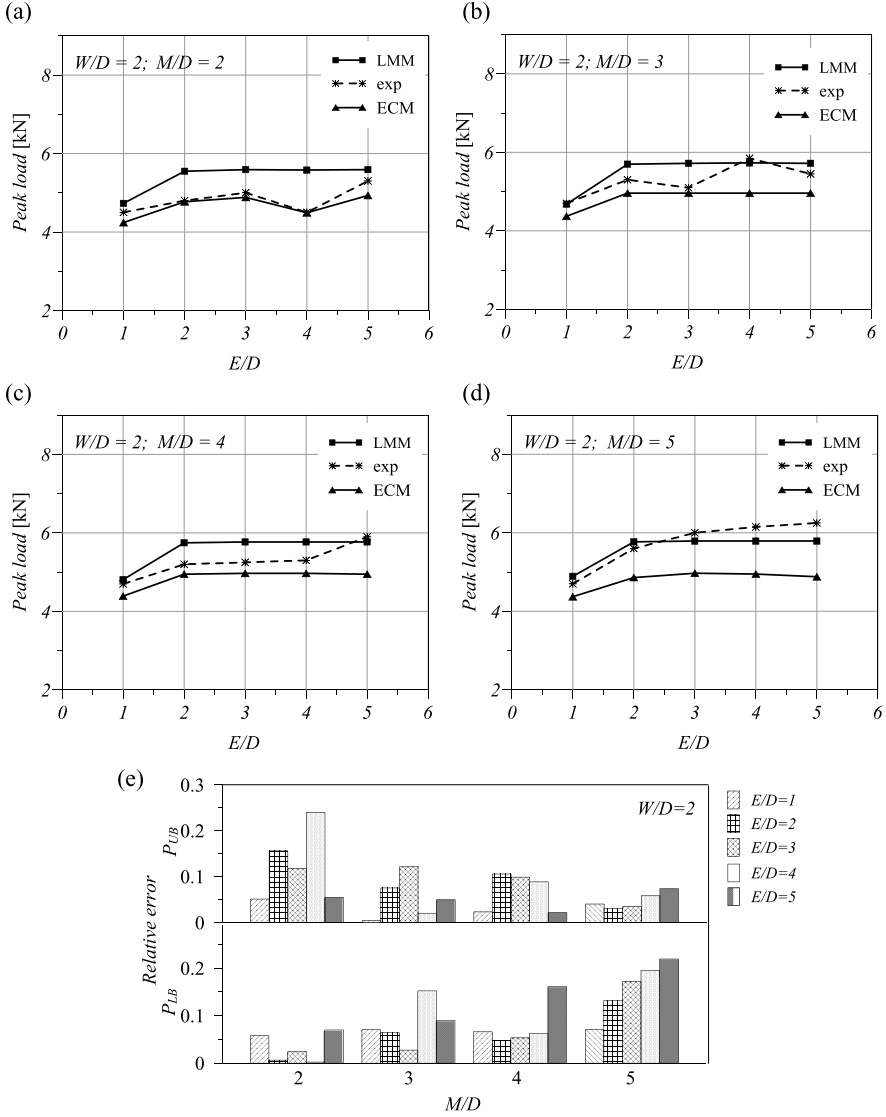


Fig. 11 Peak load values of the specimens with two serial fastener holes for $W/D = 2$. (a) $M/D = 2$; (b) $M/D = 3$; (c) $M/D = 4$; (d) $M/D = 5$; (e) Plot of the relative error given by the two procedures

For the third configuration (joint with two serial pins), the experimental and predicted peak load values are reported in Figs. 11, 12 for fixed values of W/D and different values of E/D and M/D .

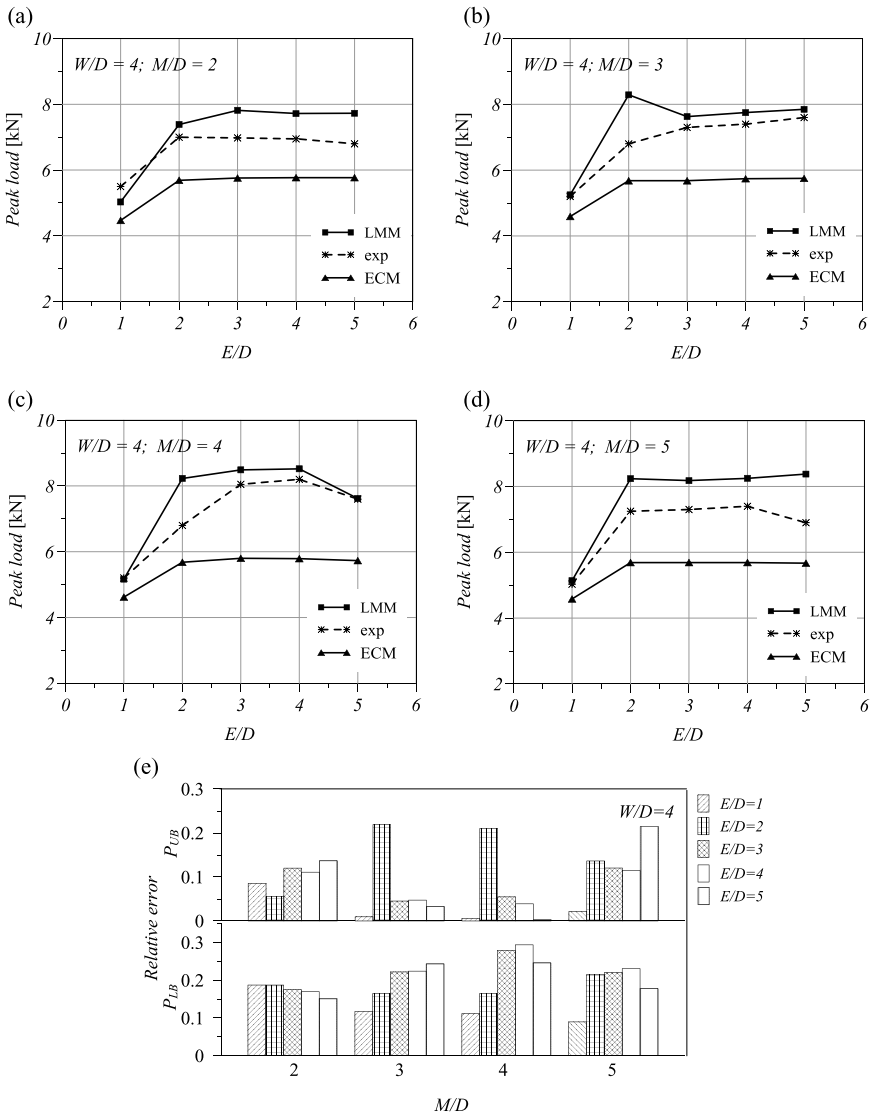


Fig. 12 Peak load values of the specimens with two serial fastener holes for $W/D = 4$. (a) $M/D = 2$; (b) $M/D = 3$; (c) $M/D = 4$; (d) $M/D = 5$; (e) Plot of the relative error given by the two procedures

Finally Figs. 13, 14 and 15 report the experimental and predicted peak load values for the two parallel pins joint (fourth configuration), for fixed values of M/D and different values of E/D and K/D .

By inspection of the obtained results, in 23 over the 177 examined cases the experimental peak value lies outside the range numerically predicted, such cases are

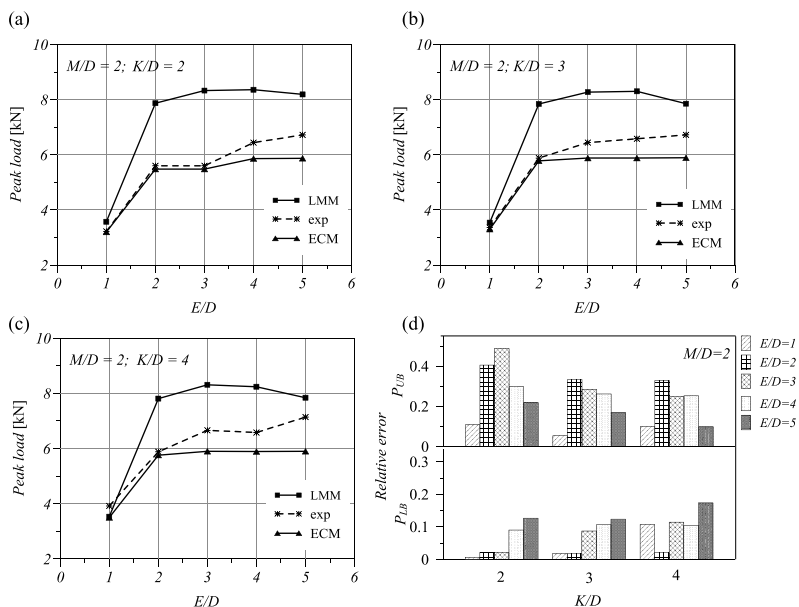


Fig. 13 Peak load values of the specimens with two parallel fastener holes for $M/D = 2$. (a) $K/D = 2$; (b) $K/D = 3$; (c) $K/D = 4$; (d) Plot of the relative error given by the two procedures

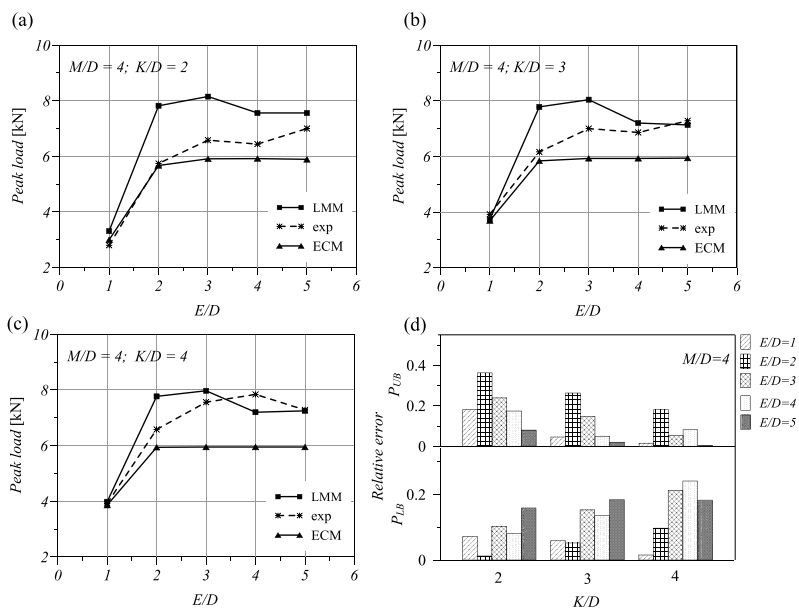


Fig. 14 Peak load values of the specimens with two parallel fastener holes for $M/D = 4$. (a) $K/D = 2$; (b) $K/D = 3$; (c) $K/D = 4$; (d) Plot of the relative error given by the two procedures

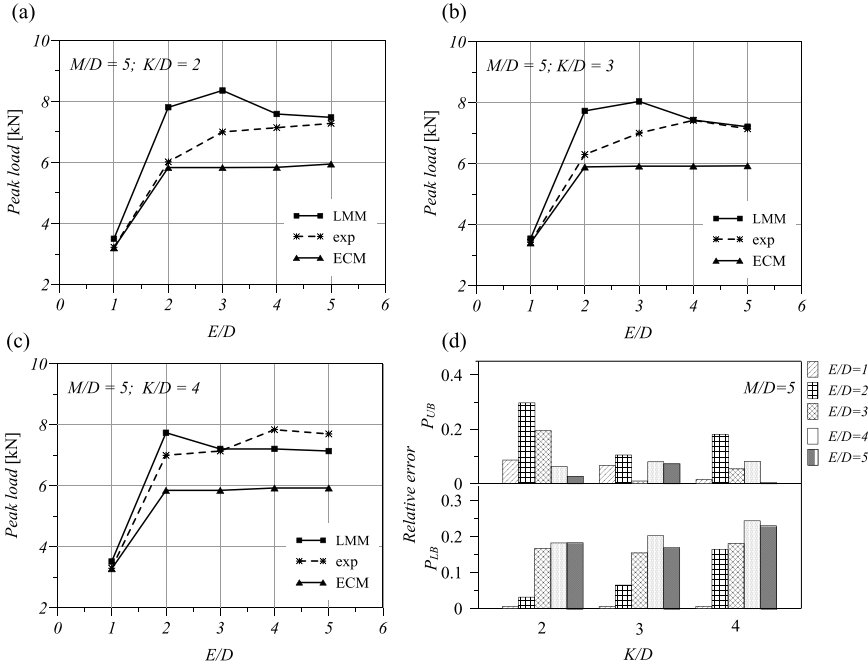


Fig. 15 Peak load values of the specimens with two parallel fastener holes for $M/D = 5$. (a) $K/D = 2$; (b) $K/D = 3$; (c) $K/D = 4$; (d) Plot of the relative error given by the two procedures

simply wrong predictions of the adopted methodology. A threefold reasons induced the authors to accept such circumstance: (i) the overall percentage of wrong predictions is about 13 %, i.e. a low value from an engineering point of view; (ii) the relative error, also in such cases, is very low; (iii) the wrong predictions are mainly concerned, in 21 over 23 cases, the upper bound to the peak load value, a value not directly affecting the design choices as the lower bound does. The P_{LB} being in practice always correct. Nevertheless, further investigations to this concern might be necessary.

Finally, Fig. 16 shows, for eight of the 177 analyzed specimens (i.e. two for each different laminate configuration considered), the plots of the upper and the lower bounds to the peak load versus the iterations number. In all cases a monotonic and rapid convergence of the methods is assured by a condition for convergence given in Ponter et al. [29]. Such circumstance, essential for any iterative based procedure, is met in all the examined cases but here is not reported for brevity.

The proposed methodology seems, at least for all the examined specimens, definitely able to predict, with good accuracy, the peak load of a multi-pin joints laminate.

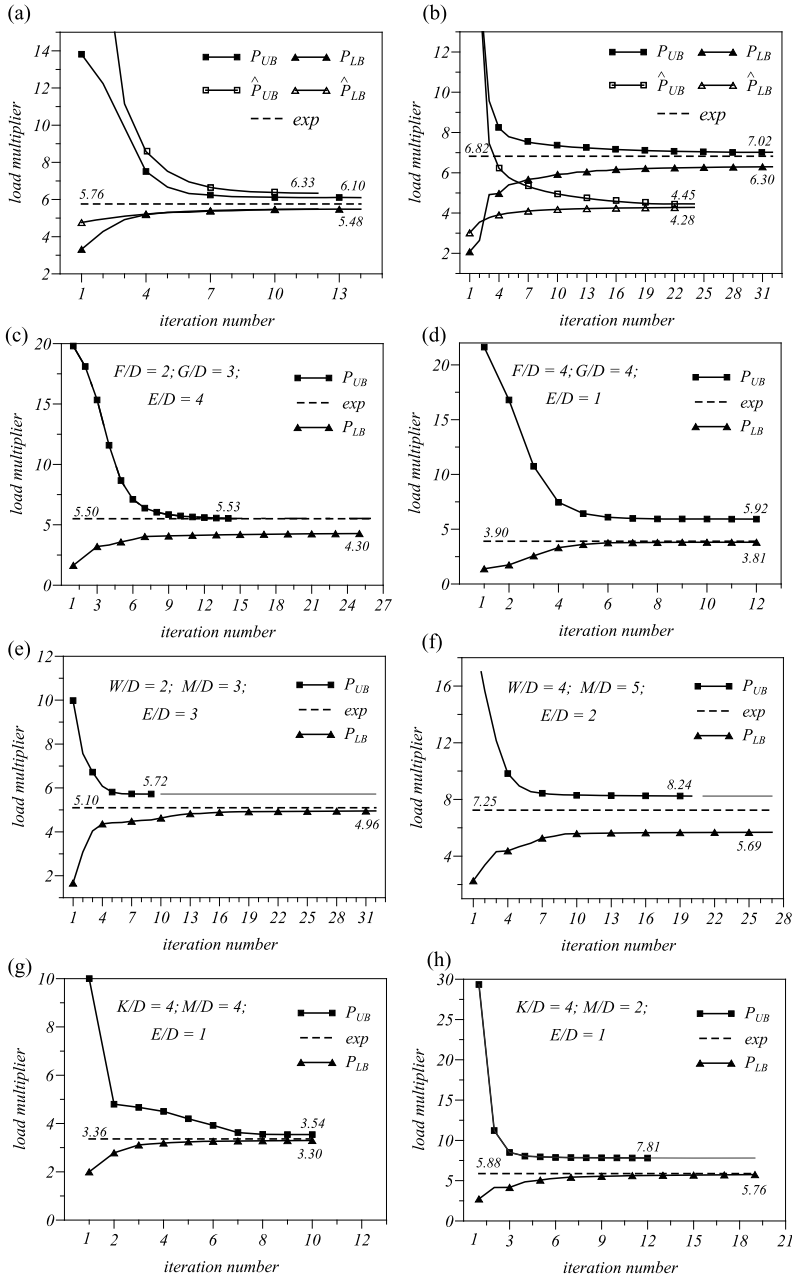


Fig. 16 Values of the upper (P_{UB}) and lower (P_{LB}) bounds to the collapse load multiplier versus iteration number. LMM prediction, solid lines with rectangular markers; ECM prediction, solid lines with triangular markers; collapse experimental threshold, finer dashed lines. (a) and (b): multilayer specimens E3 and A3 with single fastener hole; (c) and (d): multilayer specimens with three fastener holes; (e) and (f): woven specimens with two serial fastener holes; (g) and (h): woven specimens with two parallel fastener holes

Table 5 Failure modes: experimental data against predicted ones for the multi-layer specimens with one fastener hole

<i>LSS</i>	<i>E/D</i>	<i>W/D</i>	LMM	exp ^a
[90°/±45°] _S	5	5	B	B
[90°/0°/90°] _S	4	3	B	B
[0°/90°/0°] _S	2	5	S	S
[0°/90°/0°] _S	2	4	S	S
[±45°] _{2S}	3	4	B	N

^aAfter Okutan Baba [22]

Table 6 Failure modes: experimental data against predicted ones for the multi-layer specimens [0°/90°/±45°]_S with three fastener holes

<i>E/D</i>	<i>G/D</i> = 3		<i>G/D</i> = 4		<i>G/D</i> = 5	
	LMM	exp ^a	LMM	exp ^a	LMM	exp ^a
<i>F/D</i> = 2						
1	S/S/B	S/S/B	S/S/B	S/S/B	S/S/B	S/S/B
2	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
3	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
4	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
5	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
<i>F/D</i> = 4						
1	S/S/B	S/S/B	S/S/B	S/S/B	S/S/B	S/S/B
2	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
3	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
4	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
5	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
<i>F/D</i> = 6						
1	S/S/B	S/S/B	S/S/B	S/S/B	S/S/B	S/S/B
2	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
3	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
4	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B
5	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B	B/B/B

^aAfter Karakuzu et al. [14]

6.3 Failure Modes Prediction

In order to gain a deeper comprehension of the mechanical joint behaviour, beside the evaluation of the peak load it is of equally importance the prediction of the joint failure mode. Typically, with reference to a single fastener/pin hole, three main/primary failure modes can be individuated (bearing, net-tension, shear-out), other modes (tear-out, cleavage, ...) being a combination of them, also called secondary modes. When a multi-pin fastening is considered the real failure modes can be more complex for the mutual interaction of the fastenings affected by their ge-

Table 7 Failure modes: experimental data against predicted ones for the woven specimens with two serial fastener holes

E/D	$M/D = 2$		$M/D = 3$		$M/D = 4$		$M/D = 5$	
	LMM	exp ^a	LMM	exp ^a	LMM	exp ^a	LMM	exp ^a
$W/D = 2$								
1	N/N	N/N	N/S	N/N	N/S	N/N	N/-	N/N
2	N/-	N/-	N/-	N/-	N/-	N/-	N/-	N/-
3	N/-	N/-	N/-	N/-	N/-	N/-	N/-	N/-
4	N/-	N/-	N/-	N/-	N/-	N/-	N/-	N/-
5	N/-	N/-	N/-	N/-	N/-	N/-	N/-	N/-
$W/D = 4$								
1	B/SN	B/S	B/S	B/S	B/S	B/S	B/S	B/S
2	BS/BS	B/B	B/BS	B/B	B/BS	B/B	B/BS	B/B
3	B/B	B/B	B/BN	B/B	B/B	B/B	B/B	B/B
4	B/BN	B/B	B/BN	B/B	B/B	B/B	B/B	B/B
5	B/BN	B/B	B/BN	B/B	B/B	B/B	B/B	B/B

^aAfter Karakuzu et al. [12]**Table 8** Failure modes: experimental data against predicted ones for the woven specimens with two parallel fastener holes

E/D	$K/D = 2$		$K/D = 3$		$K/D = 4$	
	LMM	exp ^a	LMM	exp ^a	LMM	exp ^a
$M/D = 2$						
1	S	S	S	S	S	S
2	S	BN	S	BS	S	BS
3	BS	BN	BS	BS	BS	BS
4	BNS	BN	BS	BS	BS	BS
5	BN	BN	B	B	B	B
$M/D = 4$						
1	S	S	S	S	S	S
2	S	B	S	B	S	B
3	B	B	B	B	B	B
4	B	B	B	B	B	B
5	B	B	B	B	B	B
$M/D = 5$						
1	S	S	S	S	S	S
2	S	B	S	B	S	B
3	B	B	B	B	BS	B
4	B	B	B	B	B	B
5	B	B	B	B	B	B

^aAfter Karakuzu et al. [13]

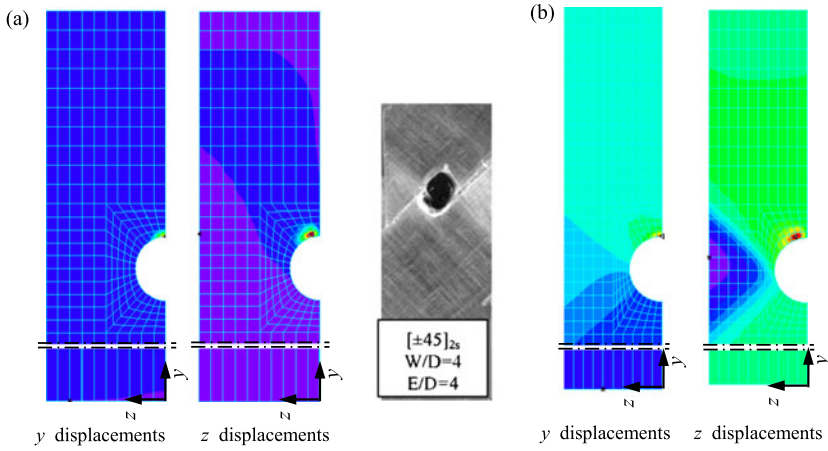


Fig. 17 Pin-loaded plate of Fig. 4: (a) predicted collapse mechanism of bearing type obtained for specimen #E4 with $W/D = 4$; (b) predicted collapse mechanism of net-tension type for specimen #E4 with $W/D = 3$ against experimental one (after Okutan Baba [22])

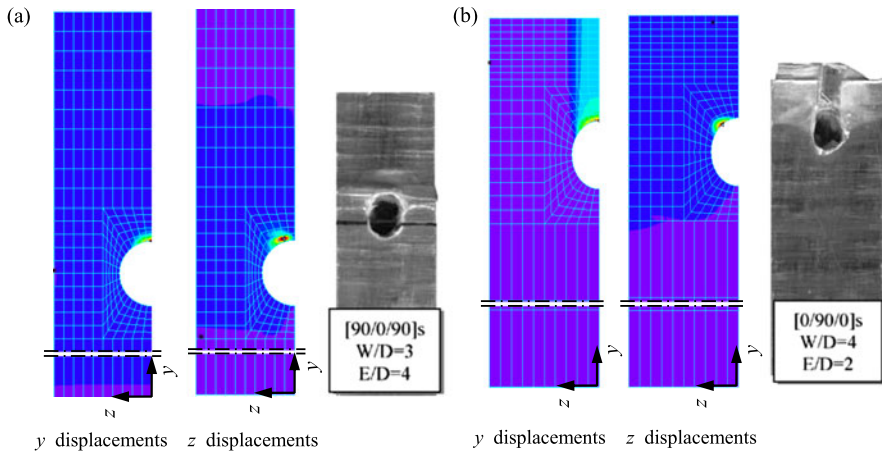


Fig. 18 Pin-loaded plate of Fig. 4: (a) predicted collapse mechanism of bearing type against experimental one for specimen #D7; (b) predicted collapse mechanism of shear-out type against experimental one for specimen #C2 (after Okutan Baba [22])

ometrical distribution (see e.g. [37]) and the possibility of a numerical prediction becomes more important.

The prediction of the failure mode is herein pursued numerically making use of the LMM which builds the collapse mechanism the joint exhibits when the loads attain their peak value or, more exactly, they reach the evaluated upper bound value to such peak. The obtained results are organized either in tabular or in band plots format. In particular, Tables 5, 6, 7 and 8 report the comparison between the experi-

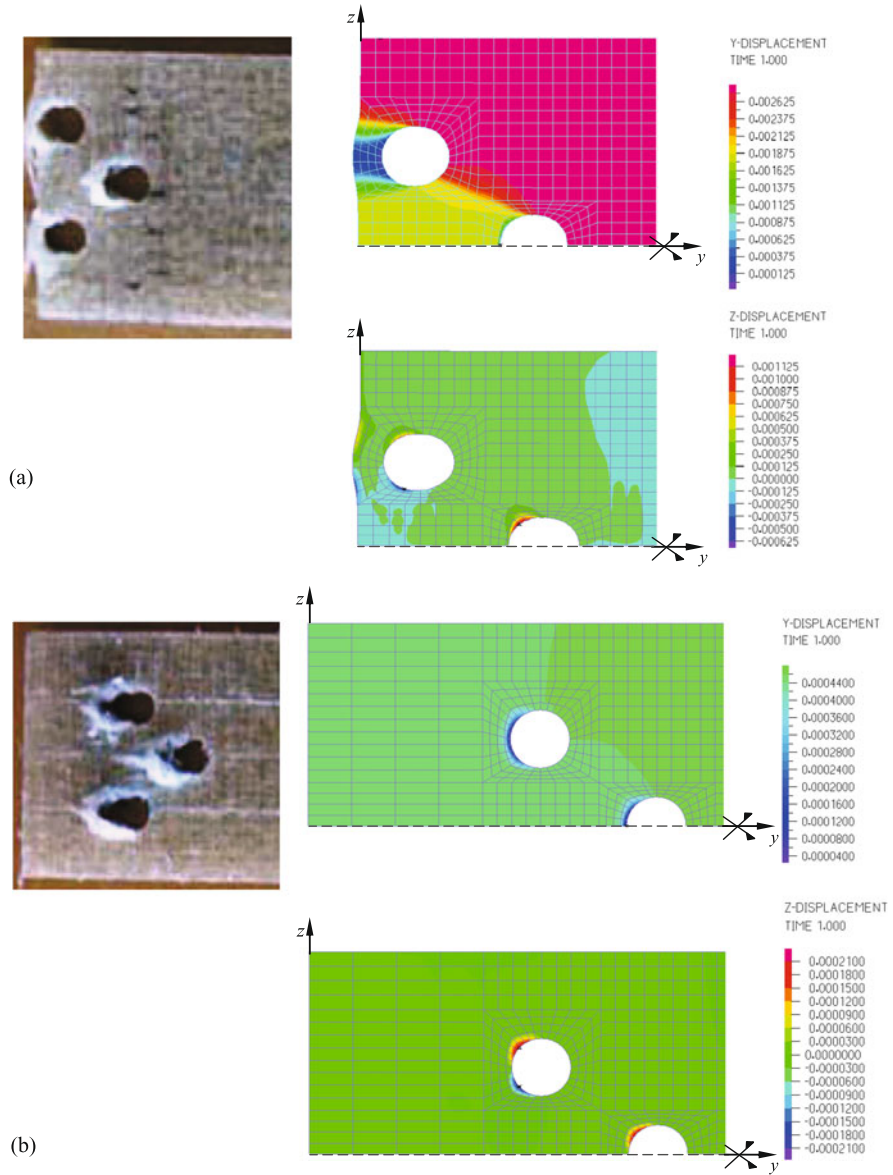


Fig. 19 Three-pins-loaded plate of Fig. 5. Failure modes predicted by numerical simulations against experimentally detected ones (after Karakuzu et al. [14]): (a) Shear-out at outer holes and Bearing at inner hole for $F/D = 2$, $G/D = 3$, $E/D = 1$; (b) Bearing at outer and inner holes for $F/D = 2$, $G/D = 3$, $E/D = 4$

mental findings and the numerical predictions in terms of collapse/failure modes for all the geometries analyzed, all grouped with respect to the four considered config-

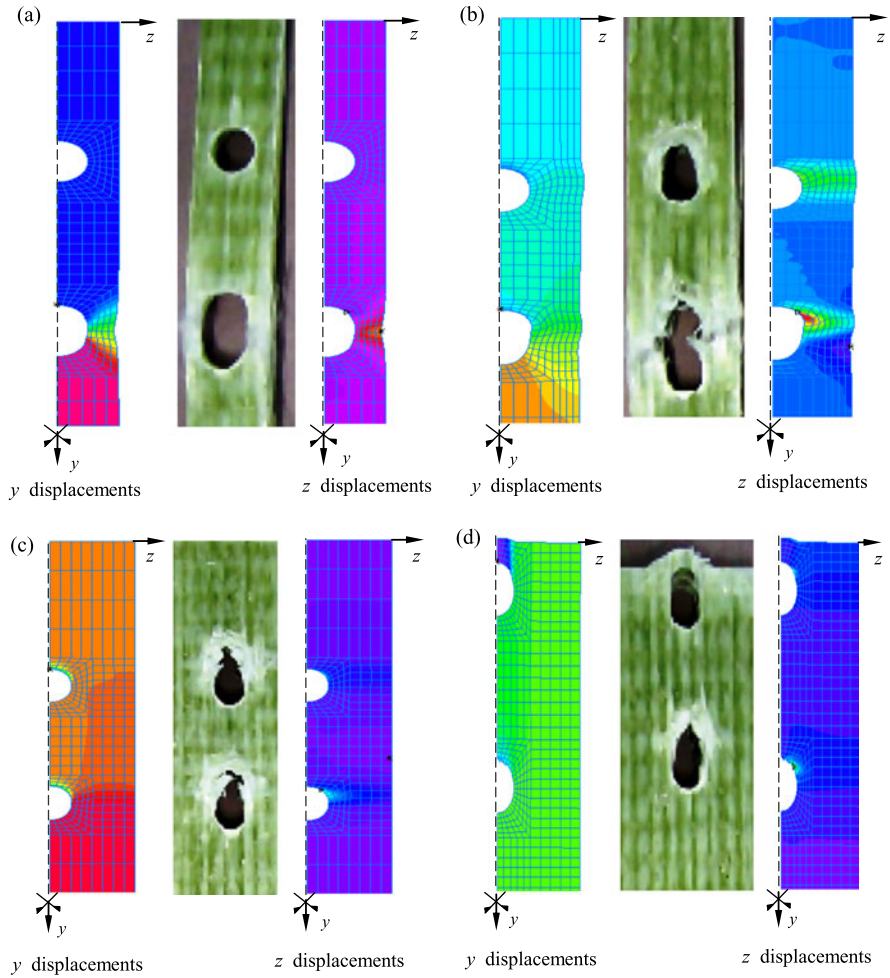


Fig. 20 Two-serial pins-loaded plate of Fig. 5. Failure modes predicted by numerical simulations against experimentally detected ones (after Karakuzu et al. [12]): (a) Net-tension at inner hole and no failure at outer hole for $W/D = 2$, $M/D = 4$, $E/D = 4$; (b) Net-tension at inner hole and bearing at outer hole for $W/D = 3$, $M/D = 4$, $E/D = 5$; (c) Bearing at inner hole and bearing at outer hole for $W/D = 4$, $M/D = 4$, $E/D = 5$; (d) Bearing at inner hole and shear-out at outer hole for $W/D = 5$, $M/D = 4$, $E/D = 1$

urations. In the above tables, as usual in the relevant literature, the symbol B means bearing, N net-tension, S shear-out, while when more symbols appear separated by slash (/) the first refers to the failure mode of the outer hole the second to that of the inner hole; eventually the symbol (–) means that no failure occurred at the considered hole.

In Figs. 17–21 the photographs of some tested specimens, at the joint failure stage, are reported together with the band plots, on the FE mesh, of the node dis-

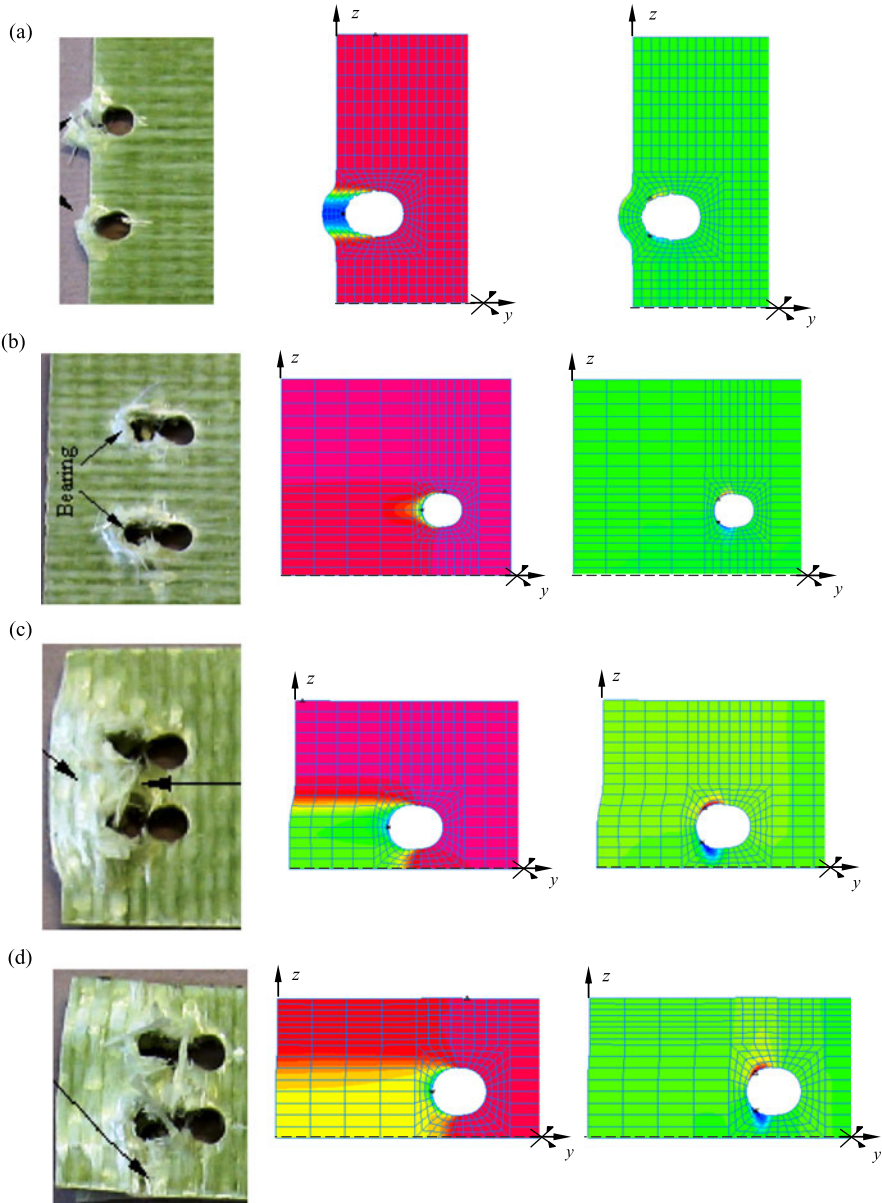


Fig. 21 Two-parallel pins-loaded plate of Fig. 5. Failure modes predicted by numerical simulations against experimentally detected ones (after Karakuzu et al. [13]): (a) Shear-out for $K/D = 4$, $M/D = 4$, $E/D = 1$; (b) Bearing for $K/D = 4$, $M/D = 4$, $E/D = 5$; (c) Bearing and Shear-out for $K/D = 3$, $M/D = 2$, $E/D = 3$; (d) Bearing and net-tension for $K/D = 2$, $M/D = 2$, $E/D = 4$

placement rates components (in y and z directions) at collapse. The latter, evaluated at last converged iteration of the LMM procedure, locate the distribution of the displacement rates the real structure exhibits at a state of incipient collapse, i.e. the one attained when the loads reach their peak value or, the computed upper bound value on it. Such distribution furnishes the searched failure mode but, obviously, only from a qualitative point of view; what it is meaningful is, in practice, only the direction of the computed displacement rates. By inspection of these figures it can be stated that the present procedure seems able to predict, in a quite good manner, the three multi-pin joints basic failure modes of bearing, shear-out and net-tension.

7 Concluding Remarks

The results of recent studies, concerning a design methodology based on numerical limit analysis and aimed at the evaluation of the load bearing capacity of single- and multi-pin joints in orthotropic composite laminates, have been presented.

The proposed methodology uses two numerical methods for limit analysis: the Linear Matching Method and the Elastic Compensation Method. The lack of associativity, postulated for the adopted Tsai-Wu-type yield criterion for orthotropic laminates, obliges to search for an upper and a lower bound to the joint peak load multiplier, the former pursued by the LMM, the latter by the ECM.

Both methods have been rephrased and adapted to the assumed yield criterion and to a layered treatment of the tackled mechanical problem. It is worth noting that the limit analysis carried out at lamina level, taking into account the stacking sequence of the laminate, appears very effective to deal with many of the through-thickness effects influencing the joint behaviour. Such layered formulation is indeed of general applicability.

The results, obtained for a remarkable number of tests on real prototypes, show a good ability of the expounded methodology to bracket the peak load detected via laboratory tests; a fairly good skillfulness to localize the collapse zone and to predict the related collapse mode; a good performance of the layered approach when dealing with different joint geometries and laminate lay-ups.

Such information are definitively very useful, at least at a preliminary stage of a design process. Concerning the tackled problem for example, limit analysis can be used for some design choices either related to the laminate setting, e.g. better lay-up, or appropriate components, for example, or the joint setting, in terms of fastener holes distribution, holes dimensions, ratios between fasteners and laminate geometrical parameters, etc. Once a good prototype has been settled more accurate and expensive numerical analyses or experimental tests can be carried out on it with considerable saving of money and time. Extension of such approach to other problems of engineering interest seem straightforward.

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