Chapter 2
Bifurcation Theory

Nonlinear systems, unlike their linear counterparts, can spontaneously break the symmetry constraints imposed by their environment for some range of values of the control parameter via a bifurcation scenario. *Bifurcation* or *branching* occurs in a nonlinear system when the state of the system depends on some parameter μ, and as that parameter varies the state branches to another state at some critical value μ_c of the parameter μ with usually a concomitant change of stability.[1] Alternatively, one may visualize this via intersection of different types of manifolds enabling the solution flow to switch from one manifold to another with a concomitant change in its stability property. The loss of hyperbolicity of the equilibrium, according to the *Hartman-Grobman Theorem* (Chap. 1), implies that the local behavior of the flow near the *bifurcation point* cannot be described by the linearized flow. On the other hand, the bifurcation process is determined by the dynamics on the *center manifold* at the bifurcation point. The *normal-forms reduction* on the center manifold proves to be convenient for a discussion of local bifurcation because, as we saw in Chap. 1, this causes a reduction in dimensionality and, therefore, proves especially helpful in the bifurcation analysis of high-dimensional systems (Crawford 1991).

When there is a multiplicity of solutions for a system, the solution sought out by the system is determined by the stability considerations. The bifurcation theory is concerned with how the multiplicity of solutions varies with the parameter μ and the stability properties of the bifurcating solutions. In this chapter, we will consider *local* bifurcation theory that addresses phenomena near a single point; for a discussion of *global* bifurcations that often involve *homoclinic* and *heteroclinic* bifurcations (see Chap. 5) see Wiggins (1988).

Bifurcations are classified according to how the stability of an equilibrium solution changes. There are two ways in which this can occur. An eigenvalue of the system linearized about equilibrium solution can pass through zero, or a pair of non-zero eigenvalues may cross the imaginary axis. The first case corresponds to a *saddle-node* or *tangent bifurcation* and describes the birth or collapse of two equilibria (like a stable node coalescing with a saddle and annihilating it). This happens

[1] Here, stability, which is controlled by the system parameters, refers to *structural stability*.

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when a manifold associated with a given equilibrium intersects itself. On the other hand, when manifolds associated with different equilibria intersect, an exchange of stability occurs—this corresponds to a *transcritical bifurcation* or *pitchfork bifurcation*. The second case corresponds to the so-called *Hopf bifurcation*[2] (Hopf 1942) and describes the birth of a family of periodic orbits (*limit cycle*) following the change in stability of a focus.

2.1 Stability and Bifurcation

The goal of bifurcation theory is to determine the existence and stability of various branches of solutions like fixed points and periodic orbits. The various equilibria emerge from one another in a continuous manner as the bifurcation parameter μ varies across the bifurcation point $\mu = \mu_c$ and the local dynamics is contained in a suitably defined *center manifold* at the bifurcation point.

Consider the first-order autonomous differential equation describing flow in a one-dimensional phase space R,

$$\frac{du}{dt} = f(u; \mu), \quad t > 0 \tag{2.1}$$

where μ is a real parameter, and f is a given analytic function of u and μ with continuous partial derivatives of all order with respect to u and μ.

The equilibrium solution $u = \bar{u}$ of equation (2.1) is found from

$$f(\bar{u}; \mu) = 0. \tag{2.2}$$

If $\partial f/\partial u = 0$ in an open neighborhood of $\mu = \mu_c$, then corresponding to one value of μ, several equilibrium solutions $\bar{u}$ may exist. This is guaranteed by the *Implicit Function Theorem*[3] which allows the solutions of equation (2.2) to become non-unique, whenever $\partial f/\partial u = 0$.

The graph of equation (2.2) is called the *branching or bifurcation diagram*. The intersecting branches are the bifurcating solutions and the points of intersection, which correspond to change of stability, are called *bifurcation points*.

The stability of equilibrium solutions changes as μ varies. Often an equilibrium solution $\bar{u}(\mu)$ will be stable for $\mu < \mu_c$ and unstable for $\mu \geq \mu_c$. Thus, as μ is increased slowly, the equilibrium solution $\bar{u}(\mu)$ becomes unstable at μ_c, the system may therefore branch to another stable solution, if available. Bifurcation the-

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[2]This bifurcation scenario was in fact recognized earlier by Andronov et al. (1966).

[3]**Theorem** *Suppose that $f(u; \mu) : R \times R \to R$ is a C^1 function satisfying*

$$f(\bar{u}; \mu_c) = 0 \quad \text{and} \quad \frac{\partial f}{\partial u}(\bar{u}; \mu_c) \neq 0.$$

Then, there exists a unique solution of the implicit equation $f(\bar{u}; \mu) = 0$ given by $\bar{u} = g(\mu)$ in some open subset W containing μ_c.

ory seeks to explore how the stability of various equilibria changes as μ is varied near μ_c. This issue, of course, depends on the nonlinear nature of the problem in an essential way.

In order to determine stability of the equilibrium solution $\bar{u}(\mu)$, consider a small perturbation $\hat{u}(t)$ about $\bar{u}(\mu)$, so that

$$u(t) = \bar{u}(\mu) + \hat{u}(t). \tag{2.3}$$

Equation (2.1) then becomes

$$\frac{d\hat{u}}{dt} = f(\bar{u} + \hat{u}; \mu). \tag{2.4}$$

Linearizing f in $\hat{u}$, equation (2.4) becomes

$$\frac{d\hat{u}}{dt} \approx f_u(\bar{u}; \mu)\hat{u} \tag{2.5}$$

from which, we have

$$\hat{u}(t) = c e^{f_u(\bar{u};\mu)t}. \tag{2.6}$$

Thus, if $f_u(\bar{u}; \mu) < 0$, $\bar{u}$ is stable and vice versa. The bifurcation point μ_c is here defined by $f_u(\bar{u}; \mu) = 0$, along with $f(\bar{u}; \mu) = 0$. Now, by the *Implicit Function Theorem*, $f(\bar{u}; \mu) = 0$ implies $\mu = \mu(\bar{u})$ whenever $f_\mu(\bar{u}; \mu) \neq 0$. We have, on differentiating $f(\bar{u}; \mu) = 0$ with respect to $\bar{u}$,

$$f_{\bar{u}} + f_\mu \frac{d\mu}{d\bar{u}} = 0. \tag{2.7}$$

Equation (2.7) shows that $d\mu/d\bar{u} = 0$ at a bifurcation point (where $f_{\bar{u}} = 0$), if $f_\mu \neq 0$ there.

In higher (say, n) dimensions, one has in place of equation (2.1)

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(\mathbf{u}; \mu). \tag{2.8}$$

Let $\bar{\mathbf{u}}(\mu)$ be an equilibrium point of equation (2.8), so that

$$\mathbf{f}\big(\bar{\mathbf{u}}(\mu); \mu\big) = \mathbf{0}. \tag{2.9}$$

The stability of $\bar{\mathbf{u}}(\mu)$ is then determined by the eigenvalues $\lambda_1(\mu), \lambda_2(\mu), \ldots, \lambda_n(\mu)$ of the Jacobian matrix

$$\mathbf{A}(\mu) \equiv \frac{\partial \mathbf{f}(\bar{\mathbf{u}}(\mu); \mu)}{\partial \bar{\mathbf{u}}}. \tag{2.10}$$

If all the eigenvalues have a negative real part, then $\bar{\mathbf{u}}(\mu)$ is stable. On the other hand, if one or more eigenvalues have a positive real part, then $\bar{\mathbf{u}}(\mu)$ is unstable.

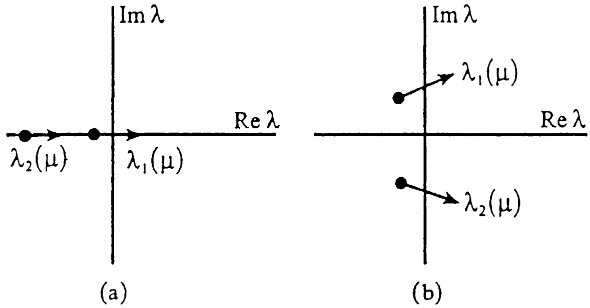


Fig. 2.1 Schematic diagrams for (**a**) saddle-node bifurcation and (**b**) Hopf bifurcation

Further, if the eigenvalues depend on the parameter μ, this stability may change as the parameter μ varies. In fact, the value of $\mu = \mu_c$, say, for which

$$\left.\begin{aligned} \operatorname{Re}\lambda_i(\mu_c) &= 0 \quad \text{for some } i \\ \operatorname{Re}\lambda_j(\mu_c) &< 0 \quad \text{for all } j \neq i \end{aligned}\right\} \tag{2.11}$$

define the bifurcation points. It is apparent that a bifurcation point can arise in two ways:

(i) $\lambda_1(\mu)$ is real-valued, $\lambda_1(\mu_c) = 0$, and $\operatorname{Re}\lambda_i(\mu_c) < 0$ for $i = 2, \ldots, n$;
(ii) $\lambda_1(\mu)$ and $\lambda_2(\mu)$ form a complex conjugate pair, so that $\lambda_1(\mu) = \bar{\lambda}_2(\mu) = \alpha(\mu) + i\beta(\mu)$, $\alpha(\mu_c) = 0$, $\beta(\mu_c) \neq 0$ and $\operatorname{Re}\lambda_i(\mu_c) < 0$ for $i = 3, \ldots, n$.

Note that for a one-dimensional system, only Case (i) can occur. Case (i) is called the *saddle-node bifurcation*. Case (ii) is called the *Hopf bifurcation*. Figure 2.1 shows the schematic diagram of the variation of the eigenvalues as μ varies through μ_c in Cases (i) and (ii) for a two-dimensional system.

Case (i) corresponds to the transition of the critical point $\bar{\mathbf{u}}(\mu)$ from a stable node into a saddle point. Case (ii) corresponds to the transition of the critical point $\bar{\mathbf{u}}(\mu)$ from a stable focus into an unstable focus and the appearance of a periodic solution.

2.2 Saddle-Node, Transcritical and Pitchfork Bifurcations

Let us consider first a few examples that exhibit the various bifurcation scenarios.

Example 2.1 Consider the equation

$$\frac{du}{dt} = f(u;\mu) = \mu - u^2. \tag{2.12}$$

The critical points of equation (2.12) correspond to

$$\mu - \bar{u}^2 = 0 \tag{2.13}$$

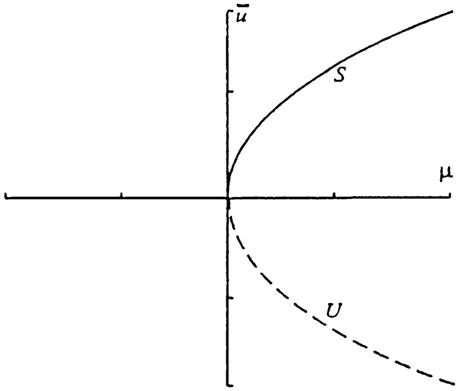


Fig. 2.2 Saddle-mode bifurcation

which are

$$\bar{u}_1 = \sqrt{\mu}, \qquad \bar{u}_2 = -\sqrt{\mu}. \tag{2.14}$$

These two branches of critical points intersect at the bifurcation point at $\mu = 0$. The existence of two branches (2.14), near $\mu = 0$ and $u = 0$, is allowed by the *Implicit Function Theorem*, because $f_u = -2u = 0$ at $u = 0$.

We have, from equation (2.12), further

$$f_u(\bar{u}_i; \mu) = -2\bar{u}_i, \quad i = 1, 2. \tag{2.15}$$

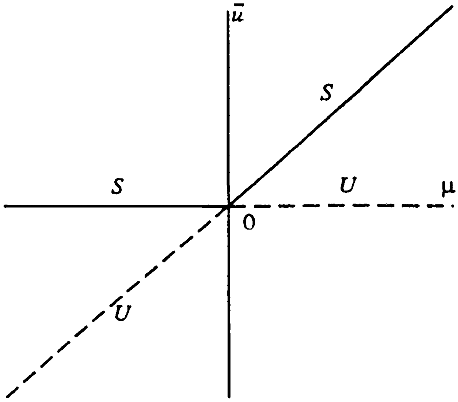
Therefore, the critical point is stable or unstable, according as $\bar{u}_i > 0$, which implies that $\bar{u}_1(\bar{u}_2)$ is stable (unstable). Thus, a single branch of critical points undergoes a transition from a stable to an unstable state, and there is an exchange of stability at $\bar{u} = 0$. Note that, here $f_\mu = 1 \neq 0$ at the bifurcation point, so (2.7) implies at $d\mu/d\bar{u} = 0$ there. As μ is varied, the two critical points $\bar{u}_1$ and $\bar{u}_2$ move toward each other, collide and destroy one another. This is an example of *saddle-node bifurcation* (see Fig. 2.2). Equation (2.12) is the canonical form of the saddle-node bifurcation. Saddle-node bifurcation provides the mechanism by which critical points (or equilibria) are created or destroyed.

Example 2.2 Consider the *logistic equation* (see Chap. 6),

$$\frac{du}{dt} = f(u; \mu) = \mu u - u^2. \tag{2.16}$$

Equation (2.16) is used as a simple model to describe population growth of a given species, the number of individuals of which is represented by u (May 1976). When u is sufficiently small the population grows or dies exponentially according as $\mu \gtrless 0$. If the population grows then, after sometime, it will have become so large that food shortage or predator activity takes effect and the growth rate will drop. The nonlinear term in equation (2.16) represents this saturation of the exponential population growth.

Fig. 2.3 Transcritical bifurcation



The critical points of equation (2.16) are found from

$$\mu\bar{u} - \bar{u}^2 = 0 \tag{2.17}$$

which are

$$\bar{u}_1 = 0, \qquad \bar{u}_2 = \mu. \tag{2.18}$$

These two branches of critical points intersect at the bifurcation point $\mu = 0$. The existence of two branches (2.18), near $\mu = 0$ and $u = 0$, is allowed by the *Implicit Function Theorem*, because $f_u = \mu - 2u = 0$ at $\mu = 0$ and $u = 0$.

We have, from equation (2.16),

$$f_u(\bar{u}_1; \mu) = \mu, \qquad f_u(\bar{u}_2; \mu) = -\mu. \tag{2.19}$$

Therefore, the critical point $\bar{u}_1$ is stable if $\mu < 0$, and unstable if $\mu > 0$, whereas $\bar{u}_2$ is stable if $\mu > 0$ and unstable if $\mu < 0$. Thus, the two branches have opposite stabilities and exchange stability at the bifurcation point $\mu = 0$. Note that, here $f_\mu = u = 0$ at the bifurcation point, so (2.7) implies that $d\mu/d\bar{u}$ may be non-zero there. This is an example of *transcritical bifurcation* (see Fig. 2.3).[4] Equation (2.16) is the canonical form of transcritical bifurcation.

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[4]Equation (2.16) has the following exact solution,

$$u(t) = \begin{cases} \frac{\mu}{1+(\mu/u_0-1)e^{-\mu t}}, & \mu \neq 0 \\ \frac{u_0}{1+u_0 t}, & \mu = 0 \end{cases}$$

where $u_0 \equiv u(t=0)$. This shows that, as $t \to \infty$,

$$u(t) \to \begin{cases} \mu, & \mu > 0 \\ 0, & \mu \leq 0, \end{cases}$$

as can also be appreciated from equation (2.16), or Fig. 2.3, for that matter!

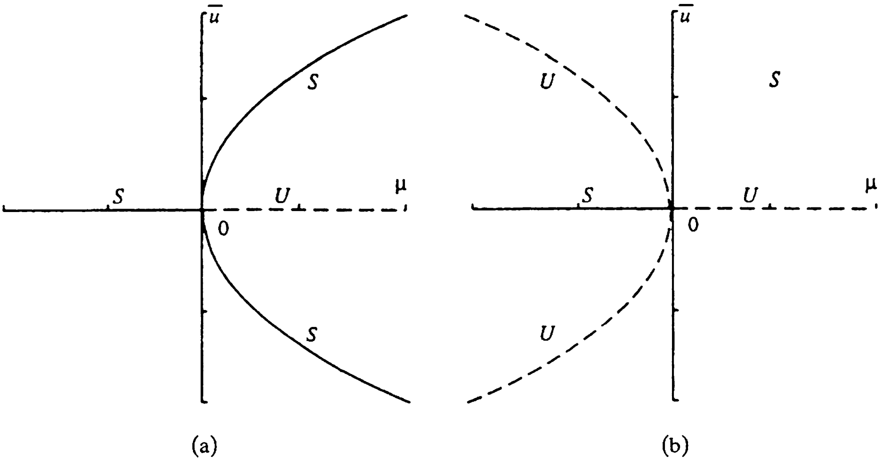

Fig. 2.4 Pitchfork bifurcation, (**a**) supercritical pitchfork, (**b**) subcritical pitchfork

Example 2.3 Consider the equation,

$$\frac{du}{dt} = f(u, \mu) = \mu u - u^3. \tag{2.20}$$

Equation (2.20) was introduced by Landau (1944) to describe the effect of nonlinearities on linear hydrodynamic instability, the amplitude of a perturbation being represented by u.

The critical points of equation (2.20) are found from

$$\mu \bar{u} - \bar{u}^3 = 0 \tag{2.21}$$

which are

$$\bar{u}_1 = 0, \qquad \bar{u}_2 = \sqrt{\mu}, \qquad \bar{u}_3 = -\sqrt{\mu}. \tag{2.22}$$

Equation (2.22) shows that, if $\mu \le 0$, there is only one branch, while, if $\mu > 0$, there are three branches. These three branches of critical points intersect at the bifurcation point at $\mu = 0$. The existence of three branches (2.22) near $\mu = 0$ and $u = 0$ is allowed by the *Implicit Function Theorem*, because $f_u = \mu - 3u^2 = 0$ at $\mu = 0$ and $u = 0$.

We have, from equation (2.20),

$$f_u(\bar{u}_1; \mu) = \mu, \qquad f_u(\bar{u}_{2,3}; \mu) = -2\mu. \tag{2.23}$$

Therefore, the critical point $\bar{u}_1$ is stable if $\mu < 0$, and unstable if $\mu > 0$, whereas $\bar{u}_{2,3}$ are stable if $\mu > 0$, and unstable if $\mu < 0$. These two branches have opposite stabilities and exchange stability at the bifurcation point $\mu = 0$. Note that, here $f_\mu = u = 0$ at the bifurcation point, but $d\mu/d\bar{u}$ is still zero there. This is an example of *supercritical pitchfork bifurcation* (see Fig. 2.4(a)). One obtains a *subcritical pitchfork bifurcation* (see Fig. 2.4(b)) if one changes the minus sign in equation (2.20) to a positive sign.[5] (The bifurcation is called *subcritical/supercritical* when new equilibria occur for the values of the parameter at which the original equilibrium is stable/unstable.) Equation (2.20) is the canonical form of the pitchfork bifurcation. Pitchfork bifurcation is generic to problems that have symmetry (note that equation (2.20) is invariant under the change of variables $u \to -u$).

Thus, the bifurcation points for transcritical and pitchfork bifurcations are determined by locating the points where the branches of critical points intersect, while saddle-node bifurcations are found by locating the points where $d\mu/d\bar{u} = 0$.

These results can now be summarized by the following theorem:

Theorem 2.1 *Consider a first-order autonomous equation*,

$$\frac{du}{dt} = f(u;\mu), \quad t > 0. \tag{2.24}$$

Suppose that $\bar{u} = \mu = 0$ is a bifurcation point and let $f(u;\mu)$ be an analytic function of u and μ in a neighborhood of $u = \mu = 0$ so that

$$f(0;0) = f_u(0;0) = 0. \tag{2.25}$$

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[5]Equation (2.20) has the following exact solution,

$$u(t) = \begin{cases} \frac{\sqrt{\mu}}{\sqrt{1+(\mu/u_0^2-1)e^{-2\mu t}}}, & \mu \neq 0 \\ \frac{u_0}{\sqrt{1+2u_0^2 t}}, & \mu = 0 \end{cases}$$

where $u_0 \equiv u(t=0)$. This shows that, as $t \to \infty$,

$$u(t) \to \begin{cases} \sqrt{\mu}, & \mu > 0 \\ 0, & \mu = 0, \end{cases}$$

the first of which describes the advent of a *supercritical equilibrium* following an initial exponential growth of the perturbation. Both branches above can be appreciated from equation (2.20), or Fig. 2.4, for that matter!

Equation (2.20), with a positive sign instead, has the following exact solution,

$$u(t) = \begin{cases} \frac{\sqrt{-\mu}}{\sqrt{1-(\mu/u_0^2+1)e^{-2\mu t}}}, & \mu \neq 0 \\ \frac{u_0}{\sqrt{1-2u_0^2 t}}, & \mu = 0. \end{cases}$$

When $\mu < 0$, this shows that,

$$u(t) \to 0 \quad \text{as } t \to \infty, \quad \text{if } |u_0| < \sqrt{-\mu}$$

$$u(t) \to \infty \quad \text{as } t \to \frac{1}{\sqrt{-2\mu}} \ln\left(\frac{u_0^2}{u_0^2+\mu}\right), \quad \text{if } |u_0| > \sqrt{-\mu},$$

which describe the possibility of a *subcritical instability* whenever the initial amplitude exceeds a critical threshold. Observe that the equilibrium states $\bar{u}_{2,3} = \pm\sqrt{-\mu}$ are now unstable.

Then,

(i) *if* $f_\mu(0;0) \neq 0$, *there exists*, *in some neighborhood of the bifurcation point* $\bar{u} = \mu = 0$, *a single branch of critical points which undergoes a saddle-node bifurcation at this point*;

(ii) *if* $f_u(0;0) = 0$ *and* $D \equiv f_{\mu\mu}(0;0) f_{uu}(0;0) - f_{u\mu}^2(0;0) > 0$, *the point* $\bar{u} = \mu = 0$ *is an isolated bifurcation point*; *if* $D < 0$, *there are two branches of critical points which intersect and exchange stability at the bifurcation point* $\bar{u} = \mu = 0$; *in the latter case*, *the bifurcation is either transcritical or pitchfork.*

Proof Since $f(u;\mu)$ is analytic, we have, in the neighborhood of the bifurcation point $\bar{u} = \mu = 0$, on using (2.25), the Taylor series,

$$f(u;\mu) = \alpha\mu + \frac{1}{2}a\mu^2 + b\mu u + \frac{1}{2}cu^2 + O\big(u^3, u^2\mu, u\mu^2, \mu^3\big) \tag{2.26}$$

where,

$$\left.\begin{aligned} \alpha &\equiv f_\mu(0;0), & a &\equiv f_{\mu\mu}(0;0), \\ b &\equiv f_{u\mu}(0;0), & c &\equiv f_{uu}(0;0) \end{aligned}\right\}. \tag{2.27}$$

Thus,

$$\left.\begin{aligned} f_u(u;\mu) &= b\mu + cu + O\big(u^2, u\mu, \mu^2\big) \\ f_\mu(u;\mu) &= \alpha + a\mu + bu + O\big(u^2, u\mu, \mu^2\big) \end{aligned}\right\}.$$

(i) In this case $\alpha \neq 0$, and the *Implicit Function Theorem* provides for a solution $\mu = \mu(\bar{u})$ of

$$f(\bar{u};\mu) = 0 \tag{2.28}$$

which is an analytic function of $\bar{u}$ and such that $\mu(0) = 0$. On using (2.25), (2.26) then gives

$$\left.\begin{aligned} \mu &= -\frac{c}{2\alpha}\bar{u}^2 + O\big(u^3\big) \\ f_u &= c\bar{u} + O\big(\bar{u}^2\big) \end{aligned}\right\}. \tag{2.29}$$

Equation (2.29) shows that if $c \neq 0$, f_u changes sign and stability is exchanged at the bifurcation point $u = 0$ near which the branch of critical points has the shape of a parabola. Besides, $d\mu/d\bar{u} = 0$ at the bifurcation point, as is necessary, since $f_\mu(0,0) = \alpha \neq 0$ here. Thus, this case corresponds to a saddle-node bifurcation.

(ii) For the case $\alpha = 0$, and $D \equiv ac - b^2 > 0$, the quadratic form

$$\frac{1}{2}a\mu^2 + b\mu u + \frac{1}{2}cu^2$$

cannot vanish for any real values of μ and u except $\mu = u = 0$. Therefore, the point $\mu = u = 0$, which is a solution of (2.28), is an isolated bifurcation point.

In order to consider the case $D<0$, let us first assume that $c\neq 0$; it then proves to be appropriate to introduce

$$u\equiv \mu\nu \tag{2.30}$$

and define

$$g(\nu;\mu)\equiv\frac{f(u;\mu)}{\mu^2}=\frac{1}{2}a+b\nu+\frac{1}{2}c\nu^2+O(\mu). \tag{2.31}$$

The critical points of equation (2.24) are then given by

$$g(\bar{\nu};\mu)=0. \tag{2.32}$$

Since,

$$g_\nu(\bar{\nu};0)=c\bar{\nu}+b=\pm\sqrt{-D}\neq 0$$

(which follows from (2.33) below) the *Implicit Function Theorem* provides for two distinct solutions $\bar{\nu}_1(\mu)$ and $\bar{\nu}_2(\mu)$ of (2.32) which are analytic functions of μ and such that $\bar{\nu}_1(0)=\tilde{\nu}_1$ and $\bar{\nu}_2(0)=\tilde{\nu}_2$, where $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are solutions of

$$g(\bar{\nu};0)=0$$

and are given by

$$\tilde{\nu}_{1,2}=\frac{-b\pm\sqrt{-D}}{c}. \tag{2.33}$$

This establishes the existence of two distinct branches of critical points $\bar{u}_1=\mu\bar{\nu}_1(\mu)$ and $\bar{u}_2=\mu\tilde{\nu}_2(\mu)$ which intersect at the bifurcation point $\mu=0$ and have slopes there given by $\tilde{\nu}_1$ and $\tilde{\nu}_2$.

Noting further that

$$f_u(\bar{u};\mu)=b\mu+c\bar{u}=\mu(c\nu+b)=\pm\mu\sqrt{-D}+O\left(\mu^2\right)$$

we see that the two branches above have opposite stabilities, and exchange stability at the bifurcation point. Thus, this case corresponds to a transcritical bifurcation.

On the other hand, for the case $c=0$, $D<0$ implies $b\neq 0$. In this case, it proves to be appropriate to introduce instead

$$\mu\equiv u\omega(u) \tag{2.34}$$

and define

$$h(u;\omega)\equiv\frac{f(u;u\omega)}{u^2}=\frac{1}{2}a\omega^2+b\omega+O(u). \tag{2.35}$$

The critical points of (2.24) are then given by

$$h(\bar{u},\bar{\omega})=0. \tag{2.36}$$

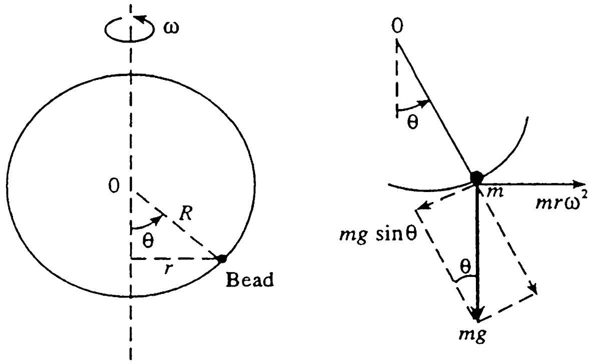


Fig. 2.5 Bead on a rotating hoop

which are

$$\bar{\omega}_{1,2} = O(\bar{u}), -\frac{2b}{a} + O(\bar{u}). \tag{2.37}$$

Since,

$$h_\omega(0, \bar{\omega}) = a\bar{\omega} + b = \pm b \neq 0$$

the *Implicit Function Theorem* provides for two distinct solutions $\bar{\omega}_1(\bar{u})$ and $\bar{\omega}_2(\bar{u})$ of (2.36) which are analytic functions of $\bar{u}$ and such that $\bar{\omega}_1(0) = 0, \bar{\omega}_2(0) = -2b/a$. This establishes the existence of two distinct branches of critical points $\mu = \bar{u}\bar{\omega}_1(\bar{u})$ and $\mu = \bar{u}\bar{\omega}_2(\bar{u})$ which intersect at the bifurcation point $\bar{u} = 0$. We have for the first branch $\mu = O(\bar{u}^2)$ or $\mu \approx \beta\bar{u}^2$, say, which is a parabola. On the other hand, we have for the second branch $\mu = O(\bar{u})$ or $\mu \approx -(2b/a)\bar{u}$, which is a straight line. Now, the slopes of the above two branches, from (2.7), are given by

$$f_u(\bar{u}; \mu) = -f_\mu(\bar{u}; \bar{u}\bar{\omega})\frac{d\mu}{d\bar{u}} = -2b\mu + O(\mu^2), b\mu + O(\mu^2). \tag{2.38}$$

So the above two branches have opposite stabilities and exchange stabilities at the bifurcation point $\bar{u} = 0$. Further, the parabolic branch is either always stable (if $b\beta > 0$) or always unstable (if $b\beta < 0$) so that one has a supercritical pitchfork bifurcation, if $b\beta > 0$, or a subcritical pitchfork bifurcation, if $b\beta < 0$. □

Example 2.4 Consider a bead of mass m sliding without friction on a circular wire or hoop of radius R (Fig. 2.5). The hoop is constrained to rotate about a vertical diameter with constant angular velocity ω. The equation of motion of the bead subject to its weight mg and the centrifugal force $mr\omega^2$ is

$$mR\theta'' = mR\omega^2 \sin\theta \cdot \cos\theta - mg\sin\theta. \tag{2.39}$$

Equilibrium prevails, therefore, when

$$mR\omega^2 \sin\bar{\theta} \cdot \cos\bar{\theta} - mg\sin\bar{\theta} = 0.$$

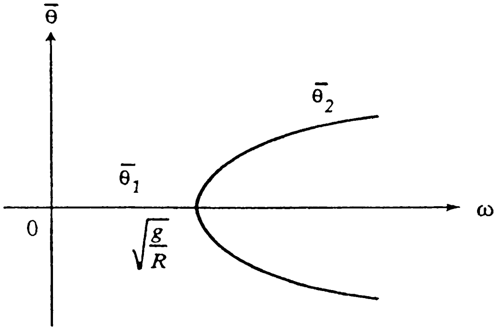


Fig. 2.6 Equilibrium solutions for a bead sliding without friction on a circular wire

which yields two equilibria,

$$\bar{\theta}_1 = 0, \qquad \bar{\theta}_2 = \cos^{-1}\left(\frac{g}{R\omega^2}\right). \tag{2.40}$$

In the neighborhood of the bifurcation point $\omega_c \equiv \sqrt{g/R}$, (2.40) becomes

$$\bar{\theta}_1 = 0, \qquad \bar{\theta}_2^2 \approx \frac{4}{\sqrt{g/R}}\left(\omega - \sqrt{\frac{g}{R}}\right) \tag{2.41}$$

so that the branch corresponding to $\bar{\theta}_2$ is locally a parabola near the bifurcation point.

In Fig. 2.6, the two equilibrium solutions are sketched as functions of ω. Equations (2.40) and (2.41) show that, if $\omega \le \sqrt{g/R}$, the only equilibrium solution is $\bar{\theta}_1 = 0$; however, if $\omega > \sqrt{g/R}$, there are three possible equilibrium solutions. The equilibrium solutions bifurcate at the bifurcation point $\omega_c \equiv \sqrt{g/R}$, and the solution sought out by the system is determined by stability considerations.

In order to determine the stability of $\bar{\theta}_1$, consider a small perturbation $\hat{\theta}$ about $\bar{\theta}_1$, and write

$$\theta = \bar{\theta}_1 + \hat{\theta} \tag{2.42}$$

and linearizing in $\hat{\theta}$, equation (2.39) becomes

$$mR\hat{\theta}'' \approx \left(mR\omega^2 - mg\right)\hat{\theta}$$

or

$$\hat{\theta}'' + \left(\frac{g}{R} - \omega^2\right)\hat{\theta} = 0 \tag{2.43}$$

from which, it is obvious that $\bar{\theta}_1$ is stable if $\omega < \sqrt{g/R}$ and unstable if $\omega > \sqrt{g/R}$.

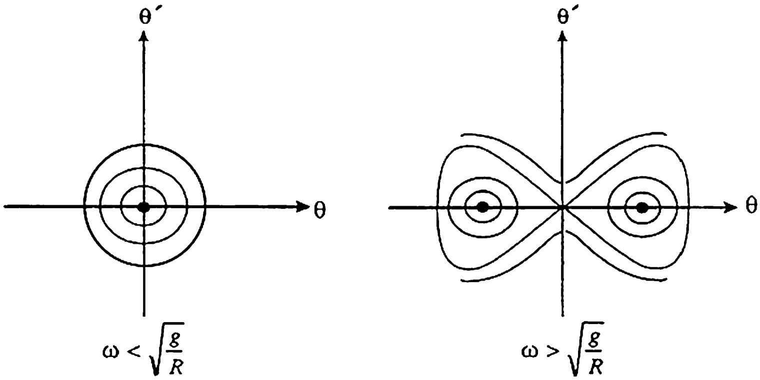


Fig. 2.7 The bifurcation of the stable center at the origin into an unstable saddle as the angular velocity ω increases

Similarly, the stability of θ_2 may be determined by writing

$$\theta = \bar{\theta}_2 + \hat{\theta} \tag{2.44}$$

and linearizing equation (2.39) again in $\hat{\theta}$; we then obtain

$$mR\hat{\theta}'' \approx -m\left[\frac{(R\omega^2)^2 - g^2}{R\omega^2}\right]\hat{\theta}$$

or

$$\hat{\theta}'' + \left[\frac{(R\omega^2)^2 - g^2}{R^2\omega^2}\right]\hat{\theta} = 0 \tag{2.45}$$

from which, it is obvious that $\bar{\theta}_2$ is stable, if $\omega > \sqrt{g/R}$ and unstable, if $\omega < \sqrt{g/R}$.

Thus, if ω starts from zero and is slowly increased, the bead will remain at the bottom of the hoop (corresponding to $\bar{\theta}_1$) until the critical value $\omega_c \equiv \sqrt{g/R}$ is reached. As ω becomes larger than ω_c, $\bar{\theta}_1$ becomes unstable and the bead will quickly bifurcate to a branch corresponding to $\bar{\theta}_2$ (see Fig. 2.6).

In the (θ, θ')-phase plane, as ω increases and passes through $\omega_c \equiv \sqrt{g/R}$, the stable center at the origin bifurcates into an unstable saddle at the origin and two non-zero stable centers located on the θ-axis (see Fig. 2.7). This is an example of pitchfork bifurcation.

2.3 Hopf Bifurcation

In order to permit the occurrence of *Hopf bifurcation*, the first-order system (2.8) must have dimension $n \geq 2$. Let us consider here, for illustration, only the planar case $n = 2$, when equation (2.8) becomes

$$\left.\begin{aligned}\frac{du}{dt} &= f(u, v; \mu)\\ \frac{dv}{dt} &= g(u, v; \mu)\end{aligned}\right\} \tag{2.46}$$

where f and g are analytic functions of u, v and μ.

The critical point $(\bar{u}(\mu), \bar{v}(\mu))$ of equation (2.46) corresponds to

$$\left.\begin{aligned}f\big(\bar{u}(\mu), \bar{v}(\mu); \mu\big) = 0\\ g\big(\bar{u}(\mu), \bar{v}(\mu); \mu\big) = 0\end{aligned}\right\} \tag{2.47}$$

and its stability is determined by the eigenvalues $\lambda_1(\mu)$ and $\lambda_2(\mu)$ of the Jacobian matrix

$$\mathbf{A}(\mu) = \begin{bmatrix} f_u(\bar{u}, \bar{v}; \mu) & f_v(\bar{u}, \bar{v}; \mu) \\ g_u(\bar{u}, \bar{v}; \mu) & g_v(\bar{u}, \bar{v}; \mu) \end{bmatrix}. \tag{2.48}$$

One has for a Hopf bifurcation,

$$\lambda_1(\mu) = \overline{\lambda_2(\mu)} = \alpha(\mu) + i\beta(\mu) \tag{2.49}$$

where assuming, without loss of generality, the bifurcation point is $\mu = 0$, one has

$$\alpha(0) = 0, \qquad \beta(0) \neq 0. \tag{2.50}$$

Equation (2.50) implies that, in the neighborhood of $\mu = 0$, $\det \mathbf{A}(\mu) \neq 0$, so that, by the *Implicit Function Theorem*, $\bar{u}(\mu)$ and $\bar{v}(\mu)$ are analytic functions of μ in a neighborhood of $\mu = 0$.

Putting,

$$\hat{u} \equiv u - \bar{u}, \qquad \hat{v} \equiv v - \bar{v} \tag{2.51}$$

and expanding f and g in powers of $\hat{u}$ and $\hat{v}$, equation (2.46) becomes

$$\begin{bmatrix} \frac{d\hat{u}}{dt} \\ \frac{d\hat{v}}{dt} \end{bmatrix} = \mathbf{A}(\mu) \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} + \begin{bmatrix} F(\hat{u}, \hat{v}; \mu) \\ G(\hat{u}, \hat{v}; \mu) \end{bmatrix} \tag{2.52}$$

where F and G are $O(\hat{u}^2, \hat{v}^2, \hat{u}\hat{v})$, as $\hat{u}$ and $\hat{v} \to 0$, and are analytic functions of $\hat{u}$ and $\hat{v}$.

To facilitate further discussion, let us suppose that $\mathbf{A}(\mu)$ has the following canonical form:

$$\mathbf{A}(\mu) = \begin{bmatrix} \alpha(\mu) & \beta(\mu) \\ -\beta(\mu) & \alpha(\mu) \end{bmatrix} \tag{2.53}$$

so that equation (2.52) becomes

$$\left.\begin{aligned}\frac{d\hat{u}}{dt} &= \alpha(\mu)\hat{u} + \beta(\mu)\hat{v} + F(\hat{u}, \hat{v}; \mu)\\ \frac{d\hat{v}}{dt} &= -\beta(\mu)\hat{u} + \alpha(\mu)\hat{v} + G(\hat{u}, \hat{v}; \mu)\end{aligned}\right\} \tag{2.54}$$

Equation (2.54) shows that the origin $\hat{u} = 0$, $\hat{v} = 0$ in the $\hat{u}, \hat{v}$-plane is a focus (see Footnote 9, Chap. 1) whose stability is determined by the sign of $\alpha(\mu)$. Since $\alpha(0) = 0$, this stability changes, as μ passes through zero.

In order to facilitate discussion, let us introduce

$$z = \hat{u} + i\hat{v}. \tag{2.55}$$

so that equation (2.54) becomes

$$\frac{dz}{dt} = \big[\alpha(\mu) - i\beta(\mu)\big]z + N(z, \bar{z}; \mu) \tag{2.56}$$

where

$$N(z, \bar{z}; \mu) \equiv F(\hat{u}, \hat{v}; \mu) + iG(\hat{u}, \hat{v}; \mu) \sim O\big(|z|^2\big) \quad \text{as } |z| \to 0.$$

It is now a rather simple matter to show that (see Appendix), using two successive near-identity analytic transformations of the form,

$$\xi = z + S(z, \bar{z}; \mu) \tag{2.57}$$

where $S \sim O(|z|^2)$, as $|z| \to 0$, (2.56) can be reduced to the *normal form*,

$$\frac{d\xi}{dt} = \big[\alpha(\mu) - i\beta(\mu)\big]\xi + \big[\gamma(\mu) + i\delta(\mu)\big]|\xi|^2\xi + O\big(|\xi|^4\big) \tag{2.58}$$

where $\gamma(\mu)$ and $\delta(\mu)$ are analytic functions of μ. Equation (2.58) implies that the nonlinear term $N(z, \bar{z}; \mu)$ in equation (2.56) can be transformed to remove all quadratic terms and all cubic terms except one, namely the term $|\xi|^2\xi$. Note that the latter term is the lowest-order nonlinear term which has the same phase as ξ, and is, therefore, the most dominant term producing resonance, as $|\xi| \to 0$.

Theorem 2.2 *The first-order equation*

$$\frac{d\xi}{dt} = \big[\alpha(\mu) - i\beta(\mu)\big]\xi + \big[\gamma(\mu) + i\delta(\mu)\big]|\xi|^2\xi \tag{2.59}$$

for μ in a neighborhood of zero, has a family of periodic solutions of period $2\pi/|\beta(0)|$, as $\mu \to 0$, which are stable (or unstable), when $\alpha(\mu) > 0$ (or $\alpha(\mu) < 0$).

Proof Introduce the polar coordinates (R, ϕ) by

$$\xi = Re^{i\phi} \tag{2.60}$$

so that equation (2.59) becomes

$$\frac{dR}{dt} = \alpha(\mu)R + \gamma(\mu)R^3 \tag{2.61}$$

$$\frac{d\phi}{dt} = -\beta(\mu) + \delta(\mu)R^2 \tag{2.62}$$

Assuming that $\gamma(0) \neq 0$ so that $\gamma(\mu)$ is non-zero in a neighborhood of $\mu = 0$, equation (2.61) has two critical points,

$$\bar{R} = 0, \tag{2.63}$$

$$\bar{R}^2 = -\frac{\alpha(\mu)}{\gamma(\mu)}. \tag{2.64}$$

The stability of these solutions is determined by the sign of $\tilde{\lambda}(\mu)$, where

$$\tilde{\lambda} = \frac{\partial}{\partial \bar{R}}\left(\alpha\bar{R} + \gamma\bar{R}^3\right) \quad \text{at } \alpha\bar{R} + \gamma\bar{R}^3 = 0 \tag{2.65}$$

or

$$\tilde{\lambda} = \begin{cases} \alpha, & \text{for } \bar{R} = 0 \\ -2\alpha, & \text{for } \bar{R}^2 = -\frac{\alpha}{\gamma}. \end{cases} \tag{2.66}$$

Equation (2.66) shows that the two branches (2.63) and (2.64) have opposite stabilities. Observe that branch (2.64), (noting that $\alpha(\mu) \sim \alpha'(0)\mu$, as $\mu \to 0$, with the generic condition $\alpha'(0) \neq 0$),[6] is a pitchfork bifurcation, (see Fig. 2.8).

Further, equation (2.62), for branch (2.64), leads to

$$\phi = \phi_0 + \omega(\mu)t \tag{2.67}$$

where,

$$\omega(\mu) \equiv -\beta - \frac{\delta\alpha}{\gamma}.$$

Thus, branch (2.64) corresponds to a periodic solution, with period $2\pi/|\omega(\mu)|$ or $2\pi/|\beta(0)|$, as $\mu \to 0$.

[6] $\alpha'(0) \neq 0$ implies that the eigenvalues cross the imaginary axis with non-zero speed so that the stability of the fixed point $(0, 0)$ changes at a finite speed, as μ passes through zero. This transversality condition leads to the existence of a unique center manifold passing through $(u, v) = (0, 0)$, $\mu = 0$ in $R^2 \times R$ (Marsden and McCracken 1976).

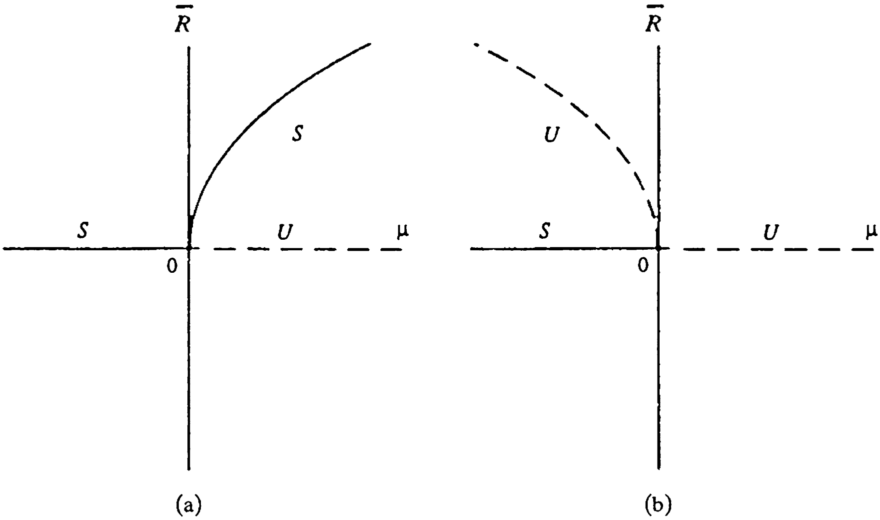


Fig. 2.8 The bifurcation diagram for a Hopf bifurcation: (**a**) supercritical and (**b**) subcritical

Therefore, in a supercritical Hopf bifurcation, as μ varies through the bifurcation point $\mu = 0$, the branch $\bar{R} = 0$ changes from a stable focus to an unstable focus and leads to a periodic solution. Note that the pitchfork bifurcation at $\mu = 0$, for equation (2.61), corresponds to a Hopf bifurcation for the full system (2.59). Thus, Hopf bifurcation is generically a pitchfork bifurcation. □

However, in a subcritical Hopf bifurcation, as μ varies through the bifurcation point $\mu = 0$, the branch $\bar{R} = 0$ changes from a stable focus to an unstable focus but goes into no periodic solution. Indeed, subcritical Hopf bifurcations provide a mechanism for the onset of chaos (like in the Lorenz model, see Sect. 6.6).

Example 2.5 Consider the first-order system of equations,

$$\left.\begin{aligned} x' &= -y - x\left(x^2 + y^2 - \mu\right) \\ y' &= x - y\left(x^2 + y^2 - \mu\right). \end{aligned}\right\} \tag{2.68}$$

The linearized version of (2.68) is

$$\left.\begin{aligned} x' &= \mu x - y \\ y' &= x + \mu y \end{aligned}\right\} \tag{2.69}$$

for which,

$$x, y \sim e^{\lambda t}, \qquad \lambda = \mu \pm i. \tag{2.70}$$

If $\mu < 0$, then $\mathrm{Re}(\lambda) < 0$, and $(0, 0)$ is a stable focus; if $\mu = 0$, then $\lambda = \pm i$, and $(0, 0)$ is a center; if $\mu > 0$, then $\mathrm{Re}(\lambda) > 0$, and $(0, 0)$ is an unstable focus. Further,

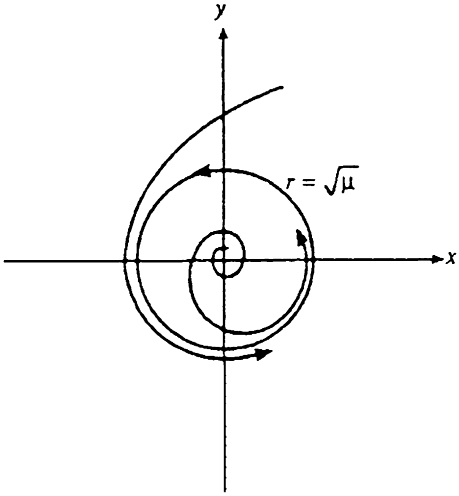


Fig. 2.9 The limit cycle for the system (2.68), when $\mu > 0$

$\frac{\partial \lambda(\mu)}{\partial \mu}|_{\mu=0} = 1 \neq 0$ so that the eigenvalues cross the imaginary axis with non-zero speed.

The nonlinear problem (2.68) can be solved exactly by putting

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{2.71}$$

We then obtain from equation (2.68),

$$r' = r\big(\mu - r^2\big) \tag{2.72}$$

$$\theta' = 1. \tag{2.73}$$

We have, from equation (2.73),

$$\theta = t + t_0. \tag{2.74}$$

For the case $\mu > 0$, equation (2.72) has the solution,

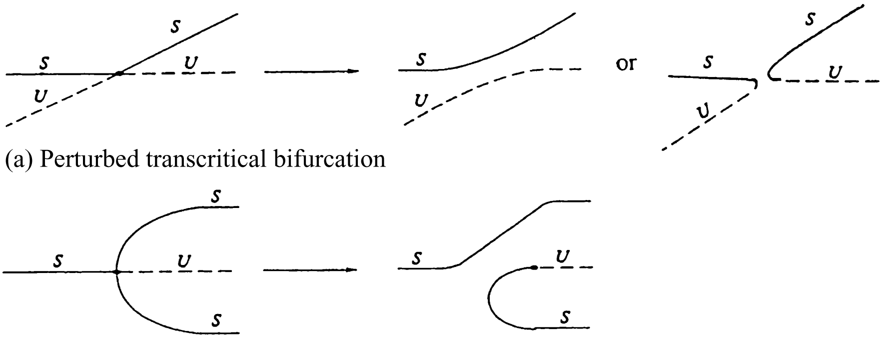
$$r = \frac{\sqrt{\mu}}{\sqrt{1 + (\mu/r_0^2 - 1)e^{-2\mu t}}} \tag{2.75}$$

where,

$$t = 0 : r = r_0. \tag{2.76}$$

Equation (2.75) shows the existence of a limit cycle at $r = \sqrt{\mu}$, for $\mu > 0$ (see Fig. 2.9), via a supercritical pitchfork bifurcation.

This is, of course, consistent with the linear result (2.70) that (0, 0) is an unstable focus for $\mu > 0$.



(a) Perturbed transcritical bifurcation

(b) Perturbed pitchfork bifurcation

Fig. 2.10 Breaking of bifurcations on perturbation

Thus, as μ passes through the bifurcation point $\mu = 0$, the origin changes from a stable focus to an unstable focus and there appears a new periodic solution bifurcating from $\mu = 0$.

2.4 Break-up of Bifurcations Under Perturbations

It may be noted that saddle-node bifurcation and Hopf bifurcation are the generic bifurcations in dynamical systems. Under perturbations (or slight imperfections in the system itself), transcritical bifurcation and pitchfork bifurcation break into saddle-node bifurcation, as shown in Fig. 2.10 (see Exercises 2.1 and 2.2). As shown in Fig. 2.10, the breakdown of a pitchfork bifurcation leads to one equilibrium state that evolves smoothly, as the control parameter increases, and to another disconnected state that exists above a critical value of the parameter. The disconnected state can only be reached via a discontinuous jump in the parameter and disappears catastrophically below a critical value of the parameter.

2.5 Bifurcation Theory for One-Dimensional Maps

Area-preserving maps sometimes provide a superior alternative to differential equations approach because of the ease with which the iterations can be carried out over hundreds of thousands of mapping periods (see Chap. 5). Here, one constructs a discrete map from the flow generated by a continuous time system by sampling the flow, at discrete times $t_n = t_0 + n\tau$ $(n = 0, 1, 2, \ldots)$,

$$x_{n+1} = f(x_n) \tag{2.77}$$

where,

$$x_n \equiv x(t_n).$$

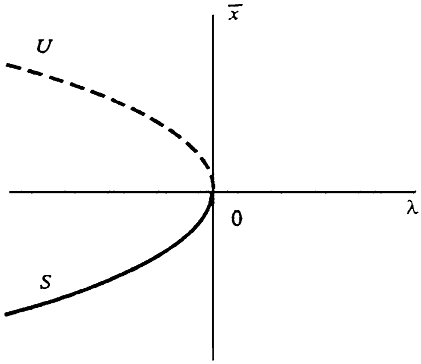


Fig. 2.11 Saddle-node bifurcation (*dashed curve* represents unstable fixed points)

We will now give a brief account of the bifurcation theory for one-dimensional maps of the form,

$$f_\lambda(x) = f(x,\lambda) : R \times R \to R. \tag{2.78}$$

To simplify the discussion, we take the fixed point to be at $\bar{x} = 0$ and the bifurcation to occur at $\lambda = 0$.

Theorem 2.3 *Let $f : R \times R \to R$ be a one-parameter family of C^2 maps satisfying*

$$\left.\begin{array}{ll} f(0,0) = 0, & \dfrac{\partial f}{\partial x}(0,0) = 1, \\ \dfrac{\partial^2 f}{\partial x^2}(0,0) > 0, & \dfrac{\partial f}{\partial \lambda}(0,0) > 0. \end{array}\right\} \tag{2.79}$$

Then, there exists intervals $(\lambda_1, 0)$, $(0, \lambda_2)$ and a number $k > 0$ such that:

(i) *there are two fixed points in $(-k, k)$, with $\bar{x} \gtrless 0$ being unstable/stable, if $\lambda \in (\lambda_1, 0)$;*
(ii) *there are no fixed points in $(-k, k)$, if $\lambda \in (0, \lambda_2)$ (saddle-node bifurcation, Fig. 2.11).*

Theorem 2.4 *Let $f : R \times R \to R$ be a one-parameter family of C^2 maps satisfying*

$$\left.\begin{array}{ll} f(0,\lambda) = 0, & \dfrac{\partial f}{\partial x}(0,0) = 1, \\ \dfrac{\partial^2 f}{\partial x \partial u}(0,0) > 0, & \dfrac{\partial^2 f}{\partial x^2}(0,0) > 0. \end{array}\right\} \tag{2.80}$$

Then, there exist two branches of fixed points for $\lambda \approx 0$. The first branch is $\bar{x} = 0$, $\forall\lambda$ and is stable/unstable, if $\lambda \lesseqgtr 0$, while the second branch has $\bar{x}(\lambda) \neq 0$, if $\lambda \neq 0$, with $\bar{x}(0) = 0$, and is stable/unstable, if $\lambda \gtrless 0$ (transcritical bifurcation, Fig. 2.12).

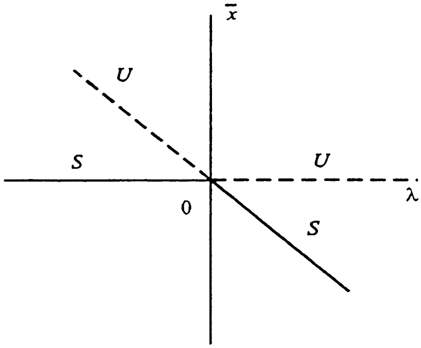


Fig. 2.12 Transcritical bifurcation (*dashed lines* represent unstable fixed points)

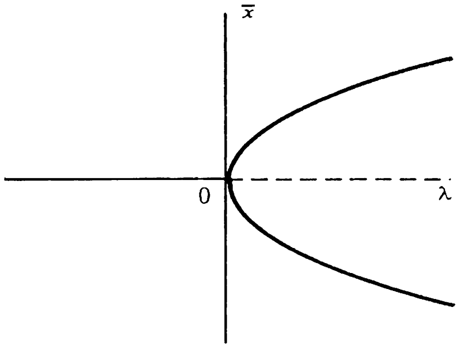


Fig. 2.13 Pitchfork bifurcation (*dashed line* represents unstable fixed points)

Theorem 2.5 *Let $f : R \times R \to R$ be a one-parameter family of C^3 maps satisfying,*

$$\left.\begin{aligned} f(-x,\lambda) &= -f(x,\lambda), & \frac{\partial f}{\partial x}(0,0) &= 1, \\ \frac{\partial^2 f}{\partial x \partial u}(0,0) &> 0, & \frac{\partial^3 f}{\partial x^3}(0,0) &< 0. \end{aligned}\right\} \tag{2.81}$$

Then there exist intervals $(\lambda_1, 0)$, $(0, \lambda_2)$ and a number $k > 0$, such that:

(i) *there is a single stable fixed point at the origin in $(-k, k)$, if $\lambda \in (\lambda_1, 0)$;*
(ii) *there are three fixed points in $(-k, k)$, the one at the origin is unstable while the other two are stable, if $\lambda \in (0, \lambda_2)$ (pitchfork bifurcation, Fig. 2.13).*

See Rasband (1990) for proofs of these Theorems.

Iterated maps, unlike continuous flows, can possess non-hyperbolic fixed points with the eigenvalue equal to -1. This leads to some bifurcation scenarios, like the *period-doubling bifurcations* (see Chap. 6), that are unique to iterated maps.

Appendix: The Normal Form Reduction

Consider the reduction to *normal* form of the equation (Grimshaw 1990),

$$\frac{dz}{dt} = \left[\alpha(\mu) - i\beta(\mu)\right]z + N(z, \bar{z}; \mu) \tag{2.82}$$

where,

$$N(z, \bar{z}; \mu) \equiv \frac{1}{2}n_1 z^2 + n_2 z\bar{z} + \frac{1}{2}n_3\bar{z}^2 + O\left(|z|^3\right).$$

For this purpose, let us introduce a *near-identity* transformation,

$$\omega = z + Q(z, \bar{z}; \mu) \tag{2.83}$$

with the inverse,

$$z \approx \omega - Q(\omega, \bar{\omega}; \mu) \tag{2.84}$$

where,

$$Q(z, \bar{z}; \mu) = \frac{1}{2}q_1 z^2 + q_2 z\bar{z} + \frac{1}{2}q_3\bar{z}^2.$$

We have from (2.83) and (2.84),

$$\frac{d\omega}{dt} = \frac{dz}{dt} + (q_1 z + q_2\bar{z})\frac{dz}{dt} + (q_2 z + q_3\bar{z})\frac{d\bar{z}}{dt}$$

which, on using (2.82)–(2.84), becomes

$$\frac{d\omega}{dt} = (\alpha - i\beta)\omega + \frac{1}{2}\hat{n}_1\omega^2 + \hat{n}_2\omega\bar{\omega} + \frac{1}{2}\hat{n}_3\bar{\omega}^2 + O\left(|\omega|^3\right) \tag{2.85}$$

where,

$$\hat{n}_1 \equiv n_1 + (\alpha - i\beta)q_1, \qquad \hat{n}_2 \equiv n_2 + (\alpha + i\beta)q_2, \qquad \hat{n}_3 \equiv n_3 + (\alpha + 3i\beta)q_3.$$

Now, since $\beta(0) \neq 0$, we may choose q_1, q_2 and q_3 in such a way that

$$\hat{n}_1 = \hat{n}_2 = \hat{n}_3 = 0 \tag{2.86}$$

in a neighborhood of the bifurcation point $\mu = 0$.

Using (2.86), equation (2.85) becomes

$$\frac{d\omega}{dt} = (\alpha - i\beta)\omega + M(\omega, \bar{\omega}; \mu) \tag{2.87}$$

where,

$$M(\omega, \bar{\omega}; \mu) \equiv \frac{1}{3}m_1\omega^3 + m_2\omega^2\bar{\omega} + m_3\omega\bar{\omega}^2 + \frac{1}{3}m_4\bar{\omega}^3 + O\left(|\omega|^4\right).$$

In order to reduce equation (2.87) further, let us introduce another near-identity transformation,

$$\xi = \omega + R(\omega, \bar{\omega}; \mu) \tag{2.88}$$

with the inverse,

$$\omega = \xi - R(\xi, \bar{\xi}; \mu) \tag{2.89}$$

where,

$$R(\omega, \bar{\omega}; \mu) \equiv \frac{1}{3} r_1 \omega^3 + r_2 \omega^2 \bar{\omega} + r_3 \omega \bar{\omega}^2 + \frac{1}{3} r_4 \bar{\omega}^3.$$

We have, from (2.88) and (2.89),

$$\frac{d\xi}{dt} = \frac{d\omega}{dt} + \left(r_1 \omega^2 + 2r_2 \omega \bar{\omega} + r_3 \bar{\omega}^2\right) \frac{d\omega}{dt} + \left(r_2 \omega^2 + 2r_3 \omega \bar{\omega} + r_4 \bar{\omega}^2\right) \frac{d\bar{\omega}}{dt}$$

which, on using (2.87)–(2.89), becomes

$$\frac{d\xi}{dt} = (\alpha - i\beta)\xi + \frac{1}{3}\hat{m}_1 \xi^3 + \hat{m}_2 \xi^2 \bar{\xi} + \hat{m}_3 \xi \bar{\xi}^2 + \frac{1}{3}\hat{m}_4 \bar{\xi}^3 + O(|\xi|^4) \tag{2.90}$$

where,

$$\left.\begin{array}{ll} \hat{m}_1 = m_1 + 2(\alpha - i\beta) r_1, & \hat{m}_2 = m_2 + 2\alpha r_2, \\ \hat{m}_3 = m_3 + 2(\alpha + i\beta) r_3, & \hat{m}_4 = m_4 + 2(\alpha + 2i\beta) r_4. \end{array}\right\}$$

Now, since $\beta(0) \neq 0$, we may choose again r_1, r_3 and r_4 in such a way that

$$\hat{m}_1 = \hat{m}_3 = \hat{m}_4 = 0 \tag{2.91}$$

in a neighborhood of the bifurcation point $\mu = 0$. However, r_2 cannot be chosen in any way to make $\hat{m}_2 = 0$ because, $\alpha(0) = 0$. So, let us choose $r_2 = 0$ giving

$$\hat{m}_2 = m_2 = \gamma(\mu) + i\delta(\mu), \text{ say.} \tag{2.92}$$

Using (2.91) and (2.92), equation (2.90) becomes

$$\frac{d\xi}{dt} = \left[\alpha(\mu) - i\beta(\mu)\right]\xi + \left[\gamma(\mu) + i\delta(\mu)\right]|\xi|^2 \xi + O\left(|\xi|^4\right) \tag{2.93}$$

as advertized in the text.

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