Chapter 2
Coordinate Transformations

Abstract For explicit solutions of engineering problems it is necessary to choose an appropriate coordinate system. In fact, the choice of coordinates should depict the geometry of the problem so that the corresponding mathematical formulation simplifies as much as possible. How to rewrite the equations of continuum theory in arbitrary coordinates is subject of *tensor calculus*, which is usually no part of the mathematics syllabus in engineering education. For this reason we start in this chapter with *tensor algebra* and the representation of tensors in arbitrary coordinate systems, before we elaborate on continuum theory any further. As the emphasis in this book is on practical mathematics, the preferred way of writing tensors will be in index notation. However, many textbooks and the scientific literature use symbolic notation instead. Therefore we will always present both, although at the expense of mathematical stringency.

Capt. Terrell: Chekov, are you sure these are the correct coordinates?
Chekov: Captain, this is the garden spot of Ceti Alpha Six!

Star Trek II, The Wrath of Khan

2.1 A Personal Remark

Typically scientists working in continuum theory split into two extremely hostile ideological factions: the supporters of symbolic tensor calculus and the friends of index notation. The former emphasize the absolute character laws of nature and constitutive equations should have. In other words these laws should be stated independently of an observer and, therefore, independently of a coordinate system. As to whether this is possible, at least in principle, or, in other words, as to whether the tensorial relations that force the laws of nature and constitutive relations into a

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mathematical form are correct, is a highly philosophical question which, in the end, can only be decided experimentally. However, for this purpose measurements in space and time need to be performed (by an observer). This is exactly what is implicitly emphasized by index notation. Certainly symbolic notation is aesthetically pleasing. This becomes immediately visible if it is juxtaposed to the cumbersome form of the index calculus. However, if the objective is to solve concrete engineering problems the symbolic way of writing is only of limited use. The situation is similar to the world of fashion: A tuxedo with a top hat may be appropriate at the Met but it is hopelessly unsuitable for gardening.

Indeed, the engineer-to-be should be capable to read the pertinent technical literature as well as use the results therein to perform further calculations. Therefore we shall present both lines of reasoning in this book and learn to appreciate the corresponding advantages—but, hopefully, without turning into ideologists.

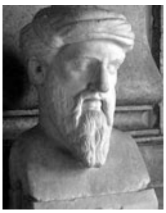
2.2 First Definitions and Notions in Index Notation

We will initially start from an index point-of-view that circumvents the notion of a (unit) vector base. In this spirit we consider a three-dimensional *Cartesian* coordinate grid consisting of three straight lines, x_1, x_2, x_3, which are orthogonal to each other. For brevity we shall denote them by x_k and note that a Latin case index runs from 1 to 3. Moreover, we consider other three-dimensional coordinate lines which may be curvilinear and denote them by z^i. Clearly points in space should be identifiable by either set of lines. In other words invertible relations of the following kind must hold:

$$z^i = z^i(x_1, x_2, x_3) \equiv z^i(x_k) \quad \text{and} \quad x_k = x_k\left(z^1, z^2, z^3\right) \equiv x_k\left(z^i\right). \tag{2.2.1}$$

Mathematically speaking such invertible relations are also known as *isomorphisms*. The distance s between two points (1) and (2) with the corresponding Cartesian coordinates $\overset{(1)}{x}_i$ and $\overset{(2)}{x}_i$ can easily be calculated following Pythagoras:

$$s = \sqrt{\left(\overset{(1)}{x}_1 - \overset{(2)}{x}_1\right)^2 + \left(\overset{(1)}{x}_2 - \overset{(2)}{x}_2\right)^2 + \left(\overset{(1)}{x}_3 - \overset{(2)}{x}_3\right)^2}. \tag{2.2.2}$$



Pythagoras of Samos lived around 580–500 BC. He was a versatile man engaged in philosophy, mysticism, mathematics, astronomy, music, healing arts, wrestling, and politics. In 532 BC he leaves Samos, flees from the local tyrant, and moves to southern Italy. In Croton he founds his famous philosophical-religious school: the so-called Pythagoreans. Surely he was not the first to know about the Pythagorean theorem but, maybe, he was one of the first interested in its proof.

Note that, in general, the corresponding formula does not hold in curvilinear coordinates, z^i:

$$s \neq \sqrt{\left(\overset{(1)}{z^1} - \overset{(2)}{z^1}\right)^2 + \left(\overset{(1)}{z^2} - \overset{(2)}{z^2}\right)^2 + \left(\overset{(1)}{z^3} - \overset{(2)}{z^3}\right)^2}. \tag{2.2.3}$$

However, if the coordinate points (1) and (2) are infinitesimally close, it becomes possible to derive a relation similar to Eq. (2.2.2). We first define:

$$\mathrm{d}x_i = \overset{(1)}{x_i} - \overset{(2)}{x_i} \quad \text{or} \quad \mathrm{d}z^k = \overset{(1)}{z^k} - \overset{(2)}{z^k} \tag{2.2.4}$$

and obtain for the total differential by using Eq. $(2.2.1)_2$:

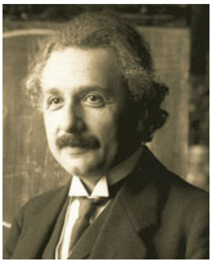
$$\mathrm{d}x_i = \frac{\partial x_i}{\partial z^1}\mathrm{d}z^1 + \frac{\partial x_i}{\partial z^2}\mathrm{d}z^2 + \frac{\partial x_i}{\partial z^3}\mathrm{d}z^3. \tag{2.2.5}$$

This can be expressed in shorthand notation:

$$\mathrm{d}x_i = \frac{\partial x_i}{\partial z^k}\mathrm{d}z^k, \tag{2.2.6}$$

if we agree *to sum up automatically from 1 to 3 (or up to 2 for planar problems) whenever an index appears twice in a product.* In the present case this concerns the index k. This is known as *EINSTEIN's summation rule in the literature.* We also refer to *k as a bound index* in contrast to *free indices*, i.e., those that do not appear twice (in the present case the letter "*i*"). The infinitesimal distance ds between the two infinitesimally close points (1) and (2) can now be calculated if we insert Eqs. (2.2.4) and (2.2.6) into (2.2.2):

$$\mathrm{d}s = \sqrt{\mathrm{d}x_i \mathrm{d}x_i} = \sqrt{\frac{\partial x_i}{\partial z^k}\frac{\partial x_i}{\partial z^j}\mathrm{d}z^k \mathrm{d}z^j}. \tag{2.2.7}$$



Albert EINSTEIN was born on March 14, 1879 in Ulm (South Germany) and died on April 18th, 1955 in Princeton. He is certainly the most eminent scientist of the twentieth century. Similar to NEWTON or MAXWELL he enriched physics by many fundamental discoveries from different fields. The development of General Relativity and the tensor calculus that was used therein for describing space–time is probably his most popular contribution. However, this did not win him the NOBEL price. On the contrary: This award was given to him for something much less "obscure," namely for his interpretation of the photo-electric effect.

Exercise 2.2.1: *Line element*

Go through the proof of Eq. (2.2.7). By doing so learn to understand the meaning of the different indices by writing down all terms according to EINSTEIN's summation rule. Realize that a double summation occurs in Eq. (2.2.7).

Obviously it is "almost" possible to write the infinitesimal distance in terms of products in $\mathrm{d}z^k$. Well, almost, since Eq. (2.2.7) teaches us that one has to multiply these products by derivatives of the Cartesian coordinates x_i w.r.t. the curvilinear coordinates z^k. These derivatives have a special name. They are known as the (covariant) components of the *metric tensor* $\boldsymbol{g}$ and defined as follows:

$$g_{kj} = \frac{\partial x_i}{\partial z^k}\frac{\partial x_i}{\partial z^j}. \tag{2.2.8}$$

The word *metric* stems from the Greek word μετρώ for measuring. This makes immediate sense because this quantity renders it possible to determine distances, cf., Eq. (2.2.7), provided the differences of coordinates $\mathrm{d}z^k$ are known. We now rewrite Eq. (2.2.7) in the following compact form:

$$\mathrm{d}s = \sqrt{g_{kj}\mathrm{d}z^k\mathrm{d}z^j}. \tag{2.2.9}$$

Note that in order to determine g_{kj} for a particular coordinate transformation it is sufficient to determine only "half of it," for example, for the sequence of indices $(k, j) = (1,1), (1,2), (1,3), (2,2), (2,3), (3,3)$. This is due to the fact that the metric tensor is symmetric:

$$g_{kj} = \frac{\partial x_i}{\partial z^k}\frac{\partial x_i}{\partial z^j} = \frac{\partial x_i}{\partial z^j}\frac{\partial x_i}{\partial z^k} = g_{jk}. \tag{2.2.10}$$

As a typical example for curvilinear coordinates, the corresponding metric tensor, and the line element we consider the case of cylindrical coordinates: Here a point in space is characterized by its radial distance, r, the polar angle, ϑ, and the height, $z{:}(r,\ \vartheta,\ z)$, $r \in [0, \infty)$, $\vartheta \in [0, \pi)$, $z \in (-\infty, +\infty)$. Just like Cartesian coordinate lines cylindrical ones are orthogonal to each other. However, in contrast to those not all of them are "straight." In particular the lines describing a constant radial distance are of circular shape (cf., Fig. 2.1). The following relations between Cartesian and cylindrical coordinates follow by simple geometric considerations from Fig. 2.1:

$$\begin{aligned} z^1 &\equiv r = \sqrt{x_1^2 + x_2^2}, & x_1 &= r\cos\vartheta = z^1 \cos z^2, \\ z^2 &\equiv \vartheta = \arctan\frac{x_2}{x_1}, & x_2 &= r\sin\vartheta = z^1 \sin z^2, \\ z^3 &= z = x_3, & x_3 &= z = z^3. \end{aligned} \tag{2.2.11}$$

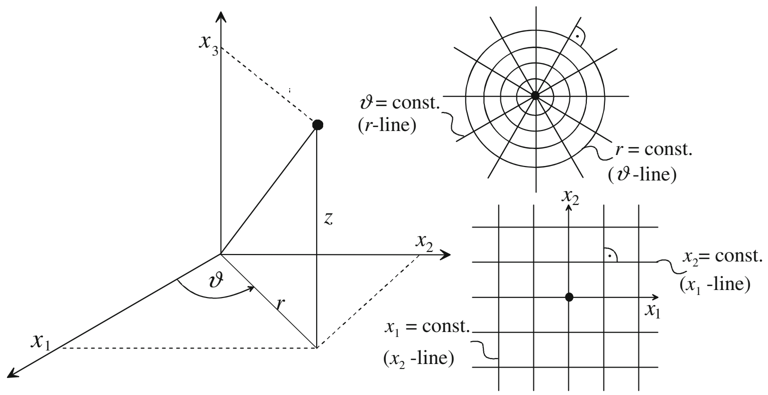


Fig. 2.1 Cylindrical coordinates

This in turn yields (for example):

$$\begin{aligned} g_{11} &= \frac{\partial x_1}{\partial z^1}\frac{\partial x_1}{\partial z^1} + \frac{\partial x_2}{\partial z^1}\frac{\partial x_2}{\partial z^1} + \frac{\partial x_3}{\partial z^1}\frac{\partial x_3}{\partial z^1} = \cos^2\, z^2 + \sin^2\, z^2 + 0 = 1, \\ g_{12} &= \frac{\partial x_1}{\partial z^1}\frac{\partial x_1}{\partial z^2} + \frac{\partial x_2}{\partial z^1}\frac{\partial x_2}{\partial z^2} + \frac{\partial x_3}{\partial z^1}\frac{\partial x_3}{\partial z^2} \\ &= -z^1 \sin\, z^2 \cos\, z^2 + z^1 \sin\, z^2 \cos\, z^2 = 0. \end{aligned} \tag{2.2.12}$$

and in the same manner we show that:

$$g_{kj} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2.13}$$

By means of Eqs. (2.2.9/2.2.13) it becomes now easily possible to obtain the line element:

$$\begin{aligned} \mathrm{d}s &= \sqrt{g_{kj}\mathrm{d}z^k\mathrm{d}z^j} \\ &= \sqrt{\begin{pmatrix} \mathrm{d}r & \mathrm{d}\vartheta & \mathrm{d}z \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \mathrm{d}r \\ \mathrm{d}\vartheta \\ \mathrm{d}z \end{pmatrix}} = \sqrt{\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + \mathrm{d}z^2}. \end{aligned} \tag{2.2.14}$$

In this context we have used the rules of matrix multiplication. Note that this was done for practical reasons and not because of necessity. The same result can be obtained by "expanding" the double sum, i.e., by writing down each term and suitable combination. The latter is always possible, even if sums of order higher than two are concerned, i.e., triple summations, etc. However, in general, the beautiful matrix notation is then no longer possible.

Exercise 2.2.2: ***Metric tensor for spherical coordinates***

Use geometric arguments in context with Fig. 1.3 to show that the following relations hold between Cartesian and spherical coordinates, $(x_1,\ x_2,\ x_3)$ and $(r,\ \vartheta,\ \varphi)$, $r \in [0, \infty)$, $\vartheta \in [0, \pi)$, $\varphi \in [0, 2\pi)$, respectively:

$$\begin{aligned}
z^1 &\equiv r = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad x_1 = r \sin\vartheta \cos\varphi = z^1 \sin z^2 \cos z^3,\\
z^2 &\equiv \vartheta = \arccos \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \quad x_2 = r \sin\vartheta \sin\varphi = z^1 \sin z^2 \sin z^3,\\
z^3 &\equiv \varphi = \arctan \frac{x_2}{x_1}, \quad x_3 = r \cos\vartheta = z^1 \cos z^2.
\end{aligned} \tag{2.2.15}$$

Discuss the shape of coordinate lines of a constant radial distance, azimuthal as well as polar angle. Show that these lines are perpendicular to each other. Moreover, show that the components of the metric tensor read:

$$g_{kj} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\vartheta \end{pmatrix}. \tag{2.2.16}$$

As a trivial example for an application of the formula for the line element (2.2.9) we calculate the circumferential length U of a circle C_R of radius R. In this case we have $\mathrm{d}r = 0$ and $\mathrm{d}z = 0$ and obtain:

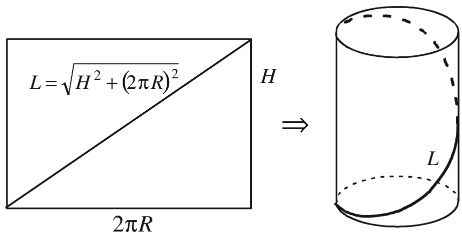
$$U = \oint_{C_R} \mathrm{d}s = \int_0^{2\pi} R\mathrm{d}\vartheta = 2\pi R. \tag{2.2.17}$$

As a more complex example for Eq. (2.2.14) we consider the situation shown in Fig. 2.2: The objective is to determine the length L of the diagonal in a rectangle of height H and width $2\pi R$. Obviously this can be obtained by using the Pythagorean theorem:

$$L = \sqrt{H^2 + (2\pi R)^2}, \tag{2.2.18}$$

which has nothing to do with Eq. (2.2.14). However, now transform the rectangle into a three-dimensional object, namely the mantle of a cylinder, as shown in Fig. 2.2. This way the former diagonal is also transformed into a three-dimensional curve.

Fig. 2.2 Transformation of a straight line in the plane into a three-dimensional curve



It makes sense to calculate the length of the curve with cylindrical coordinates. We first note that the radial distance of the curve on the mantle does not change:

$$r = R = \text{const.} \tag{2.2.19}$$

Consequently Eq. (2.2.14) becomes:

$$\mathrm{d}r = 0 \Rightarrow \mathrm{d}s = \sqrt{R^2 \mathrm{d}\vartheta^2 + \mathrm{d}z^2}. \tag{2.2.20}$$

Now we assume that the height z changes linearly with the polar angle ϑ:

$$z = A\vartheta + B. \tag{2.2.21}$$

Of course we have:

$$z(\vartheta = 0) = 0 \quad \text{and} \quad z(\vartheta = 2\pi) = H, \tag{2.2.22}$$

and therefore the constants A and B become:

$$A = \frac{H}{2\pi} \quad \text{and} \quad B = 0. \tag{2.2.23}$$

If we insert this in Eq. (2.2.20) we obtain:

$$\mathrm{d}z = \frac{H}{2\pi}\mathrm{d}\vartheta \Rightarrow s = \int\limits_{\vartheta=0}^{\vartheta=2\pi} \sqrt{R^2 + \left(\frac{H}{2\pi}\right)^2}\, \mathrm{d}\vartheta = \sqrt{R^2 + \left(\frac{H}{2\pi}\right)^2}\, 2\pi, \tag{2.2.24}$$

in other words, a result identical to Eq. (2.2.18).

Exercise 2.2.3: *Line element in spherical coordinates*

Show with the result for the metric tensor from Exercise 2.2.2 that the line element ds in spherical coordinates is given by:

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r^2 \sin^2 \vartheta \mathrm{d}\varphi^2 + r^2 \mathrm{d}\vartheta^2}. \tag{2.2.25}$$

Use the equation to show that the length of the equatorial circumference as well as the length of any great circle of a sphere of radius R is given by:

$$U = 2\pi R. \tag{2.2.26}$$

Exercise 2.2.4: ***Decoration***

A producer of hats for the Loge of the Metric Tensor Fetishists faces the problem of providing enough string for the decoration of length L on a piece of felt of quarter-circle-shape. The felt is then rolled into a fashionable cone hat as shown in Fig. 2.3. By doing so the radial distance r of the string grows proportionally with the polar angle ϑ starting from zero to the radius R of the quarter circle.

Calculate the length L of the string

(a) in plane polar coordinates (Fig. 2.3, left)

(b) in three-dimensional cylindrical coordinates (Fig. 2.3, right) and show that in both cases $L \approx 1.324R$.

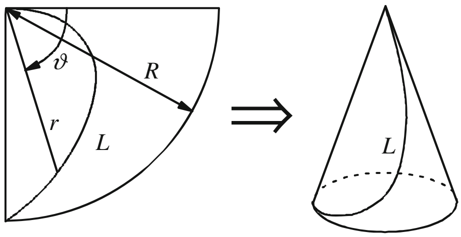


Fig. 2.3 Making of a decoration

Clearly the same length must result, independently of the choice of coordinates. Discuss the pros and cons of both methods, also by comparison with the previously discussed problem of the diagonal.

We summarize what we have learned so far: Cartesian coordinates are characterized by straight lines and form a square-like mesh in the plane and a cube-like mesh in three dimensions. Polar coordinates (or in other words cylindrical coordinates of the plane), however, form a spider-like web such that the lines of constant radius are given by concentric circles around the origin, whereas the lines of constant polar angle are straight and run through the origin. So one set of lines is curved, but both sets are *perpendicular* to each other, just like the Cartesian case. However, in general, coordinate meshes do not have to intersect at a 90° angle. An example is shown in Fig. 2.4: Next to the traditional Cartesian system denoted by x a scissors-like coordinate system is drawn and denoted by z. It obviously takes two angles, α and β, in order to characterize the orientation of the z system. In Exercise 2.3.1 it will be shown that the corresponding metric tensor is *not* diagonal unlike the previous cases of polar, cylindrical, and spherical coordinates. This is due to the fact that these were *orthogonal* coordinate systems whereas the scissors-like system is not. In fact, we shall see shortly that components of the metric tensor can be interpreted as scalar products between vectors pointing in the direction of the coordinate lines. If these are orthogonal to each other their scalar products must vanish.

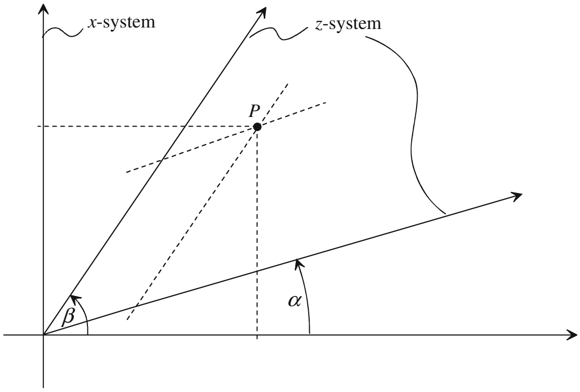


Fig. 2.4 Skew coordinate system

2.3 Vector Interpretation of the Metric

We now return to the problem of calculating the distance between two infinitesimally close points, which was already investigated component-wise in Eq. (2.2.6). In the absolute language of vectors we denote the infinitesimal distance between both points by $\mathrm{d}\boldsymbol{x}$. Moreover, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ (or short $\boldsymbol{e}_i,\ i = 1,\ 2,\ 3$) stand for three Cartesian unit vectors satisfying so-called conditions of *orthonormality*:

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}, \tag{2.3.1}$$

where the so-called *Kronecker symbol* a.k.a. *unit tensor* has been used:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \tag{2.3.2}$$

If we now observe the relations (2.2.1) between the coordinates x_i and z^k we may write:

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}x_i \boldsymbol{e}_i = \frac{\partial x_i}{\partial z^k} \mathrm{d}z^k \boldsymbol{e}_i, \tag{2.3.3}$$

where Eq. (2.2.6) has been used again and Einstein's summation rule has been extended to expressions related to coordinate lines and vectors. Reshuffling terms in (2.3.3) yields:

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}z^k \boldsymbol{g}_k, \quad \boldsymbol{g}_k = \frac{\partial x_i}{\partial z^k} \boldsymbol{e}_i. \tag{2.3.4}$$

Leopold KRONECKER was born on December 7, 1823 in Liegnitz (Silesia) and died on December 29, 1891 in Berlin. In the spring of 1841 KRONECKER starts to study mathematics at the University of Berlin. There he attends lectures by the famous mathematicians DIRICHLET, JACOBI, and STEINER. However, he is also interested in philosophy and participates in the lectures offered by SCHELLING. Another famous mathematician, Ernst Eduard KUMMER, becomes his teacher and mentor. It is therefore not too surprising that KRONECKER inherits KUMMER's chair in 1883. It is curious to note that during a mathematical debate on the *infinite* he insisted on proofs of related theorems within a *finite* number of steps, i.e., by avoiding the concept of quantities of size epsilon. This is why David HILBERT referred to him as *Verbotsdiktator* (forbidding dictator). This name, however, would also suit to characterize the effect of the symbol named after him.

Figure 2.5 illustrates for the planar case (which is simpler to draw) how we have to interpret the new defined vectors $\boldsymbol{g}_k$. The figure shows how a tangent vector to the line $z^1 =$ const., namely $\boldsymbol{g}_2$, is obtained:

$$\begin{aligned} \mathrm{d}\boldsymbol{x} &= \lim_{\Delta \boldsymbol{x} \to 0} \Delta \boldsymbol{x} = \lim_{\Delta x_i \to 0} \Delta x_i \boldsymbol{e}_i \\ &= \lim_{\Delta \boldsymbol{x} \to 0} \frac{x_i(z^1, z^2 + \Delta z^2, z^3) - x_i(z^1, z^2, z^3)}{\Delta z^2} \Delta z^2 \boldsymbol{e}_i = \frac{\partial x_i}{\partial z^2} \boldsymbol{e}_i \mathrm{d}z^2. \end{aligned} \tag{2.3.5}$$

Consequently, for this case we have:

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}z^2} = \frac{\partial x_i}{\partial z^2} \boldsymbol{e}_i \equiv \boldsymbol{g}_2. \tag{2.3.6}$$

It is slightly irritating that $\boldsymbol{g}_2$ (and not $\boldsymbol{g}_1$) denotes the tangent vector to the lines $z^1 =$ const. and *vice versa*. A solution to this puzzle is offered in Fig. 2.1: In the Cartesian case lines for which $x_1 =$ const. holds (say) are parallel to the abscissa x_2 and we also call them x_2-lines for short. Consequently, (see again Fig. 2.1) lines for which $r =$ const. are ϑ-lines and *vice versa*. However, words are always vague and confusing. What is important, though, is the equation that tells us how to calculate the new vector base $\boldsymbol{g}_k$, i.e., Eq. $(2.3.4)_2$. The terminology used for these vectors is, in the end, arbitrary.

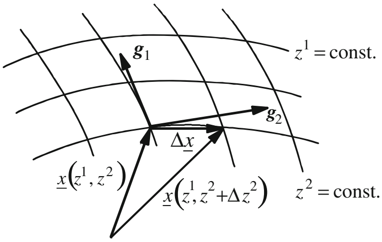


Fig. 2.5 Tangent vectors to the lines of skew/curvilinear coordinates

For the case of three dimensions all arguments hold analogously. Therefore we may say that, in general, $\boldsymbol{g}_k$ denotes tangent vectors to the lines $z^j = \text{const.}$ However, note that these are *not* necessarily unit vectors. We now use them to calculate the following scalar product and rearrange terms slightly:

$$\begin{aligned} \boldsymbol{g}_k \cdot \boldsymbol{g}_l &= \left(\frac{\partial x_i}{\partial z^k}\boldsymbol{e}_i\right) \cdot \left(\frac{\partial x_j}{\partial z^l}\boldsymbol{e}_j\right) = \frac{\partial x_i}{\partial z^k}\frac{\partial x_j}{\partial z^l}\boldsymbol{e}_i \cdot \boldsymbol{e}_j \\ &= \frac{\partial x_i}{\partial z^k}\frac{\partial x_j}{\partial z^l}\delta_{ij} = \frac{\partial x_i}{\partial z^k}\frac{\partial x_i}{\partial z^l} \equiv g_{kl}, \end{aligned} \tag{2.3.7}$$

where the KRONECKER symbol of Eq. (2.3.2) has been used. Note that the effect of the KRONECKER symbol consists of replacing one of its bound indices in the remaining expression of a product with its other index. In order to prove this statement all sums must be expanded first. Then all of the occurring terms can be simplified by observing Eq. (2.3.2).

We conclude that the scalar product between the two tangent vectors yields the components of the metric tensor. The basic definition of the scalar product of two vectors involves the cosine of the angle they enclose. Therefore non-diagonal components of the metric must vanish, if the curvilinear coordinates are *orthogonal* as, for example, in the case of cylindrical or spherical transformations. Consequently, their explicit calculation is unnecessary, albeit possible, as for example demonstrated in Eq. $(2.2.12)_2$.

As a specific example we consider the case of plane polar coordinates for which the tangent vectors to the coordinate lines can be calculated explicitly. We use Eqs. (2.2.11) in context with $(2.3.4)_2$ to obtain:

$$\begin{aligned} \boldsymbol{g}_1 &= \frac{\partial x_1}{\partial z^1}\boldsymbol{e}_1 + \frac{\partial x_2}{\partial z^1}\boldsymbol{e}_2 = \cos\vartheta\,\boldsymbol{e}_1 + \sin\vartheta\,\boldsymbol{e}_2 \equiv \boldsymbol{e}_r, \\ \boldsymbol{g}_2 &= \frac{\partial x_1}{\partial z^2}\boldsymbol{e}_1 + \frac{\partial x_2}{\partial z^2}\boldsymbol{e}_2 = -r\sin\vartheta\,\boldsymbol{e}_1 + r\cos\vartheta\,\boldsymbol{e}_2 \equiv r\boldsymbol{e}_\vartheta. \end{aligned} \tag{2.3.8}$$

In this equation we have introduced the commonly used unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\vartheta$ of polar coordinates. They are shown in Fig. 2.6. In particular, the second chain of equations shows that the tangent vectors $\boldsymbol{g}_i$ do not necessarily need to be normalized.

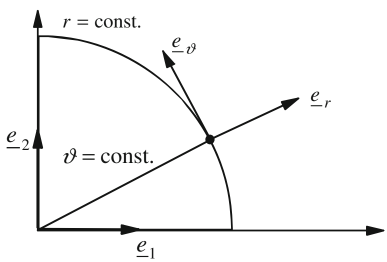


Fig. 2.6 Plane polar coordinates and the corresponding (unit) tangent vectors

Exercise 2.3.1: ***Metric of a plane skew coordinate system***

Use geometric arguments to show that for the scissor-like coordinates of Fig. 2.4 the following relations hold:

$$x_1 = \cos\alpha\, z^1 + \cos\beta\, z^2,\; x_2 = \sin\alpha\, z^1 + \sin\beta\, z^2, \tag{2.3.9}$$

and the metric tensor reads:

$$g_{kj} = \begin{pmatrix} 1 & \cos(\alpha-\beta) \\ \cos(\alpha-\beta) & 1 \end{pmatrix}. \tag{2.3.10}$$

Discuss and interpret the case $\alpha - \beta = \pi/2$. Show that the tangent vectors are given by:

$$\boldsymbol{g}_1 = \cos\alpha\, \boldsymbol{e}_1 + \sin\alpha\, \boldsymbol{e}_2,\; \boldsymbol{g}_2 = \cos\beta\, \boldsymbol{e}_1 + \sin\beta\, \boldsymbol{e}_2. \tag{2.3.11}$$

Now use this result in context with the interpretation of metric components as scalar products and rederive Eq. (2.3.10).

Exercise 2.3.2: ***Metric for spherical coordinates in vector notation***

Recall the equations for spherical coordinates discussed in Exercise 2.2.2 and use them in context with Eq. $(2.3.4)_2$ to show that the tangent vectors read:

$$\begin{aligned} \boldsymbol{g}_1 &= \sin\vartheta\, \cos\varphi\, \boldsymbol{e}_1 + \sin\vartheta\, \sin\varphi\, \boldsymbol{e}_2 + \cos\vartheta\, \boldsymbol{e}_3, \\ \boldsymbol{g}_2 &= r\cos\vartheta\, \cos\varphi\, \boldsymbol{e}_1 + r\cos\vartheta\, \sin\varphi\, \boldsymbol{e}_2 - r\sin\vartheta\, \boldsymbol{e}_3, \\ \boldsymbol{g}_3 &= -r\sin\vartheta\, \sin\varphi\, \boldsymbol{e}_1 + r\sin\vartheta\, \cos\varphi\, \boldsymbol{e}_2. \end{aligned} \tag{2.3.12}$$

Use these results to reconfirm the expression for the metric tensor shown in Eq. (2.2.16). Also confirm that the tangent vectors can be linked to the unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta$, and $\boldsymbol{e}_\varphi$ of Fig. 1.3 as follows:

$$\boldsymbol{g}_1 = \boldsymbol{e}_r, \quad \boldsymbol{g}_2 = r\, \boldsymbol{e}_\vartheta, \quad \boldsymbol{g}_3 = r\sin\vartheta\, \boldsymbol{e}_\varphi. \tag{2.3.13}$$

Exercise 2.3.3: ***Elliptic coordinates***

The following relations hold between the so-called elliptic coordinates z^1, z^2 and planar Cartesian coordinates x_1, x_2:

$$\begin{aligned} &x_1 = c \cosh z^1 \cos z^2,\ x_2 = c \sinh z^1 \sin z^2 , \\ &\quad z^1 \in [0, \infty),\ z^2 \in [0, 2\pi) . \end{aligned} \tag{2.3.14}$$

Rearrange these equations to show that of constant values of z^1 and z^2 can be interpreted as confocal ellipses and hyperbolae. Identify both axes of the ellipses in terms of the parameter c shown in Eq. (2.3.14). Discuss the limit for which the ellipse degenerates into a crack. Determine the length of the crack. Show that the metric is given by:

$$g_{ij} = \frac{c^2}{2} \begin{pmatrix} \cosh(2z^1) - \cos(2z^2) & 0 \\ 0 & \cosh(2z^1) - \cos(2z^2) \end{pmatrix}, \tag{2.3.15}$$

and confirm that the tangential vectors can be written as:

$$\begin{aligned} \boldsymbol{g}_1 &= c \sinh z^1 \cos z^2\, \boldsymbol{e}_1 + c \cosh z^1 \sin z^2\, \boldsymbol{e}_2, \\ \boldsymbol{g}_2 &= -c \cosh z^1 \sin z^2\, \boldsymbol{e}_1 + c \sinh z^1 \cos z^2\, \boldsymbol{e}_2. \end{aligned} \tag{2.3.16}$$

Use the interpretation of metric coefficients as scalar products to rederive Eq. (2.3.15).

2.4 Co- and Contravariant Components

In this section we introduce the notions of *co-* and *contravariant* components which are important in context with the representation of vectors and tensors in arbitrary curvilinear coordinate systems. For this purpose we consider the situation shown in Fig. 2.7: The vector $\boldsymbol{A}$ is first decomposed w.r.t. a Cartesian coordinate system called x. In this frame it is characterized by the components $\underset{(x)}{A_1}$ and $\underset{(x)}{A_2}$. In order to point out that these are components w.r.t. a Cartesian frame they carry the additional suffix "(x)."

Besides the Cartesian system a skew coordinate system called z is depicted. The vector $\boldsymbol{A}$ can be decomposed in two ways w.r.t. that system. One possibility is to project the vector *parallel* to the coordinate lines z^1 and z^2: Fig. 2.7 (right). This way we obtain the components $\underset{(z)}{A^1}$ and $\underset{(z)}{A^2}$. They are components w.r.t. the skew coordinate system, and this is emphasized by a suffix "(z)." They are also known as *contravariant* components or, in other words, we speak of the contravariant representation of the vector $\boldsymbol{A}$ in the z-system. This way of representation is characterized by *upper indices* at the vector symbol. Indeed, without knowing, we have already used this notation in context with the coordinate lines z^i from Sect. 2.2.

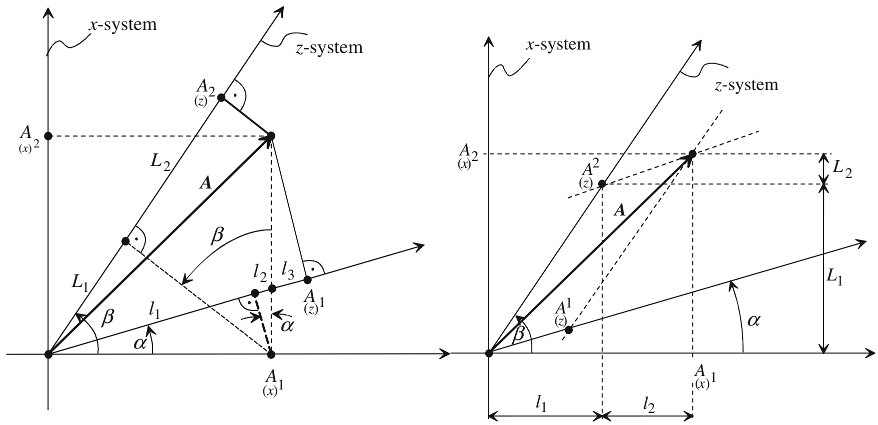


Fig. 2.7 Co- and contravariant components

However, there is a second possibility of how to represent vectors in skew curvilinear coordinate systems: The vector $\boldsymbol{A}$ can also be projected *perpendicularly* to the z-axes. This is indicated in Fig. 2.7 (left). In this manner we obtain $\underset{(z)}{A_1}$ and $\underset{(z)}{A_2}$, which are called *covariant* components and identified by lower indices.

Note that in Cartesian coordinate systems, i.e., in the x-system, it is impossible to distinguish co- and contravariant components. It is for that reason that we have used x_i for the components of the position vector above. However, the notation x^i would be equally appropriate.

Next we shall prove a formula that allows us to transform a certain set of coordinates—co-, contravariant, or Cartesian—into another one. The proof will be illustrated for the special case of two dimensions. However, the formula also holds for the 3D-case: The components $\underset{(x)}{A_i}$ of a vector $\boldsymbol{A}$ w.r.t. a Cartesian coordinate system, x, can be converted into co- and contravariant components, $\underset{(z)}{A_i}$ or $\underset{(z)}{A^i}$, w.r.t. a skew or curvilinear coordinate system, z, by differentiation of the coordinate transformation from Eq. (2.2.1):

$$\underset{(z)}{A^i} = \frac{\partial z^i}{\partial x_k}\underset{(x)}{A_k}, \quad \underset{(z)}{A_i} = \frac{\partial x_k}{\partial z^i}\underset{(x)}{A_k}\,. \tag{2.4.1}$$

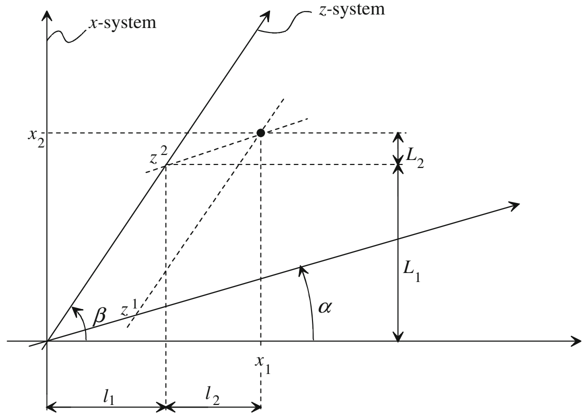


Fig. 2.8 A position represented in a Cartesian and a skew coordinate system

For the proof we consider the systems x and z in Fig. 2.8 and conclude that the coordinate transformation of Eq. (2.2.1) can be written explicitly as (also see Exercise 2.3.1):

$$\begin{aligned} x_1 &= \cos\alpha\, z^1 + \cos\beta\, z^2, \quad z^1 = H\,[\sin\beta\, x_1 - \cos\beta\, x_2], \\ x_2 &= \sin\alpha\, z^1 + \sin\beta\, z^2, \quad z^2 = H\,[-\sin\alpha\, x_1 + \cos\alpha\, x_2], \\ H &= [\cos\alpha\, \sin\beta - \sin\alpha\, \cos\beta]^{-1} \equiv [\sin(\beta-\alpha)]^{-1}. \end{aligned} \tag{2.4.2}$$

A closer examination of Fig. 2.7 shows that the contravariant components can be written as:

$$\underset{(z)}{A^1} = H\left[\sin\beta \underset{(x)}{A_1} - \cos\beta \underset{(x)}{A_2}\right], \quad \underset{(z)}{A^2} = H\left[-\sin\alpha \underset{(x)}{A_1} + \cos\alpha\, \underset{(x)}{A_2}\right], \tag{2.4.3}$$

whereas we find for the covariant ones:

$$\underset{(z)}{A_1} = \cos\alpha \underset{(x)}{A_1} + \sin\alpha\, \underset{(x)}{A_2}, \quad \underset{(z)}{A_2} = \cos\beta \underset{(x)}{A_1} + \sin\beta \underset{(x)}{A_2}\,. \tag{2.4.4}$$

By differentiation of Eq. (2.4.2) the validity of Eq. (2.4.1) is easily established. Of course, being a planar problem, the summation runs only from 1 to 2.

Exercise 2.4.1: ***Transformation formulae from a plane Cartesian to a skew coordinate system***

Use the auxiliary quantities l_1, l_2, l_3, L_1, L_2 in Fig. 2.7 in order to confirm Eqs. (2.4.3), (2.4.4), and (2.4.1).

Next we multiply in Eq. (2.4.1) $\underset{(z)}{A}^i$ by the metric g_{ni}, observe Eq. (2.2.8) and obtain:

$$g_{ni}\underset{(z)}{A}^i = \frac{\partial x_l}{\partial z^n}\frac{\partial x_l}{\partial z^i}\frac{\partial z^i}{\partial x_k}\underset{(x)}{A_k} = \frac{\partial x_l}{\partial z^n}\delta_{lk}\underset{(x)}{A_k} = \frac{\partial x_k}{\partial z^n}\underset{(x)}{A_k} = \underset{(z)}{A_n}\,. \tag{2.4.5}$$

Here we have used the *chain rule* after the second equality sign (or, figuratively speaking, "cancelled out" ∂z^i). This generates a Kronecker symbol, δ_{lk}, first and then Eq. $(2.4.1)_2$ was observed. Of course, the Kronecker symbol is nothing else but the unit matrix in component form, i.e., we may write:

$$\frac{\partial x_l}{\partial z^n}\delta_{lk}\underset{(x)}{A_k} = \frac{\partial x_k}{\partial z^n}\underset{(x)}{A_k} \quad \text{or} \quad \delta_{lk}\underset{(x)}{A_k} = \underset{(x)}{A}\,_l, \tag{2.4.6}$$

which, consequently, transforms the index l in Eq. (2.4.5) into the index k, or vice versa. Eq. (2.4.1) was applied once more after the last equality sign of Eq. (2.4.5). This time, however, for the covariant components $\underset{(z)}{A_n}$.

We conclude that by means of the covariant components g_{lk} of the metric tensor it becomes possible to convert the contravariant index k into a covariant one, l. This process is also known as *contraction* in textbooks on tensors: Multiplication with the covariant metric components g_{lk} *lowers* the index k. However, it is also possible to *raise* indices. To this end we now introduce the inverse to g_{lk} by:

$$g^{lk} = \frac{\partial z^l}{\partial x_p}\frac{\partial z^k}{\partial x_p}, \tag{2.4.7}$$

and may write:

$$\underset{(z)}{A}^i = g^{ij}\underset{(z)}{A_j}\,, \quad \underset{(z)}{A_i} = g_{ij}\underset{(z)}{A}^j\,. \tag{2.4.8}$$

The first equation shows that contraction of an expression with g^{ij} will raise the covariant index j to a contravariant index i.

Exercise 2.4.2: ***Contravariant metric components***

Convince yourself by explicit expansion and use of the chain rule that the contravariant components g^{lk} shown in Eq. (2.4.7) truly constitute the inverse of the covariant form g_{lk}.

Exercise 2.4.3: ***Transforming covariant into contravariant components***

Follow the arguments of the text and show analogously that covariant components can be converted into contravariant ones according to $(2.4.8)_1$.

In fact, we may manipulate the partial derivatives in Eq. (2.4.1) as if they were fractions, i.e., write:

$$\underset{(x)}{A_k} = \frac{\partial x_k}{\partial z^i} \underset{(z)}{A^i}, \quad \underset{(x)}{A_k} = \frac{\partial z^i}{\partial x_k} \underset{(z)}{A_i}\,. \tag{2.4.9}$$

All of this is a consequence of the chain rule. If, for example, we multiply Eq. $(2.4.1)_1$ by the expression $\partial x_m / \partial z^i$, we obtain:

$$\frac{\partial x_m}{\partial z^i} \underset{(z)}{A^i} = \frac{\partial x_m}{\partial z^i} \frac{\partial z^i}{\partial x_k} \underset{(x)}{A_k} = \frac{\partial x_m}{\partial x_k} \underset{(x)}{A_k} = \delta_{mk} \underset{(x)}{A_k} = \underset{(x)}{A_m}, \tag{2.4.10}$$

i.e., by renaming the indices $m \to k$ exactly Eq. $(2.4.9)_1$. Equation $(2.4.9)_2$ can be validated in an analogous manner.

It was mentioned in context with Eq. (2.2.9) that the metric tensor allows to calculate distances with curvilinear coordinates. We will now show that it can also be used to determine the length of a vector. For this purpose we start from the basic definition for the length of a vector, namely with the scalar product. If the vector is represented by Cartesian components we may write:

$$\begin{aligned} A = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}} &= \sqrt{(\underset{(x)}{A_i}\, \boldsymbol{e}_i) \cdot (\underset{(x)}{A_j}\, \boldsymbol{e}_j)} = \sqrt{\underset{(x)}{A_i}\, \underset{(x)}{A_j}\, \boldsymbol{e}_i \cdot \boldsymbol{e}_j} \\ &= \sqrt{\underset{(x)}{A_i}\, \underset{(x)}{A_j}\, \delta_{ij}} = \sqrt{\underset{(x)}{A_i}\, \underset{(x)}{A_i}}, \end{aligned} \tag{2.4.11}$$

where use was made of Eqs. (2.3.1/2.3.2) and the properties of the Kronecker symbol. By observing Eq. $(2.4.9)_1$ and the basic definition of the metric tensor (2.2.8) this yields:

$$A = \sqrt{\frac{\partial x_i}{\partial z^k} \underset{(z)}{A^k} \frac{\partial x_i}{\partial z^l} \underset{(z)}{A^l}} = \sqrt{\frac{\partial x_i}{\partial z^k} \frac{\partial x_i}{\partial z^l} \underset{(z)}{A^k}\, \underset{(z)}{A^l}} = \sqrt{g_{kl}\, \underset{(z)}{A^k}\, \underset{(z)}{A^l}}, \tag{2.4.12}$$

or by $(2.4.9)_2$ and Eq. (2.4.7) for the inverse metric:

$$A = \sqrt{\frac{\partial z^k}{\partial x_i} \underset{(z)}{A_k} \frac{\partial z^l}{\partial x_i} \underset{(z)}{A_l}} = \sqrt{\frac{\partial z^k}{\partial x_i} \frac{\partial z^l}{\partial x_i} \underset{(z)}{A_k}\, \underset{(z)}{A_l}} = \sqrt{g^{kl}\, \underset{(z)}{A_k}\, \underset{(z)}{A_l}}. \tag{2.4.13}$$

Note that bound indices in the sense of the summation convention can be used only once (consequently we have to distinguish between k and l). Moreover, the index calculus allows us to check easily if the summation convention has been applied correctly: Bound indices in a tensor equation always have to appear in pairs, i.e., one of them is covariant and the other one is contravariant (see, for example, the index k in g^{kl} connecting to $\underset{(z)}{A_k}$). This property becomes also evident in the following third alternative equation for the length of a vector:

$$
\begin{aligned}
A &= \sqrt{\frac{\partial x_i}{\partial z^k}\underset{(z)}{A^k}\frac{\partial z^l}{\partial x_i}\underset{(z)}{A_l}} = \sqrt{\frac{\partial x_i}{\partial z^k}\frac{\partial z^l}{\partial x_i}\underset{(z)}{A^k}\,\underset{(z)}{A_l}} \\
&= \sqrt{\delta_k^l\,\underset{(z)}{A^k}\,\underset{(z)}{A_l}} = \sqrt{\underset{(z)}{A^l}\,\underset{(z)}{A_l}}.
\end{aligned}
\tag{2.4.14}
$$

Analogously to the case of vectors co- and contravariant components can also be introduced for tensors. If we consider the absolute tensor quantity $\boldsymbol{B}$, which could represent the stress tensor $\boldsymbol{\sigma}$ or the strain tensor $\boldsymbol{\varepsilon}$ (say), we can write analogously to Eq. (2.4.1):

$$
\begin{aligned}
&\underset{(z)}{B^{ij}} = \frac{\partial z^i}{\partial x_k}\frac{\partial z^j}{\partial x_l}\underset{(x)}{B_{kl}}, \quad \underset{(z)}{B_{ij}} = \frac{\partial x_k}{\partial z^i}\frac{\partial x_l}{\partial z^j}\underset{(x)}{B_{kl}}, \\
&\underset{(z)}{B^i{}_j} = \frac{\partial z^i}{\partial x_k}\frac{\partial x_l}{\partial z^j}\underset{(x)}{B_{kl}}, \quad \underset{(z)}{B^j{}_i} = \frac{\partial x_k}{\partial z^i}\frac{\partial z^j}{\partial x_l}\underset{(x)}{B_{kl}}\,.
\end{aligned}
\tag{2.4.15}
$$

For obvious reasons the components in the last two Eq. of (2.4.15) are also known as *mixed components* of the tensor $\boldsymbol{B}$. As in the case of vectors all indices can be raised and lowered by means of the co- and contravariant metric components of Eq. (2.4.8). For example:

$$
\underset{(z)}{B^{ij}} = g^{ik}g^{jl}\underset{(z)}{B_{kl}}, \; \underset{(z)}{B_{ij}} = g_{ik}g_{jl}\,\underset{(z)}{B^{kl}}\,. \tag{2.4.16}
$$

And we may treat the derivatives like ordinary fractions following Eq. (2.4.9):

$$
\begin{aligned}
&\underset{(x)}{B_{kl}} = \frac{\partial x_k}{\partial z^i}\frac{\partial x_l}{\partial z^j}\underset{(z)}{B^{ij}}, \quad \underset{(x)}{B_{kl}} = \frac{\partial z^i}{\partial x_k}\frac{\partial z^j}{\partial x_l}\underset{(z)}{B_{ij}}, \\
&\underset{(x)}{B_{kl}} = \frac{\partial x_k}{\partial z^i}\frac{\partial z^j}{\partial x_l}\underset{(z)}{B^i{}_j}, \quad \underset{(x)}{B_{kl}} = \frac{\partial z^i}{\partial x_k}\frac{\partial x_l}{\partial z^j}\underset{(z)}{B^j{}_i}\,.
\end{aligned}
\tag{2.4.17}
$$

Again the proof is based on successive application of the chain rule. Observe that bound indices always appear in co-/contravariant pairs.

Exercise 2.4.4: ***The components of the metric tensor as co-/contravariant components of the unit tensor***

Show by using the definitions (2.2.8), (2.4.7) and by application of the chain rule to ${\delta^i}_j = \partial z^i / \partial z^j$ that the following relations hold for the Kronecker symbol (which was originally defines in a Cartesian frame):

$$g^{ij} = \frac{\partial z^i}{k} \frac{\partial z^j}{\partial x_l} \delta_{kl}, \quad g_{ij} = \frac{\partial x_k}{\partial z^i} \frac{\partial x_l}{\partial z^j} \delta_{kl},$$
$${\delta^i}_j = \frac{\partial z^i}{\partial x_k} \frac{\partial x_l}{\partial z^j} \delta_{kl}, \quad \delta_i^j = \frac{\partial x_k}{\partial z^i} \frac{\partial z^j}{\partial x_l} \delta_{kl}. \tag{2.4.18}$$

Interpret these equations by using (2.4.15).

Exercise 2.4.5: ***The co-/contravariant components of the position vector in cylindrical and spherical coordinates***

Show by using the Eqs. (2.4.9), (2.2.11), and (2.2.15) that the following relations hold for the co-contravariant components of the position vector, $\boldsymbol{x}$, in cylindrical and spherical coordinates, respectively:

$$\underset{(z)}{x}^1 = r\,, \quad \underset{(z)}{x}^2 = 0\,, \quad \underset{(z)}{x}^3 = z; \qquad \underset{(z)}{x_1} = r, \;\; \underset{(z)}{x_2} = 0, \;\; \underset{(z)}{x_3} = 0. \tag{2.4.19}$$

Explain the difference between coordinate lines and the position vector.

Exercise 2.4.6: ***The co-/contravariant components of the stress tensor in cylindrical coordinates***

Use Eqs. $(2.4.15)_1$ and (2.2.11) to show that the contravariant components of the stress tensor in polar coordinates (i.e., plane cylindrical coordinates) read:

$$\begin{aligned}
\underset{(z)}{\sigma}^{11} &= \cos^2\vartheta\; \sigma_{xx} + 2\,\sin\vartheta\,\cos\vartheta\; \sigma_{xy} + \sin^2\vartheta\; \sigma_{yy}, \\
\underset{(z)}{\sigma}^{12} &= \tfrac{1}{r}\left[-\sin\vartheta\,\cos\vartheta\; \sigma_{xx} + \left(\cos^2\vartheta - \sin^2\vartheta\right) \sigma_{xy} + \sin\vartheta\,\cos\vartheta\; \sigma_{yy}\right], \\
\underset{(z)}{\sigma}^{22} &= \tfrac{1}{r^2}\left(\sin^2\vartheta\; \sigma_{xx} - 2\,\sin\vartheta\,\cos\vartheta\; \sigma_{xy} + \cos^2\vartheta\, \sigma_{yy}\right).
\end{aligned} \tag{2.4.20}$$

The symbols $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}, \sigma_{yy}, \sigma_{yz}, \sigma_{zz}$ denote the components for plane stress in Cartesian coordinates $x_1, x_2, x_3 \equiv x, y, z$. Moreover, use Eqs. $(2.4.15)_2$ and (2.2.11) and derive corresponding expressions for the covariant components of the stress tensor in cylindrical coordinates. How do the co-/contravariant expressions for the stress tensor fit into the world of Mohr's circle in 2D ?

Exercise 2.4.7: ***The co-/contravariant components of the stress tensor in spherical coordinates***

Use Eqs. (2.2.15) and (2.4.15) and derive expressions for the co-/contravariant components of the stress tensor in spherical coordinates for given Cartesian stress components σ_{xx}, σ_{xy}, σ_{xz}, σ_{yy}, σ_{yz}, and σ_{zz}, radial distance r, and the two angles φ and ϑ.

Christian Otto Mohr was born on October 8, 1835 in Wesselburen in Holstein (Germany) and died on October 2, 1918 in Dresden. Despite being a full-blood engineer he also showed a certain appreciation to useful mathematical concepts. Examples of this are MOHR's circles, which allowed for an intuitive representation of the components of the various stress tensors without knowing tensor calculus. Another one is MOHR's analogy, a graphical method for solving the differential equation of deflective beams for geometrically difficult cases. MOHR held chairs for technical mechanics in Stuttgart and Dresden and proved himself to be a didactically sensitive professor whenever teaching strength of materials.

2.5 Co- and Contravariant from the Perspective of Vectors

By using the base vectors $\boldsymbol{g}_k$ from Eq. $(2.3.4)_2$ we may write for an arbitrary vector $\boldsymbol{A}$:

$$\boldsymbol{A} = \underset{(z)}{A^k} \boldsymbol{g}_k = \underset{(z)}{A^k} \frac{\partial x_i}{\partial z^k} \boldsymbol{e}_i. \tag{2.5.1}$$

On the other hand we have:

$$\boldsymbol{A} = \underset{(x)}{A_j} \boldsymbol{e}_j. \tag{2.5.2}$$

Consequently we conclude that:

$$\underset{(x)}{A_i} = \underset{(z)}{A^k} \frac{\partial x_i}{\partial z^k} \Rightarrow \underset{(z)}{A^k} = \frac{\partial z^k}{\partial x_i} \underset{(x)}{A_i}. \tag{2.5.3}$$

In other words: In the representation $\boldsymbol{A} = \underset{(z)}{A^k} \boldsymbol{g}_k$ the quantities $\underset{(z)}{A^k}$ must be interpreted as the contravariant components of the vector $\boldsymbol{A}$. We now define another base $\boldsymbol{g}^l$ (a.k.a. as the *dual base*) according to:

$$\boldsymbol{g}^l = \frac{\partial z^l}{\partial x_j} \boldsymbol{e}_j \tag{2.5.4}$$

and use it for decomposing the vector $\boldsymbol{A}$:

$$\boldsymbol{A} = \boldsymbol{g}^l \underset{(z)}{A_l} = \frac{\partial z^l}{\partial x_j} \boldsymbol{e}_j \underset{(z)}{A_l} \,. \tag{2.5.5}$$

By comparison with Eq. (2.5.2) we obtain:

$$\underset{(x)}{A_i} = \frac{\partial z^l}{\partial x_i} \underset{(z)}{A_l} \quad \Rightarrow \quad \underset{(z)}{A_l} = \frac{\partial x_i}{\partial z^l} \underset{(x)}{A_i} \,. \tag{2.5.6}$$

Consequently the quantities $\underset{(z)}{A_l}$ are truly the covariant components of $\boldsymbol{A}$ and we may write:

$$\boldsymbol{A} = \underset{(z)}{A_l}\, \boldsymbol{g}^l = \underset{(z)}{A^k} \boldsymbol{g}_k. \tag{2.5.7}$$

For the scalar product of the two sets of base vectors, $\boldsymbol{g}^l$ and $\boldsymbol{g}_k$, we find:

$$\begin{aligned} \boldsymbol{g}^l \cdot \boldsymbol{g}_k &= \left(\frac{\partial z^l}{\partial x_j} \boldsymbol{e}_j \right) \cdot \left(\frac{\partial x_i}{\partial z^k} \boldsymbol{e}_i \right) = \frac{\partial z^l}{\partial x_j} \frac{\partial x_i}{\partial z^k} \boldsymbol{e}_j \cdot \boldsymbol{e}_i \\ &= \frac{\partial z^l}{\partial x_j} \frac{\partial x_i}{\partial z^k} \delta_{ji} = \frac{\partial z^l}{\partial x_i} \frac{\partial x_i}{\partial z^k} = \frac{\partial z^l}{\partial z^k} = \delta^l{}_k. \end{aligned} \tag{2.5.8}$$

If we recall Eq. (2.3.7), i.e., the relation $\boldsymbol{g}_k \cdot \boldsymbol{g}_l = g_{kl}$ for the scalar product and the analogous condition:

$$\begin{aligned} \boldsymbol{g}^l \cdot \boldsymbol{g}^k &= \left(\frac{\partial z^l}{\partial x_j} \boldsymbol{e}_j \right) \cdot \left(\frac{\partial z^k}{\partial x_i} \boldsymbol{e}_i \right) = \frac{\partial z^l}{\partial x_j} \frac{\partial z^k}{\partial x_i} \boldsymbol{e}_j \cdot \boldsymbol{e}_i \\ &= \frac{\partial z^l}{\partial x_j} \frac{\partial z^k}{\partial x_i} \delta_{ji} = \frac{\partial z^l}{\partial x_j} \frac{\partial z^k}{\partial x_j} \equiv g^{lk}, \end{aligned} \tag{2.5.9}$$

we find by using Eq. (2.5.7) after scalar multiplication by $\boldsymbol{g}^m$:

$$\boldsymbol{g}^m \cdot \left(\boldsymbol{g}^l \underset{(z)}{A_l} \right) = \boldsymbol{g}^m \cdot \left(\underset{(z)}{A^k} \boldsymbol{g}_k \right) \quad \Rightarrow \quad \underset{(z)}{A^m} = g^{ml} \underset{(z)}{A_l}, \tag{2.5.10}$$

or by $\boldsymbol{g}_m$:

$$\boldsymbol{g}_m \cdot \left(\underset{(z)}{A_l}\, \boldsymbol{g}^l \right) = \boldsymbol{g}_m \cdot \left(\underset{(z)}{A^k} \boldsymbol{g}_k \right) \quad \Rightarrow \quad \underset{(z)}{A}{}_m = g_{mk} \underset{(z)}{A^k} \,. \tag{2.5.11}$$

Note that we have run across these formulae before in Eq. (2.4.8).

> **Exercise 2.5.1:** ***Base vectors for skew coordinates***
>
> Recall the results of Exercise 2.3.1. Use the corresponding coordinate transformation $z^k = z^k(x_i)$ and calculate the base vectors $\boldsymbol{g}^l$. Verify the orthogonality conditions $\boldsymbol{g}^l \cdot \boldsymbol{g}_k = \delta^l_k$ and depict the $\boldsymbol{g}$-vectors in the z-coordinate system.

Thus the vector $\boldsymbol{A}$ can be represented in the Cartesian base $\boldsymbol{e}_j$ (see Eq. 2.5.2) as well as in the skew-curvilinear bases $\boldsymbol{g}_k$ and $\boldsymbol{g}^l$, which are not normalized (Eq. 2.5.7). The same holds for tensorial quantities. As an example we consider the tensor of second order of Eq. (2.4.15), $\boldsymbol{B}$. The following representation is valid in the Cartesian base:

$$\boldsymbol{B} = \underset{(x)}{B_{kl}}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l \tag{2.5.12}$$

and the following ones in the skew-curvilinear case:

$$\begin{aligned} \boldsymbol{B} &= \underset{(z)}{B^{kl}}\, \boldsymbol{g}_k \otimes \boldsymbol{g}_l,\ \boldsymbol{B} = \underset{(z)}{B_{kl}}\, \boldsymbol{g}^k \otimes \boldsymbol{g}^l, \\ \boldsymbol{B} &= \underset{(z)_l}{B^k{}_l}\, \boldsymbol{g}_k \otimes \boldsymbol{g}^l,\ \boldsymbol{B} = \underset{(z)}{B_k{}^l}\, \boldsymbol{g}^k \otimes \boldsymbol{g}_l.. \end{aligned} \tag{2.5.13}$$

We conclude that the number of possible ways of representation in skew-curvilinear bases increases significantly with the order of the tensors. A second order tensor offers already four different possibilities. However, a second order tensor is by no means the highest type encountered in continuum theory. For example, the stiffness tensor $\boldsymbol{C}$ of linear elasticity is of fourth order and allows for sixteen co-/contravariant representations. Indeed, it looks quite innocent in a purely Cartesian frame:

$$\boldsymbol{C} = \underset{(x)}{C}{}_{klmn}\, \boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_m \otimes \boldsymbol{e}_n. \tag{2.5.14}$$

The notion of the *tensor product* or *dyad*, i.e., the symbol "$\otimes$" deserves an explanation. Even though it is as necessary as a Mercedes star it is customarily used in the literature, the main reason being to distinguish a product between vectors (or tensors) in absolute notation from the scalar and the vector product, which are identifiable by the symbols "$\cdot$" and "$\times$". Mathematically speaking, two vectors (first order tensors), $\boldsymbol{A}$ and $\boldsymbol{B}$, are mapped onto a number (zeroth order tensor) by writing $\boldsymbol{A} \cdot \boldsymbol{B}$, onto another (axial) vector by $\boldsymbol{A} \times \boldsymbol{B}$, and onto a second order tensor by $\boldsymbol{A} \otimes \boldsymbol{B}$. Just like the scalar or vector product "$\otimes$" can also be introduced axiomatically by defining a corresponding algebra. However, this will not be detailed any further in this book and the reader is referred to the more mathematically oriented literature cited below.

Exercise 2.5.2: ***The trace of a second order tensor***

The *trace* of a tensor of second order is defined in Cartesian coordinates as follows:

$$\operatorname{tr}\boldsymbol{B} = \underset{(x)}{B_{kk}}\,. \tag{2.5.15}$$

Use Eq. (2.4.17) to show that:

$$\operatorname{tr}\boldsymbol{B} = g^{ij}\underset{(z)}{B_{ij}} = g_{ij}\underset{(z)}{B^{ij}} = \underset{(z)}{B^{i}}{}_{i} = \underset{(z)}{B_{j}^{j}}\,. \tag{2.5.16}$$

Interpret the last two expressions in terms of bound co-/contravariant indices.

2.6 Physical Components of Vectors and Tensors

Exercises 2.4.6 and 2.4.7 have shown that the co-/contravariant components of vectors and tensors do not necessarily all have the same physical dimension, for example that of a stress. We have to concede that in engineering terms co-/contravariant components of a vector or a tensor are, in general, rather unphysical quantities.

However, in the case of orthogonal coordinate transformations whose coordinate lines are perpendicular to each other it becomes possible to recover the physical notion of Cartesian components by introducing so called *physical components*. The key to their definition lies in the fact that the metric tensor of orthogonal coordinate transformations is diagonal:

$$\boldsymbol{g}_m \cdot \boldsymbol{g}_l = \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{bmatrix}, \quad \boldsymbol{g}^m \cdot \boldsymbol{g}^l = \begin{bmatrix} g^{11} & 0 & 0 \\ 0 & g^{22} & 0 \\ 0 & 0 & g^{33} \end{bmatrix}. \tag{2.6.1}$$

Exercise 2.6.1: ***Diagonal property of the metric for orthogonal coordinate transformations***

Prove Eq. (2.6.1) and show that the metric tensor for coordinate transformations with perpendicularly oriented coordinate lines has only components along its diagonal.

To do so use the definition (2.2.8) for the metric tensor and recall that the derivatives $\partial x_i/\partial z_k$ are the components of tangent vectors to the coordinate lines z_k: Eq. (2.3.4).

In that case the length of a vector (cf., Eqs. 2.4.12/2.4.13) can be expressed as a sum of quadratic terms:

$$A = \sqrt{\sum_{i=1}^{3} g_{\underline{ii}} (\underset{(z)}{A^i})^2} = \sqrt{\sum_{i=1}^{3} g^{\underline{ii}} (\underset{(z)}{A_i})^2}. \tag{2.6.2}$$

In order not to break with EINSTEIN's summation convention we had to use an explicit summation sign. Hence the expressions $g_{\underline{ii}}$ and $g^{\underline{ii}}$ do not consist of three terms. In fact they stand for only one diagonal element of the co-/contravariant components of the metric tensor, namely no. "i". We have emphasized the exception to the summation convention by an underscore, and agree that underlined indices appearing twice in a product will not be summed up.We now define so-called *physical vector components*, $A_{\langle i \rangle}$, by:

$$A_{\langle i \rangle} = \sqrt{g_{\underline{ii}}}\, \underset{(z)}{A^i} = \sqrt{g^{\underline{ii}}}\, \underset{(z)}{A_i}\,. \tag{2.6.3}$$

If the square of these components is summed up we immediately obtain the length of the vector $\boldsymbol{A}$:

$$A = \sqrt{A_{\langle i \rangle} A_{\langle i \rangle}}. \tag{2.6.4}$$

Note that physical components can also be defined for tensors of second and higher order:

$$B_{\langle ij \rangle} = \sqrt{g_{\underline{ii}}} \sqrt{g_{\underline{jj}}}\, \underset{(z)}{B^{ij}} = \sqrt{g^{\underline{ii}}} \sqrt{g^{\underline{jj}}}\, \underset{(z)}{B_{ij}}\,. \tag{2.6.5}$$

Exercise 2.6.2: *The length of a vector in physical components*

Combine Eqs. (2.6.1) and (2.6.2) to prove Eq. (2.6.4). Clarify the meaning of underlined indices. Also try to calculate the length through scalar products according to Eqs. (2.6.2) and (2.5.7), which need to be specialized to orthogonal coordinate transformations before:

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{A} &= \left(\boldsymbol{g}_l \underset{(z)}{A^l}\right) \cdot \left(\boldsymbol{g}_k \underset{(z)}{A^k}\right) = \underset{(z)}{A^k}\, \underset{(z)}{A^k}\, g_{\underline{kk}}, \\ \boldsymbol{A} \cdot \boldsymbol{A} &= \left(\boldsymbol{g}^l \underset{(z)}{A_l}\right) \cdot \left(\boldsymbol{g}^k \underset{(z)}{A_k}\right) = \underset{(z)}{A_k}\, \underset{(z)}{A_k}\, g^{\underline{kk}}. \end{aligned} \tag{2.6.6}$$

Exercise 2.6.3: ***Physical components of the stress*** **tensor** ***in cylindrical coordinates***

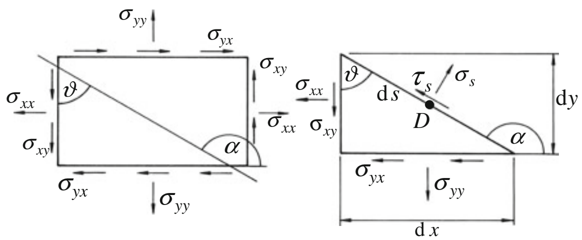
Use the results from Exercise 2.4.6 and derive the following expressions for the physical components of the stress tensor in cylindrical coordinates. In particular, verify that all physical components have the physical dimension of stress:

$$
\begin{aligned}
\sigma_{\langle rr\rangle} &= \cos^2\vartheta\,\sigma_{xx} + 2\sin\vartheta\cos\vartheta\,\sigma_{xy} + \sin^2\vartheta\,\sigma_{yy},\\
\sigma_{\langle r\vartheta\rangle} &= -\sin\vartheta\cos\vartheta\,\sigma_{xx} + \left(\cos^2\vartheta - \sin^2\vartheta\right)\sigma_{xy} + \sin\vartheta\cos\vartheta\,\sigma_{yy},\\
\sigma_{\langle rz\rangle} &= \cos\vartheta\,\sigma_{xz} + \sin\vartheta\,\sigma_{yz}\,,\\
\sigma_{\langle\vartheta\vartheta\rangle} &= \sin^2\vartheta\,\sigma_{xx} - 2\sin\vartheta\cos\vartheta\,\sigma_{xy} + \cos^2\vartheta\,\sigma_{yy},\\
\sigma_{\langle\vartheta z\rangle} &= -\sin\vartheta\,\sigma_{xz} + \cos\vartheta\,\sigma_{yz},\\
\sigma_{\langle zz\rangle} &= \sigma_{zz}\,.
\end{aligned}
\tag{2.6.7}
$$

Recall the problem of equilibrium of forces shown in Fig. 2.9. The analysis will lead to the following equations for MOHR's circle:

$$
\begin{aligned}
\sigma_s &= \tfrac{1}{2}\left(\sigma_{xx} + \sigma_{yy}\right) - \tfrac{1}{2}\left(\sigma_{yy} - \sigma_{xx}\right)\cos(2\vartheta) + \sigma_{xy}\sin(2\vartheta),\\
\tau_s &= \tfrac{1}{2}\left(\sigma_{yy} - \sigma_{xx}\right)\sin(2\vartheta) + \sigma_{xy}\cos(2\vartheta).
\end{aligned}
\tag{2.6.8}
$$

Fig. 2.9 A reminder of MOHR's circle in 2D



How is this result related to Eq. (2.6.7)? Use the following trigonometric theorems to answer this question:

$$
\begin{aligned}
&\sin^2\vartheta = \tfrac{1}{2}[1 - \cos(2\vartheta)], \quad \cos^2\vartheta = \tfrac{1}{2}[1 + \cos(2\vartheta)],\\
&\sin(2\vartheta) = 2\sin\vartheta\,\cos\vartheta.
\end{aligned}
\tag{2.6.9}
$$

Exercise 2.6.4: ***Physical components of the metric tensors***

Show that for orthogonal coordinates systems the physical components of the metric tensor read:

$$g_{\langle ij \rangle} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6.10}$$

Exercise 2.6.5: ***The* von *Mises flow rule***

In Cartesian coordinates the *stress deviator* $\underset{(x)}{\Sigma}{}_{ij}$ is defined as follows:

$$\underset{(x)}{\Sigma}{}_{ij} = \underset{(x)}{\sigma}{}_{ij} - \frac{1}{3} \underset{(x)}{\sigma}{}_{kk}\, \delta_{ij}. \tag{2.6.11}$$

Show that this quantity is trace-free:

$$\underset{(x)}{\Sigma}{}_{ll} = 0. \tag{2.6.12}$$

Recall the postulate by von Mises according to which a metal starts flowing if the following threshold value, σ_{y}, is reached:

$$\sigma_{\mathrm{y}}^2 = \frac{3}{2} \underset{(x)}{\Sigma}{}_{ij} \underset{(x)}{\Sigma}{}_{ij}\,. \tag{2.6.13}$$

σ_{y} is a material specific quantity and known as the *yield stress*. Explain why it makes sense to remove the trace of state of stress from a strength criterion, at least for polycrystalline metals. Show that in the case of plane stress it is possible to write:

$$\sigma_{\mathrm{y}}^2 = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\sigma_{xy}^2. \tag{2.6.14}$$

Moreover, specialize the von Mises criterion to a 1D tensile bar with a tensile stress σ as well as to a block subjected to a shear load τ and show that:

$$\tau = \frac{1}{\sqrt{3}}\sigma. \tag{2.6.15}$$

Transform from the Cartesian system x to an arbitrary system z to show that:

$$\underset{(x)}{\sigma}{}_{kk} = g_{ij} \underset{(z)}{\sigma}{}^{ji} = g^{ji} \underset{(z)}{\sigma}{}_{ij} = \underset{(z)}{\sigma}{}_{i}{}^{i} \tag{2.6.16}$$

and, in a similar manner, that we may also write:

$$\underset{(z)}{\Sigma}_{ij} = \underset{(z)}{\sigma}_{ij} - \frac{1}{3}\underset{(z)}{\sigma}{}^{k}_{k}\, g_{ij}, \quad \underset{(z)}{\Sigma}{}^{ij} = \underset{(z)}{\sigma}{}^{ij} - \frac{1}{3}\underset{(z)}{\sigma}{}^{k}_{k}\, g^{ij} \tag{2.6.17}$$

and:

$$\sigma_{\mathrm{y}}^2 = \frac{3}{2}\underset{(z)}{\Sigma}_{ij}\underset{(z)}{\Sigma}{}^{ji} = \frac{3}{2} g_{ir} g_{js}\underset{(z)}{\Sigma}{}^{rs}\underset{(z)}{\Sigma}{}^{ji} = \frac{3}{2} g^{ir} g^{js}\underset{(z)}{\Sigma}_{ij}\underset{(z)}{\Sigma}_{rs}\,. \tag{2.6.18}$$

Prove the following relations by assuming orthogonal coordinates:

$$\underset{(x)}{\sigma}_{kk} = \underset{(z)}{\sigma}_{\langle kk\rangle}, \; \underset{(z)}{\Sigma}_{\langle ij\rangle} = \underset{(z)}{\sigma}_{\langle ij\rangle} - \frac{1}{3}\underset{(z)}{\sigma}_{\langle kk\rangle}\,\delta_{\langle ij\rangle} \tag{2.6.19}$$

and:

$$\sigma_{\mathrm{y}}^2 = \frac{3}{2}\underset{(z)}{\Sigma}_{\langle ij\rangle}\underset{(z)}{\Sigma}_{\langle ij\rangle}\,. \tag{2.6.20}$$

Finally specialize to plane polar coordinates and show that:

$$\sigma_{\mathrm{y}}^2 = \sigma_{\langle rr\rangle}^2 + \sigma_{\langle\vartheta\vartheta\rangle}^2 - \sigma_{\langle rr\rangle}\sigma_{\langle\vartheta\vartheta\rangle} + 3\sigma_{\langle r\vartheta\rangle}^2. \tag{2.6.21}$$

In context with physical coordinates a few remarks or rules for computing combinations between scalar and tensor products are in order. For simplicity we assume that the following vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, and the (second order) tensors, $\boldsymbol{C}$ and $\boldsymbol{D}$, are expressed in an orthogonal coordinate system with unit vectors $\boldsymbol{e}_i = \sqrt{g^{\underline{ii}}}\boldsymbol{g}_i$ so that physical coordinates can be used. Then by definition of the scalar product we may write:

$$\begin{aligned}
&\boldsymbol{e}_i \cdot \left(\boldsymbol{e}_j \otimes \boldsymbol{e}_k\right) = \delta_{ij}\boldsymbol{e}_k, \quad \left(\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right) \cdot \boldsymbol{e}_k = \boldsymbol{e}_i\delta_{jk}, \\
&\boldsymbol{e}_i \cdot \left(\boldsymbol{e}_j \otimes \boldsymbol{e}_k\right) \cdot \boldsymbol{e}_l = \delta_{ij}\delta_{kl}, \quad \left(\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right) : \left(\boldsymbol{e}_k \otimes \boldsymbol{e}_l\right) = \delta_{il}\delta_{jk}.
\end{aligned} \tag{2.6.22}$$

Richard Edler von Mises was born on April 19, 1883 in Lemberg (now Ukraine) and died on July 14, 1953 in Boston. From 1909 to 1918 he was a professor of applied mathematics in Straßburg (now France) where he investigated problems of solid and fluid mechanics, aerodynamics, statistics, and probability theory. During WW I he was on the Austrian side where he built and flew his own battle plane. After the war he went to Berlin until the Nazis forced him into exile in 1933, first to Istanbul and then on to Harvard.

It should be noted that some authors use the following relation for the double scalar product instead:

$$\left(\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right)..(\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = \delta_{ik}\delta_{jl}. \tag{2.6.23}$$

Also note that all of these complications can be avoided if the index notation is used exclusively from the very start.

Exercise 2.6.6: ***Scalar and double scalar products in vector notation***

Use Eq. (2.6.22) to show that:

$$\begin{aligned} &\boldsymbol{A} \cdot \boldsymbol{C} = A_{\langle i\rangle} C_{\langle ik\rangle} \boldsymbol{e}_k, \ \boldsymbol{C} \cdot \boldsymbol{B} = C_{\langle ij\rangle} B_{\langle j\rangle} \boldsymbol{e}_i, \\ &\boldsymbol{A} \cdot \boldsymbol{C} \cdot \boldsymbol{B} = A_{\langle i\rangle} C_{\langle ij\rangle} B_{\langle j\rangle}, \ \boldsymbol{C} : \boldsymbol{D} = C_{\langle ij\rangle} D_{\langle ji\rangle}. \end{aligned} \tag{2.6.24}$$

Moreover, show that for four vectors $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{E}$, and $\boldsymbol{F}$ we may write:

$$(\boldsymbol{A} \otimes \boldsymbol{B}) : (\boldsymbol{E} \otimes \boldsymbol{F}) = (\boldsymbol{A} \cdot \boldsymbol{F})\,(\boldsymbol{B} \cdot \boldsymbol{E}). \tag{2.6.25}$$

2.7 A Touch of Differential Geometry

The Einstein convention requires summation from 1 to 2 for plane problems or from 1 to 3 for spatial ones. So far corresponding tensor equations could be applied to both cases. However, in what follows we will deliberately concentrate on two-dimensional situations. Our intention is to capture the *geometry of curved surfaces* mathematically. Such a tool will later be of great importance when these surfaces have a real meaning in terms of continuum physics. For example, they can represent the boundary between a liquid (or gaseous) and a solid region, such as the wall of pressure vessel or the skin of a soap bubble. In general such surfaces are curved and by no means planar. In order to become familiar with the mathematics pertinent to their characterization (i.e., differential geometry) some training is required. However, as we shall see, it is just a straightforward continuation of our remarks on metric tensors and tangent vectors.

In order to quantify the amount of curvature of a surface in three-dimensional space two curvilinear surface coordinates are used, called z^α, $\alpha = 1{,}2$. They behave analogously to the curvilinear coordinates z^i from Eq. (2.2.1) the only difference being that there are only two of them. We emphasize this by small Greek indices, for example, α. A position vector $\boldsymbol{x}$ on the surface is described by its three Cartesian coordinates x_i which, in turn, are functions of the surface coordinates:

$$x_i = x_i\left(z^1, z^2\right) \equiv x_i(z^\alpha), \quad i = 1,\ 2,\ 3. \tag{2.7.1}$$

Two tangent vectors, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$, can now be defined analogously to Eq. (2.3.4)$_2$:

$$\boldsymbol{\tau}_\alpha = \frac{\partial x_i(z^\gamma)}{\partial z^\alpha}\boldsymbol{e}_i, \quad \alpha = 1,\ 2. \tag{2.7.2}$$

Obviously their components in (three-dimensional) Cartesian coordinates are then given by:

$$\tau^i_\alpha = \frac{\partial x_i(z^\gamma)}{\partial z^\alpha}. \tag{2.7.3}$$

In this case it does not matter if we write the index i at the top or at the bottom of the symbol τ because it refers to a Cartesian representation. However, the positioning of the index α does matter: Just as in the case of index k from Eq. (2.3.4)$_2$ it signals covariance.

Note that just like the case of $\boldsymbol{g}_k$ the tangent vectors $\boldsymbol{\tau}_\alpha$ are no unit vectors. They still need to be normalized in order to define a unit vector $\boldsymbol{n}$ normal to the surface:

$$\boldsymbol{n} = \frac{\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2}{|\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2|}. \tag{2.7.4}$$

Moreover, just like in Eq. (2.5.4), it is possible to define dual tangent vectors $\boldsymbol{\tau}^\beta$ by:

$$\boldsymbol{\tau}^\beta = \frac{\partial z^\beta}{\partial x_j}\boldsymbol{e}_j. \tag{2.7.5}$$

Because of the chain rule the following orthogonality conditions hold (cf., (2.5.8)):

$$\boldsymbol{\tau}^\beta \cdot \boldsymbol{\tau}_\alpha = \delta^\beta_\alpha. \tag{2.7.6}$$

We are now in a position to define co- and contravariant surface metrics in complete analogy to the Eqs. (2.2.10), (2.3.7), (2.4.7), and (2.5.8/2.5.9):

$$g_{\alpha\beta} = \frac{\partial x_i}{\partial z^\alpha}\frac{\partial x_i}{\partial z^\beta} \equiv \boldsymbol{\tau}_\alpha \cdot \boldsymbol{\tau}_\beta, \quad g^{\alpha\beta} = \frac{\partial z^\alpha}{\partial x_j}\frac{\partial z^\beta}{\partial x_j} \equiv \boldsymbol{\tau}^\alpha \cdot \boldsymbol{\tau}^\beta. \tag{2.7.7}$$

In order to obtain a measure for the local curvature of the surface we first define the so-called *curvature tensor* (with the Cartesian components n_i of the normal $\boldsymbol{n}$ from Eq. (2.7.4)):

$$b_{\alpha\beta} = \frac{\partial^2 x_i}{\partial z^\alpha \partial z^\beta} n_i. \tag{2.7.8}$$

This definition deserves a comment: Calculus teaches us that the extreme values of a curve $y = y(x)$ within the plane are governed by the second derivative $\mathrm{d}^2 y / \mathrm{d}^2 x$, i.e., the local curvature in a point x. This explains the second derivatives in Eq. (2.7.8). Moreover, a plane surface is, of course, not curved. Therefore we expect that the curvature tensor vanishes. And, indeed, the scalar product (note the summation implied by the index i) in Eq. (2.7.8) or, in other words the projection of the curvature onto the normal vector, becomes zero in this case. Moreover, note that by virtue of Eq. (2.7.3) we may write for the curvature in Eq. (2.7.8) as well:

$$\frac{\partial^2 x_i}{\partial z^\alpha \partial z^\beta} = \frac{\partial \tau^i_\alpha}{\partial z^\beta} \Rightarrow \frac{\partial^2 x_i}{\partial z^\alpha \partial z^\beta} \boldsymbol{e}_i = \frac{\partial \tau^i_\alpha}{\partial z^\beta} \boldsymbol{e}_i \equiv \frac{\partial \tau^i_\alpha \boldsymbol{e}_i}{\partial z^\beta} = \frac{\partial \boldsymbol{\tau}_\alpha}{\partial z^\beta}. \tag{2.7.9}$$

Note that because of their constancy the Cartesian unit vectors $\boldsymbol{e}_i$ are not affected by partial differentiation. The curvature may also be interpreted as the change of tangent vectors with the lines of coordinates, which is less intuitive. In vector notation the covariant components of the curvature tensor can therefore be written as:

$$b_{\alpha\beta} = \frac{\partial \boldsymbol{\tau}_\alpha}{\partial z^\beta} \cdot \boldsymbol{n}. \tag{2.7.10}$$

And, if it pleases, the curvature tensor can also be written in complete invariant form:

$$\boldsymbol{b} = b_{\alpha\beta} \boldsymbol{\tau}^\alpha \otimes \boldsymbol{\tau}^\beta = \frac{\partial \boldsymbol{\tau}_\alpha}{\partial z^\beta} \cdot \boldsymbol{n}\, \boldsymbol{\tau}^\alpha \otimes \boldsymbol{\tau}^\beta. \tag{2.7.11}$$

It is customary to define the *mean curvature* through the mean trace of the curvature tensors (cf., Exercise 2.5.2):

$$K_\mathrm{m} = \frac{1}{2} \operatorname{tr} \boldsymbol{b} \Rightarrow K_\mathrm{m} = \frac{1}{2} b^\alpha_\alpha = \frac{1}{2} g^{\alpha\beta} b_{\alpha\beta}. \tag{2.7.12}$$

Obviously this is an invariant, a scalar, since we have seen in Exercise 2.5.2 that, independently of the coordinate system, a trace will always yield the same value.

Exercise 2.7.1: ***Differential geometry of a spherical surface***

Investigate the surface of a sphere of radius R. To this end identify as surface coordinates $z^1 = \vartheta$, $z^2 = \varphi$ and recall the transformation rules for spherical coordinates from Exercise 2.2.2. Use the definition (2.7.3) and calculate the components of the tangent vectors, τ^i_ϑ and τ^i_φ, w.r.t. a Cartesian base in the center of the sphere:

$$\begin{aligned} &\tau^1_\vartheta = R\cos\vartheta\ \cos\varphi\ , \quad \tau^2_\vartheta = R\cos\vartheta\ \sin\varphi\ , \quad \tau^3_\vartheta = -R\sin\vartheta, \\ &\tau^1_\varphi = -R\sin\vartheta\ \sin\varphi\ , \quad \tau^2_\varphi = R\sin\vartheta\ \cos\varphi\ , \quad \tau^3_\varphi = 0. \end{aligned} \tag{2.7.13}$$

Show that both vectors are orthogonal to each other. Are they related to the base vectors $\boldsymbol{g}_1, \boldsymbol{g}_2, \boldsymbol{g}_3$ of Exercise 2.3.1? Depict them on the surface of a sphere together with surface coordinate lines. Are the tangent vectors normalized? Show by using Eq. (2.7.7) that the surface metric $g_{\alpha\beta}$ is given by:

$$g_{\alpha\beta} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \vartheta \end{pmatrix}. \tag{2.7.14}$$

Calculate its inverse $g^{\alpha\beta}$. Show by using Eq. (2.7.4) that the Cartesian components of the unit normal n_i to the sphere are given by:

$$n_1 = \sin\vartheta \ \cos\varphi \ , \quad n_2 = \sin\vartheta \ \sin\varphi \ , \quad n_3 = \cos\vartheta. \tag{2.7.15}$$

Use them to calculate the curvature tensor based on the definition shown in Eq. (2.7.8). Finally show that the mean curvature is given by $K_{\mathrm{m}} = -1/R$. Try to interpret the minus sign by using terms like "convex" or "concave."

Exercise 2.7.2: ***Differential geometry of a cylindrical surface***

Investigate now the mantle surface of a circular cylinder of radius R. For this purpose choose $z^1 = \vartheta$, $z^2 = z$ as surface coordinates. Recall the transformations for cylindrical coordinates from Sect. 2.2 and follow the procedures of Exercise 2.7.1. Show first that the tangent vectors are given by:

$$\begin{aligned} &\tau_\vartheta^1 = -R\sin\vartheta, \quad \tau_\vartheta^2 = R\cos\vartheta, \quad \tau_\vartheta^3 = 0, \\ &\tau_z^1 = 0, \quad \tau_z^2 = 0, \quad \tau_z^3 = 1. \end{aligned} \tag{2.7.16}$$

Use them and calculate the surface metric:

$$g_{\alpha\beta} = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.7.17}$$

Show that the unit normal in Cartesian coordinates is given by:

$$n_1 = \cos\vartheta \ , \quad n_2 = \sin\vartheta \ , \quad n_3 = 0. \tag{2.7.18}$$

Use the results and prove that the mean curvature is given by $K_{\mathrm{m}} = -1/(2R)$. Interpret the factor $\frac{1}{2}$ and compare it to the result for a spherical surface.

2.8 Would You Like to Know More?

The book by Schade and Neemann [1] is a real treasure chest of mathematical formulae (which makes it easier to read since it is written in German) for true disciples of the index calculus. In particular one should look at Sects. 4.2.2 and

4.2.4 for the concepts of "metric," and "co-/contravariance." The books by Itskov [2] and Bertram [3] insist on a mathematically more stringent approach and emphasize the absolute tensor calculus. Particularly worth reading in context with the present section are Chap. 1 and Sect. 3.2 (for differential geometry) in the first and Sects. 1.1 and 1.2 in the second book. Tensor algebra and tensor analysis are also treated in concise form in Irgens [4], Chap. 12, in index as well as in absolute notation, and also in Liu [5], Appendix A.1.

In general the tensor concepts presented so far have been known for a long time. Consequently it is also worth while to study the "classics." In this context the article by Ericksen [6] in the Encyclopedia of Physics, Sects. I, II, and (in parts) III should be mentioned first. Moreover, the book by Green and Zerna [7] is to be recommended, in particular Sects. 1.1 to 1.10. Finally Chaps. 1, 12 and, notably, Sect. 13 (with many exotic coordinate transformations) in the book by Flügge [8] should be pointed out.

Several notions, such as Mohr's circle or yield stress, were used in this section without further explanations. In this context it may be useful to study textbooks on strength of materials, e.g., Hibbeler [9], Sects. 10.3 and 10.7, or Gross et al. [10], Sects. 2.2.3 and 3.3.

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