

Chapter 2

The Science of Machines

The great events that Carnot touched and shaped never caused him to lose sight of the adaptation of the science of mechanics to the science of machines that he thought to initiate when still an obscure young engineer. Fortunately for the historian, the documentation is adequate to permit following his conceptions from their genesis in the entry he prepared for the prize contest set by the Academy of sciences in April 1777 right through their development during his lifetime into the subject matter of a new branch of science in the 1820s. Since his own ideas were expressed in their most individual and unadorned form in his first publication, the *Essai sur les machines en général* of 1783, it will be best to present them through the medium of the detailed analysis of that work that occupies the current chapter. Thereupon it will be informative to look first back and then forward. The stages through which Carnot formulated his approach may be observed in the successive memoirs he submitted to the Academy in 1778 and 1780. Twenty years later, an ostensibly retired statesman, he extended and developed the subject in *Principes fondamentaux de l'équilibre et du mouvement*. Beyond this, it is one of the purposes of the present monograph to exhibit that Sadi Carnot's *Réflexions sur la puissance motrice du feu*, published in 1824, the year after his father's death, may properly be read not only as the foundation of thermodynamics, but also as the culmination of a methodologically and conceptually coherent series of Carnot essays on the science of machines. After surveying the literature of that subject a little more at large, it will be appropriate finally to consider the relevance of Carnot's mathematical writings to his mechanics and his work in science.

2.1 Summary of *Essai sur les machines en général*

There is no difficulty in understanding why the scientific community should have ignored Carnot's *Essai sur les machines en général* in the 1780s. His book does not read like the rational mechanics of the eighteenth century. It had long since

become normal to compose treatises of mechanics addressed to a professional public in the language of mathematical analysis; though Carnot reasoned no less rigorously than did contemporary mathematical argument, he conducted the discussion verbally, conceived the mathematical expressions he did employ in a geometric or trigonometric rather than algebraic spirit, and usually went on to explain in words what the formulas contained. The genre was apparently of an altogether lower order than that of d'Alembert and Lagrange or Euler and the Bernoulli family. Judging by the style alone, prolix and naive, a contemporary reader might easily have supposed the book to be among the many negligible writings that retailed merely elementary mechanics under one pretext or another.

Yet, the essay, despite its title, could never have served the purpose of a practical manual for designing or employing actual machinery. Carnot proceeded on the basis of a highly abstract definition of a machine: it was an intermediary body serving to transmit motion between two or more primary bodies that do not act directly one on another. Carnot lodged his complaint against the existing mode in mechanics right at the outset. Since its problems were normally limited to analyzing the interactions among primary bodies, the practice had tacitly arisen of abstracting from the mass of the machine itself as if it were inertia free. While simplifying problems in mechanics, however, that method of treatment, carried over from the geometrization of simple machines in statics, complicated a dynamical study of machines. It was standard procedure to deduce from the laws of mechanics the particular rules of equilibrium and motion in each class of simple machines, i.e., cords, the lever, the crank, the pulley, the wedge, the screw, and the inclined plane. No general principles existed, however, that were capable of containing the conditions of motion and equilibrium in all machines. In order to establish such principles, the mechanist needed to treat machines like other bodies and account for their mass in his analysis. Only then would the science of machines in general be feasible, and then it would come down to resolution of the following problem:

Given the virtual motion of any system of bodies (i.e., that which each of the bodies would describe if it were free), find the real motion it will assume in the next instant in consequence of the mutual interaction of the bodies considered as they exist in nature, i.e., endowed with the inertia that is common to all the parts of matter.¹

Already it will appear that, verbally expressed—though the argument was, the *Essai sur les machines en général* was beyond the comprehension of readers without formal education. It presupposed the competence of scientifically literate persons, but was not couched in language that would attract their interest. The essay would have been accessible to trained engineers, in other words, and since the author himself was of a first generation of them, the impediment was that it could have been appreciated only in a profession that barely existed.

A difficulty about the content complements that about the form. The *Essai sur les machines en général* mingled novelty of approach with the most elementary aspects

¹*Essai sur les machines en général* (Carnot 1786, § X, p 21).

of mechanics in such an unassuming fashion that it is not easy to distinguish in retrospect what was new from what was obvious. Indeed, to treat Carnot's scientific career in point of discovery would be relatively unprofitable. Innovations there were, to be sure, and far from negligible. But they did not occur on the frontiers of unsolved problems in the science of mechanics. They occurred behind the front lines, so to say, and rather than consider Carnot to have been a seeker and finder in the conventional scientific sense, it will be better to approach his work as the critic might that of a writer who should introduce a new idiom into a literature in order to bring existing, largely unnoticed resources to bear on different purposes. What Carnot added to mechanics accrued largely in virtue of what he thought to do with it.

Before entering into his reasoning, a reader needs to be reminded of certain contemporaneous physical conventions about the structure of matter. He will not get beyond the first few pages of the *Essai sur les machines en général* without encountering the distinction between hard bodies and elastic bodies that entered mechanics in the early stages of collision theory in the seventeenth century and disappeared into the theory of elasticity in the nineteenth. Carnot's work came into that development somewhat past its midpoint and inherited as assumptions the positions adopted by Maupertuis.² Perfectly hard bodies were held to be indeformable and perfectly elastic bodies to contain forces capable of restoring their initial shape and volume after compression or shock of impact. For completeness there should have been the third category of soft bodies, deformable and incapable of self-restitution. In the seventeenth century, John Wallis had addressed one of the classic papers inaugurating these studies to the problem of the behavior of inelastic bodies in collision.³ He wrote of hard bodies but, paradoxically enough, soft bodies would behave in the same way in collision since they would not rebound. Resilience or its absence being the significant alternatives, elasticity and hardness sufficed for the idealized conditions. The polarity was unsymmetrical, however, because hardness did not admit of degree whereas the degrees of elasticity could be attributed to a complement of softness.

In modeling nature itself, hard bodies were taken to be fundamental, their properties a manifestation of the impenetrability of matter inhering in the ultimate corpuscles. In actual bodies, solid matter was imagined to consist in such corpuscles connected to each other by rods or shafts—rigid, inextensible, and incompressible rods in hard bodies, springs in elastic bodies. In order to follow Carnot's thought, it will further be important to appreciate the way in which other physical states related to the model, for that was not in the usual notion of progression from solid through liquid to gas. Instead, liquids were fluids congruent with hardness in mechanical

²For a helpful account, see *The significance of "hard bodies" in the history of scientific thought* (Scott 1959, pp 199–210).

³On that one can see *A Summary Account of the General Laws of Motion*, 26 November 1668 (Wallis).

properties in that they were incompressible though deformable, and gases were fluids mechanically congruent with elasticity in deformability and resilience.

These distinctions created dynamical difficulties for Carnot that might arouse impatience in his modern reader. Historically, however, it would be misleading to consider them false problems, for working through them was precisely what led Carnot in the direction of physics of work, power, and ultimately energy, wherein they did indeed become obsolete. The embarrassment grew out of the famous issue in eighteenth-century dynamics about whether momentum (MV) or “live force” (MV^2) should be admitted to be the fundamental quantity conserved in interchanges of motion. Ignoring (as is legitimate for present purposes) metaphysical aspects, it was a limitation upon the employment of the principle of live force that although seventeenth-century collision theory conserved its quantity in the interaction of perfectly elastic bodies, in the supposedly more fundamental case of hard bodies, MV^2 was conserved only when motion was communicated smoothly (“by insensible degrees” is the usual phrase), and not in impact or collision.

Looking back at Carnot’s writing, one can see that a treatise in which the quantity now called work (force · distance) was to be designated as the measure of what machines accomplish was in some sense bound to presuppose its equivalence to kinetic energy ($\frac{1}{2}MV^2$). How Carnot saw in advance that he must move towards such a convertibility can only be conjectured. Dimensionally it derived from the equivalence in the law of fall of “ mgh ” to “ $\frac{1}{2}mv^2$ ”, known since Galileo as the means of equating the velocity that a body would generate in falling a certain distance with the force required to carry it back. Beyond that, the relation had to do with his object in thinking of machines as massive bodies in action and not mere extensions of rigidity serving the geometrical transfer of inertial motion, or in simpler words with his having been an engineer. Certainly, too, his ideas carried over the role of live-force conservation from hydrodynamics into what he would have regarded as the deeper problem of hard-body interactions.

Following the publication of Daniel Bernoulli’s famous treatise in 1738,⁴ hydrodynamics was the subject in which the principle of live force was basic and indispensable in the solution of engineering problems. There is thematic evidence running through all Carnot’s writings on mechanics that the hydraulic application of principles and findings, although ostensibly a special case, actually held a major if not a primary place in his thinking.⁵ It would have been likely enough at the outset, though not strictly correct, to think of motion being communicated by liquids in accordance with the principle of continuity and to suppose from the common incompressibility of liquids and hard bodies that the conservation of live force might profitably be taken to be the principle governing the latter types of interaction, in nature the most fundamental.

⁴*Hydrodynamica sive de viribus et motibus fluidorum commentarii* (Bernoulli D).

⁵Scholars have often cited the analogy between fluid flow and heat flow in discussing the influence of Lazare’s work upon Sadi’s, though, as will appear, the present monograph makes a much wider claim for the continuity between the work of father and son. See below Chapter 2, pp 44–45, Chapter 3, pp 77–78, Chapter 5, pp 130–131.

At any rate, whether for those reasons or others, the strategy of Carnot's *Essai sur les machines en général* was decided by the necessity to outflank the discrepancy between continuous and discontinuous change of motion in hard-body interaction and to define the sense in which the principle of live force might equally well be employed in continuous or discontinuous interchange of motion between elastic or inelastic bodies. To that end Carnot tacitly made use of distinctions that, although he never saw so far, later developed into those between potential and kinetic energy, work and power, input and output, scalar and vector quantity, reversible and irreversible process.

In his avowed point of view Carnot robustly admitted to the engineer's inaptitude for metaphysics. There had always been, he acknowledged, two distinct approaches to the science of mechanics, the experiential and the rational. The former, to which he adhered, took its departure from those basic notions that we draw from our gross experience of nature and to which we give the names body, power, equilibrium, and motion. These ideas neither could nor needed to be defined. They were primary, and other conceptions were derivative, such as velocity and the various types of force in terms of which laws of motion were framed. The second approach was one that would make mechanics a purely rational science. Its adherents began with hypotheses, deduced the laws that bodies in motion would exhibit if the hypotheses were correct, and then compared their conclusions with phenomena. What they thus gained in elegance was at the cost of incomparably greater difficulty, for nothing was more embarrassing in exact science, and especially in mechanics, than an effort to formulate definitions entirely free of ambiguity. That was something he would not and need not attempt⁶ in his *Essai sur les machines en général* (Carnot 1786, pp 105–106).

Carnot's reluctance with respect to definitions makes the *Essai sur les machines en général* hard to follow for persons schooled in nineteenth- and twentieth-century terminology. Only by the use he made of terms is it borne in on the modern reader what he meant by them. Perhaps Carnot's ambiguity may have been felt at the time, for in reworking the material 20 years later for the *Principes de l'équilibre et du mouvement*,⁷ he gave a glossary. Even those terms that are dimensionally

⁶Carnot did not pose the choice between with the two schools of mechanics as explicitly Jouguet implies—i.e., that what distinguishes them is that one takes mass to be fundamental and force for a derivative notion, and the other takes force to be fundamental and the laws of motion and impact for derivative. Carnot's one distinction may indeed come down to that, but he did not say so, and Jouguet like other commentators on his mechanics confuses his *Essai sur les machines en général* of 1783 with the *Principes fondamentaux de l'équilibre et du mouvement* of 1803 (Carnot 1803a). On Jouguet see his *Lectures de mécanique. La mécanique enseignée par les auteurs originaux* (Jouguet II, p 72).

⁷Perhaps it will be useful to reproduce the dimensional glossary from Carnot's *Principes fondamentaux de l'équilibre et du mouvement* (Carnot 1803a, § 18, p 13). Denoting mass by m , linear distance by e (espace), and time by t , then

“Any quantity of the form or reducible to the form $\frac{e}{t}$ is called velocity”.

“Any quantity of this form, $m \frac{e}{t}$ is called quantity of motion”.

what might be expected did not carry the full meaning of later usage. For example, the simplest, “vitesse”, might better be rendered speed than velocity, for Carnot disposed of no convention for combining intensity and direction of quantity in a single expression, but always specified the sense in which a velocity or force was to be reckoned. The present discussion will follow Carnot’s own usage, translating “vitesse” usually as velocity in order to avoid awkwardness, and understanding that by “force” he usually meant quantity-of-motion (momentum) in moving bodies. Given his point of view, only secondarily did he need to allude to Newtonian force, the product of mass times acceleration. When he did, he normally employed the phrase widely used in the eighteenth century, “force-motrice”, or “motive-force”, dimensionally identical with “dead force” or statical pressure.

From among the basic principles of mechanics proper, Carnot chose two for an introductory criticism that also established his own standpoint. The first stated for the general law of equilibrium in weight-driven machines the condition that the center of gravity of the system be at the lowest point possible. The second, which Carnot called “Descartes’ famous law of equilibrium”; will be unfamiliar under that designation both to the student of modern mechanics and the student of Descartes, who in fact made no such statement. The principle was that two forces in equilibrium were in inverse ratio to their “vitesses” (by which Carnot meant the motions they produce) at the instant when one prevailed infinitesimally over the other, thus initiating a “small” motion.

Carnot preferred to rely on the center of gravity principle. True, it was open to certain objections. For one, its applicability appeared to be limited to machines driven by weight. That defect, however, Carnot dismissed as merely apparent. It was always possible to reduce the operation of other forces to that of gravity by replacing their agency in principle with that of a weight acting over a pulley. This imaginary transformation of the system may at first annoy the modern reader as a somewhat sophomoric evasion of the difficulty, but it would be better to see in it the engineer’s way of reducing an abstract problem to his own terms. Anyone who has been to engineering school will have resorted to similar devices, although it would be more germane to recall for a moment the analytic role of the experience and manipulation of weight in the mechanics of an Archimedes, a Stevin, or a Galileo. Even Lagrange was not above it.

“Any quantity of this form, $\frac{e}{t^2}$ is called accelerating or retarding force”.

“Any quantity of this form, $m \frac{e}{t^2}$ is called motive force”.

“Any quantity of this form, $m \frac{e}{t}$ or of this, $m \frac{e}{t^2}$ is called simply force or power.” [The ambiguity here reflects the historical situation. Although the concept of power (puissance) as the rate work or energy changes with respect to time was implicit in Carnot’s analysis, he could also use the word as a synonym for force, often with the implication of latent force as the ability to exert effort].

“Any quantity of this form, $m \frac{e^2}{t^2}$ is named motive live force, moment of motive force or moment-of-activity”.

“Any quantity of this form, $m \frac{e^2}{t}$ is named moment of the quantity of motion, or quantity of action”.

The significance for Carnot of the last two quantities will appear in what follows.

A further objection was more serious. There were exceptions to the statement that equilibrium required that the center of gravity be at the lowest possible point, as indeed there were to all assertions involving maxima and minima. There might be provisional or temporary equilibrium in other configurations of the system. In order to obviate that difficulty, Carnot employed a type of argument that his reader soon comes to recognize for characteristic. He restated the principle by appealing to the operation of an ideal machine. The form is arbitrary. No forces are applied other than weights. No motion has been imparted. In the quiescent state, the sum of the resistances of the fixed supports estimated vertically must equal the weight of the system. Suppose now that a “small” motion begins. A portion of the weight must have gone into producing movement, and only with the remainder are the fixtures loaded. The difference between weight of the system and load on the fixed supports will be the force depressing the center of gravity at a rate (here Carnot used the word “vitesse” where we should say acceleration, a term he rarely employed) equal to that difference divided by the mass of the system. It follows that if the center of gravity did not descend, there would be equilibrium:

To demonstrate that several weights applied to any machine whatever are in equilibrium, it suffices to prove that if the machine be left to itself, the center of gravity of the system will not descend.⁸

What is noteworthy is the mode of reasoning, which combined the ideal with the operational and did so in a negative and restrictive way. In a treatise purporting to rest upon experience, a reader might expect that a demonstration would consist in a generalization of experiments. Not at all—it was based upon the exclusion in an idealized, unrealizable thought experiment of what we recognize from our general experience would not occur in the behavior of real objects. Nothing was new about the logic: Stevin, in obtaining the law of the inclined plane from the exclusion of perpetual motion, had adapted the geometric proof from the absurd to physics. The assertion was that a proposition was correct if its contrary entailed consequences that were physically unthinkable. Nothing was new about the model: the motions of systems of mass–points had long been the object of an analysis that had idealized bodies and turned rational mechanics into a form of mathematics. But, as in other aspects of Carnot’s work, what made the difference was the use he repeatedly drew out of such arguments until they became the distinctive idiom of the science of machine motion.

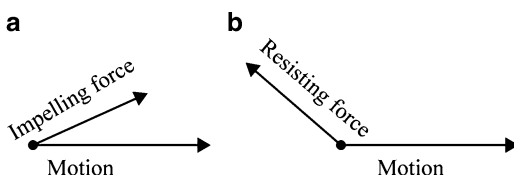
As for the so-called principle of Descartes, Carnot found in it disqualifying flaws. It was less general than the center-of-gravity principle thus transformed, from which it could be deduced by conversion of its forces into weights acting over pulleys. It applied only to systems in which no more than two forces were at work. More seriously, it envisaged only the relative amounts of the forces in equilibrium, whereas in requiring their vertical projections the center-of-gravity principle specified also the direction of those forces. (It is just in these passages that one may begin to appreciate how Carnot’s attempts to analyze the manner in

⁸Carnot 1786, § II, p 14.

which forces transmitted by shafts, cords, and pulleys would constrain and move points within systems composed of rigid members, clumsy though these constructs seem, they nevertheless belong to the pre-history of vector analysis in exhibiting awareness that the quantity of a force comprises direction as well as intensity).

Nothing in the principle of Descartes even required that two forces in order to be in equilibrium must act in opposite senses, and it could not, therefore, specify in what their opposition consisted. In raising that problem, Carnot anticipated one of the clarifications of which he was always proudest, the distinction between what he called impelling forces (“forces-sollicitantes”) and resisting forces (“forces-resistantes”). Not by a metaphysical differentiation between cause and effect would we tell which is which, but merely by the geometry of the system. As shown in the sketch, a force that made an acute angle with the direction wherein the motion occurred would be called impelling; a force that made an obtuse angle would be called resisting (Fig. 2.1).

Fig. 2.1



Finally, Carnot lodged another criticism against the principle of Descartes together with all those that invoked a “small”—i.e., infinitesimal—motion arising in the system. He had in view here the principle of virtual velocities, of course, although he did not identify it by name. This approach never specified what determined that infinitesimal motion. If it was necessary to invoke a new mechanical principle for that, then the original one was inadequate. If, on the other hand, the geometry of the system sufficed to determine the nascent motion, how did it do so? The identical objection lodged against analyses that considered two machines in two states infinitely close to each other. In specifying how the machine must move in passing from one to the other, either it was necessary to invoke a new principle, or else the determination was present in the geometry of the system, and in the latter case it was a defect of the principle that it failed to make evident the geometric conditions of the motion. In fact, however, such motions were subject to certain conditions, in definition of which Carnot was going to propose for the whole class the designation of geometric motions. As will appear, his geometric motions differed from virtual velocities in being finite, and amounted to possible or actual displacements in which no internal work was performed or energy consumed within the system.

Both of the foregoing principles applied only to equilibrium, and though Carnot alluded to d’Alembert’s principle as justification for extending equilibrium principles to motion, he made no actual use of it, preferring in practice to found the reasoning on the conservation of live force. From among laws of motion and equilibrium themselves, Carnot chose two as axiomatic. The first was the equality in opposite senses of action and reaction, which he did not identify as Newton’s

third law of motion. It applied to all bodies without exception. The second applied only to hard bodies and asserted that in their interactions, whether by impact or pressure, their relative velocity in the next instant was zero. From these two laws, two corollaries followed: first, that the intensity of interactions between two bodies depended only on their motion relative to each other, and second, that the force they exerted on each other in impact was always along a perpendicular to their common surface at the point of contact.

Carnot began the argument itself by deducing from these fundamental principles an equation stating that in the motion of a system of hard bodies the net effect of mutual interactions among the corpuscles constituting the system was zero. Applied to generalized systems of hard bodies, this equation reduced to an expression equivalent to the conservation of live force. Just there, however, Carnot encountered the inapplicability of that principle to hard-body collision. It was mainly in order to turn this difficulty that Carnot defined for purposes of analysis the class of geometric motions, i.e., displacements depending for their possibility on the geometry of a system and not on rules of dynamics based in physics. For such motions Carnot might ignore the supposed loss of live force in inelastic collision. Invoking them permitted him to transform his fundamental equation into an indeterminate of general application into which arbitrary values determining the motions might be introduced in the solution of particular cases.

In later terms, this analysis amounted to a derivation of conservation of moment-of-momentum or torque from conservation of energy or work. Since what Carnot required for a theory of machines was precisely the license to disregard the internal constraints and interactions of systems of bodies and forces, he made conservation of moment-of-momentum (“moment of the quantity-of-motion”, in his words) the fundamental principle in mechanics generally. From it he deduced the Principle of Least Action and restatement of conservation of live force in an altered form expressing that what machines transmit is power rather than motion. Indeed, the central thrust of Carnot’s contribution to mechanics was tacitly to transform the analysis of motion into the analysis of power.

The second part of the *Essai sur les machines en général* applied Carnot’s adaptation of the principles of mechanics to the operation of machines themselves. The most important definition was of the quantity now called work. Carnot called it “Moment-of-Activity” and identified it clearly as the basis for measuring input against output in machine processes. Recurring to the principle of moment-of-momentum, Carnot deduced from it the most original of his propositions: that, in the case of a machine transmitting motion smoothly, the work done by the impelling forces equals the work done by the resisting forces. The principle, since named after Carnot, follows directly: characteristically, he thought of it and stated it restrictively to the effect that the work done by a system of moving parts equals the work done on it only if percussion or turbulence be eliminated in the transmission of motion (or power). A concluding scholium inveighed against the chimaera of perpetual motion, discussed the factors of mechanical advantage, developed the conditions for efficient design of machinery with particular emphasis on hydraulic power, and made the distinction already outlined between the experiential and rational approaches to the science of mechanics.

Such were the main heads and findings of Carnot's first publication. Readers interested largely in a qualitative summary of what he did may find this outline sufficient and prefer to pass on directly to Chapter 3, which discusses the background of the *Essai sur les machines en général* and the ensuing development of the science of machines. Readers more immediately concerned with the inwardness of eighteenth-century mechanics itself may wish to follow the detailed précis and analysis of the argument that occupies the remainder of the present chapter.

2.2 Geometric Motions

Imagine, Carnot charged his reader, any system of hard bodies in which the virtual motion is modified into some other motion in consequence of the internal constraints and interactions among the bodies. The example that he meant but did not specify was in consequence of their being assembled into a machine. The general problem of mechanics was to find that other motion, and Carnot began his resolution by summing the interaction of contiguous corpuscles, m' and m'' . From the two basic axioms (equality of action and reaction and zero relative motion following collision) and designating by

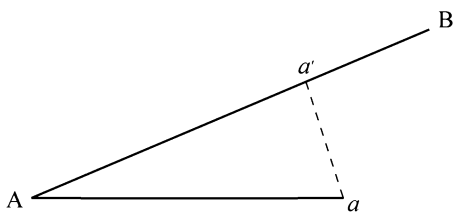
F'	The action of m'' on m' , i.e., the “force” (quantity of motion) that m'' imparts to m' ,
F''	The reaction of m' against m'' ,
V' and V''	The respective velocities immediately after impact,
q' and q''	The angles between the directions respectively of V' and F' and those of V'' and F'' ,

Carnot obtained the expression⁹

$$\int F' V' \cos q' + \int F'' V'' \cos q'' = 0$$

⁹Perhaps it will be helpful to give here Carnot's explanation of this method of representing the projection of a force or any directed quantity upon another in terms of the elementary trigonometry of a right triangle (Fig. 2.2).

Fig. 2.2



for the interaction of neighboring corpuscles taken two by two; which according to his wont he also stated verbally:

That is to say, that the sum of the products of the quantities of motion impressed on each other by the corpuscles separated by each of these little inextensible wires or incompressible rods [...] multiplied by the velocity of the corpuscle on which it is impressed, evaluated in the direction of that force, is equal to zero.¹⁰

Carnot next considered the system in motion as a whole in order to analyze the internal interactions of the constituent parts, and adopted designations that he thereafter used quite consistently for the several quantities. He called

The mass of each corpuscle	m
Its virtual velocity	W
Its actual velocity	V
The velocity that it <i>loses</i> , ¹¹ in the sense that W is the resultant of V and of this velocity which is	U
The force (quantity of motion) that each contiguous particle imparts to m and from which derives all the motion it receives from the system	F
The angle between the directions of W and V	X
The angle between the directions of W and U	Y
The angle between the directions of V and U	Z
The angle between the directions of V and F	q

Aa represents the initial quantity and AB the direction in which it is to be “estimated”. Thus, the projection $\overline{Aa'}$ of the force Aa estimated in the direction $\overline{AB}$ is given by “ $Aa \cos aAB$ ”. The explanation is from the *Principes fondamentaux de l'équilibre et du mouvement* (Carnot 1803a, § 26, p. 16). In the *Essai sur les machines en général* he did not stop to explain these elementary devices but simply employed them. (The interest lies mainly in the example they afford of his geometric and trigonometric way of visualizing relations. This comes out initially in the 1778 memoir for the *Académie des sciences*. (See: Gillispie 1971, Appendix B esp., § 52–60)). In the expression given in the text above, the velocity of m' imparted by the force F' estimated in the direction of F' is “ $V' \cos q'$ ”. Since F' and F'' denote quantity of motion, what this expression says in the notation of a modern primer would be $\sum M' V'^2 + \sum M'' V''^2$. Carnot in this book used the sign “ Σ ” indifferently for integration and summation.

¹⁰Carnot 1786, § XV, p 26.

¹¹It was one of Carnot’s central conventions that when a system is set in motion, the difference between virtual and actual velocity in any component part is “lost to” or as the sign may be “gained from” the mutual interactions or constraints within the system. The usage comes from d’Alembert’s mechanics. Carnot used the terms in the *Essai sur les machines en général* of 1783 and explained them in the *Principes fondamentaux de l'équilibre et du mouvement* of 1803 (Carnot 1803a, § pp 41–43; pp. 24–26 and Pl. I, Fig. 4), and did so in the following manner. If a body M tends to move with the virtual velocity $\overline{MW}$, but is constrained instead to take on the velocity $\overline{MV}$, then if a parallelogram of velocities is constructed, the velocity $\overline{MW}$ maybe resolved into two components. of which one, $\overline{MV}$, will be the actual or remaining velocity and the other $\overline{MU}$ will be called the velocity “lost” by the body M . Prolonging MW in the opposite direction will give us

Given these designations, Carnot by means of simple algebra obtained from the previous equation an expression that he called his first fundamental equation (E), as follows:

$$\int mVU \cos Z = 0 \quad (E) \quad (2.1)$$

Only later in the argument did he point out that this expression was formally identical with conservation of live force. (Since $U \cos Z = V$ the expression reduces to $\int mV^2 = 0$). He could not simply invoke that principle directly, however, given his assumption about the inapplicability of the principle to hard-body collision. He was bound by the generality in which he had set himself the problem to convert the expression into one applicable to interactions of elastic body and of hard body (the more so as the latter were generally taken for the term of comparison in nature) whether motion was communicated by impact or by insensible degrees.

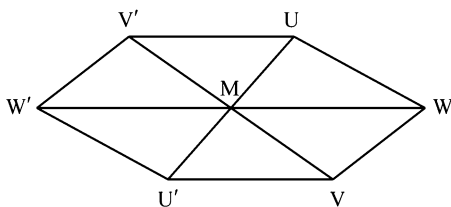
To that end, Carnot introduced at this, the critical juncture of his argument, the notion that he always regarded as his most significant contribution, to mathematics as well as to mechanics: the idea of geometric motions, i.e., displacements that depended for their possibility only on the geometry of a system quite independently of the science of dynamics. Carnot's geometric motions became virtual displacements in later mechanics. The idea was an interesting one, therefore; it played a distinctive part in his own analysis; more signally than any other element it exhibits that a kind of operational economy in Carnot's reasoning was what differentiated it from the classical mode of analytic mechanics.

Imagine, Carnot asked in regard to a generalized system of hard bodies, that just as the system is struck, its actual motion be stilled and it be made instead to describe two successive movements arbitrary in character but subject to the condition that they be equal in velocity and opposite in direction. The configuration of the system being given, it was clear that such an effect could be accomplished in infinitely many ways, and (what was essential) by purely geometric operations.

In the *Essai sur les machines en général* Carnot did not at once achieve a definition of this class of motions that was both simple and adequate to the use he needed to make of them (although about the use itself, as will appear, he was perfectly clear). Thus, he laid down first that motions would be geometric if they

the complementary case, a construction for velocity "gained", $\overline{MU'}$ in configurations in which the constraints augment initial virtual velocity (Fig. 2.3).

Fig. 2.3



involved the constituent bodies of a system in no displacement relative to each other. The converse would not be true, however, for motions that did involve such displacements might be geometric. For example, in the case of a machine composed of two weights suspended in equilibrium over a wheel and axle, a motion consisting of the descent of one from a height equal to the circumference of the wheel compensated by the raising of the other through a length equaling the circumference of the axle would be geometric since the equal and opposite movement could occur. Moreover, Carnot's purpose required him in a later passage to extend the concept to include motions of a system that meet the condition of reversibility in virtue of eliminating from consideration the components that (in later terminology) do no work. Perhaps, therefore, it will be legitimate for the sake of clarity to give the definition he did achieve in reworking the material in 1803 for the *Principes de l'équilibre et du mouvement*, wherein he suppressed allusions to these imaginary and reversible displacements, and laid down:

Any motion will be called geometric if, when it is impressed upon a system of bodies, it has no effect on the intensity of the actions that they do or can exert on each other when any other motion is impressed upon them.¹²

In the *Essai sur les machines en général* however, Carnot was thinking primarily in terms of bodies and displacements when it was a question of internal interactions in a system and only derivatively in terms immediately translatable as work and energy. Applying the concept of geometric motion to a generalized system of hard

¹²Carnot (1803a), § 136, p 108. It will help clarify movement tangential to the surface both stages in his thought to give a few examples.

(a) From the *Essai sur les machines en général* of 1783 (Carnot 1786):

1. Two globes in contact. An impulse displacing both in the same direction along the line of centers would produce a geometric motion; an impulse separating them along the line of centers would not.
2. Several bodies attached by flexible but inextensible wires to a common center. Any motion in which all remain equidistant from the center is geometric even if they shift among themselves; any motion altering the length of a radius is not.
3. A body moving on a curved surface. A movement tangential to the surface would be geometric; any departure from the tangent would not.

In each instance, the justification is that the equal and opposite motion is possible in the first case and not in the second (*Ivi*, 28–30n).

(b) From the *Principes de l'équilibre et du mouvement* of 1803:

1. Two bodies, A and B, are fixed to either extremity of a lever arm; the arm is rotated around a fixed point C. Each body assumes an angular velocity proportional to AC and BC respectively; since neither of these velocities influences the action of one body on the other whether by gravity or in any other manner the rotation of the lever is a geometric motion (Carnot 1803a § 139, p 111).

bodies, he noted that by definition the relative velocity of neighboring corpuscles would be zero during the initial instant of such a motion. Designating by

- u The absolute velocity of the corpuscle m in the initial instant of a geometric motion
- U As before, the velocity “lost” to the internal interactions,
- z The angle between u and U ,

then, since the corpuscles of the system would not tend to displace relative to each other in consequence of the velocity “ u ” alone, the mutual interactions within the system would be the same whether it was considered that “ m ” was animated solely by the velocity lost “ U ” or by the combined velocities “ u ” and “ U ”. But if all the corpuscles were animated by the velocity “ U ” alone, equilibrium would necessarily obtain. Therefore, the real velocity after interaction would be “ u ”, and by reasoning similar to that which yielded the first fundamental equation,

$$\int mVU \cos Z = 0, \text{ (E)} \quad (2.1)$$

Carnot had his second fundamental equation,

$$\int muU \cos z = 0, \text{ (F)} \quad (2.2)$$

In which he has replaced an actual, physical velocity “ V ” with an idealized geometric velocity “ u ”.

Let us summarize, then, more explicitly than Carnot in the toils of his argument did himself, what precisely he thought to have gained by transposing the problem into terms of geometric motions. In the first place, that step justified him in extending the principle of conservation of live force to hard-body interactions, whether sudden or smooth, and that is how he interpreted Equation (F) (see Eq. 2.2). As he observed later in the *Principes de l'équilibre et du mouvement*, all bodies are susceptible of geometric motion, whether hard, soft, or elastic, whether solid or fluid, for the reason that by definition these motions had no bearing on the internal interactions of bodies and were entirely independent of these dynamical distinctions (Carnot 1803a, § 144, pp 114–115). Hence, Equation (F) (see Eq. 2.2) extended to hard-body interaction, and, since it contained as a special case Equation (E) (see Eq. 2.1), which was itself a statement of conservation of live force, it authorized Carnot to take that principle as fundamental in the realm of geometric motion. And that, in the second place, is all he needed, since among the geometric motions of which a system was capable would be the real motion that it assumed upon any actual impulse. In the third place, therefore, Equation (F) (see Eq. 2.2) represented the solution to the problem originally posed—i.e., given the virtual motion of any system of hard bodies, to find the actual motion of the system upon communication of external forces.

It will be well to specify what Carnot meant by a solution. Since Equation (F) (see Eq. 2.2) was indeterminate, it would hold whatever the value of “ u ”. Provided that the motion be geometric, particular values and arbitrary directions might be attributed to the indeterminate according to the conditions of the specific problem, and it would then always be possible to formulate equations between the unknowns. For an example, Carnot produced the problem of a system of bodies unchanging in relative positions and containing no attachments to external fixtures. The solution could be drawn from Equation (F) (see Eq. 2.2) by supposing arbitrarily that all points of the system were subjected to a geometric motion such that their velocities “ u ” were parallel to a given right line. Since “ u ” was then constant, Equation (F) (see Eq. 2.2) became

$$\int mU \cos z = 0,$$

which stated that the sum of the components of the forces “lost” to the constraints in the arbitrary direction of “ u ” was zero, and hence the resultant force was the same as if each body had been free. This was a “well-known principle”, observed Carnot, not identifying it.

A second example was that of the same system made to rotate around a given axis, so that each of the points described a circle in a plane perpendicular to the axis. The movement being geometric, “ R ” being the radius of “ m ” and “ A ” a constant for all points, it was clear that

$$u = AR$$

and that Equation (F) (see Eq. 2.2) became

$$\int mRU \cos z = 0.$$

That is to say, the sum of the moments relative to any axis of the forces “lost” to the mutual interactions was zero—another “well-known principle” said Carnot, not identifying it in that place for conservation of angular momentum or moment-of-momentum.¹³ Of that, to which he recurred in a fundamental way, more needs to be said in the next section. What is characteristic to notice here is that such results were what Carnot meant by solutions to the problem. Contrary to what might have been expected of a young engineer, there were no numerical problems and solutions in the *Essai sur les machines en général* just as there were no formulations not applicable generally to any type of machine.

¹³On that please consult *Essai sur les machines en général* (Carnot 1786, § XVII, p 33).

2.3 Moment-of-Momentum

Having obtained a general solution for his fundamental problem of mechanics, Carnot now needed to state it in a form suited to drawing out the consequences for the theory of machines. To that end he turned to demonstrating its equivalence to the law called in later mechanics that of the moment-of-momentum, which involved much more for him than the example just cited concerning moments of forces relative to different axes of rotation. Carnot named the principle moment of the quantity-of-motion. Adducing it in terms of his geometric motions, he employed it for the same purpose that (according to Truesdell) physicists usually have done, in order “to obviate the need to specify the mutual forces among the particles of a rigid or deformable body.”¹⁴

Referring still to a generalized dynamical system of bodies, wherein “ m ” is the mass of each body and “ V ” its velocity, suppose there be impressed upon it a geometric motion of velocity “ u ” which in its direction makes the angle “ y ” with the direction of the velocity “ V ”. The quantity

$$muV \cos y$$

would then be the moment of the quantity of motion “ mV ” with respect to the geometric motion “ u ” and its sum, $\sum muV \cos y$ would be the moment of the quantity-of-motion of the system with respect to the geometric motion imparted to it. Retaining the notation of the basic problem and expressing it by

$\int muW \cos x$	The moment of the quantity-of-motion of the system before impact,
$\int muV \cos y$	The moment of the quantity-of-motion of the system after impact,
$\int muU \cos z$	The moment of the quantity-of-motion “lost” in impact,

Carnot showed by a simple trigonometric argument that

$$\int muU \cos z = 0$$

and hence

$$\int muW \cos x = \int muV \cos y$$

¹⁴Whence the law of moment of momentum? in his *Essays in the History of Mechanics* (Truesdell 1968b, p 243).

so that he could state a fundamental theorem:

In the impact of hard bodies, whether the impact be immediate or be transmitted by means of any springless machine [machine quelconque sans ressort], it is always true that with respect to any geometric motion—

1. The moment of the quantity-of-motion lost by the system as a whole is equal to zero.
2. The moment of the quantity-of-motion lost by any portion of the bodies of the system is equal to the moment of the quantity-of-motion gained by the remaining portion.
3. The moment of the quantity of real motion of the general system immediately after impact is equal to the moment of the quantity-of-motion of the same system immediately before impact.¹⁵

These three propositions were identical at bottom, being simply interchangeable ways of stating the solution contained in Carnot's fundamental Equation (F) (see Eq. 2.2). Nevertheless, the third was the most important to Carnot, for it was from that way of looking at the problem that he later in the argument drew out his own injunction to continuity in the transmission of power. Already in this passage, however, the principle appeared to him to be the most valuable in all the science of mechanics. For here was a quantity that did indeed remain unaltered in any impact, whether direct or indirect. The quantity was not what Descartes had thought it to be, the sum of the quantities of motion. That conservation holds only for particular directions and only when the system is free. Neither was it the sum of the live forces. That quantity is conserved only in the gradual transmission of motion. Here was this further quantity, however, that could not be diminished by obstacles interposed in the way of the motions of the system, nor yet by the machines that transmitted the motions, nor finally by percussions that might intervene; and that was the moment of the quantity-of-motion of the system in general with respect to any geometric motion it could perform. That principle (Carnot went on) contained all the laws of equilibrium and motion of hard bodies. He would next show that it might equally well be extended to other bodies whatever their nature or degree of elasticity.

In the inquiry just cited, Truesdell remarks that specialists in classical mechanics consider their subject to be one based on three fundamental laws: "the conservation or balance of force, torque, and work, or, in other terms, of linear momentum, moment of momentum, and energy" (Truesdell 1968a, pp 241–242). It would falsify the way it looked to Carnot himself to say that he saw the science so categorically. Yet it would not falsify what was actually in his reasonings to say that he found the first of little use for his purposes, felt that he could assume the third while requiring to give it more prominence than it was currently receiving, and while not claiming to originate the second, did believe that he had given an original argument for a principle the generality and profundity of which was then far from understood, and which he required in order to draw consequences that were his own. Even so it was not his originality on which he insisted but the conviction the argument gave to a rationality of mechanical practice that he felt should have been possible at any time. For Carnot was proud of his subject itself rather than vain of what he individually contributed to it.

¹⁵Carnot (1786), § XXII, pp 43–44.

Carnot justified his strong statement of conservation of moment of the quantity-of-motion with a series of corollaries and remarks that deduced from it principles he had been assuming and expressed them in a form applicable to the motion of machines. In the first he demonstrated that among the geometric motions of which a system was capable, the one that would actually occur would give a minimum value for the sum of the products of each of the masses by the square of the velocity lost—though what he really meant, he hastened to point out was that the differential would be zero,

$$d \int mU^2 = 0,$$

which could also be true of a maximum or, under certain conditions, of a value that was neither a maximum nor a minimum. (No doubt this possibility was what he had had in mind at the outset in noting the exceptions to which the conventional statement of the lowest center of gravity principle was liable (Carnot 1786, § II, p 12)). Carnot explicitly recognized the analogy of this proposition with Maupertuis' Principle of Least Action. The remark is revealing of the background of his own thinking, for, excepting d'Alembert, Carnot in the *Essai sur les machines en général*, mentioned no other writer on mechanics more recent than Descartes. Departing from d'Alembert's point of view that the fundamental phenomena are those of inertial motion, the quantity of which constitutes force in his thinking, he directed the analysis to determining what forces could do, their action—though for Carnot this was an engineering concept and never a metaphysical one.

In the second corollary, Carnot turned to live force and stated that in hard-body impact the sum of live forces before collision equals the sum after collision plus the sum of the live forces that would obtain if the velocity remaining to each moving part were equal to what it had lost. The proof consists in a simple and obvious trigonometric transformation of Carnot's first fundamental Equation (E) (see Eq. 2.1)

$$\int mVU \cos Z = 0$$

into the expression

$$\int mW^2 = \int mV^2 + \int mU^2.$$

The analogy, Carnot here pointed out, between his Equation (E) (see Eq. 2.1) and the conservation of live force thus amounted virtually to a demonstration. So also did the analogy of that equation with the same conservation in systems of hard bodies of which the motion changed insensibly, for then, “ U ” becoming infinitesimal, “ U^2 ” became an infinitesimal of the second order. He developed and established these points in the remaining two corollaries.

In the third corollary, Carnot turned attention to a subject that he had not yet discussed and that may be called Newtonian force only if it be disclaimed immediately that he saw it in such guise. For Carnot, the case was the one he had just mentioned, that of a system of hard bodies in which motion was communicated by insensible degrees. It was “motive force” that would have that effect, “force–motrice”, and since he had not previously considered it, he required additional notation according to which

The mass of each body is	m
Its velocity is	V
Its motive force is	p
The angle between V and p is	R
Its velocity in any geometric motion of the system is	u
The angle between u and p is	r
The angle between V and u is	y
The element of time is	dt

Then the Equations (E) and (F) were shown to take the respective forms¹⁶:

$$\int mVpdt \cos R - \int mVdV = 0$$

$$\int mupdt \cos r - \int mud (V \cos y) = 0$$

Considering, then, a body freely describing a trajectory in uniformly-accelerated motion, he integrated the former of those equations over time and space and obtained the conservation of live force in the form

$$\int mV^2 = \int mK^2 + \int mV'^2$$

where “ V ” was final velocity, “ K ” initial velocity, and “ V' ” velocity acquired under a constant motive force in an undetermined time.

Finally, Carnot in his fourth corollary derived from the conservation of moment of the quantity-of-motion a proof that his fundamental indeterminate Equation (F) (see Eq. 2.2)

$$\int muU \cos z = 0$$

¹⁶The terms “ $pdt \cos R - dV$ ” represented the velocity lost by m in the direction of V in consequence of the interactions of the bodies, and were substituted for “ $U \cos Z$ ” in Equation (E) (see Eq. 2.1). Similarly “ $pdt \cos r - d(V \cos y)$ ” was the velocity lost by m in the direction of u , and substituted for “ $U \cos z$ ” in Equation (F) (see Eq. 2.2).

governed motion and equilibrium not only in the case of hard body (that he had already proved in deriving it), but that it was a general law holding good for all bodies whatsoever. The proof being a digression from the theory of machines itself, it will not be necessary to follow it in detail. It turned on pointing out that in hard-body problems, Equation (E) (see Eq. 2.1) became one of the determinates of the indeterminate (F) (see Eq. 2.2) since by definition in that case, $u = V$ (i.e., the real motion is geometric). For other types of body, expressions drawn from their nature would be needed in order to establish determinate values. For example, in respect of the force F that bodies exerted on each other, if in other bodies it was n times unity, then it would be possible to put $\frac{U}{n}$ in place of U and express Equation (E) (see Eq. 2.1) in the form

$$\frac{n}{1-n} \int m V U \cos Z = \int m U^2.$$

In the case of perfect elasticity, in which convention would put $n = 2$ this expression reduced to

$$\int m W^2 = \int m V^2,$$

which, of course, was conservation of live force. That conservation was for elastic collision what the Equation (E) (see Eq. 2.1) was for hard bodies, i.e., the characteristic determinant, which was just what he had undertaken to demonstrate at the outset.

Three remarks completed Carnot's formulation of the principles of equilibrium and motion themselves. The first referred the expressions he had derived to Cartesian coordinates, in which they appeared much simpler. Indicating by single, double, and triple primes the components of the quantities referred to three mutually perpendicular axes, and considering the conditions of equilibrium, Equation (E) (see Eq. 2.1) reduced to $0 = 0$; the fundamental Equation (F) (see Eq. 2.2) became

$$\int m u' W' + \int m u'' W'' + \int m u''' W''' = 0;$$

and the general indeterminate for equilibrium in the case of motive forces became

$$\int m u' p' + \int m u'' p'' + \int m u''' p''' = 0.$$

A second remark was more significant of Carnot's own style of thought. Traditionally mechanics had not only abstracted from the massiness of machines, but had treated obstacles and fixtures like the fulcrum of a lever either as stationary or as outside the system. Strictly speaking, however, no object in nature could be considered stationary, and no body was without mass. Rather than thus limiting the scope of the analysis, Carnot proposed a different mode of visualizing, one

involving approximations to the material reality. He would let the fixtures and obstacles affecting the motions of a system be considered bodies of infinite mass and density though mobile in principle and hence susceptible of geometric motions. He would let bodies that merely communicated motion without offering any resistance to changes in the state of the system be considered bodies of infinitesimal mass and density. Of the two notions, the former seems the more significant, not for any practical difference it made in mechanics, but because this mode of accountability for mass bears a striking ancestral resemblance to that of the accountability for heat in Sadi Carnot's analysis of a heat engine drawing upon and discharging into reservoirs and sinks of heat of infinite capacity. By means of the latter notion, on the other hand, the mathematical theory of each of the classic types of machine might be derived from Equation (F) (see Eq. 2.2).

Finally, Carnot began turning the discussion toward the object ultimately in view, the application of the principles of equilibrium and motion in the form just given them to the theory of machines. For that purpose, he found the word "power" (*puissance*) more natural than "force", and it would appear that Carnot's discussion marks an early stage of the process in which that word began making its transition from the usages of ordinary life into the service of exact science and engineering. Nothing could be more fundamental, of course, than the existence of bodies acting on and modifying the motions of other bodies in accordance with first principles. Cases would arise, however, in which it would prove convenient to abstract from the mass of bodies for the sake of considering only the effort being made by them—the pull of the cord against the load, the thrust of the wind against the sail, the drag of the current upon the boat, the resistance of the body to impulsion, the friction between the bearing and the moving part. When it was a question of effect, the size or nature of the agent made no difference, whether the source of power were man, beast, or machine; wind, water current, or weight. As for machines, they were considered to be assemblages of immaterial obstacles and moving parts capable of transmitting action without internal reaction. Carnot took them to be, in other words, bodies of infinitesimal density and mass, to which were applied forces external to the system—powers in a word—but Carnot was consistent with his own realism. Machines in fact possessed mass, and he would not neglect it. He would simply consider its effect as he did the other forces or powers exerted by agents external to the system. Any force was to be taken as a quantity of motion "lost" by the agent that exerted it, whatever that agent might be. If, then, that force was designated " F " its quantity was evidently the same as that expressed by the product " mU " in all the foregoing discussions; and if " Z " was the angle between the direction of that force and any geometric velocity " u " imparted to the system, then the fundamental Equation (F) (see Eq. 2.2) might be expressed as follows:

$$\int Fu \cos Z = 0 \text{ (AA) .} \quad (2.3)$$

That was the form in which Carnot employed it in the discussion of machines in general that ensued in Part II, wherein when he wrote of force he was thinking of what he also called power, and tacitly taking its quantity over time.

It is clear, therefore, that we have reached the crucial turn that Carnot gave to the arguments of classical mechanics in what was said of machines. Classically machines were taken to be agents for the transmission of motion. The quantity of motion was an expression of force and involved mass. Abstracting from the mass of machines in order to study motions, the formulations of mechanics took no account of the motions of machines themselves. Substituting the notion of power for that of force, Carnot was able to transpose the notion of machines into that of agents for the transmission of power. Instead of neglecting the mass of the machines, he considered it to be one of the powers affecting motions. The problem became, therefore, adaptation of the laws of motion to laws of power and specification of the optimum conditions for transmission of power, and it was in order to prepare the ground that Carnot had brought the concept of motions “lost” and “gained” into the purview of the classical conservation principles and models of the structure of matter.

2.4 Moment-of-Activity—the Concept of Work

In all the foregoing, which constituted Part I of the *Essai sur les machines en général*, Carnot had not proposed to add to the science of mechanics but to state its principles in a form and arrange them in a sequence applicable to the science of “machines properly speaking” that he next inaugurated in Part II. There the writing went more directly to the point, the tone became more businesslike, and the yield of the successive propositions was more evident in the analysis they afforded of the transmission of power.

The analysis began with Carnot’s distinction between forces applied to a machine in motion according to whether their direction made an acute or an obtuse angle with the direction of motion of the point of application. The former he called “moving” or “impelling” forces, the second being the adjective usually employed. The latter he called “resisting” forces. Elsewhere in the *Essai sur les machines en général*, Carnot observed mildly that the definition would avert dispute about whether forces were to be considered causes or effects (Carnot 1786, p iv). Here it sufficed to notice that impelling forces could become resisting and vice versa if the direction of the motion changed, and that in any power system each of the forces would be impelling or resisting with respect to any given geometric motion imparted to the system according to the angle between the respective directions.

Carnot needed those metaphysically indifferent definitions for themselves. He also needed the distinction between the two classes of force for its auxiliary importance in his discussion of what in retrospect appears the most significant recognition in all his writing on mechanics and on mathematics, that of the importance to the science of power of the quantity that is now called “work” and that he called “moment-of-activity”. At first glance it might appear that extending as he did the distinction between impelling and resisting force into one between moments-of-activity “consumed” and “produced” by forces applied to a system

arrested him on the threshold instead of in the presence of a unified concept indistinguishable in substance from that of work. Such was not the case, however—though a little complex in the phraseology, the distinction was simply that between work done on or by a system.

Let a force “ P ,” he laid down, be moving with a velocity “ u ” (for, though he did not say so, such a movement could only be geometric), and let the angle between “ P ” and “ u ” be “ z ” then the quantity “ $P \cos z \, dt$ ” (where “ dt ” is the differential of time) would be what Carnot defined as the *Moment-of-Activity* consumed by the force “ P ” during the time “ dt ”. Characteristically enough, Carnot was thinking about the gravitational instance for his example. Let a weight fall the distance “ H ” in time “ t ”. Then, “ $dH = u \cos z \, dt$ ”, and the moment-of-activity consumed during time “ dt ” would be “ $P \int dH$ ” or “ PH ”—the product of force by distance: foot-pounds in the Anglo-American dimensions of elementary physics.

Now, it is very difficult to establish categorical priorities in the history of mechanics. It is equally difficult, however, to find specialists in mechanics prior to Lazare Carnot who explicitly singled out that quantity to be the measure of what forces and powers accomplish and a unit in terms of which the fundamental conservations were to be stated. It is significant, moreover, that he stated it as the sum of the moment-of-activity consumed in successive instants—i.e., as what was accomplished by a process occurring in time. He was thinking about power, and not merely its dimensional equivalent, the product of displacement by motive force.

For it is clear that he did know what he was doing. Considering an entire system of forces applied to a machine in motion, the moment-of-activity consumed by the totality of the forces would be the difference between the sum of moments-of-activity consumed by the impelling forces and the sum of moments-of-activity consumed by the resisting forces. Since impelling forces made an acute angle and resisting forces an obtuse angle with motions, the cosines would be positive in the former case and negative in the latter, and the sign of moments-of-activity of impelling forces would thus be positive and of resisting forces negative.

Turning now to the reciprocal point of view, the moment-of-activity *produced* by a force was to be understood as the product of its magnitude evaluated in a direction *contrary* to its velocity multiplied by the distance that the point of application traversed in an element of time. Obviously, then, the moments-of-activity *consumed* and *produced* were equal quantities of opposite signs, and the difference between them was identical in the terminology of power to that in mechanics proper between moments-of-the-quantity-of-motion gained or lost with respect to a given geometric motion. In a final definition, Carnot combined his two classes of force and with them his two classes of moment-of-activity into a single designation. He called the moment-of-activity *exerted* by a force that which was *consumed* by an impelling force or *produced* by a resisting force. Since he did not know which was which, he would in practice always take the cosine of the smaller of the two supplementary angles between the force and the motion of the point of application, so that the moment-of-activity exerted by a force would always be a positive quantity.

Clearly then, this notion contained what is later to be found in the employment given by the science of physics to the concept of work. Having laid down these

definitions for the mechanics of machines, Carnot proceeded to state what he called its fundamental theorem:

Whatever the state of rest or motion of any system of forces applied to a machine, if a geometric motion be imparted to it without altering the forces, the sum of the product of each by the velocity of the point of application, evaluated at the first instant and in the direction of the force, will be zero.¹⁷

That statement was, of course, simply the adaptation to a system of forces or powers of the principle of conservation of the moment-of-the-quantity-of-motion for a system of bodies in motion. Expressed in the following form,

$$\int Fu \cos z = 0,$$

it was evidently identical with the Equation (AA) (see Eq. 2.3) into which he had transformed his fundamental indeterminate (F) (See Eq. 2.2) for the purpose of applying it to the transmission of power (Carnot 1786, § XXX, p 63). Indeed, it was obvious (Carnot remarked immediately) that strictly speaking this fundamental theorem was nothing more than the principle of Descartes with its defects repaired—i.e., generalized beyond equilibrated pairs to any system of forces with regard both to their direction and intensity.

Following the fundamental theorem, Carnot set out a series of six corollary propositions. The first three concerned the conditions of equilibrium of systems of machines and weights of various sorts on which geometric motions were impressed, and the fourth asserted the balance of torques around each of three mutually perpendicular axes along which forces in equilibrium could be resolved. Except for redeeming his opening promise to give a rigorous derivation of the original lowest center-of-gravity principle of equilibrium, Carnot did not seem much interested in these propositions, for he gave them in summary fashion and mainly for the sake of completeness.

The fifth was different: “Special law concerning machines of which the motion changes by insensible degrees” he called it, and now at last he had come to what he did certainly recognize to be his own, though still without insisting on his originality. The law ran thus:

*In a machine of which the motion is changing by insensible degrees, the moment-of-activity consumed in a given time by the impelling forces equals the moment-of-activity exerted in the same time by the resisting forces.*¹⁸

That would be true if it could be proved that the moment-of-activity consumed by all the forces of the system during the given time was zero. If “*F*” designated each of the forces, “*V*” its velocity, and “*z*” the angle between them, then the requirement

¹⁷“THÉORÈME FONDAMENTAL. Principe général de l’équilibre & du mouvement dans les machines. XXXIV” (Carnot 1786, p 68).

¹⁸Carnot (1786), § XLI, pp 75–76.

was that $\int F V \cos Z dt = 0$; but by the fundamental theorem just stated, it was clear that $\int F V \cos Z = 0$. Hence the corollary followed from the general principle of equilibrium in machines.

It was not, however, in this almost tautological proof that Carnot's interesting contribution consisted, nor yet in the statement of the special law, although he did recognize it to be "the most important in all the theory of the motions of machines properly speaking". For he introduced the principle of continuity in the transmission of power by way of exemplifying an application of this special law, and then developed its consequences in the scholium that concluded the *Essai sur les machines en général* (Carnot 1786, § XLIX–LXIV, pp 81–89).

When working out his own thoughts Carnot usually preferred reasoning on weights and asked the reader to suppose that they be the powers applied to a machine and that

- m be the mass of each of the bodies,
- M designate the total mass of the system,
- g be the force of gravity,
- V be the actual velocity of a body m ,
- K be its initial velocity,
- t be the time elapsed since the motion began,
- H be the height through which the center of gravity of the system falls in t time, and
- W be the velocity acquired in the height H .

Two types of force were involved in the operation of the machine, the pull of gravity and the inertial resistance of the various bodies to change of motion. By definition, the moment-of-activity *consumed* in the whole system by the action of gravity during time t was

$$MgH,$$

which was equivalent to

$$\frac{1}{2}MW^2.$$

Considering now the forces of inertia, the velocity of m being V and becoming in the next instant $V + dV$, its force of inertia in the direction of V would be

$$m \frac{dV}{dt}$$

Reverting again to definitions, the moment-of-activity exerted by the forces of inertia during time dt was

$$m \frac{dV}{dt} V dt, \text{ or simplifying, } mVdV.$$

The moment-of-activity *consumed*, therefore, during time t by the forces of inertia was

$$\int mVdV$$

Integrating and completing the integral, it became

$$\frac{1}{2}mV^2 - \frac{1}{2}mK^2$$

(Carnot sometimes adopted the device of including the constant of integration in the term for initial velocity). Therefore, the moment-of-activity *consumed* in time t by all the bodies of the system would be

$$\frac{1}{2} \int mV^2 - \frac{1}{2} \int mK^2.$$

Now, by the conditions of the problem, inertia was a resisting and gravity an impelling force. Thus, by the special law just stated in the corollary, it would be true that

$$MW^2 = \int mV^2 - \int mK^2$$

or

$$\int mV^2 = \int mK^2 + MW^2.$$

As usual, Carnot also told verbally what he had just done:

*In a weight-driven machine of which the motion is changing by insensible degrees, the sum of the live forces of the system after a given time is equal to the sum of the initial live forces plus the sum of the live forces that would obtain if all the bodies of the system were animated by a common velocity equal to that due to the height through which the center of gravity of the system has fallen.*¹⁹

It followed immediately that (a) in a weight-driven machine in uniform motion, the center of gravity of the system remained at constant height (since then $V = K$, $W^2 = 0$, $H = 0$); (b) No matter in what manner a weight was raised to a certain height, the forces producing that effect must have been such as to consume a moment-of-activity equal to the product of the weight by the height; (c) To produce any movement by insensible degrees in a system of bodies, the forces (Carnot said “powers”) to produce that effect must have consumed a moment-of-activity equal to half the quantity by which the sum of the live forces of the system was increased.

¹⁹Carnot (1786), § XLII, pp 77–78.

From these last two propositions it further followed that in order to raise a weight M from rest to a height H while imparting to it a velocity V , the forces employed to that end must consume a moment-of-activity equal to

$$MgH + \frac{1}{2}MV^2.$$

And it seems worth following Carnot's reader through these elementary steps since they end with this expression dimensionally and notationally identical to an energy statement. Conceptually, of course, the range was extremely restricted, but there the expression was, nonetheless, and it would be a generalization and not an alteration of this type of thinking that would extend that range. So far, indeed, all that was new was the type of thinking, for dimensionally and algebraically the equivalence of those two terms was a relation as old and elementary as Leibniz's early discussions on the true measure of force. What Carnot had contributed was couching the relations in terms of the operation of machines, inertial masses transmitting powers over distances, and thinking of live force as the capacity for consuming moments-of-activity of machines in motion, or in later words, of forces doing work over distances and in time.

Now, however, we come to the consequence that was his own both in substance as well as genre. The previous reasoning assumed that motion in the system changed by insensible degrees. Suppose, however, that some discontinuity, some impact or percussion, intervened. Then let

- h be the height from which the center of gravity has fallen at the moment of percussion,
- X be the sum of the live forces immediately before percussion,
- Y be the sum of the live forces immediately after percussion,
- Q be the moment-of-activity that the live forces must consume throughout the entire movement,
- q be the moment-of-activity that the live forces must consume up to the moment of percussion.

Finally, suppose for the sake of simplicity that the movement of the system began from and ended in rest. It would be evident from the relation just established that

$$q = Mgh + \frac{1}{2}X,$$

and similarly that the moment-of-activity to be consumed following percussion would be

$$Q = Mgh + \frac{1}{2}X - \frac{1}{2}Y.$$

Now Carnot appealed back to his statement of what he had held in Part I of the *Essai sur les machines en général* to be the fundamental law of motion and equilibrium in mechanics proper, the conservation of the moment-of-the-quantity-of-motion (Carnot 1786, § XXII, pp 43–45) and specifically to its first corollary, which found that of all the motions of which a system was capable, that which would actually occur would be the geometric motion such that the sum of the live forces of each of the masses would be a minimum²⁰ (Carnot 1786, § XXIII, pp 45–48). Therefore, since $X > Y$ the moment-of-activity to be consumed in raising M to the height H was larger than if there had been no percussion, for in the latter case the equation would simply be

$$Q = MgH.$$

It followed that the same impelling forces could raise the same weight to a greater height if percussion were avoided than if it occurred. Carnot deferred an explicit statement of this, his Principle, to the concluding summary. For any machine, the condition of maximum efficiency is that $q = Q$:

Now, in order to fulfill that condition, I say, first, that any impact or sudden change is to be avoided, for it is easy to apply to all imaginable cases the reasoning developed in that of weight-driven machines; whence it follows that whenever there is impact, there is simultaneously a loss of moment-of-activity on the part of the impelling forces, a loss so real that their effect is necessarily diminished [...]. It is then with good reason that we have proposed that in order to make machines produce the greatest possible effect, they must never change their [state of] motion except by insensible degrees. We must except only those that by their very nature are subject in their operation to various percussions, as are most mills. But even in this case it is clear that all sudden changes should be avoided that are not essential to the constitution of the machine.²¹

Such was Carnot's Principle. It is characteristic that he should have worked it out by example, claimed a generality that he did not actually demonstrate, and then exhibited his awareness that in actual practice the condition of perfect efficiency is an ideal to be approximated so far as the nature of the process allows. To the demonstration he added a final corollary on hydraulic machines, observing merely that since a fluid might be regarded as an infinity of solid but detached corpuscles—and since he had already proved what was usually taken to be merely experimental truth, i.e., the conservation of live force in incompressible fluids wherein motion changes insensibly (that is without splashing or turbulence)—everything that he had demonstrated for systems of hard bodies held equally good for masses composed of incompressible fluids (Carnot 1786, § XLVIII, p 80). But that extension was more important than the afterthought it appears to be in the formal structure, and it will be easier to appreciate its significance and indeed the quality of Carnot's performance in the mechanics of machines if we accompany him into the scholium in which he enlarged more informally, and more naturally, on what he hoped his *Essai sur les machines en général* would accomplish.

²⁰This is the principle that Carnot recognized as analogous to Maupertuis' Law of Least Action.

²¹Carnot (1786), § LIX, pp 91–92.

2.5 Practical Conclusions

In the concluding discussion containing the statement of Carnot's principle just quoted, he relaxed his formality and spoke his mind. According to Corollary V, the moment-of-activity consumed in a given time by impelling forces equals that exerted in the same time by the resisting forces in a machine that changes gradually in its state of motion. Actually, he observed, that proposition contained nearly all the applicable part of the theory of machines. In practice most working devices were powered by agents—animals, springs, weights—that exerted “dead force” continuously. Most real machines, furthermore, once set in motion soon reached a steady pace of operation such that the forces required to keep the process going balanced the elements of resistance. It now appears, therefore, that this corollary had been the goal of the argument all along, and we would not be forcing his meaning to call it his Work Principle. Indeed, we would not altogether be forcing his language, for in one example of the equivalence between moment-of-activity and effect, he remarked it propos of an arrangement for raising a weight a certain distance, that no machine could be designed by which it would be possible “with the same work [travail] (that is to say the same force and the same velocity)” to lift the object higher in the given time (Carnot 1786, § LIV, p 85).

In turning now to practice, Carnot maintained the generality of his discussion. Designating by

$$Q \quad \left\{ \begin{array}{l} \text{the work done} \\ \text{the moment-of-activity consumed} \end{array} \right\}$$

by impelling forces and by

$$q \text{ that } \left\{ \begin{array}{l} \text{done on} \\ \text{exerted by} \end{array} \right\}$$

the resisting forces, then Corollary V the Work Principle, could be symbolized by

$$Q = q.$$

It followed that there were two related sets of conditions for the most efficient possible operation of machines. The first was that the greatest possible mechanical advantage be secured and the second that no motion be wasted. In order to clarify the former condition, the relation could also be expressed

$$FVt = q,$$

where F was the resultant of all impelling forces and V the resultant of their velocities. Achieving the maximum effect would involve varying those three factors. A fourth variable would be the direction in which the impelling forces acted,

but it was obvious (though Carnot labored the point in one of his trigonometric excursions) that for best results the force ought always to be applied in the direction of the velocity. As for the intensity of force, the time of application, and speed of operation, no general rules applied and experience would need to govern. For example, a man might turn a crank one foot in diameter for eight hours a day making an effort equivalent to twenty-five pounds at a rate of one revolution every two seconds. If he were to work faster, his output would suffer for he could no longer maintain the twenty-five pound exertion. If he turned the handle more slowly, he increased F proportionately less than V , and the moment-of-activity would diminish. Every source of power would have its own maximum, of which Carnot could say only that it was determined by its physical constitution and that experience alone would find it.

No hidden resources of power lay in the capacities of machines, therefore, and what theory offered was identification of the factors among which economy would dictate the wise proportions to observe. If time were not important, force might be economized at its expense. But (the reader may feel a bit impatiently) so elaborate a discussion can hardly have been required to exemplify the ancient maxim that what is gained in force is lost in time or speed. Carnot clothed the point in new terminology, but gave no new findings, and what must be appreciated is that he evidently thought that rigorous demonstration would help and to that end expressed himself with an engineer's passion and urgency. These matters, so obvious to him, must need the conclusive proof he was giving, else why the persistence of chimerical schemes for perpetual motion under one pretext or another, why the continuing waste of ingenuity and money? It would falsify Carnot's book to underemphasize that, perhaps its most deeply felt point.

The novel considerations that Carnot brought forward, however, pertained mainly to the second set of requirements for maximum efficiency, the avoidance of waste motions. Instead of generalizing the reasoning that had established his Principle in the example of weight-driven machines, so as to cover all possible cases, Carnot turned to developing its relevance for hydraulic machines, and did so in a series of paragraphs that seem particularly striking when compared to the reasoning in the memoir by his son, *Réflexions sur la puissance motrice du feu*, where heat is treated like a fluid in flow.²²

Looking at the ordinary waterwheel from an ideal point of view, Carnot observed that its design embodied two faults. First, the water was generally allowed to fall onto the blades and thus to transmit motion by percussion. Second, after striking the blade the water ran off with a velocity that was entirely lost to the process. In order that a hydraulic machine be as efficient as possible, its design should be such that the water would lose all its motion to the mechanism, and should do so without turbulence or splashing. It made no difference what form the machine might take. As always, Carnot was dealing in the analysis with idealized machines. He recognized

²²This analogy has often been noticed, e.g., Charles Brunold in his *L'entropie. Son rôle dans le développement historique de la thermodynamique* (Brunold 1930, pp 37–40).

that descending to actuality might make it desirable to depart from these conditions. For simplicity of construction, nothing was likely to prove better than wheels turned by the impact of water.

Practically, moreover, the closer the design approached to satisfying one of the conditions, that is absorbing all the motion of the water in the wheel, the greater would be the impact and the greater the loss of power to percussion. The lesser the impact, conversely, the less would be the proportion of the power of the water transferred to the wheel. The shrewd designer would, therefore, regard these conditions not as goals to be attained but as norms to be approached in the degree that circumstances render the one saving or the other relatively more important. Indeed, it was an ultimate condition of efficiency in machines of all types that no motion be produced extraneous to their purpose, and readers of Sadi Carnot's memoir will also recognize this injunction adapted to the heat engine in his requirement that no differential of temperature be admitted that does not measure itself in a change of volume in the gas confined in a cylinder. As his father originated the example, it showed the most efficient pump to be the one that delivered water into the reservoir at velocity zero.

Expanding in his closing remarks on the concept of moment-of-activity, Carnot introduced a final illustration of its utility that carried his perception of its significance beyond the study of machines into generalized physics. Suppose the problem to be one of a system of bodies mutually attracting one another by forces that vary as any power of the distance. (Only at this late stage did he move into a mechanics of which the model was clearly Newtonian, and then only as an object of an analysis developed out of other considerations and other problems than those of central force systems.) Suppose the system was impelled to move from some given configuration to another. No matter, then, what the sequence in which the individual bodies were displaced; no matter, further, what route they took, provided only that no percussion intervened; no matter, finally, what sort of machines effected the transformations—quite independently of all such incidental means to the end, the moment-of-activity that the external agents consumed would always be the same, assuming the system to be at rest in its initial and final states.

Now, then, this certainly might be taken for a model of the kind of analysis that the physics of work and energy has found useful ever since those topics became explicit in the 1820s, '30s, and '40s. Its most recognizable offspring again was the heat cycle of Sadi Carnot, which considered a system in view of what had been done to or by it in shifting from an initial to a final state. The family resemblance was most marked in the abstractedness of the system, in the notion that process consisted in the transition between successive "states", in the restriction that this transition be gradual and continuous, in the requirement that all changes be reversible (which for Lazare Carnot was still to say that all motions be geometric), in the indifference (given those conditions) to the details (rate, route, or order of displacements), in the relevance of this extreme schematization to the actuality of operating tools, engines, and machinery. Clearly, the relationship between the science of machines and thermodynamics was similar to that between Lazare and Sadi Carnot. It was one of parentage.

Lazare and Sadi Carnot

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