Robust Optimal Control of Continuous Linear Quadratic System Subject to Disturbances

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Abstract In this chapter, the robust optimal control of linear quadratic system is considered. This problem is first formulated as a minimax optimal control problem. We prove that it admits a solution. Based on this result, we show that this infinite-dimensional minimax optimal control problem can be approximated by a sequence of finite-dimensional minimax optimal parameter selection problems. Furthermore, these finite-dimensional minimax optimal parameter selection problems can be transformed into semi-definite programming problems or standard minimization problems. A numerical example is presented to illustrate the developed method.

1 Introduction

A fundamental problem of theoretical and practical interest, that lies at the heart of control theory, is the design of controllers that yield acceptable performance for a family of plants under various types of inputs and disturbances [1]. This problem is often referred to as a robust optimal control problem. Normally, there are two kinds of criteria to achieve robust controller design. One is based on a statistical description, i.e., the criterion of the expectations of the cost and the constraints is adopted [17]. For the other one, the worst-case performance criterion is adopted [2–4, 9–12, 15, 18].

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The dynamical systems can be classified into two kinds—discrete dynamical system and continuous dynamical system. For the robust optimal control of linear discrete dynamical system with quadratic cost function, there are many results available [2–5, 9–12, 15, 18]. If disturbances lie in an ellipsoid, then it is shown in [3] that such an optimal control problem without constraints is equivalent to a semi-definite programming (SDP) problem. If the optimal control problem is subject to constraints on the state and control, it can be relaxed (see [3]) as a second-order cone programming (SOCP). If disturbances lie in a polyhedral, then such a robust optimal control problem becomes computationally highly demanding, (see [2, 12]). For other results on such robust optimal control problems, see, for example, [2, 3, 10–12, 18]. For robust optimal control governed by continuous dynamical system, a computational scheme is developed in [16]. By introducing a linear operator and resorting to its norm, the original minimax optimal control problem can be transformed into a standard optimal control problem. This method depends crucially on the special form of the cost function. If the cost function is with the terminal cost, then this method does not work.

In this chapter, we consider a class of robust optimal control problems governed by continuous dynamical systems subject to constraints on the admissible controls and the disturbances. The cost function involves not only a quadratic integral cost, but also a terminal cost expressed in the form of quadratic terminal state. Furthermore, we will use piecewise functions, rather than orthornormal basis as in [16], to approximate admissible control functions. We first show that this robust optimal control problem admits a solution. Based on this result, we show that this infinite-dimensional minimax optimal control problem can be approximated by a sequence of finite-dimensional minimax optimal parameter selection problems. Then, we show that these minimax optimal parameter selection problems can be transformed into SDPs. We also show that these minimax optimal parameter selection problems can also be transformed into standard minimization problems. Thus, gradient-based optimization methods can be applied. To illustrate our developed method, a numerical example is presented.

2 Problem Formulation

Consider the continuous linear dynamical system

$$\begin{aligned} &\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) + C(t)\,w(t)\,, t \in [0, T]\,, \\ &x(0) = x^0, \end{aligned} \tag{1}$$

where T is the given terminal time, $x(t) \in \mathbb{R}^n$ is the state at time t, x^0 is a given initial state, $u(t) \in \mathbb{R}^m$ is the input at time t, $w(t) \in \mathbb{R}^r$ is the uncertainty at time t, and A, B and C are matrices with appropriate dimension.

Let

$$\mathscr{W} = \left\{ w \in L^2 \left([0, T], \mathbb{R}^r \right) : \|w\|_{L^2}^2 = \int_0^T (w(t))^T w(t)\, dt \le \zeta^2 \right\}, \tag{2}$$

and

$$\mathscr{U} = \left\{ u \in L^2 \left([0, T], \mathbb{R}^m \right) : \|u\|_{L^2}^2 = \int_0^T (u(t))^T u(t)\, dt \le \eta^2 \right\}. \tag{3}$$

A function u is said to be an admissible control if $u \in \mathscr{U}$. Note that $\mathscr{W}$ and $\mathscr{U}$ are weakly closed in $L^2([0, T], \mathbb{R}^r)$ and $L^2([0, T], \mathbb{R}^m)$, respectively. For brevity, they are simply referred to as weakly closed.

Now our robust optimal control problem can be stated as follows.

Problem (P). Choose $(u^*, w^*) \in \mathscr{U} \times \mathscr{W}$ such that

$$\begin{aligned} J\left(u^*, w^*\right) = \min_{u \in \mathscr{U}} \max_{w \in \mathscr{W}} J(u, w) = (x(T))^T P x(T) + \\ \int_0^T (x(t))^T Q(t) x(t) + (u(t))^T R(t) u(t)\, dt, \end{aligned} \tag{4}$$

where P, $Q(t)$ and $R(t)$ are all positive definite matrices with appropriate dimensions.

To proceed, we assume that the matrices $A(t)$, $B(t)$, $C(t)$, $Q(t)$ and $R(t)$ are continuous matrix-valued functions defined on $[0, T]$.

3 Existence Theorem

Note that for each given $t \in [0, T]$, P, $Q(t)$ and $R(t)$ are all positive definite matrices. Let $F(t, \tau)$ be the $n \times n$ state transition matrix that satisfies

$$\begin{aligned} \dot{F}(t, \tau) &= A(t) F(t, \tau), \\ F(\tau, \tau) &= I, \end{aligned} \tag{5}$$

where I is the identity matrix. Then, for each given u and w, the solution of (1) can be expressed as

$$x(t|u, w) = F(t, 0) x_0 + \int_0^t F(t, \tau) B(\tau) u(\tau)\, d\tau + \int_0^t F(t, \tau) C(\tau) w(\tau)\, d\tau. \tag{6}$$

Since P and $Q(t)$ are positive definite matrices for each given $t \in [0, T]$, $J(u, w)$ is strictly convex with respect to x. From (6), it follows that x is linear with respect to w. Thus, $J(u, w)$ is strictly convex with respect to w. For each given $u \in \mathscr{U}$, since $\mathscr{W}$ is a weakly sequentially compact subset of $L^2([0, T], \mathbb{R}^r)$, there exists a $w(u)$

such that

$$J(u, w(u)) = \max_{w \in \mathscr{W}} J(u, w).$$

Let

$$\mathscr{G}(u) = \int_0^T (u(t))^T R(t) u(t) dt.$$

Note that for each given $t \in [0, T]$, $R(t)$ is positive definite, it is easily to verify that $\mathscr{G}(u)$ is a strictly convex function with respect to u. Now we have the following lemmas.

Lemma 1. *If $u_n \rightharpoonup u$ as $n \to \infty$, ($u_n \rightharpoonup u$ means that u_n converges to u weakly in $L^2([0, T], \mathbb{R}^m)$). Then,*

$$u \in \mathscr{U} \text{ and } \mathscr{G}(u) \leq \underline{\lim}_{n \to \infty} \mathscr{G}(u_n). \tag{7}$$

If $u_n \to u$ as $n \to \infty$, ($u_n \to u$ means that u_n converges to u in the norm of $L^2([0, T], \mathbb{R}^m)$), where $\{u_n\} \subset \mathscr{U}$, then

$$u \in \mathscr{U} \text{ and } \lim_{n \to \infty} \mathscr{G}(u_n) = \mathscr{G}(u). \tag{8}$$

Proof. Suppose that $u_n \rightharpoonup u$. Clearly, $u \in \mathscr{U}$, as $\mathscr{U}$ is a weakly closed set in $L^2([0, T], \mathbb{R}^m)$. By the convexity of $\mathscr{G}(u)$, we have

$$\mathscr{G}(u_n) \geq \mathscr{G}(u) + \langle D\mathscr{G}(u), u_n - u \rangle = \mathscr{G}(u) + 2\int_0^T (u_n(t) - u(t))^T R(t) u(t) dt. \tag{9}$$

Note that $\{u_n\} \subset \mathscr{U}$ and $R(\cdot)$ is continuous on $[0, T]$, we can show that

$$\int_0^T (u_n(t))^T R(t) u_n(t) dt$$

is bounded uniformly with respect to n. Thus, $\underline{\lim}_{n \to \infty} \mathscr{G}(u_n)$ exists. Since $R(\cdot)$ is continuous on $[0, T]$ and $u \in L^2([0, T], \mathbb{R}^m)$, it follows that $R(\cdot) u(\cdot) \in L^2\left([0, T], \mathbb{R}^{n \times m}\right)$. Thus,

$$\lim_{n \to \infty} \int_0^T (u_n(t))^T R(t) u(t) dt = \int_0^T (u(t))^T R(t) u(t) dt \tag{10}$$

as $u_n \rightharpoonup u$. Therefore, (7) holds.
Suppose that $u_n \to u$, *i.e.*,

$$\|u_n - u\|_{L^2} \to 0. \tag{11}$$

Clearly, $u \in \mathscr{U}$. Since $\{u_n\} \subset \mathscr{U}$ and $R(\cdot)$ is continuous on $[0, T]$, there exists a constant $\varkappa$ such that

$$\|R(\cdot)\, u_n(\cdot)\|_{L^2} \leq \varkappa \text{ for all } n = 1, 2, \ldots,$$

and

$$\|R(\cdot)\, u(\cdot)\|_{L^2} \leq \varkappa.$$

Thus,

$$\begin{aligned}
|\mathscr{G}(u_n) - \mathscr{G}(u)| &\leq \left| \int_0^T (u_n(t) - u(t))^T R(t)\, u_n(t)\, dt \right| + \\
&\quad \left| \int_0^T (u_n(t) - u(t))^T R(t)\, u(t)\, dt \right| \\
&\leq \varkappa \|u_n - u\|_{L^2} + \left| \int_0^T (u_n(t) - u(t))^T R(t)\, u(t)\, dt \right| \\
&\leq 2\varkappa \|u_n - u\|_{L^2}.
\end{aligned} \tag{12}$$

Since $u_n \to u$, it follows that $\lim\limits_{n \to \infty} \mathscr{G}(u_n) = \mathscr{G}(u)$. This completes the proof.

Define

$$\mathscr{F}(u, w) = (x(T|u, w))^T P x(T|u, w) + \int_0^T (x(t|u, w))^T Q(t)\, x(t|u, w)\, dt,$$

We have the following lemma.

Lemma 2. *Suppose that $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$ as $n \to \infty$, where $\{u_n\} \subset \mathscr{U}$ and $\{w_n\} \subset \mathscr{W}$. Then,*

$$\lim_{n \to \infty} \mathscr{F}(u_n, w_n) = \mathscr{F}(u, w), \tag{13}$$

where $u \in \mathscr{U}$ and $w \in \mathscr{W}$.

Proof. Since $\mathscr{U}$ and $\mathscr{W}$ are weakly closed, $u \in \mathscr{U}$ and $w \in \mathscr{W}$. By the continuity of $A(t)$, $F(t, \cdot)$ is continuous on $[0, t]$ for each $t \in [0, T]$ Note that

$$\begin{aligned}
&|x(t|u_n, w_n) - x(t|u, w)| \\
&= \left| \int_0^t F(t, \tau)\, B(\tau)\, (u_n(\tau) - u(\tau))\, d\tau + \int_0^t F(t, \tau)\, C(\tau)\, (w_n(\tau) - w(\tau))\, d\tau \right| \\
&\leq \left| \int_0^t F(t, \tau)\, B(\tau)\, (u_n(\tau) - u(\tau))\, d\tau \right| + \left| \int_0^t F(t, \tau)\, C(\tau)\, (w_n(\tau) - w(\tau))\, d\tau \right| \\
&= \left| \int_0^T \tilde{F}(t, \tau)\, B(\tau)\, (u_n(\tau) - u(\tau))\, d\tau \right| + \left| \int_0^T \tilde{F}(t, \tau)\, C(\tau)\, (w_n(\tau) - w(\tau))\, d\tau \right|,
\end{aligned}$$

where

$$\tilde{F}(t,\tau) = \begin{cases} F(t,\tau), & if \tau \le t, \\ 0_{n\times n} & \text{else.} \end{cases}$$

Clearly, $\tilde{F}(t,\tau)B(\tau)$ and $\tilde{F}(t,\tau)C(\tau)$ are continuous on $[0,T]$ except at the point $\tau = t$ and hence $\tilde{F}(t,\tau)B(\tau) \in L^2\left([0,T],\mathbb{R}^{n\times m}\right)$ and $\tilde{F}(t,\tau)C(\tau) \in L^2\left([0,T],\mathbb{R}^{n\times r}\right)$. Thus, for each $t \in [0,T]$, we have

$$\lim_{n\to\infty} x_n(t|u_n,w_n) = x(t|u,w). \tag{14}$$

On the other hand,

$$\begin{aligned}
|x(t|u_n,w_n)| &= \left|F(t,0)x_0 + \int_0^t F(t,\tau)B(\tau)u_n(\tau)d\tau + \right.\\
&\quad \left.\int_0^t F(t,\tau)C(\tau)w_n(\tau)d\tau\right| \\
&\le |F(t,0)x_0| + \left|\int_0^t F(t,\tau)B(\tau)u_n(\tau)d\tau\right| + \\
&\quad \left|\int_0^t F(t,\tau)C(\tau)w_n(\tau)d\tau\right| \\
&\le |F(t,0)x_0| + \left[\sum_{i=1}^m \left(\int_0^t \left((F(t,\tau)B(\tau))_i\right)^2 d\tau\right)\right]^{1/2} \\
&\quad \left[\sum_{i=1}^m \int_0^T \left(u_{n,i}(\tau)\right)^2 d\tau\right]^{1/2} \\
&\quad + \left[\sum_{i=1}^r \int_0^t \left((F(t,\tau)C(\tau))_i\right)^2 d\tau\right]^{1/2} \\
&\quad \left[\sum_{i=1}^r \int_0^T \left(w_{n,i}(\tau)\right)^2 d\tau\right]^{1/2},
\end{aligned}$$

where $(F(t,\tau)B(\tau))_i$ is the i-th element of $F(t,\tau)B(\tau)$. By the continuity of $\int_0^t \left((F(t,\tau)B(\tau))_i\right)^2 d\tau$, $\int_0^t \left((F(t,\tau)C(\tau))_i\right)^2 d\tau$ and $F(t,0)x_0$, there exists a ρ such that

$$\begin{aligned}
\rho = \max_{i=1,\ldots,m;\, j=1,\ldots,r;\, t\in[0,T]} &\left\{\int_0^t \left((F(t,\tau)B(\tau))_i\right)^2 d\tau,\right. \\
&\left.\int_0^t \left((F(t,\tau)C(\tau))_j\right)^2 d\tau, |F(t,0)x_0|\right\}.
\end{aligned}$$

It follows that

$$|x\,(t|u_n, w_n)| \le \rho + \rho^{1/2}\left(\|u_n\|_{L^2} + \|w_n\|_{L^2}\right) \le \rho + \rho^{1/2}\,(\zeta + \eta)\,,\ \forall t \in [0, T].$$

Since $Q\,(t)$ is continuous on $[0, T]$ and is positive definite for each $t \in [0, T]$, we have, for any $t \in [0, T]$,

$$0 \le (x\,(t|u_n, w_n))^T\, Q\,(t)\, x\,(t|u_n, w_n) \le \max_{i,j=1,\dots,n;t\in[0,T]} \left|Q_{i,j}\,(t)\right| \left(\rho + \rho^{1/2}\,(\zeta + \eta)\right)^2.$$

Therefore, by Lebesgue Dominated Convergence Theorem (Theorem 2.6.4 in [14]), it holds that

$$\begin{aligned} &\lim_{n\to\infty} \int_0^T (x\,(t|u_n, w_n))^T\, Q\,(t)\, x\,(t|u_n, w_n)\, dt \\ &= \int_0^T \lim_{n\to\infty} (x\,(t|u_n, w_n))^T\, Q\,(t)\, x\,(t|u_n, w_n)\, dt \\ &= \int_0^T (x\,(t|u, w))^T\, Q\,(t)\, x\,(t|u, w)\, dt. \end{aligned} \tag{15}$$

By virtue of (14) with $t = T$ and (15), we obtain

$$\lim_{n\to\infty} \mathscr{F}\,(u_n, w_n) = \mathscr{F}\,(u, w).$$

This completes the proof.

From Lemma 1 and Lemma 2, we have the following lemma.

Lemma 3. *If $u_n \rightharpoonup u$ and $w_n \rightharpoonup w$, where $\{u_n, w_n\} \subset \mathscr{U} \times \mathscr{W}$, then,*

$$(u, w) \in \mathscr{U} \times \mathscr{W} \text{ and } J\,(u, w) \le \underline{\lim}_{n\to\infty}\, J\,(u_n, w_n).$$

If $u_n \to u$ and $w_n \rightharpoonup w$, where $\{u_n, w_n\} \subset \mathscr{U} \times \mathscr{W}$, then,

$$(u, w) \in \mathscr{U} \times \mathscr{W} \text{ and } J\,(u, w) = \lim_{n\to\infty} J\,(u_n, w_n).$$

Now we have the following main theorem in this section.

Theorem 1. *Consider Problem (P). Then, there exists a $(u^*, w^*) \in \mathscr{U} \times \mathscr{W}$ such that*

$$J\left(u^*, w^*\right) = \min_{u\in\mathscr{U}} \max_{w\in\mathscr{W}} J\,(u, w). \tag{16}$$

Proof. Note that $L^2([0,T],\mathbb{R}^r)$ is reflexive and $\mathscr{U}$ is a compact and convex set. It follows that $\mathscr{U}$ is weakly sequentially compact. To prove (16), it suffices, by Proposition 38.12 in [19], to prove that

$$J(u) = \max_{w \in \mathscr{W}} J(u,w)$$

is weakly sequentially lower semi-continuous. That is to say, we only need to prove

$$J(u) \le \underline{\lim}_{n\to\infty} J(u_n) \text{ when } u_n \rightharpoonup u. \tag{17}$$

Suppose that $u_n \rightharpoonup u$. From Lemma 3, we know that

$$J(u,w) \le \underline{\lim}_{n\to\infty} J(u_n,w), \text{ for any } w \in \mathscr{W}.$$

Clearly,

$$\max_{w\in\mathscr{W}} J(u_n,w) \ge J(u_n,w).$$

It follows that

$$J(u,w) \le \underline{\lim}_{n\to\infty} J(u_n,w) \le \underline{\lim}_{n\to\infty} \max_{w\in\mathscr{W}} J(u_n,w), \text{ for any } w \in \mathscr{W}.$$

Thus,

$$J(u) = \max_{w\in\mathscr{W}} J(u,w) \le \underline{\lim}_{n\to\infty} \max_{w\in\mathscr{W}} J(u_n,w) = \underline{\lim}_{n\to\infty} J(u_n, w(u_n)) = \underline{\lim}_{n\to\infty} J(u_n).$$

This completes the proof.

4 Problem Approximation

Consider a monotonically non-decreasing sequence $\{S^p\}_{p=1}^{\infty}$ of finite subsets of $[0,T]$. For each p, let n_p+1 points of S^p be denoted by $t_0^p, t_1^p, \ldots, t_{n_p}^p$. These points are chosen such that

$$t_0^p = 0, t_{n_p}^p = T, \text{ and } t_{k-1}^p < t_k^p, k = 1,2,\ldots,n_p.$$

Thus, associated with each S^P there is the obvious partition $\mathscr{I}^p$ of $[0,T]$ defined by

$$\mathscr{I}^p = \{I_k^p : k = 1,\ldots,n_p\},$$

where $I_k^p = [t_{k-1}^p, t_k^p)$.

We choose S^P such that $\lim_{p\to\infty} S^p$ is dense in $[0, T]$, that is

$$\lim_{p\to\infty} \max_{k=1,\ldots,n_p} \left|I_k^p\right| = 0,$$

where $\left|I_k^p\right| = t_k^p - t_{k-1}^p$, the length of the *kth* interval.

Let

$$u^p(t) = \sum_{k=1}^{n_p} \sigma^{p,k} \chi_{I_k^p}(t), \tag{18}$$

$$w^p(t) = \sum_{k=1}^{n_p} \theta^{p,k} \chi_{I_k^p}(t), \tag{19}$$

and

$$\sigma^p = [(\sigma^{p,1})^T, \ldots, (\sigma^{p,n_p})^T]^T \text{ and } \theta^p = [(\theta^{p,1})^T, \ldots, (\theta^{p,n_p})^T]^T,$$

where

$$\sigma^{p,k} = \left[\sigma_1^{p,k}, \ldots, \sigma_m^{p,k}\right]^T, \text{ and } \theta^{p,k} = \left[\theta_1^{p,k}, \ldots, \theta_r^{p,k}\right]^T,$$

χ_I denotes the indicator function of I defined by

$$\chi_I(t) = \begin{cases} 1, & t \in I, \\ 0, & \text{elsewhere.} \end{cases}$$

Define

$$\Pi^p = \left\{\sigma^p \in \mathbb{R}^{mn_p} : \left(\sigma^p\right)^T U^P \sigma^p \le \eta^2\right\}, \tag{20}$$

$$\Xi^p = \left\{\theta^p \in \mathbb{R}^{rn_p} : \left(\theta^p\right)^T W^P \theta^p \le \zeta^2\right\}, \tag{21}$$

$$\mathscr{U}^p = \left\{u^p(t) = \sum_{k=1}^{n_p} \sigma^{p,k} \chi_{I_k^p}(t) : \sigma^p \in \Pi^p\right\},$$

and

$$\mathscr{W}^p = \left\{w^p(t) = \sum_{k=1}^{n_p} \theta^{p,k} \chi_{I_k^p}(t) : \theta^p \in \Xi^p\right\},$$

where

$$U^P = diag\left(\left|I_1^p\right| I_{m\times m}, \left|I_2^p\right| I_{m\times m}, \ldots, \left|I_{n_p}^p\right| I_{m\times m}\right),$$

and

$$W^P = diag\left(\left|I_1^p\right| I_{r\times r}, \left|I_2^p\right| I_{r\times r}, \ldots, \left|I_{n_p}^p\right| I_{r\times r}\right).$$

It is clear that $\mathscr{U}^p \subseteq \mathscr{U}$ and $\mathscr{W}^p \subseteq \mathscr{W}$. Furthermore, we have the following lemma.

Lemma 4. *For any $u \in \mathscr{U}$ and $w \in \mathscr{W}$, let*

$$u^p(t) = \sum_{j=1}^{n_p} \sigma^{p,j} \chi_{I_j^p}(t) \tag{22}$$

and

$$w^p(t) = \sum_{j=1}^{n_p} \theta^{p,j} \chi_{I_j^p}(t), \tag{23}$$

where

$$\sigma^{p,j} = \frac{1}{\left|I_j^p\right|} \int_{I_j^p} u(t)dt$$

and

$$\theta^{p,j} = \frac{1}{\left|I_j^p\right|} \int_{I_j^p} w(t)dt.$$

Then, $u^p \in \mathscr{U}^p$ and $w^p \in \mathscr{W}^p$. Furthermore,

$$u^p \to u \text{ and } w^p \to w. \tag{24}$$

Proof. Note that

$$\begin{aligned}\int_0^T \left(u^p(t)\right)^T u^p(t)dt &= \int_0^T \left(\sum_{j=1}^{n_p} \sigma^{p,j} \chi_{I_j^p}(t)\right)^T \left(\sum_{j=1}^{n_p} \sigma^{p,j} \chi_{I_j^p}(t)\right) dt \\ &= \sum_{j=1}^{n_p} \int_{I_j^p} \left(\sigma^{p,j}\right)^T \sigma^{p,j} dt = \sum_{j=1}^{n_p} \frac{1}{\left|I_j^p\right|} \int_{I_j^p} u^T(t)dt \int_{I_j^p} u(t)dt \\ &\leq \sum_{j=1}^{n_p} \frac{1}{\left|I_j^p\right|} \left|I_j^p\right| \int_{I_j^p} u^T(t)u(t)dt = \int_0^T u^T(t)u(t)dt. \end{aligned} \tag{25}$$

Thus, $u^p \in \mathscr{U}^p$. In a similar way, we can show that $w^p \in \mathscr{W}^p$. From Lemma 6.4.1 of [14], we have

$$u^p(t) \to u(t), \text{ for almost all } t \in [0, T],$$

and

$$w^p(t) \to w(t), \text{ for almost all} t \in [0, T].$$

Note that $\{u^p\} \times \{w^p\} \subset \mathscr{U} \times \mathscr{W}$ and $u \times w \in \mathscr{U} \times \mathscr{W}$. We have $\|u^p\|_{L^2}^2 \leq \eta^2$ and $\|w^p\|_{L^2}^2 \leq \zeta^2$ for all $p = 1, \ldots,$ while $\|u\|_{L^2}^2 \leq \eta^2$ and $\|w\|_{L^2}^2 \leq \zeta^2$. Since T is a finite number, the conclusion of the lemma follows readily.

With $u \in \mathscr{U}^p$ and $w \in \mathscr{W}^p$, the dynamical system (1) becomes

$$\begin{aligned} \dot{x}(t) &= A(t)\,x(t) + B(t)\sum_{k=1}^{n_p}\sigma^{p,k}\chi_{I_k^p}(t) + C(t)\sum_{k=1}^{n_p}\theta^{p,k}\chi_{I_k^p}(t), \\ x(0) &= x^0, \end{aligned} \tag{26}$$

and $J(u, w)$ becomes

$$\tilde{J}\left(\sigma^p, \theta^p\right) = (x(T))^T P x(T) + \int_0^T \Bigg\{(x(t))^T Q(t)\,x(t) + \left(\sum_{k=1}^{n_p}\sigma^{p,k}\chi_{I_k^p}(t)\right)^T R(t)\left(\sum_{k=1}^{n_p}\sigma^{p,k}\chi_{I_k^p}(t)\right)\Bigg\}\, dt.$$

Now we define the following minimax optimal parameter selection problem.
Problem $\left(P_p\right)$: For the given dynamical system (26), choose $(\sigma^{p,*}, \theta^{p,*}) \in \Pi^p \times \Xi^p$ such that

$$\tilde{J}\left(\sigma^{p,*}, \theta^{p,*}\right) = \min_{\sigma^p \in \Pi^p}\max_{\theta^p \in \Xi^p} \tilde{J}\left(\sigma^p, \theta^p\right).$$

Remark 1. Following a similar argument given for the proof of Theorem 1, we can show that for Problem $\left(P_p\right)$, there exists a $(\sigma^{p,*}, \theta^{p,*}) \in \Pi^p \times \Xi^p$ such that

$$J\left(\sigma^{p,*}, \theta^{p,*}\right) = \min_{\sigma^p \in \Pi^p}\max_{\theta^p \in \Xi^p} \tilde{J}\left(\sigma^p, \theta^p\right). \tag{27}$$

Theorem 2. *Suppose that* (u^*, w^*) *and* $(\sigma^{p,*}, \theta^{p,*})$ *are the optimal solutions of Problem* (P) *and Problem* $\left(P_p\right)$*, respectively. That is,*

$$J\left(u^*, w^*\right) = \min_{u \in \mathscr{U}}\max_{w \in \mathscr{W}} J(u, w) \text{ and } \tilde{J}\left(\sigma^{p,*}, \theta^{p,*}\right) = \min_{\sigma^p \in \Pi^p}\max_{\theta^p \in \Xi^p} \tilde{J}\left(\sigma^p, \theta^p\right).$$

Then,

$$\lim_{p \to \infty} \tilde{J}\left(\sigma^{p,*}, \theta^{p,*}\right) = J\left(u^*, w^*\right). \tag{28}$$

Proof. Suppose that (28) is not true. Then, there exists an $\varepsilon_0 > 0$ and a sub-sequence $\{\sigma^{p_k,*}, \theta^{p_k,*}\}$ such that

$$\left|\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k,*}\right)-J\left(u^*,w^*\right)\right|\geq\varepsilon_0. \tag{29}$$

Let $\tilde{u}^{p_k,*}(t)$ and $\tilde{w}^{p_k,*}(t)$ be the piecewise constant functions corresponding to $\sigma^{p_k,*}$ and $\theta^{p_k,*}$ given by (18) and (19), respectively. Clearly, $(\tilde{u}^{p_k,*},\tilde{w}^{p_k,*})\in\mathscr{U}\times\mathscr{W}$ for any k. Since $\mathscr{U}\times\mathscr{W}$ is weakly sequentially compact in $L^2\left([0,T],\mathbb{R}^m\right)\times L^2\left([0,T],\mathbb{R}^r\right)$, it follows that $\{(\tilde{u}^{p_k,*},\tilde{w}^{p_k,*})\}_{k=1}^{\infty}$ contains a subsequence, which is denoted by the same sequence, and a $(\tilde{u}^*,\tilde{w}^*)\in\mathscr{U}\times\mathscr{W}$ such that

$$\left(\tilde{u}^{p_k,*},\tilde{w}^{p_k,*}\right)\rightharpoonup\left(\tilde{u}^*,\tilde{w}^*\right). \tag{30}$$

Let u^{*,p_k} and w^{*,p_k} be constructed from u^* and w^* according to (22) and (23), respectively, as follows:

$$u^{*,p_k}(t)=\sum_{j=1}^{n_{p_k}}\sigma^{*,p_k,j}\chi_{I_j^{p_k}}(t),$$

and

$$w^{*,p_k}(t)=\sum_{j=1}^{n_{p_k}}\theta^{*,p_k,j}\chi_{I_j^{p_k}}(t),$$

where

$$\sigma^{*,p_k,j}=\frac{1}{\left|I_j^{p_k}\right|}\int_{I_j^{p_k}}u^*(t)dt$$

and

$$\theta^{*,p_k,j}=\frac{1}{\left|I_j^{p_k}\right|}\int_{I_j^{p_k}}w^*(t)dt.$$

From Lemma 4, we have

$$u^{*,p_k}\to u^*\text{ and }w^{*,p_k}\to w^*.$$

Note that $u^{*,p_k}\to u^*$ and $\tilde{w}^{p_k,*}\rightharpoonup\tilde{w}^*$, it follows from Lemma 3 that

$$J\left(u^{*,p_k},\tilde{w}^{p_k,*}\right)\to J\left(u^*,\tilde{w}^*\right)\text{ as }k\to\infty.$$

Thus, for any $\varepsilon>0$, there exists a constant $K\in\mathbb{N}$, such that

$$J\left(u^{*,p_k},\tilde{w}^{p_k,*}\right)\leq J\left(u^*,\tilde{w}^*\right)+\varepsilon,\forall k>K.$$

Hence, we have

$$\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k,*}\right)=\min_{\sigma^{p_k}\in\Pi^{p_k}}\tilde{J}\left(\sigma^{p_k},\theta^{p_k,*}\right)\leq\tilde{J}\left(\sigma^{*,p_k},\theta^{p_k,*}\right)=J\left(u^{*,p_k},\tilde{w}^{p_k,*}\right)$$
$$\leq J\left(u^*,\tilde{w}^*\right)+\varepsilon\leq\max_{w\in\mathscr{W}}J\left(u^*,w\right)+\varepsilon=J\left(u^*,w^*\right)+\varepsilon.$$

Letting $k\to\infty$, we obtain

$$\overline{\lim_{k\to\infty}}\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k,*}\right)\leq J\left(u^*,w^*\right)+\varepsilon.$$

Since ε is arbitrary, it holds that

$$\overline{\lim_{k\to\infty}}\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k,*}\right)\leq J\left(u^*,w^*\right).\tag{31}$$

On the other hand, we note that $\tilde{u}^{p_k,*}(t)\rightharpoonup\tilde{u}^*$ and $w^{*,p_k}\to w^*$. Then, from Lemma 3, we have

$$J\left(u^*,w^*\right)=\min_{u\in\mathscr{U}}J\left(u,w^*\right)\leq J\left(\tilde{u}^*,w^*\right)\leq\underline{\lim_{k\to\infty}}\tilde{J}\left(\sigma^{p_k,*},\theta^{*,p_k}\right)\tag{32}$$
$$\leq\underline{\lim_{k\to\infty}}\max_{\theta^{p_k}\in\Xi^{p_k}}\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k}\right)=\underline{\lim_{k\to\infty}}\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k,*}\right).$$

Combining (31) and (32), it holds that

$$\lim_{k\to\infty}\tilde{J}\left(\sigma^{p_k,*},\theta^{p_k,*}\right)=J\left(u^*,w^*\right).\tag{33}$$

(33) is a contradictory to (29). Thus, (28) is true. This completes the proof.

Next, we have the following theorem.

Theorem 3. *Suppose that* $(\sigma^{p,*},\theta^{p,*})$ *is the optimal solution of Problem* $\left(P_p\right)$. *Let* $u^{p,*}(t)$ *and* $w^{p,*}(t)$ *be the piecewise constant functions corresponding to* $\sigma^{p,*}$ *and* $\theta^{p,*}$ *given by (18) and (19), respectively. If there exists a subsequence* $\{u^{p_k,*},w^{p_k,*}\}$ *such that* $u^{p_k,*}\to\bar{u}$ *and* $w^{p_k,*}\to\bar{w}$, *as* $k\to\infty$. *Then,* $(\bar{u},\bar{w})$ *is also an optimal solution of Problem* (P).

Proof. Since $u^{p_k,*}\to\bar{u}$ and $w^{p_k,*}\to\bar{w}$, as $k\to\infty$, we have

$$\lim_{k\to\infty}J\left(u^{p_k,*},w^{p_k,*}\right)=J\left(\bar{u},\bar{w}\right).\tag{34}$$

For any $u\in\mathscr{U}$, let u^{p_k} be constructed from u according to (22). Then, by Lemma 4, $u^{p_k}\in\mathscr{U}$ and $u^{p_k}\to u$ as $k\to\infty$. Now, as $w^{p_k,*}\to\bar{w}$, we have

$$\lim_{k\to\infty}J\left(u^{p_k},w^{p_k,*}\right)=J\left(u,\bar{w}\right).\tag{35}$$

From (34) and (35), it follows that for any $\varepsilon>0$, there exists a $K\in\mathbb{N}$ such that for any $k>K$, we have

$$J\left(u^{p_k,*}, w^{p_k,*}\right) - \varepsilon \leq J\left(\bar{u}, \bar{w}\right) \leq J\left(u^{p_k,*}, w^{p_k,*}\right) + \varepsilon.$$

Thus,

$$J\left(\bar{u}, \bar{w}\right) \leq J\left(u^{p_k,*}, w^{p_k,*}\right) + \varepsilon \leq J\left(u^{p_k}, w^{p_k,*}\right) + \varepsilon.$$

Letting $k \to \infty$, we have

$$J\left(\bar{u}, \bar{w}\right) \leq J\left(u, \bar{w}\right) + \varepsilon.$$

Since ε is arbitrary, it holds that

$$J\left(\bar{u}, \bar{w}\right) \leq J\left(u, \bar{w}\right), \forall u \in \mathscr{U}. \tag{36}$$

In a similar way, we can prove that

$$J\left(\bar{u}, w\right) \leq J\left(\bar{u}, \bar{w}\right), \ \forall w \in \mathscr{W}. \tag{37}$$

Combining (36) and (37), we conclude that $(\bar{u}, \bar{w})$ is an optimal solution of Problem (P).

Remark 2. By a close examination of Theorems 2, we see that it suggests a method for solving Problem (P). First, choose an integer $p \geq 2$, select a partition of the interval $[0, T]$ and solve Problem $\left(P_p\right)$. Then, increasing p and using the solution obtained from the previous step as the initial guess, solve Problem $\left(P_p\right)$ again. This process is repeated until the change of the optimal value of the cost function is within a desired tolerance.

5 Sub-problem Solution

From Remark 2, we know that Problem (P) can be solved by solving a sequence of Problem $\left(P_p\right)$. However, Problem $\left(P_p\right)$ is still a minimax optimal parameter selection problem, and hence, it is hard to solve. In this section, we will develop a computational method to solve Problem $\left(P_p\right)$.

For each given σ^p and θ^p, the solution of (26) can be rewritten as:

$$x(t) = F(t, 0)\, x^0 + \tilde{F}_1(t)\, \sigma^p + \tilde{F}_2(t)\, \theta^p,$$

where $\tilde{F}_1 \in \mathbb{R}^{n \times m n_p}$ and $\tilde{F}_2 \in \mathbb{R}^{n \times r n_p}$ are, respectively, defined by

$$\tilde{F}_1(t) = \left[\int_0^t F(t, \tau)\, B(\tau)\, \chi_{I_1^p}(\tau) d\tau, \int_0^t F(t, \tau)\, B(\tau)\, \chi_{I_2^p}(\tau) d\tau, \ldots, \right.$$
$$\left. \int_0^t F(t, \tau)\, B(\tau)\, \chi_{I_{n_p}^p}(\tau) d\tau \right]$$

and

$$\tilde{F}_2(t) = \left[\int_0^t F(t,\tau)\, C(\tau)\, \chi_{I_1^p}(\tau)d\tau, \int_0^t F(t,\tau)\, C(\tau)\, \chi_{I_2^p}(\tau)d\tau, \ldots, \int_0^t F(t,\tau)\, C(\tau)\, \chi_{I_{n_p}^p}(\tau)d\tau\right].$$

Thus, $\tilde{J}(\sigma^p, \theta^p)$ can be rewritten as

$$\tilde{J}\left(\sigma^p, \theta^p\right) = \left(\sigma^p\right)^T G_1 \sigma^p + \left(\theta^p\right)^T G_2 \theta^p + 2h_1^T \sigma^p + 2h_2^T \theta^p + 2\left(\sigma^p\right)^T G_3 \theta^p + c_0, \tag{38}$$

where

$$G_1 = \left(\tilde{F}_1(T)\right)^T P \tilde{F}_1(T) + \int_0^T \left(\tilde{F}_1(t)\right)^T Q(t)\, \tilde{F}_1(t)\, dt + \int_0^T \left(\bar{F}(t)\right)^T R(t)\, \bar{F}(t)\, dt,$$

$$\bar{F}(t) = \left[\chi_{I_1^p}(t) I_{r\times r}, \chi_{I_2^p}(t) I_{r\times r}, \ldots, \chi_{I_{n_p}^p}(\tau) I_{r\times r}\right],$$

$$G_2 = \left(\tilde{F}_2(T)\right)^T P \tilde{F}_2(T) + \int_0^T \left(\tilde{F}_2(t)\right)^T Q(t)\, \tilde{F}_2(t)\, dt,$$

$$G_3 = \left(\tilde{F}_1(T)\right)^T P \tilde{F}_2(T) + \int_0^T \left(\tilde{F}_1(t)\right)^T Q(t)\, \tilde{F}_2(t)\, dt,$$

$$h_1 = \left(F(T,0)\, x^0\right)^T P \tilde{F}_1(T) + \int_0^T \left(F(t,0)\, x^0\right)^T Q(t)\, \tilde{F}_1(t)\, dt,$$

$$h_2 = \left(F(T,0)\, x^0\right)^T P \tilde{F}_2(T) + \int_0^T \left(F(t,0)\, x^0\right)^T Q(t)\, \tilde{F}_2(t)\, dt,$$

and

$$c_0 = \left(F(T,0)\, x^0\right)^T P F(T,0)\, x^0 + \int_0^T \left(F(t,0)\, x^0\right)^T Q(t)\, F(t,0)\, x^0 dt.$$

It is easy to verify that G_1 and G_2 are positive definite. Let $\tilde{\sigma}^p = G_1^{1/2}\sigma^p + G_1^{-1/2} h_1$. Then, Problem $\left(P_p\right)$ becomes Problem $\left(\bar{P}_p\right)$ with a difference of a constant in the cost, which is defined as follows:
Problem$\left(\bar{P}_p\right)$: choose $(\tilde{\sigma}^{p,*}, \theta^{p,*}) \in \tilde{\Pi}^p \times \varXi^p$ such that

$$\bar{J}\left(\tilde{\sigma}^{p,*}, \theta^{p,*}\right) = \min_{\tilde{\sigma}^p \in \tilde{\Pi}^p} \max_{\theta^p \in \varXi^p} \bar{J}\left(\tilde{\sigma}^p, \theta^p\right) = \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p + \left(\theta^p\right)^T G_2 \theta^p + 2\tilde{h}_2^T \theta^p + 2\left(\tilde{\sigma}^p\right)^T \tilde{G}_3 \theta^p, \tag{39}$$

where $\tilde{G}_3 = G_1^{-1/2} G_3$, $\tilde{h}_2 = h_2 - G_3^T G_1^{-1} h_1$ and

$$\tilde{\Pi}^p = \left\{ \tilde{\sigma}^p \in \mathbb{R}^{mn_p} : \left(G_1^{-1/2} \tilde{\sigma}^p - h_1 \right)^T U^P \left(G_1^{-1/2} \tilde{\sigma}^p - h_1 \right) \leq \eta^2 \right\}.$$

Now we have the following theorem.

Theorem 4. *Problem* $\left(\bar{P}_p\right)$ *is equivalent to the following SDP:*

$$\min_{\tilde{\sigma}^p, \lambda, z} z$$

subject to

$$\begin{bmatrix} I & \tilde{\sigma}^p & \tilde{G}_3 \\ (\tilde{\sigma}^p)^T & z - \zeta^2 \lambda & -\tilde{h}_2^T \\ \tilde{G}_3^T & -\tilde{h}_2 & \lambda W^P - G_2 + \tilde{G}_3^T \tilde{G}_3 \end{bmatrix} \succeq 0, \tag{40}$$

and

$$\begin{bmatrix} I & \left(U^P\right)^{1/2} G_1^{-1/2} \tilde{\sigma}^p - \left(U^P\right)^{1/2} h_1 \\ \left(\left(U^P\right)^{1/2} G_1^{-1/2} \tilde{\sigma}^p - \left(U^P\right)^{1/2} h_1 \right)^T & \eta^2 \end{bmatrix} \succeq 0. \tag{41}$$

Proof. Problem $\left(\bar{P}_p\right)$ can be re-written as:

$$\min z$$

subject to

$$z - \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p - \left(\theta^p\right)^T G_2 \theta^p - 2\tilde{h}_2^T \theta^p - 2\left(\tilde{\sigma}^p\right)^T \tilde{G}_3 \theta^p \geq 0, \ \forall \theta^p : \left(\theta^p\right)^T W^P \theta^p \leq \zeta^2, \tag{42}$$

$$\left(G_1^{-1/2} \tilde{\sigma}^p - h_1 \right)^T U^P \left(G_1^{-1/2} \tilde{\sigma}^p - h_1 \right) \leq \eta^2. \tag{43}$$

Clearly, (43) is equivalent to (41). For the proof of the equivalence between (42) and (40), it is referred to [3].

With the increase of the partition number n_p, the size of (40) becomes very large. It is well-known that solving a large SDP can be computationally expensive. For an alternative approach, we show that Problem $\left(\bar{P}_p\right)$ is equivalent to an optimiztion problem solvable by gradient-based optimization methods. We have the following theorem.

Theorem 5. *Problem* $\left(\bar{P}_p\right)$ *is equivalent to the following optimization problem, which is referred to as Problem* (GP).

$$\min_{\lambda,\tilde{\sigma}^p} \hat{J}\left(\lambda, \tilde{\sigma}^p\right) = \zeta^2\lambda + \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p + \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right)^T \left(\lambda W^P - G_2\right)^\dagger \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right), \tag{44}$$

subject to the constraints

$$\left(G_1^{-1/2}\tilde{\sigma}^p - h_1\right)^T U^P \left(G_1^{-1/2}\tilde{\sigma}^p - h_1\right) \leq \eta^2, \tag{45}$$

$$\lambda \geq \lambda_0 = \max_{k=1,\ldots,n_p-1,\, j=1,\ldots,m} \frac{\gamma_{km+j}}{\left|I_k^p\right|}, \tag{46}$$

and

$$\left(I - \left(\lambda W^P - G_2\right)\left(\lambda W^P - G_2\right)^\dagger\right)\left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right) = 0, \tag{47}$$

where $\left(\lambda W^P - G_2\right)^\dagger$ *is the pseudo-inverse of* $\lambda W^P - G_2$, γ_i, $i = 1, \ldots, n_p$ *are the eigenvalues of the matrix* G_2.

To prove this theorem, we need the following lemma.

Lemma 5. *(Theorem 4.3 in [6]) Consider a symmetric matrix* $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$. *Then,* $M \succeq 0$ *if and only if*

$$A \succeq 0, \left(I - AA^\dagger\right) B = 0,\ C - B^T A^\dagger B \succeq 0.$$

Proof of Theorem 5: From [3], we know that the (42) can be replaced by the following homogeneous inequality.

$$\begin{aligned} &t^2\left(z - \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p\right) - \left(\theta^p\right)^T G_2\theta^p - 2t\tilde{h}_2^T\theta^p - 2t\left(\tilde{\sigma}^p\right)^T \tilde{G}_3\theta^p \geq 0, \\ &\quad \forall\theta^p : \left(\theta^p\right)^T W^P\theta^p \leq \zeta^2 t^2. \end{aligned} \tag{48}$$

(48) can be re-written as

$$\begin{aligned} &\left[\left(\tilde{\sigma}^p\right)^T, t\right]\begin{bmatrix} -G_2 & -\tilde{h}_2 - \left(\tilde{\sigma}^p\right)^T \tilde{G}_3 \\ -\tilde{h}_2^T - \tilde{G}_3^T\tilde{\sigma}^p & z - \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p \end{bmatrix}\begin{bmatrix} \tilde{\sigma}^p \\ t \end{bmatrix} \geq 0, \forall\theta^p : \left[\left(\tilde{\sigma}^p\right)^T, t\right] \\ &\qquad\qquad\qquad\qquad \begin{bmatrix} -W^P & \\ & \zeta^2 \end{bmatrix}\begin{bmatrix} \tilde{\sigma}^p \\ t \end{bmatrix} \geq 0. \end{aligned} \tag{49}$$

Thus, (42) is satisfied if and only if there exists a $\lambda \geq 0$ such that

$$\begin{bmatrix} \lambda W^P - G_2 & -\tilde{h}_2 - \left(\tilde{\sigma}^p\right)^T \tilde{G}_3 \\ -\tilde{h}_2^T - \tilde{G}_3^T\tilde{\sigma}^p & z - \zeta^2\lambda - \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p \end{bmatrix} \succeq 0. \tag{50}$$

According to Lemma 5, (50) is satisfied if and only if

$$\lambda W^P - G_2 \succeq 0, \tag{51}$$

$$\left(I - \left(\lambda W^P - G_2\right)\left(\lambda W^P - G_2\right)^{\dagger}\right)\left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right) = 0, \tag{52}$$

and

$$z - \zeta^2\lambda - \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p - \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right)^T \left(\lambda W^P - G_2\right)^{\dagger} \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right) \geq 0. \tag{53}$$

The condition (51) leads to (46). (53) is equivalent to

$$\zeta^2\lambda + \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p + \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right)^T \left(\lambda W^P - G_2\right)^{\dagger} \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right) \leq z. \tag{54}$$

Combining (43), (51), (52) and (54), we obtain that Problem $\left(\bar{P}_p\right)$ is equivalent to Problem (GP). This completes the proof.

Remark 3. The constraint (47) in Problem (GP) can be removed if $\lambda > \lambda_0$. This is because

$$I - \left(\lambda W^P - G_2\right)\left(\lambda W^P - G_2\right)^{-1} = 0.$$

Furthermore, we claim that there exists a $\bar{\lambda}$ such that the constraint $\lambda \geq \lambda_0$ in Problem (GP) can be replaced by $\lambda_0 \leq \lambda \leq \bar{\lambda}$. The reason is as follows.

$$\begin{aligned}\lim_{\lambda\to\infty} \hat{J}\left(\lambda, \tilde{\sigma}^p\right) &= \lim_{\lambda\to\infty} \zeta^2\lambda + \left(\tilde{\sigma}^p\right)^T \tilde{\sigma}^p + \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right)^T \left(\lambda W^P - G_2\right)^{\dagger} \\ &\qquad \left(\tilde{h}_2 + \left(\tilde{\sigma}^p\right)^T \tilde{G}_3\right) \\ &\geq \lim_{\lambda\to\infty} \zeta^2\lambda = \infty \text{ for any } \tilde{\sigma}^p \in \tilde{\Pi}^p.\end{aligned}$$

Now we can divide the interval $\left[\lambda_0, \bar{\lambda}\right]$ as two parts $[\lambda_0, \lambda_0+\epsilon]$ and $\left[\lambda_0+\epsilon, \bar{\lambda}\right]$ during the process of solving Problem (GP), where ϵ is a small constant. Thus, Problem (GP) can be solved by solving two sub-optimization problems, *i.e.*,

$$\min_{\lambda,\tilde{\sigma}^p} \hat{J}\left(\lambda, \tilde{\sigma}^p\right) = \min\left\{\min_{\lambda_0\leq\lambda\leq\lambda_0+\epsilon} \min_{\tilde{\sigma}^p} \hat{J}\left(\lambda, \tilde{\sigma}^p\right), \min_{\lambda_0+\epsilon\leq\lambda\leq\bar{\lambda}} \min_{\tilde{\sigma}^p} \hat{J}\left(\lambda, \tilde{\sigma}^p\right)\right\}.$$

For each fixed $\lambda \in [\lambda_0, \lambda_0+\epsilon]$, $\tilde{\sigma}^p$ is obtained by only solving a convex optimization with quadratic cost function and a quadratic constraint. During the minimization process of $\min_{\lambda_0+\epsilon\leq\lambda\leq\bar{\lambda}} \min_{\tilde{\sigma}^p} \hat{J}\left(\lambda, \tilde{\sigma}^p\right)$, $\left(\lambda W^P - G_2\right)^{\dagger}$ in (44) can be replaced by $\left(\lambda W^P - G_2\right)^{-1}$. In this case, the gradient $\frac{\partial \hat{J}(\lambda,\tilde{\sigma}^p)}{\partial\lambda}$ is easily obtained. By direct verification, we know that $\hat{J}(\lambda, \tilde{\sigma}^p)$ is convex with respect to λ and $\tilde{\sigma}^p$, respectively. In view of this property, a bi-iterative method can be applied. First λ is fixed and $\hat{J}(\lambda, \tilde{\sigma}^p)$ is minimized with respect to $\tilde{\sigma}^p$. Then, $\tilde{\sigma}^p$ is fixed as the one obtained in the previous step, and $\hat{J}(\lambda, \tilde{\sigma}^p)$ is minimized with respect to λ. This process is repeated until the change of the cost is within the given tolerance. The remaining question is how to solve the minimization problem $\min_{\lambda_0 \le \lambda \le \lambda_0 + \epsilon} \min_{\tilde{\sigma}^p} \hat{J}(\lambda, \tilde{\sigma}^p)$. If $\lambda \to \lambda_0$, then the matrix $\lambda W^P - G_2$ becomes singular. Thus, it is important to develop an efficient computation method for this case. It is a future research problem. Nevertheless, Theorem 5 does offer a possible way to solve Problem $\left(P_p\right)$.

6 Numerical Experiment

In the following computation, the computer routines are implemented in a Matlab environment and the software packages SeDuMi [13] and YALMIP [8] are used.

To illustrate our developed method, let us consider the following example, where the dynamical system is given by

$$\begin{aligned} \dot{x}_1(t) &= 2x_1 + x_2 + u_1 + w_1, \\ \dot{x}_2(t) &= x_2 + u_2 + w_2, \end{aligned} \tag{55}$$

with the given initial condition

$$x_1(0) = 1, x_2(0) = -1.$$

$$\mathscr{W}_\zeta = \left\{ w \in L^2\left([0,1], \mathbb{R}^2\right) : \|w\|_{L^2}^2 = \int_0^1 (w(t))^T w(t)\, dt \le \zeta^2 \right\},$$

and

$$\mathscr{U}_\eta = \left\{ u \in L^2\left([0,1], \mathbb{R}^2\right) : \|u\|_{L^2}^2 = \int_0^1 (u(t))^T u(t)\, dt \le \eta^2 \right\}.$$

Since

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},$$

it is clear that

$$F(t,\tau) = \begin{bmatrix} e^{2(t-\tau)} & (t-\tau)\, e^{2(t-\tau)} \\ 0 & e^{2(t-\tau)} \end{bmatrix}.$$

Now we consider the following optimization problem

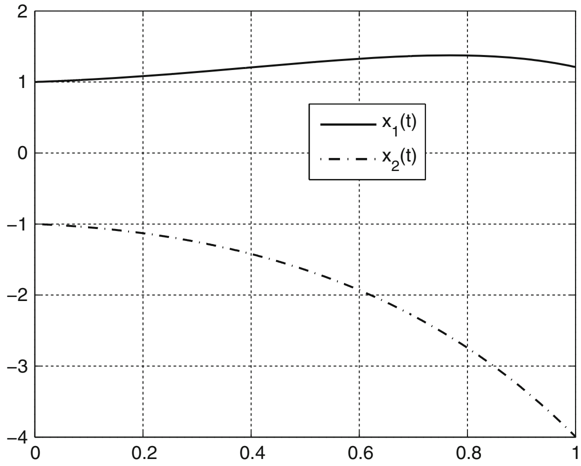

Fig. 1 The optimal state without the disturbance

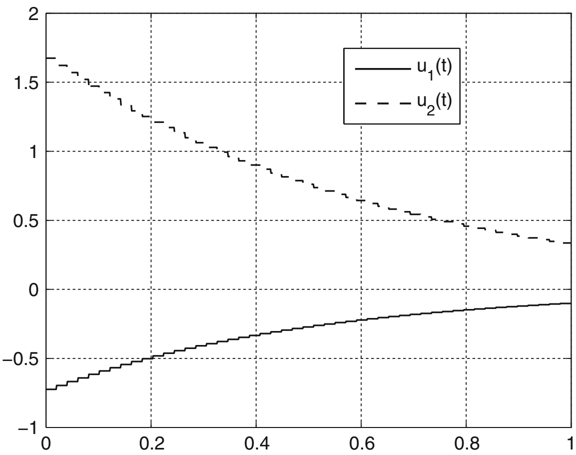

Fig. 2 The optimal control without the disturbance

$$\min_{u \in \mathscr{U}_\eta} \max_{w \in \mathscr{W}_\zeta} J(u, w) = (x(1))^T x(1) + 0.05 \int_0^1 (u(t))^T u(t)\, dt.$$

We first consider the case when there is no disturbances, *i.e.*, $w_1 = w_2 = 0$ in (55). For this case, we use MISER [7] with 50 equally spaced knots to solve it. The state and the optimal control obtained are depicted in Figs. 1, 2. The corresponding optimal cost is 17.4711.

For the case with disturbances, we let $\zeta = 0.1$ and $\eta = 1$. During the computational process, all the partitions of [0, 1] are equally spaced. Then, the computational results of Problem $\left(\bar{P}_p\right)$ with different n_p are given in Table 1. From Table 1, we can

Table 1 The computational results of problem $(\bar{P}_p)$ with different n_p

	z	λ
$n_p = 25$	0.508171679707355	29.210423777966025
$n_p = 50$	0.288402123372642	28.78547097896896
$n_p = 100$	0.288402115236866	28.789910948034414

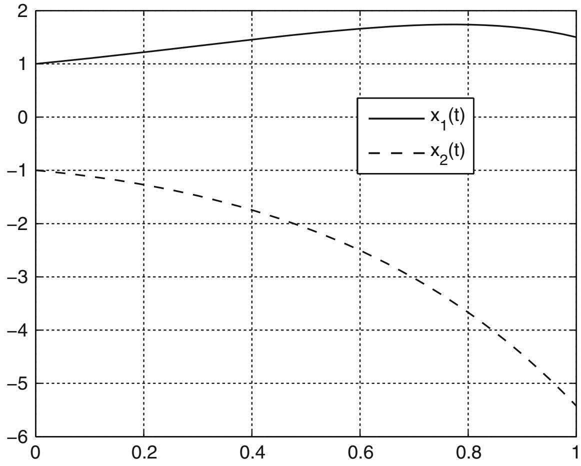

Fig. 3 The optimal state $x(t)$ with disturbances

see that when the number of sub-intervals is doubled from 50 to 100, the change of z is smaller than 10^{-6}. Thus, we take $n_p = 50$ as the optimal solution. The corresponding optimal states without disturbances is plotted in Fig. 3. The corresponding piecewise constant optimal control elements are depicted in Figs. 4, 5. The obtained optimal cost in this case is 33.3236. Clearly, there is a significant increase of the optimal cost value when disturbances are presented.

Now let us examine how the optimal cost is changed with reference to different ζ and η. Taking $n_p = 100$ with the partition points being equally spaced, the results of Problem $(\bar{P}_p)$ corresponding to different ζ and η are presented in Table 2. From Table 2, we can see that for the same ζ, the larger the η, the smaller the z is obtained. On the other hand, for the same η, the larger the ζ, the larger the z is obtained. From our computation, we know that a large z always gives rise to a large optimal cost of Problem (P). Thus, a large ζ will lead to a large cost function value of Problem (P) and a large η will lead to a small cost function value of Problem (P). This is expected, because if the feasible set is enlarged, then the optimal cost is reduced.

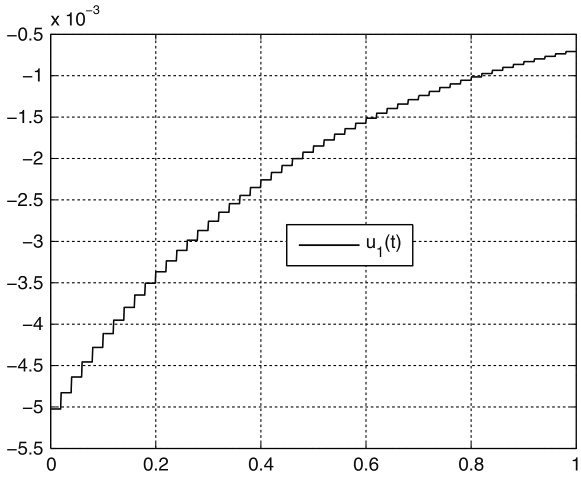

Fig. 4 The optimal control $u_1(t)$ of Problem (P) with disturbance

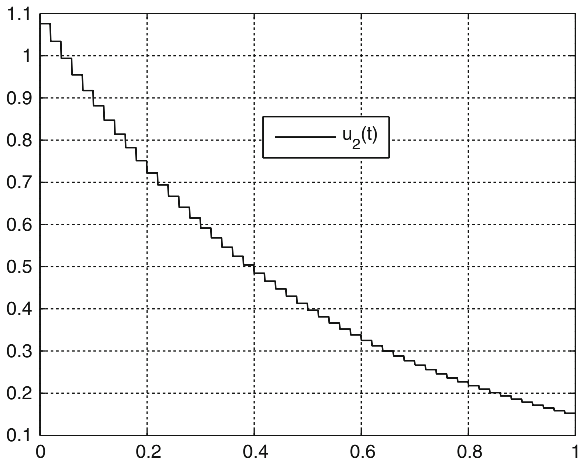

Fig. 5 The optimal control $u_2(t)$ of Problem (P) with disturbance

Table 2 The computational results of problem $(\bar{P}_p)$ with different ζ and η

ζ	η	z
0.1	1	0.288402115236866
0.2	1	1.151966252722664
0.5	1	7.196915158138601
0.1	5	0.184774427787928
0.2	5	0.737455469828136
0.5	5	5.829605274519559

7 Concluding Remarks and Future Research

In this chapter, we have shown that an infinite-dimensional minimax optimal control problem can be approximated by a sequence of finite-dimensional minimax optimal parameter selection problems. Furthermore, these finite dimensional minimax optimal parameter selection problems can be transformed into either SDPs or standard minimization problems. For SDP, it is easily solved by available software packages. However, an efficient method for Problem (GP) is still not available since the matrix $\lambda W^P - G_2$ becomes singular at the point $\lambda = \lambda_0$. It remains an open question.

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