

Chapter 2

One-Variable Equations and Inequalities: The Unity of Computational Experiment and Formal Demonstration

2.1 Introduction

This chapter concerns Type II applications of computing technology to one-variable equations and inequalities. It shows how in the context of TEMP there is a need not only to justify the results of such applications by using theory but also to use theory in order to fill in missing parts left from experimentation. Consequently, the chapter argues for the importance of mathematical knowledge the teachers of mathematics need in order to provide students with Style II assistance in the context of TEMP. The chapter demonstrates how TEMP may include a transition from problem solving to problem posing so that the ACA framework can support Type II technology applications to mathematics teaching. Finally, the chapter shows how the use of technology in the context of experimentation with mathematical concepts can develop entries into the history of mathematics through which connections between the classic concepts and their representations through the modern tools can be established.

2.2 Solving Equations as a Combination of Experiment and Theory

In this section, a combination of computational experiment and its mathematical demonstration will be considered in the context of solving algebraic equations in one variable. The use of graphs as basic tools of the experiment will be considered from different points of view. Sometimes, an equation may have a simple form so that its single solution can be easily guessed and checked. In that case, graphing not only allows one to see the uniqueness of the solution but it suggests how to turn mathematical visualization into formal proof; in other words, to use visual imagery in support of deductive reasoning. Sometimes, in addition to showing the absence of solutions, graphing prompts a direction in which an analytical proof of this visual demonstration can be developed. Such an approach has ancient roots as

evidenced by the following historically remarkable statement made by Archimedes¹ (1912) in a letter to Eratosthenes², “Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method has previously given us some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge” (p. 13). Nowadays, in the United States, this statement has been echoed in an expectation by Common Core State Standards (2010) for “mathematically proficient students... [who] are able to use technological tools to explore and deepen their understanding of concepts” (p. 7). It also rings in the most recent recommendations by Conference Board of the Mathematical Sciences (2012) for secondary school mathematical teacher preparation in the U.S.: “Reasoning from known results and definitions supports retention of knowledge in a mathematical domain by giving it structure and connecting new knowledge to prior knowledge” (p. 56). In Australia (National Curriculum Board 2008), the role of digital technologies is considered important for these tools allow students to “readily explore various aspects of the behavior of a function or relation numerically, graphically, geometrically and algebraically... can make previously inaccessible mathematics accessible, and enhance the potential for teachers to make mathematics interesting to more students, including the use of realistic data and examples” (p. 9). Likewise, one of the teaching principles included in another secondary mathematics curriculum materials (Ministry of Education, Singapore 2006) states, “The opportunity to deal with empirical data and use mathematical tools for data analysis should be part of the learning at all levels” (p. 4).

Following are eight examples of using the classic Archimedes’ approach and modern educational ideas in the context of solving algebraic equations. Most of the problems discussed, however, are not technology-immune as far as equation solving is understood as the root finding alone. Yet, they were chosen to introduce basic skills and strategies needed to deal with problems discussed in other chapters of the book. Furthermore, considering the end result of solving an equation to be more than just root finding, TEMP encourages learners to understand the meaning of mathematical techniques and problem-solving strategies. Finally, some examples open a window to ideas related to problems with parameters and, in addition, demonstrate an educational value of parameterization as a method of posing new problems and understanding why analytic methods used in solving equations without parameters work. As Common Core State Standards for Mathematics (Common Core State Standards 2010), when arguing for the inclusion in the U. S. secondary school curriculum rich mathematical tasks, put it, “mathematical understanding is the ability to justify... *why* a particular mathematical statement is true or where a mathematical rule comes from” (p. 4, italics in the original). In this regard, it

¹ Archimedes—a Greek mathematician of the 3rd century B.C.—is, by some accounts (e.g., Rohlin 2013), the greatest mathematician of all times.

² Eratosthenes—a Greek scholar of the third century B.C.

is worth noting Poincaré's³ mathematical education remark that students “wish to know not merely whether all the syllogisms of a demonstration are correct, but why they link together in this order rather than another” (cited in Hadamard⁴ 1996, p. 104). At the same time, “if students... have not probed the role of various conditions in making a theorem or technique work, then they are unlikely to use it appropriately” (Mason and Watson 2001, p. 11). With this in mind, the examples of this section, by drawing on the notion of TEMP, include the discussion of each demonstration to enable collateral learning opportunities.

Example 2.2.1: Exponential functions: from experiment to “tricks” to theory.

Consider the equation

$$3^x + 4^x = 7^x. \quad (2.1)$$

The task is to find a solution to Eq. (2.1), prove its uniqueness, formulate a similar equation and solve it under the same requirements.

One can note that due to the equality $3+4=7$, the value $x=1$ is a solution to Eq. (2.1). Alternatively, by graphing the function $y=3^x+4^x-7^x$ and locating its graph's intersection with the x -axis, one can find that $x=1$ is such an intersection and, therefore, it is a solution to Eq. (2.1). Similarly, one can graph separately the left-and right-hand sides of (2.1) to see that the two graphs intersect at the point $x=1$. This completes the experimental part of the task as no other solutions can be discovered through these informal means.

The theoretical part of the task begins with the question: Are there other solutions? In answering this question, Style II assistance is required. Towards this end, a teacher can suggest dividing both sides of Eq. (2.1) by 4^x to get the equation

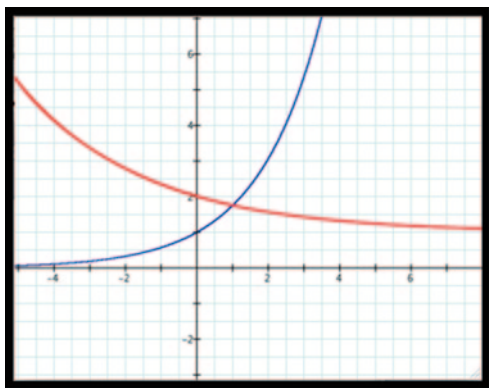
$$\left(\frac{3}{4}\right)^x + 1 = \left(\frac{7}{4}\right)^x.$$

At that point, the teacher knows the meaning of this operation, but students may not get the idea without an explanation. This is the pivotal point of learning problem solving under the assistance of the ‘more knowledgeable other’—first seeing teacher's suggestions as “tricks” and then, after being explained the mathematical meanings of these tricks, integrate them into an individual problem-solving tool kit. A student with a tendency to learn at the deep structure of learning may ask the teacher whether it matters, when performing the above trick, which term out of the three— 3^x , 4^x , or 7^x —is used as the divisor. This question is aimed at the understanding of the very nature of the trick, its meaning and possible applications

³ Henri Poincaré (1854–1912)—the outstanding French mathematician and physicist known for his major contributions to practically all branches of mathematics. Recently, Poincaré's name became known to general public (Gessen 2009) after his outstanding problem in topology was solved by a Russian mathematician Grigori Perelman who declined several international awards for his work including The Fields Medal and (\$ 1 million) Clay Millennium Prize.

⁴ Jacques Salomon Hadamard (1865–1963)—a French mathematician who made important contributions to many branches of mathematics and, in particular, studied mathematical thinking process.

Fig. 2.1 Graphical demonstration of the uniqueness of the solution to (2.1)



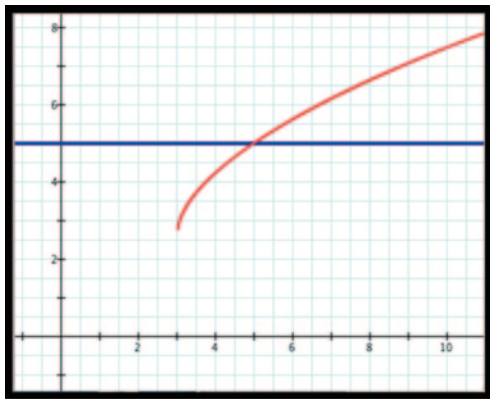
to other mathematical contexts. Of course, the student does not necessarily see his/her question in such a generalized way. But the teacher has to be prepared to see the general in the particular (Mason and Pimm 1984) when acting at the deep structure of teaching. Note, that because of the homogeneous form of Eq. (2.1), the answer, in general, is that it does not matter which term is to use as the divisor⁵. However, dividing by 4^x is more efficient as it immediately yields an equation with both sides having different types of monotone behavior.

One of the features of TEMP is that the use of technology provides support system in understanding the meanings of the “tricks” that mathematicians use to take for granted. As shown in Fig. 2.1, the graphs of the functions $f(x) = (7/4)^x$ and $g(x) = (3/4)^x + 1$, to which one arrives through the above division trick, intersect at one point only, $x = 1$. But one may ask whether visualizing a single point is due to the obvious limitation of technology. In fact, if students don’t raise such a question, the teacher may encourage this discussion in order to remind the students that formal proof of the uniqueness of such intersection follows from the properties of an exponential function $y = a^x$ which strictly increases when $a > 1$ and strictly decreases when $0 < a < 1$. Noting that $7/4 > 1$ and $3/4 < 1$ completes the proof.

In general, if $f(x)$ increases and $g(x)$ decreases, then it follows from the equality $f(x_0) = g(x_0)$ that $f(x) > f(x_0)$ and $g(x) < g(x_0)$ for $x > x_0$. Likewise, for $x < x_0$ the equality $f(x_0) = g(x_0)$ implies $f(x) < f(x_0)$ and $g(x) > g(x_0)$. A deep structure of signature pedagogy of mathematics includes teacher’s ability to provide students

⁵ Another example is the equation $a \cdot \cos \varphi + b \cdot \sin \varphi = c$ (studied in Chap. 6) that can be reduced to a homogeneous equation $a \cdot (\cos^2 \frac{\varphi}{2} - \sin^2 \frac{\varphi}{2}) + 2b \cdot \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} = c \cdot (\cos^2 \frac{\varphi}{2} + \sin^2 \frac{\varphi}{2})$, which, in turn, after dividing its both sides by $\cos^2 \frac{\varphi}{2}$ becomes a quadratic equation $(a + c)x^2 - 2bx + c - a = 0$, $x = \tan \frac{\varphi}{2}$. By the same token, dividing by $\sin^2 \frac{\varphi}{2}$ yields a quadratic equation about $\cot \frac{\varphi}{2}$ and dividing by $\sin \frac{\varphi}{2} \cos \frac{\varphi}{2}$ eventually produces a quadratic equation about either $\tan \frac{\varphi}{2}$ or $\cot \frac{\varphi}{2}$. At the same time, the equation $xy = 2(x + y)$, which, in particular, describes rectangles with area and perimeter numerically equal to each other, is not a homogeneous one and, therefore, it does matter when selecting $2xy$ as the divisor of its both sides thus yielding the equation $\frac{1}{2} = \frac{1}{x} + \frac{1}{y}$ easily solvable in whole numbers. This implies not all ideas when taken at its face value (in our case, dividing both sides of a equation by something) are not equally productive.

Fig. 2.2 Graphical demonstration of the uniqueness of the solution to (2.2)



with Style II assistance in elaborating this statement about the property of the exponential function. Indeed, let $x_1 > x_2$; then $a^{x_1} - a^{x_2} = a^{x_2} \cdot (a^{x_1 - x_2} - 1)$. How can one prove that $a^{x_1 - x_2} - 1 > 0$ when $x_1 > x_2$ and $a > 1$? Let $x_1 - x_2 = \Delta$. When Δ is an integer then, by definition, $a^\Delta = \underbrace{a \cdot a \cdot \dots \cdot a}_{\Delta \text{ times}} > 1$; when $\Delta = n/m$, then, by definition, $a^\Delta = a^{\frac{n}{m}} = \sqrt[m]{a^n} > 1$, $n, m \in \mathbb{N}$. The case when Δ is an irrational number is considered through an approximation of Δ by the sequence of corresponding rational numbers.

A similar equation, $2^x + 7^x = 9^x$, can be derived from the simple equality $2 + 7 = 9$. The rest is analogous to the method of solving (2.1): dividing both sides of the equation by 7^x to get $(2/7)^x + 1 = (9/7)^x$, graphing the functions $f(x) = (2/7)^x + 1$ and $g(x) = (9/7)^x$, and noting that due to the inequalities $2/7 < 1$ and $9/7 > 1$ the graphs of the functions don't have more than one point of intersection. That is, once again, $x = 1$ is the only solution to the newly formulated equation.

Example 2.2.2: Radicals with monotone behavior.

Consider the equation

$$\sqrt{2x-6} + \sqrt{x+4} = 5. \quad (2.2)$$

Once again, the task is to find a solution to Eq. (2.2) and prove its uniqueness. Then, as an extension, formulate a similar problem and solve it.

The guess and check approach enables one to discover that the value $x=5$ satisfies Eq. (2.2). Indeed, $\sqrt{2 \cdot 5 - 6} + \sqrt{5 + 4} = \sqrt{4} + \sqrt{9} = 2 + 3 = 5$. Note that this approach is quite naturally grounded in search for the value of x making the radicals $\sqrt{x+4}$ and $\sqrt{2x-6}$ whole numbers. Alternatively, graphing the functions $f(x) = \sqrt{2x-6} + \sqrt{x+4}$ and $g(x) = 5$ (Fig. 2.2) shows that the graphs have only one point in common. This implies that $x=5$ is the single solution to Eq. (2.2), provided that $f(x)$ assumes each of its values one time only. Indeed, Fig. 2.2 shows monotone behavior of the radicals and, thereby, allows one to conjecture that the function $f(x)$ strictly increases. Consequently, $f(x)$ coincides with the constant function $g(x)$ at one point only, $(5, 5)$. This concludes the experimental part of solving Eq. (2.2).

To formally verify the above visual conjecture regarding the monotone behavior of $f(x)$, one has to estimate $\Delta f(x) = f(x + \Delta x) - f(x)$ by proving the inequality $f(x + \Delta x) > f(x)$ for all $x \geq 3$ (making $2x - 6 \geq 0$) and $\Delta x > 0$. Indeed,

$$\begin{aligned}
 \Delta f(x) &= f(x + \Delta x) - f(x) \\
 &= \sqrt{2(x + \Delta x) - 6} + \sqrt{x + \Delta x + 4} - \sqrt{2x - 6} - \sqrt{x + 4} \\
 &= \sqrt{2(x + \Delta x) - 6} - \sqrt{2x - 6} + \sqrt{x + \Delta x + 4} - \sqrt{x + 4} \\
 &= \frac{(\sqrt{2(x + \Delta x) - 6} - \sqrt{2x - 6})(\sqrt{2(x + \Delta x) - 6} + \sqrt{2x - 6})}{\sqrt{2(x + \Delta x) - 6} + \sqrt{2x - 6}} \\
 &\quad + \frac{(\sqrt{x + \Delta x + 4} - \sqrt{x + 4})(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})}{\sqrt{x + \Delta x + 4} + \sqrt{x + 4}} \\
 &= \frac{2(x + \Delta x) - 6 - 2x + 6}{\sqrt{2(x + \Delta x) - 6} + \sqrt{2x - 6}} + \frac{x + \Delta x + 4 - x - 4}{\sqrt{x + \Delta x + 4} + \sqrt{x + 4}} \\
 &= \frac{2\Delta x}{\sqrt{2(x + \Delta x) - 6} + \sqrt{2x - 6}} + \frac{\Delta x}{\sqrt{x + \Delta x + 4} + \sqrt{x + 4}} > 0.
 \end{aligned}$$

Thus, $f(x + \Delta x) > f(x)$ for $x \geq 3$, $\Delta x > 0$, that is, $\Delta f(x) > 0 \quad \forall x \geq 3$. This completes the theoretical part of solving Eq. (2.2).

In constructing a similar equation, one has to choose the right-hand side, say, 7, split 7 in two parts, say, $7 = 4 + 3$, and then note that $\sqrt{0 + 16} = 4$ and $\sqrt{2 \cdot 0 + 9} = 3$, thus $\sqrt{x + 16} + \sqrt{2x + 9} = 7$ is an equation with a monotone left-hand side and $x = 0$ being its only root.

Example 2.2.3: Inequalities as tools in solving equations with radicals.

Consider a more complicated equation with radicals

$$\sqrt{2x + 3} - \sqrt{x^2 + 2x - 6} = x - 3. \quad (2.3)$$

The difference in complexity between (2.2) and (2.3) requires a different choice of functions to deal with. One has to keep in mind that the rigidity of thinking often stands in the way of formal mathematical demonstration. By the same token, it is the flexibility of thinking that often leads to success in problem solving through overcoming difficulties of the trial and error approach. Technology can help one make the right choice by trying to graph different combinations of functions to be later explored analytically. Such a choice often falls on the most lucid graphical representation. All this is a part of teacher's knowledge toolkit required for providing students with the Style II assistance.

So, this time, let $f(x) = \sqrt{2x + 3} - x + 3$ and $g(x) = \sqrt{x^2 + 2x - 6}$. By analogy with Eq. (2.2), one can check to see that $x = 3$ is a solution to Eq. (2.3). This fact can first be confirmed experimentally by using, for example, *The Geometer's Sketchpad* as shown in Fig. 2.4 where the graphs of $f(x)$ and $g(x)$ intersect at a single point, (3, 3). Note that the choice of software for mathematical visualization in the context of solving equations is flexible and it often depends on a problem in question. For example, Eq. (2.3) has the best representation by *The Geometer's*

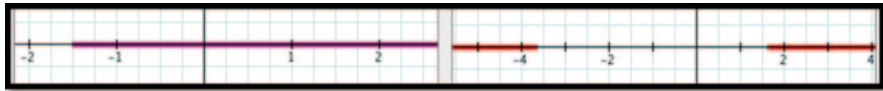


Fig. 2.3 Determining solution set for inequality (2.3)

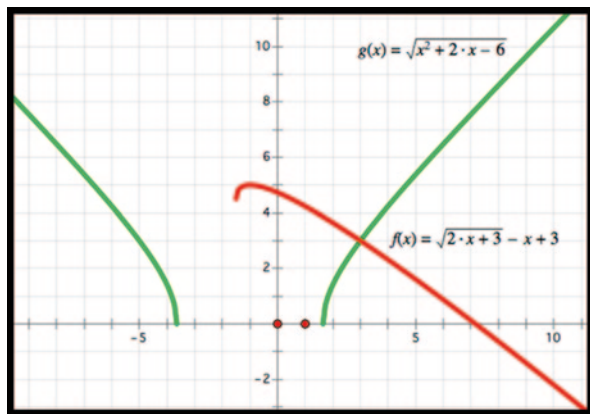
Sketchpad in comparison with the *Graphing Calculator* or *Wolfram Alpha*. Given a mathematical situation, knowing which computer program to apply in order to enable the most efficient representation constitutes an important skill needed for Type II application of technology. In the modern mathematics classroom, as some educational documents suggest, “mathematically proficient students consider the available tools when solving a mathematical problem... to explore and deepen their understanding of concepts” (Common Core State Standards 2010, p. 7) and “should be given opportunities to select and use the particular applications that may be helpful to them as they search for their own solutions to problems” (Ontario Ministry of Education 2005, p. 14). Such pretty ambitious expectations for pre-college mathematics students’ learning to use technology is one of many indicators of the richness of TEMP that teacher candidates need to acquire during their studies of mathematics for teaching.

It remains to prove that the functions $f(x)$ and $g(x)$ coincide at one point only. This time, the proof will be based on our knowledge of the domain of $f(x)$, something that was not used in Example 2.2.2. To this end, in order to prove that $f(x)$ decreases, first note that the inequalities $2x + 3 \geq 0$ and $x^2 + 2x - 6 \geq 0$ imply that $x \geq -(3/2)$ and $x \geq \sqrt{7} - 1$ or $x \leq -\sqrt{7} - 1$. The diagram of Fig. 2.3 yields $x \geq \sqrt{7} - 1$.

Therefore, the evaluation of $\Delta f(x)$ can be carried out as follows

$$\begin{aligned}
 \Delta f(x) &= f(x + \Delta x) - f(x) \\
 &= \sqrt{2(x + \Delta x) + 3} - (x + \Delta x) + 3 - \sqrt{2x + 3} + x - 3 \\
 &= \sqrt{2(x + \Delta x) + 3} - \sqrt{2x + 3} - \Delta x \\
 &= \frac{(\sqrt{2(x + \Delta x) + 3})^2 - (\sqrt{2x + 3})^2}{\sqrt{2(x + \Delta x) + 3} + \sqrt{2x + 3}} - \Delta x \\
 &= \frac{2\Delta x}{\sqrt{2(x + \Delta x) + 3} + \sqrt{2x + 3}} - \Delta x \\
 &< \Delta x \left(\frac{2}{\sqrt{2(\sqrt{7} - 1) + 3} + \sqrt{2(\sqrt{7} - 1) + 3}} - 1 \right) \\
 &= \Delta x \left(\frac{1}{\sqrt{2(\sqrt{7} - 1) + 3}} - 1 \right) \\
 &< \Delta x \left(\frac{1}{\sqrt{2(\sqrt{4} - 1) + 2}} - 1 \right) < \Delta x \left(\frac{1}{2} - 1 \right) < 0.
 \end{aligned}$$

Fig. 2.4 Graphic solution of Eq. (2.3)



Note that the numeric estimations

$$x + \Delta x > x \geq \sqrt{7} - 1$$

and

$$2(\sqrt{7} - 1) + 3 > 2(\sqrt{4} - 1) + 2 = 2(2 - 1) + 2 = 4$$

were used above to replace $1/\sqrt{2(\sqrt{4} - 1) + 2}$ with a larger value, $1/\sqrt{4} = 1/2$, which, in turn, was easier to compare with the number 1 than a cumbersome radical expression. Educational value of learning to perform numeric estimation instead of direct calculation is the development of analytic culture necessary for understanding different mathematical “tricks” used to make sense of computational experiments.

At the same time, the evaluation of $\Delta g(x)$ does not require anything beyond using the difference of squares formula used above in evaluating $\Delta f(x)$ as well. Indeed,

$$\begin{aligned} \Delta g(x) &= g(x + \Delta x) - g(x) \\ &= \sqrt{(x + \Delta x)^2 + 2(x + \Delta x) - 6} - \sqrt{x^2 + 2x - 6} \\ &= \frac{(\sqrt{(x + \Delta x)^2 + 2(x + \Delta x) - 6})^2 - (\sqrt{x^2 + 2x - 6})^2}{\sqrt{(x + \Delta x)^2 + 2(x + \Delta x) - 6} + \sqrt{x^2 + 2x - 6}} \\ &= \Delta x \frac{2x + \Delta x + 2}{\sqrt{(x + \Delta x)^2 + 2(x + \Delta x) - 6} + \sqrt{x^2 + 2x - 6}} > 0. \end{aligned}$$

This completes the theoretical part of solving Eq. (2.3).

Remark. In comparison with Example 2.2.2, Example 2.2.3 showed that whereas experimental parts in solving two different equations may be similar (hiding a

difference in complexity of the equations within software), theoretical parts do reflect this difference. It also demonstrated a difference between teaching at the surface structure and at the deep structure levels, associating computational experiment with the former way of teaching and its validation through the formal use of mathematics with the latter way of teaching. Moreover, Example 2.2.3 showed how inequalities could be used as tools when providing rigor in the process of solving equations. Put another way, learning to use inequalities as tools in the context of solving equations is an example of how collateral learning naturally occurs at the deep structures of teaching and learning.

Sometimes, the very form of an equation with radicals allows for the analytic demonstration of the uniqueness of the root without the need to select monotone functions like in the previous examples.

Example 2.2.4: Factoring as a means of solving equations with radicals.

Demonstrate that the equation

$$2\sqrt{x+2} - \sqrt{x^2+2x} = x-4 \quad (2.4)$$

has a single solution and prove its uniqueness.

The use of *Wolfram Alpha* in solving Eq. (2.4) shows immediately the uniqueness of the solution $x=4$. Next, noting that $x^2+2x = x(x+2)$, Eq. (2.4) can be rewritten in the form

$$2\sqrt{x+2} - \sqrt{x} \cdot \sqrt{x+2} = (\sqrt{x}-2)(\sqrt{x}+2)$$

whence, factoring out $\sqrt{x+2}$ yields

$$\sqrt{x+2}(2-\sqrt{x}) + (2-\sqrt{x})(2+\sqrt{x}) = 0$$

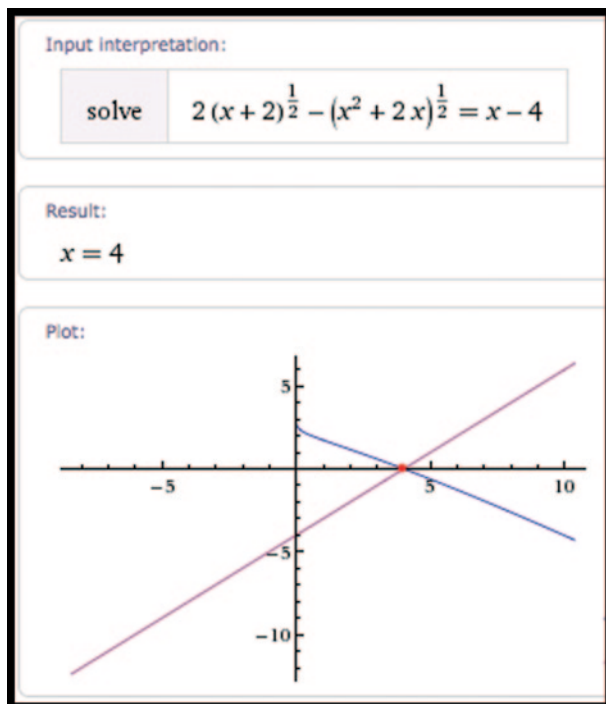
and then, in turn, factoring out $2-\sqrt{x}$, yields

$$(2-\sqrt{x})(\sqrt{x+2} + \sqrt{x}+2) = 0.$$

Hence, either $\sqrt{x} = 2$ or $\sqrt{x+2} + \sqrt{x} + 2 = 0$. The former equation yields $x=4$; the latter equation does not have solutions, as a sum of three positive numbers cannot be equal to zero. This completes the analytical (theoretical) part of solving Eq. (2.4). The success of this part was due to the specific form of the equation (both sides of which have common factors) that made algebraic manipulations leading to factoring possible. This specific form was structured by the second radical being a factor of the first one and a special combination of numerical coefficients in the terms $\sqrt{x+2}$, $\sqrt{x^2+2x}$, and $x-4$.

The recognition of these structural connections, something that started with seeing the relation $x^2+2x = x(x+2)$, is an important element of students' learning at the deep structure level and teachers being prepared to provide Style II assistance in the context of TEMP.

Fig. 2.5 Using *Wolfram Alpha* in solving (2.4)



Example 2.2.5: From computational experiment to proof to problem posing.

Demonstrate both computationally and analytically that the equation

$$2 \sin x = 5x^2 + 2x + 3 \quad (2.5)$$

does not have real solutions. Then formulate a similar equation and carry out similar demonstrations. Experimentally, construct equations, connecting trigonometric and quadratic functions, that have two, three, and four real solutions.

As shown in Fig. 2.6, the graphs of the functions $f(x) = 2 \sin x$ and $g(x) = 5x^2 + 2x + 3$ don't have points in common. This provides experimental evidence that Eq. (2.5) does not have real solutions. The computational experiment can be confirmed analytically by noting that

$$g(x) = 5x^2 + 2x + 3 = 5\left(x + \frac{1}{5}\right)^2 + \frac{14}{5} > 2 \geq 2 \sin x = f(x)$$

for all $x \in \mathbb{R}$. The resulting inequality, $g(x) > f(x)$, which holds true for all $x \in \mathbb{R}$ completes our inquiry into Eq. (2.5). Note that a similar example can be developed if one considers the quadratic function $g(x) = x^2 + bx + c = (x + b/2)^2 + c - b^2/4$ and then defines b and c from the inequality $g(x) > 2$ whence $c - (b^2/4) > 2$ or $c > (b^2/4) + 2$. For example, if $b=2$, then c may be any number greater than three and therefore, the equation $2 \sin x = x^2 + 2x + 4$, just as Eq. (2.5), does not have real solutions.

Computational Experiment Approach to Advanced
Secondary Mathematics Curriculum

Abramovich, S.

2014, XV, 325 p. 252 illus., Hardcover

ISBN: 978-94-017-8621-8