Chapter 2
Introduction: Periodic Filters and Filter Banks

Abstract In this chapter filtering of periodic signals is outlined. Periodic filters and periodic filter banks are defined. Perfect reconstruction filter banks are characterized via their polyphase matrices.

2.1 Periodic Filters

Filtering of signals is a basic tool of signal processing. This procedure has some peculiarities in the periodic case, which are discussed in this section.

2.1.1 Definition of Periodic Filters

A linear operator $\mathbf{h}:\Pi[N] \longrightarrow \Pi[N]$, $\mathbf{y} = \mathbf{hx}$, is time-invariant if the integer shift $\mathbf{x}_d \stackrel{\text{def}}{=} \{x[k+d]\}$ of the input signal produces the same shift $\mathbf{y}_d \stackrel{\text{def}}{=} \{y[k+d]\}$ of the output. Such operators are called digital periodic filters (p-filters).

Denote by $\tilde{\delta}[k] \stackrel{\text{def}}{=} \delta[k](\text{mod}N)$ the N-periodized Kronecker delta. If the input signal $\mathbf{i} = \left\{\tilde{\delta}[k]\right\}$, then the output signal $\mathbf{hi} = \{h[k]\}$, which, obviously, belongs to $\Pi[N]$, is called the *impulse response* (IR) of the p-filter $\mathbf{h}$. In the sequel, we use the notation $\mathbf{h}$ for both the p-filter and its IR $\{h[k]\}$. If the IR of a p-filter is finite up to periodization then the p-filter is referred to as the periodic finite impulse response p-filter (p-FIR p-filters). Otherwise, the p-filter is referred to as the infinite impulse response p-filter (IIR p-filters).

Any periodic signal $\mathbf{x}$ can be represented as a linear combination of impulses $x[k] = \sum_{l=0}^{N-1} x[l]\,\tilde{\delta}[k-l] = \mathbf{x} \circledast \mathbf{i}$. Then, due to the time-invariance of the filter $\mathbf{h}$, the output signal becomes $\mathbf{y} = \mathbf{hx} = \mathbf{x} \circledast \mathbf{h} \iff y[k] = \sum_{l=0}^{N-1} h[k-l]\,x[l]$. Thus, application of the p-filter to a periodic signal is implemented via the discrete circular convolution. The DFT of the IR of the p-filter $\mathbf{h}$ is called its *frequency response* (FR)

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$$\hat{\mathbf{h}} = \left\{\hat{h}[n]\right\}, \quad \hat{h}[n] = \sum_{k=0}^{N-1} h[k]\,\omega^{-nk}, \quad n \in \mathbb{Z}.$$

Periodic filtering of a signal reduces to multiplication in the frequency domain: $\mathbf{h} \circledast \mathbf{x} \Longleftrightarrow \hat{h}[n]\,\hat{x}[n], n \in \mathbb{Z}$. The frequency response of a filter can be represented in a polar form $\hat{h}[n] = \left|\hat{h}[n]\right| e^{i \arg(\hat{h}[n])}$, where the positive N-periodic sequence $\left|\hat{h}[n]\right|$ is called the *magnitude response* (MR) of $\mathbf{h}$ and the real-valued 2π-periodic sequence $\arg(\hat{h}[n])$ is called the *phase response* of $\mathbf{h}$. A p-filter is referred to as a linear phase one if its phase response is linear in n. If the IR of a p-filter $\mathbf{h}$ is symmetric or antisymmetric within the interval $k = -N/2, \ldots, N/2 - 1$ then $\mathbf{h}$ is a linear phase p-filter.

A p-filter $\mathbf{h}$ is called low-pass if it passes low frequencies of signals but attenuates or completely rejects frequencies higher than a cutoff frequency. Its FR $\hat{h}[0] \neq 0$, $\hat{h}[\pm N/2]$ is close or equal to zero. On the contrary, a high-pass p-filter $\mathbf{g}$ attenuates or rejects frequencies lower than the cutoff frequency. Its frequency response $\hat{g}[\pm N/2] \neq 0$, while $\hat{h}[0]$ is close or equal to zero. A band-pass filter passes frequencies within a certain range (passband) and attenuates (rejects) frequencies outside that range (stopband).

2.1.2 Multirate p-Filtering

Assume that $\mathbf{x} \stackrel{\text{def}}{=} \{x[k]\}$ belongs to $\Pi[N]$, $N = 2^j$, and $M = 2^m,\ m < j$. The operation $(\downarrow M)\mathbf{x} = \grave{\mathbf{x}} \stackrel{\text{def}}{=} \{x[Mk]\} \in \Pi[N/M]$ is called downsampling the signal $\mathbf{x}$ by factor of M. Assume that a signal $\mathbf{x}$ belongs to $\Pi[N/M]$. The operation

$$(\uparrow M)\mathbf{x} = \acute{\mathbf{x}}, \quad \acute{x}[k] = \begin{cases} x[l], & \text{if } k = lM; \\ 0, & \text{otherwise.} \end{cases}, \quad l \in \mathbb{Z},$$

is called upsampling the signal $\mathbf{x}$ by factor of M. Obviously, the downsampled signal $(\downarrow M)\mathbf{x} = \mathbf{x}_{0,M}$ is the zero polyphase component of the signal $\mathbf{x}$, while the upsampled signal $(\uparrow M)\mathbf{x}$ is a signal, whose all the polyphase components, except for the zero component, vanish.

If p-filtering a signal is accompanied by downsampling or upsampling then it is called multirate p-filtering. Let $\mathbf{h} = \{h[k]\}$ and $\tilde{\mathbf{h}} = \left\{\tilde{h}[k]\right\}$ be p-filters. Application of the p-filter $\bar{\mathbf{h}} = \left\{\tilde{h}[-k]\right\}$, which is the time-reversed p-filter $\tilde{\mathbf{h}}$, to a signal $\mathbf{x}$, which is followed by downsampling by factor of M, produces the signal $\mathbf{y} = \{y[k]\} \in \Pi[N/M]$, $y[k] = \sum_{l=0}^{N-1} \tilde{h}[l - Mk]\,x[l]$, which is the zero polyphase component of the signal $\mathbf{y} = \bar{\mathbf{h}} \circledast \mathbf{x}$. Application of the p-filter $\mathbf{h}$ to a signal $\check{\mathbf{y}} = \{\check{y}[k]\} \in \Pi[N/M]$, which is upsampled by factor of M, produces the signal

$$\check{\mathbf{x}} = \left\{\check{x}[k]\right\} \in \Pi[N], \quad \check{x}[k] = \sum_{l=0}^{N/M-1} h[k-Ml]\,\check{y}[l] \Longleftrightarrow \hat{\check{x}}[n] = \hat{h}[n]\,\hat{\check{y}}[Mn].$$

In the rest of the volume we mainly deal with down(up)sampling by factor $M = 2$ and denote by $\mathbf{x}_0 = \mathbf{x}_{0,2}$ and $\mathbf{x}_1 = \mathbf{x}_{1,2}$ the even and the odd polyphase components of a signal $\mathbf{x}$.

$M = 2$: **Downsampling:** The downsampled signal $\check{\mathbf{y}} = \mathbf{y}_0$ is the even polyphase component of the signal $\mathbf{y} = \{y[k]\}$, $y[k] = \sum_{l=0}^{N-1} \tilde{h}[l-k]\,x[l]$, whose DFT is $\hat{y}[n] = \hat{\tilde{h}}[-n]\,\hat{x}[n]$. Using Eq. (1.27), we get

$$\begin{aligned}\hat{\tilde{h}}[-n]\,\hat{x}[n] &= \hat{\tilde{h}}_0[-n]_1\,\hat{x}_0[n]_1 + \hat{\tilde{h}}_1[-n]_1\,\hat{x}_1[n]_1 \\ &\quad + \omega^{-n}\left(\widehat{\tilde{h}_0}[-n]_1\,\hat{x}_1[n]_1 + \hat{\tilde{h}}_1[-n]_1\,\hat{x}_0[2(n+1)]\right) \\ \Longrightarrow \hat{y}[n] &= \hat{\tilde{h}}_0[-n]_1\,\hat{x}_0[n]_1 + \hat{\tilde{h}}_1[-n]_1\,\hat{x}_1[n]_1, \quad n \in \mathbb{Z}.\end{aligned} \tag{2.1}$$

$M = 2$: **Upsampling:** Assume that p-filter $\mathbf{h}$ is applied to a signal $\check{\mathbf{y}} = \{\check{y}[k]\} \in \Pi[N/2]$, which is upsampled by factor of 2. Then,

$$\begin{aligned}\check{x}[k] = \sum_{l=0}^{N/2-1} h[k-2l]\,\check{y}[l] \Longleftrightarrow \hat{\check{x}}[n] = \hat{h}[n]\,\hat{\check{y}}[n]_1 = \left(\hat{h}_0[n]_1 + \omega^{-n}\hat{h}_1[n]_1\right)\,\hat{\check{y}}[n]_1 \\ \Longrightarrow \hat{\check{x}}_0[n]_1 = \hat{h}_0[n]_1\,\hat{\check{y}}[n]_1, \quad \hat{\check{x}}_1[n]_1 = \hat{h}_1[n]_1\,\hat{\check{y}}[n]_1.\end{aligned} \tag{2.2}$$

Interpolating p-filters If the DFT of the even polyphase component of a p-filter $\mathbf{h}$ is constant $\hat{h}_0[n]_1 \equiv C$ then the p-filter is called interpolating. In that case, Eq. 2.2 implies that the DFT of the zero polyphase component of the output signal $\check{\mathbf{x}}$ is $\hat{\check{x}}_0[n]_1 = C\hat{\check{y}}[n]_1$. This means that $\check{x}[2k] = C\check{y}[k]$, $k \in \mathbb{Z}$.
Example **Butterworth p-filters**: Denote

$$f^{2r}[n]_1 \stackrel{\text{def}}{=} \frac{\omega^n\left(\cos^{2r}\pi n/N - \sin^{2r}\pi n/N\right)}{\cos^{2r}\pi n/N + \sin^{2r}\pi n/N}. \tag{2.3}$$

Obviously the denominator of the $N/2$-periodic sequence $f^{2r}[n]_1$ is strictly positive and $f^{2r}[0]_1 = 1$. Therefore, the sequences

$$\hat{h}[n] \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(1 + \omega^{-n} f_d^{2r}[n]_1\right) = \frac{\sqrt{2}\cos^{2r}\pi n/N}{\cos^{2r}\pi n/N + \sin^{2r}\pi n/N}, \tag{2.4}$$

$$\hat{g}[n] \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}}\left(1 - \omega^{-n} f_d^{2r}[n]_1\right) = \frac{\sqrt{2}\sin^{2r}\pi n/N}{\cos^{2r}\pi n/N + \sin^{2r}\pi n/N} \tag{2.5}$$

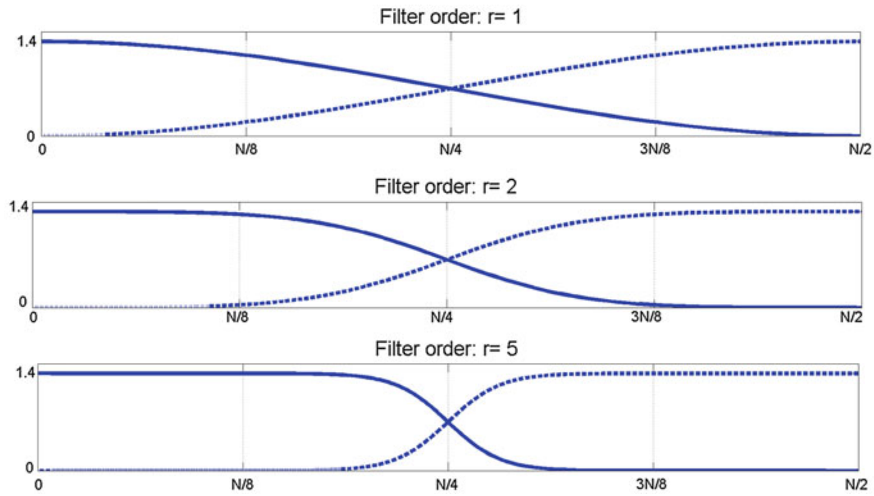

Fig. 2.1 The FR of the low-pass p-filters **h** (*solid lines*) and of the high-pass p-filters **g** (*dashed lines*) for orders $r = 1, 2, 5$

can serve as the FR of interpolating p-filters **h** and **g**. The FR $\hat{h}[0] = \sqrt{2}, /; \hat{h}[N/2] = 0$. Therefore, the p-filter **h** is low-pass. On the contrary, $\hat{g}[0] = 0, /; \hat{g}[N/2] = \sqrt{2}$. Therefore, the p-filter **g** is high-pass.

The sequences $\hat{h}[n]$ and $\hat{g}[n]$ are the magnitude squared FR of the periodized half-band low- and high-pass Butterworth filters of order r, respectively. The Butterworth filters are widely used in signal processing [1]. Figure 2.1 displays the magnitude squared FR of the half-band low- and high-pass Butterworth filters of orders $r = 1, 2, 5$. One can observe that these frequency responses, especially of the order 5 filters, are flat within their pass-band, while practically vanishing at the stop-band.

2.2 Periodic Filter Banks

Keeping in mind the forthcoming periodic wavelet and wavelet frame transforms, we introduce periodic filter banks.

2.2.1 Filter Banks

Definition 2.1 The set of p-filters $\tilde{\mathbf{H}} \stackrel{\text{def}}{=} \left\{\tilde{\mathbf{h}}^s\right\}$, $s = 0, \ldots, S-1$, which, being time-reversed and applied to an input signal $\mathbf{x} \in \Pi[N]$, produces the set of the output signals $\{\tilde{\mathbf{y}}^s\}_{s=0}^{S-1}$, which are downsampled by factor of M,

$$y^s[l] = \sum_{k=0}^{N-1} \tilde{h}^s[k - Ml]\, x[k], \quad s = 0, \ldots, S-1,\ l \in \mathbb{Z}, \tag{2.6}$$

is called the $S-$channel analysis p-filter bank.

Definition 2.2 The set of p-filters $\mathbf{H} \stackrel{\text{def}}{=} \{\mathbf{h}^s\}$, $s = 0, \ldots, S-1$, which, being applied to a set of input signals $\{\mathbf{y}^s\} \in \Pi[N/M]$, $s = 0, \ldots, S-1$, that are upsampled by factor of M, produces the output signal $\check{x}[l]) = \sum_{s=0}^{S-1} \sum_{k=0}^{N/M-1} h^s[l - Mk]\, y^s[k]$, $l \in \mathbb{Z}$, is called the $S-$channel synthesis p-filter bank.

Definition 2.3 If the upsampled signals $\mathbf{y^s}$, $s = 0, \ldots, S-1$, which are defined in Eq. 2.6, are used as an input to the synthesis p-filter bank and the output signal is $\check{\mathbf{x}} = \mathbf{x}$, then the pair of analysis–synthesis p-filter banks form a perfect reconstruction (PR) p-filter bank.

Definition 2.4 If the number of channels S equals to the downsampling factor M then the p-filter bank is said to be critically sampled. If $S > M$ then the p-filter bank is oversampled.

Critically sampled PR p-filter banks are used in the wavelet analysis, while oversampled PR p-filter banks serve as a source for discrete-time wavelet frames design.

In the volume, we deal mainly with p-filter banks, whose downsampling factor is $M = 2$ *and* $\tilde{\mathbf{h}}^0$ *and* $\mathbf{h}^0$ *are the low-pass p-filters.*

2.2.2 Characterization of p-Filter Banks

Assume that $\tilde{\mathbf{H}} \stackrel{\text{def}}{=} \left\{\tilde{\mathbf{h}}^s\right\}$, $s = 0, \ldots, S-1$, is an analysis p-filter bank with the downsampling factor of 2. Then, its application to a signal $\mathbf{x} \in \Pi[N]$ produces S signals from $\Pi[N/2]$:

$$y^s[l] = \sum_{k=0}^{N-1} \tilde{h}^s[k-2l]\, x[k], \quad s = 0, \ldots, S-1, \; l \in \mathbb{Z}. \tag{2.7}$$

It follows from Eq. (2.1) that

$$\hat{y}^s[n]_1 = \hat{\tilde{h}}^s_0[-n]_1\, \hat{x}_0[n]_1 + \hat{\tilde{h}}^s_1[-n]_1\, \hat{x}_1[n]_1, \quad n \in \mathbb{Z}. \tag{2.8}$$

Assume that $\mathbf{H} \stackrel{\text{def}}{=} \{\mathbf{h}^s\}$, $s = 0, \ldots, S-1$, is a synthesis p-filter bank with the upsampling factor of 2. Then, its application to the upsampled signals $\mathbf{y}^s \in \Pi[N/2]$ produces a signal from $\Pi[N]$:

$$\check{x}[l]) = \sum_{s=0}^{S-1} \sum_{k=0}^{N/2-1} h^s[l-2k]\, y^s[k], \quad l \in \mathbb{Z}. \tag{2.9}$$

Equation (2.2) implies that the DFTs of the polyphase components are

$$\hat{\hat{x}}_0[n]_1 = \sum_{s=0}^{S-1} \hat{h}_0^s[n]_1\, \hat{y}^s[n]_1, \quad \hat{\hat{x}}_1[n]_1 = \sum_{s=0}^{S-1} \hat{h}_1^s[n]_1\, \hat{y}^s[n]_1. \tag{2.10}$$

Equations (2.8) and (2.10) can be represented in a matrix form by

$$\begin{pmatrix} \hat{y}^0[n]_1 \\ \vdots \\ \hat{y}^{S-1}[n]_1 \end{pmatrix} = \tilde{\mathbf{P}}[-n] \cdot \begin{pmatrix} \hat{x}_0[n]_1 \\ \hat{x}_1[n]_1 \end{pmatrix}, \quad \begin{pmatrix} \hat{\hat{x}}_0[n]_1 \\ \hat{\hat{x}}_1[n]_1 \end{pmatrix} = \mathbf{P}[n] \cdot \begin{pmatrix} \hat{y}^0[n]_1 \\ \vdots \\ \hat{y}^{S-1}[n]_1 \end{pmatrix}$$

where the $S \times 2$ analysis and the $2 \times S$ synthesis polyphase matrices are, respectively,

$$\tilde{\mathbf{P}}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{\tilde{h}}_0^0[n]_1 & \hat{\tilde{h}}_1^0[n]_1 \\ \vdots & \vdots \\ \hat{\tilde{h}}_0^{S-1}[n]_1 & \hat{\tilde{h}}_1^{S-1}[n]_1 \end{pmatrix} \quad \mathbf{P}[n] \stackrel{\text{def}}{=} \begin{pmatrix} \hat{h}_0^0[n]_1 & \ldots & \hat{h}_0^{S-1}[n]_1 \\ \hat{h}_1^0[n]_1 & \ldots & \hat{h}_1^{S-1}[n]_1 \end{pmatrix}, \quad n \in \mathbb{Z}.$$

If the relations

$$\mathbf{P}[n] \cdot \tilde{\mathbf{P}}[-n] = \mathbf{I}_2, \tag{2.11}$$

where $\mathbf{I}_2$ is the 2×2 identity matrix, holds for all $n \in \mathbb{Z}$ then

$$\mathbf{P}[n] \cdot \tilde{\mathbf{P}}[-n] \cdot \begin{pmatrix} \hat{x}_0[n]_1 \\ \hat{x}_1[n]_1 \end{pmatrix} = \begin{pmatrix} \hat{x}_0[n]_1 \\ \hat{x}_1[n]_1 \end{pmatrix}.$$

Thus, Eq. (2.11) is the condition for the analysis–synthesis pair $\left\{\tilde{\mathbf{H}},\ \mathbf{H}\right\}$ of p-filter banks to form a PR p-filter bank.

Reference

1. A.V. Oppenheim, R.W. Schafer, *Discrete-Time Signal Processing*, 3rd edn. (Prentice Hall, New York, 2010)

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