

Some Recent Progress Concerning Topology of Fractals

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1 Stable Invariant Sets of Iterated Function Systems

Let (X, d) be a complete metric σ space with metric d or let $(X, \mathcal{T})$ be a compact Hausdorff topological space. For $S \subset X$ let $\bar{S}$ be the closure of S . Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X , and let $\mathcal{K}(X)$ denote the family of all nonempty compact subsets of X . If (X, d) is a metric space, let $\mathcal{C}(X)$ denote the family of all nonempty closed bounded subsets of X . The *Hausdorff distance* h :

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$\mathcal{P}(X) \times \mathcal{P}(X) \rightarrow [0, \infty]$ is defined by

$$h(B, C) := \inf\{r > 0 : B \subset N_r C, \text{ and } C \subset N_r B\}$$

for all $B, C \in \mathcal{P}(X)$, where

$$N_r B := \{x \in X : \inf_{b \in B} d(x, b) < r\}$$

stands for the r -neighbourhood of B . (Thus h is an extended-valued pseudometric [HuPap97, p. 5, Chap. 1])

If (X, d) is a metric space then h restricted to either $\mathcal{C}(X) \times \mathcal{C}(X)$ or $\mathcal{K}(X) \times \mathcal{K}(X)$ is a metric. The metric spaces $(\mathcal{C}(X), h)$ and $(\mathcal{K}(X), h)$ are *hyperspaces* [HuPap97, IIIINad99]. The hyperspace $(\mathcal{C}(X), h)$ is complete if and only if (X, d) is complete if and only if the hyperspace $(\mathcal{K}(X), h)$ is complete, see for example [Hen99, HuPap97].

If $(X, \mathcal{T})$ is a compact Hausdorff space we endow $\mathcal{K}(X)$ with the *Vietoris topology* $\mathcal{T}_V$ derived from the topology $\mathcal{T}$ and generated by subbasic sets of the form $\{A \in \mathcal{K}(X) : A \subset U\}$ and $\{A \in \mathcal{K}(X) : A \cap U \neq \emptyset\}$, where $U \in \mathcal{T}$, [IIIINad99, HuPap97]. The hyperspace $(\mathcal{K}(X), \mathcal{T}_V)$ is a compact Hausdorff space if and only if $(X, \mathcal{T})$ is a compact Hausdorff space, see [Kie02], [p. 16, Proposition 1.1.9] [IIIINad99], Chap. 1. If (X, d) is a compact metric space then the topology on the metric space $(\mathcal{K}(X), h)$ induced by the metric h is the Vietoris topology, [Kie02], [p. 21, Proposition 1.2.9]. In fact it does not matter whether we use Hausdorff metric or Vietoris topology in the hyperspace of compacta if the base space (X, d) is metric since $(\mathcal{K}(X), h)$ as topological space is identical with $(\mathcal{K}(X), \mathcal{T}_V)$ [IIIINad99] [p. 16, Theorem 3.1]. This is in contrast to $(\mathcal{C}(X), h)$ where two equivalent metrics on X might give rise to topologically different hyperspaces $\mathcal{C}(X)$, see [IIIINad99] [p. 14, Exercise 2.11].

An *iterated function system* (IFS) $\mathcal{F}$ is the space X together with a finite set of functions $f_i : X \rightarrow X, i = 1, \dots, N$, and is written as

$$\mathcal{F} = (X; f_1, \dots, f_N),$$

see for example [B93, B06]. It is usually assumed, as we do throughout this article, that the maps f_i are continuous and that X is either a complete metric space or a compact Hausdorff space. In the classical Hutchinson theory [Hut81] the f_i are contraction mappings and X is a complete metric space. In this case the IFS is said to be *hyperbolic*. In this article we do not discuss the cases where the IFS comprises either an infinite countable set of maps or a nondenumerable set of maps. We note only that it is relatively straightforward to generalize some of the ideas that we mention, related to how attractors sit in their basins of attraction, to the case where the set of functions is compact, say in the compact-open topology on X^X when X is a compact Hausdorff space. But this article is also concerned with relationships between attractors and code space, dynamics on attractors, and transformations

between attractors, derived from these relationships, and all of these notions require that the IFS comprises only finitely many maps.

The *code space* (I^∞, ρ) is the set of infinite words over the finite alphabet $I = \{1, \dots, N\}$ furnished with the *Baire metric*

$$\rho((\sigma_i)_{i=1}^\infty, (v_i)_{i=1}^\infty) := 2^{-\min\{i: \sigma_i \neq v_i\}}$$

for $(\sigma_i)_{i=1}^\infty, (v_i)_{i=1}^\infty \in I^\infty$ where, by convention, $2^{-\min \emptyset} := 0$. This metric is also referred to as the *code space metric* [B93, B06]. The topology of code space is the same as that of the classical Cantor set with the Euclidean metric; it is also the Tychonoff countable product topology of a discretely topologized finite alphabet; thus it is compact, but also of importance—see Chap. 7—the Baire metric provides a tree structure to the code space. Code space appears in many places, including in automata theory, e.g. [CalEtal97] and symbolic dynamics, e.g. [Par76].

We say that $(x_n)_{n=0}^\infty$ is an *orbit* of $x_0 \in X$ under $\mathcal{F}$ if

$$x_n := f_{i_n}(x_{n-1}), \quad n \geq 1, \quad \text{with } i_n \in \{1, \dots, N\}. \quad (1)$$

Every orbit is specified by its initial point x_0 and an infinite word $(i_1, i_2, \dots) \in I^\infty$.

Define the *Hutchinson operator* $F = F_{\mathcal{F}}$ for $\mathcal{F}$ by

$$F : \mathcal{P}(X) \rightarrow \mathcal{P}(X) \ni S \mapsto F(S) = \overline{\bigcup_{n=1}^N f_n(S)}.$$

We use the conventional notation $f(S) = \{f(s) : s \in S\}$. The Hutchinson operator maps $\mathcal{K}(X)$ into itself and, under mild assumptions, $\mathcal{C}(X)$ into itself, so we consider the discrete dynamical systems

$$F|_{\mathcal{K}(X)} : \mathcal{K}(X) \rightarrow \mathcal{K}(X) \quad \text{and} \quad F|_{\mathcal{C}(X)} : \mathcal{C}(X) \rightarrow \mathcal{C}(X).$$

The n -fold composition of F is written as F^n , and by slight abuse of notation we may drop the restrictions $|_{\mathcal{K}(X)}$ and $|_{\mathcal{C}(X)}$ when the meaning is clear from context. The orbit $(F^n(B_0))_{n=1}^\infty$ of a set $B_0 \in \mathcal{P}(X)$ under F , treated as a discrete dynamical system on $\mathcal{P}(X)$, describes the collective behaviour of orbits of points under $\mathcal{F}$. We are interested in topological questions related to fixed points of F either in $\mathcal{K}(X)$ or in $\mathcal{C}(X)$, and in the relationship between orbits of F and its fixed points. A *fixed-point* A of F is a set which is invariant under F , that is,

$$F(A) = A.$$

A fixed point of F may also be called an *invariant set* of F . These sets include many common fractals, and are closely related to random fractals. Why are they “fractal”? Because they are “self-similar” in the sense that they are unions of transformations of themselves. But note that they may be classical Euclidean entities, for example in the case that $(X, d) = (\mathbb{R}^2, d)$, such as points and line segments.

There are various notions of the “attractor” of an IFS $\mathcal{F}$. These are obtained by imposing conditions on the asymptotic behaviour of orbits. According to Bandt [Ban89] there are two basic constructions of an “attractor” A :

- *construction from the outside*

$$A = \bigcap_{k=1}^{\infty} F^k(C_0) \subset \dots \subset F^{n+1}(C_0) \subset F^n(C_0),$$

where C_0 is called an absorbing set ([Chu02] Chap. 1 p. 24, [Wig03] Definition 8.2.3 p. 108), and

- *construction from the inside*

$$A = \overline{\bigcup_{k=1}^{\infty} F^k(A_0)} \supset \dots \supset F^{n+1}(A_0) \supset F^n(A_0),$$

where $A_0 \subset A$ is called a nucleus [LasMyj96].

These two constructions, which are always possible in the case of a hyperbolic IFS, lead to three generalizations in nonhyperbolic settings: (strict) attractor, global maximal attractor, and semiattractor.

The *global maximal attractor* is the largest invariant set of $\mathcal{F}$, usually taken among compact or closed bounded sets. This concept has a long history and is one of the main objects of interest in dynamical systems [Hal88, Chu02, McG92, Val00, Aki93, CarEtal03]. In the context of IFS theory, it has been considered in [McG92, L03, Kie02] and is intimately related to construction from the outside.

A *semi-attractor* is the smallest invariant set of $\mathcal{F}$; it possesses a regeneration property and arises in processes which generalize construction from the inside. Semi-attractors were introduced and developed by Lasota and Myjak, [LasMyj00], using Kuratowski topological limits. They have a natural relationship with the notion of an “IFS with condensation set” [B93], see Chap. 2. A semiattractor is a variant of the so-called global minimal attractor (cf. [Chu02]). The theory of semiattractors is fairly complete. Its deterministic part is a manifestation of the rich theory of Markov-Feller operators, and so far no other concept of attractor has been proposed that is able to retain consistently the stochastic-deterministic duality present in the classical Hutchinson theory [Hut81, B93].

We shall say, following [BVin12a], that $A \in \mathcal{K}(X)$ is a (*strict*) *attractor* of the IFS $\mathcal{F} = (X; f_1, \dots, f_N)$ if there exists an open neighbourhood $U(A) \supset A$, such that

$$F^n(B) \xrightarrow{n \rightarrow \infty} A, \text{ for all } U(A) \supset B \in \mathcal{K}(X). \quad (2)$$

The convergence here is with respect to the Vietoris topology, or, equivalently, in the metric space case, with respect to the Hausdorff metric. Note that, due to the continuity of $F|_{\mathcal{K}(X)} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$, see [BL12], the attractor is an invariant set $A = F(A)$, a fixed point of F . The *basin* $\mathcal{B}(A)$ of a strict attractor A is the union of all open neighborhoods $U(A)$ such that (2) holds. From additivity of F , i.e., $F(B_1 \cup B_2) = F(B_1) \cup F(B_2)$ for $B_1, B_2 \in \mathcal{P}(X)$, it follows that (2) holds for

all compact $B \subset \mathcal{B}(A)$. In the language of Myjak and Lasota [LasMyj00] a strict attractor is an *asymptotically stable invariant set* for $\mathcal{F}$. It is common in papers on IFS theory to refer to a strict attractor of an IFS as, simply, an attractor of the IFS. We follow that practice here, reserving the adjective “strict” for situations, such as Chap. 5, where extra precision is needed.

When dealing with attractors, it may be possible to restrict attention to a subsystem that possesses a unique attractor, as we describe in this paragraph. We say that $B \in \mathcal{P}(X)$ is a *forward invariant set* for F if $F(B) \subset B$. A forward invariant set is known under other names as well: *positively invariant set* [Chu02], [Chap. 1.2], *subinvariant set* [LasMyj00, L03], and sporadically in the geometric context, *super-self-similar set* [Fal97]. If B is closed then we define the *restriction of $\mathcal{F}$ to B* to be the iterated function system $\mathcal{F}|_B := (B; f_1|_B, f_2|_B, \dots, f_N|_B)$. The IFS $\mathcal{F}|_B$ is also called a *subsystem* of $\mathcal{F}$. If $\mathcal{F}$ possesses an attractor $A \in \mathcal{K}(X)$ then we may consider $\mathcal{F}|_A$. This subsystem comprises an IFS on a compact Hausdorff space. More generally, given an attractor A of $\mathcal{F}$ with basin $U(A)$, it can occur that there exists an associated *attractor-block* $C = C(A) \in \mathcal{K}(X)$, namely a set $U(A) \supset C \in \mathcal{K}(X)$ with the property that $A \subset C$ and $F(C) \subset C^\circ$, the interior of C . An attractor block can always be found, associated with any given attractor, when X is a compact Hausdorff space (see [McG92, p. 32, Theorem 11.4], [Aki93, Chap. 3]), in which case the subsystem $\mathcal{F}|_{C(A)}$ comprises an IFS on a compact Hausdorff space, with unique strict attractor A and basin $C(A)$.

An IFS may possess multiple attractors or may not possess an attractor. It is known that hyperbolic and weakly contractive IFSs possess unique attractors [Hut81, Hat85, Mat93, Wic91, AndFi04]. For example, if $\mathcal{F}$ is hyperbolic then $F|_{\mathcal{K}(X)} : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is a contraction mapping and has a unique fixed point $A \in \mathcal{K}(X)$. That is, A is a fixed point of the Hutchinson operator (e.g., [BL12]). There is also a recent relatively complete theory concerning the existence and uniqueness of strict attractors for real projective IFSs [BVin12a]; see Chap. 6.

A general consistent language for handling nonunique dynamics— for example those induced by collections of maps, as above, or ill-posed problems, for example in economic optimization— has been developed around the notion of multifunction. The terminology of multifunctions is used mainly in the control theory community, see for example [AubCel84]; in dynamical systems the concept of relations is used, for example in [Aki93, McG92], although for example [LasMyj00, Val00, CarEtal03] use the multifunction concept. Early, although not the earliest, studies of nonunique dynamics appear in [Pel77, Rox65, Bro79]. We note that Ed Vrscay et al. [LaTEtal06] have considered some theory and applications of generalized IFSs defined using multifunctions.

Although $F : \mathcal{K}(X) \rightarrow \mathcal{K}(X)$ is a continuous dynamical system it is hard to apply many of the deep results of dynamical systems theory because the geometry of hyperspaces is very special (e.g. [Ban86, IllNad99]); this makes the study of dynamics on hyperspaces quite specific, see [RoF03, AndVat07, GarEtal09].

Associated with the IFS $\mathcal{F} = (X; f_1, \dots, f_N)$ is the IFS on the hyperspace, $\widehat{\mathcal{F}} = (\mathcal{K}(X); \widehat{f}_1, \dots, \widehat{f}_N)$, whose attractor is an example of a so-called *superfractal* [B06, pp. 391–394 Chap. 5.3–5.5]. The induced mappings $\widehat{f}_i$ are given by

$$\widehat{f}_i(B) := f_i(B)$$

for $B \in \mathcal{K}(X)$, $i = 1, \dots, N$. We observe that if $\widehat{\mathcal{F}}$ admits unique invariant set, then $\mathcal{F}$ also admits unique invariant set, but uniqueness is lost in the reverse direction.

Theorem 1.1 *If $\widehat{\mathcal{F}}$ possesses attractor, then $\mathcal{F}$ possesses attractor.*

Example 1.2. Let $X = \{\exp(2\pi it) : t \in [0, 1]\}$ be the circle in the complex plane. We define two maps $f_1, f_2 : X \rightarrow X$, constant one and a rotation, by putting for $x = \exp(2\pi it)$, $t \in [0, 1]$

$$\begin{aligned} f_1(x) &= 1, \\ f_2(x) &= \exp(2\pi i\theta t), \end{aligned}$$

where $\theta > 1$ is irrational. It turns out that $\mathcal{F}$ has X as a strict attractor whose basin is X ; but the hypersystem $\widehat{\mathcal{F}}$ does not possess an attractor.

In Chap. 2, following ideas of Kieninger, we observe relationships between the code space for $\mathcal{F}$, and three types of attractor of $\mathcal{F}$. This allows us to link IFS theory to symbolic dynamics [Par76]. In this direction we note that Mauldin and Urbański [MauUrb96, MauUrb03] have developed yet another notion of fractal associated with general IFSs, namely a limit set. However this invariant set, associated with countable noncompact IFS, is not closed and has a complicated topology in the sense of descriptive set theory. The Jørgensen set [Jor06] is an improvement upon their definition.

It seems that clarifying relations between various types of limit sets would be beneficial for identifying basic dynamical and geometrical properties of fractals and separating them from side-effects possible in specific classes of systems. Some attempts towards universal solutions have been presented by Rudnik [Rud92] (inverse limits), Wicks, Bedford & Fisher [Wic91, BedFis96] (topological zooming), Edalat [Eda95] (ordered domains and topologies), Kieninger [Kie02] (fibred systems and symbolic dynamics), and Leinster [Lei11] (categorical view on self similarity).

In Chap. 2 we discuss addresses of points and sets on IFS attractors using Kieninger's classification. In particular, we explain what it means for an attractor to be “point-fibred”. Such attractors play a key role in Chap. 3 where we mention the fascinating new area of fractal transformations. Point-fibred attractors also feature in Chap. 4 where we summarize some recent progress on a fundamental topological question of Kameyama. In Chap. 5 we mention recent work of McGehee and Wiandt, who have developed a beautiful theory of iterated closed relations on compact Hausdorff space: in particular we mention how, whenever an IFS has an attractor there is an associated dual “global maximal repeller”: this leads to the general investigation, in Chaps. 5 and 6, of attractor/repeller pairs associated with affine, Möbius, bi-affine and projective IFSs. Finally, in Chap. 7 we discuss some recent progress in understanding the “Chaos Game” algorithm for calculation of attractors: we obtain new results by adopting a topological point of view.

2 The Code Space Map and Kieninger's Classification of IFS Attractors

Throughout this chapter, let $\mathcal{F} = (X; f_1, \dots, f_N)$ be an IFS on a compact Hausdorff space X . Let F be the Hutchinson operator for $\mathcal{F}$ and let $I^\infty = \{1, \dots, N\}^\infty$ be the code space for $\mathcal{F}$.

Following [Kie02, p. 67], the set

$$X^\# = X^\#(\mathcal{F}) := \lim_{n \rightarrow \infty} F^n(X) = \bigcap_{n=1}^{\infty} F^n(X)$$

is called the *maximal invariant set* of $\mathcal{F}$. It is the *global maximal attractor* since X is compact. One should also note that $X^\#$ is strictly invariant (as the Mauldin-Urbański limit set [MauUrb96]) in the sense that

$$F(X^\#) = X^\# = \bigcup_{i=1}^N f_i(X^\#).$$

A fixed point $A \in \mathcal{K}(X)$ of F is called a *minimal invariant set* of $\mathcal{F}$ if $F(B) \neq B$ for all $B \in \mathcal{K}(A) \setminus \{A\}$; equivalently A is a forward invariant set minimal with respect to the inclusion ordering of the hyperspace $\mathcal{K}(X)$. We have a multivalued version of Birkhoff's theorem : $\mathcal{F}$ possesses at least one minimal invariant set, e.g., [Kie02, p. 72], , [Aki93, Chap. 4 Theorem 12 (c)], .

The *code space map* (also *coding map* or *coordinate map*) $\pi = \pi_{\mathcal{F}} : I^\infty \rightarrow \mathcal{K}(X)$ is defined by

$$\pi_{\mathcal{F}}(\sigma) = X_\sigma(\mathcal{F}) := \lim_{n \rightarrow \infty} f_{\sigma|_n}(X) = \bigcap_{n=1}^{\infty} f_{\sigma|_n}(X)$$

for all $\sigma \in I^\infty$, where $f_{\sigma|_n}$ denotes the composition $f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_n}$. The set X_σ is called a *fibre* of $\mathcal{F}$. If $C \in \mathcal{K}(X)$ is forward-invariant, i.e., $F(C) \subset C$, then $C_\sigma(\mathcal{F}) = X_\sigma(\mathcal{F}|_C)$ [Kie02, p. 85, Prop. 4.1.7 (i)]. . The maximal invariant set $X^\#(\mathcal{F})$ is the union of its fibres, $X^\#(\mathcal{F}) = \bigcup_{\sigma \in I^\infty} X_\sigma(\mathcal{F})$, [Kie02, p. 95, Proposition 4.3.2], cf. [Hut81, MauUrb96] .

Definition 2.1 [Kie02, p. 97]. *Let $\mathcal{F}$ be an iterated function system on a compact metric space X , with fibres X_σ , $\sigma \in I^\infty$, and maximal invariant set $X^\#$. Then $\mathcal{F}$ and $X^\#$ are called*

1. *minimal-fibred if*

$$C_\sigma = X_\sigma \text{ for all } \sigma \in I^\infty \text{ and all subsystems } \mathcal{F}|_C \text{ (} C \in \mathcal{K}(X), F(C) \subset C \text{);}$$

2. *strongly-fibred if*

$$\text{for each open set } U \text{ with } U \cap X^\# \neq \emptyset \text{ there is } \sigma \in I^\infty \text{ such that } X_\sigma \subset U;$$

3. *point-fibred if*

X_σ is a singleton for all $\sigma \in I^\infty$.

We can classify each attractor of $\mathcal{F}$ according to how it is fibred. Let A be a strict attractor of $\mathcal{F}$. Let $C = C(A)$ be an attractor block for A (see p. 73 for a definition). Then $\mathcal{F}|_C$ and $A = C^\#$ are minimal-fibred, and may be strongly-fibred or point-fibred; accordingly we will say that A is a *minimal-fibred attractor* of F , or *strongly-fibred attractor* of F or *point-fibred attractor* of F . It follows from the definition and [Kie02, p. 108 Corollary 4.4.4], that point-fibred implies strongly-fibred implies minimal-fibred.

Examples show that $\mathcal{F}|_C$ and $C^\#$ may be minimal-fibred while $C^\#$ is not an attractor of $\mathcal{F}|_C$. However, if $\mathcal{F}$ is strongly-fibred then it has a unique strict attractor equal to $X^\#$, and there are examples of attractors that are minimal-fibred but not strongly-fibred.

Minimal fibring occurs when the maximal invariant set is a minimal invariant set, so there always exists a minimal-fibred attractor for a subsystem of any IFS on a compact metric space. Furthermore, each minimal-fibred system can be converted into a strongly-fibred system by treating $X^\#$ as a condensation set. This may be achieved by restricting $\mathcal{F}$ to $X^\#$ and including the identity map as an extra function in the IFS, as follows. In fact, we have that $\mathcal{F}$ is minimal-fibred if and only if the IFS $\mathcal{F}_C := (X^\#; f_0, f_1, \dots, f_N)$, where $f_0 : X^\# \rightarrow X^\#$ is the identity map, has strongly-fibred maximal invariant set $X^\#$, i.e., $X^\#$ is (the unique) semi-attractor of $\mathcal{F}_C$, [Kie02, p. 108 Proposition 4.4.5].

Point-fibred systems are of special interest to us because they can be used to construct transformations between attractors, as we will review in Chap. 3. This is possible because in this case the code space map is continuous.

Theorem 2.2 [Kie02, p. 05, Proposition 4.3.22], *Let $\mathcal{F}$ be an iterated function system on a compact Hausdorff space X and let $X^\#$ be the maximal invariant set of $\mathcal{F}$. Then the code space map $\pi_{\mathcal{F}} : I^\infty \rightarrow \mathcal{K}(X)$ is upper semicontinuous and $\cup \pi_{\mathcal{F}}(I^\infty) = X^\#$.*

Moreover $\pi_{\mathcal{F}} : I^\infty \rightarrow \mathcal{K}(X)$ is continuous if and only if $X^\#$ is a point-fibred attractor of $\mathcal{F}$.

But strongly and minimal fibred systems are of great interest too, because they can be very beautiful, as illustrated in Fig. 1. In this example, which corresponds to IFS of two affine maps, we have rendered the attractor, a filled right-angle triangle, in two different ways, so as to draw attention to some of its fibres: fibres which are points are exaggerated in the left-hand image, while fibres which are line segments are exaggerated in the right-hand image. Both these sets of fibres are dense in the attractor. This attractor has the property that it is not possible to extend its basin of attraction outside the triangle itself: there is no open set containing it such that it is the attractor of the IFS restricted to the closure of the open set. This follows from [AtkEtal10], where it is shown that if an affine IFS on $\mathbb{R}^M$ has an attractor then the attractor is point-fibred. So in the present example we have convergence “from the inside” but not “from the outside”.

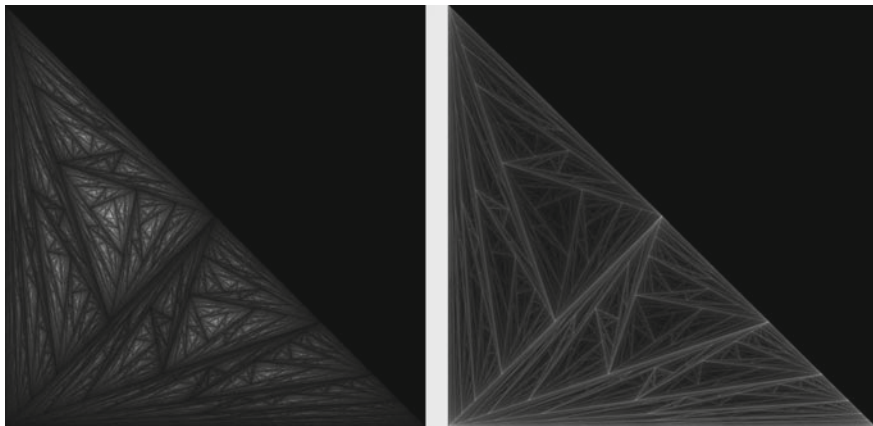


Fig. 1 The triangular attractor of a strongly-fibred (but not point-fibred) attractor of an affine IFS is rendered here using pictures of different associated invariant measures. The IFS is $(R^2; f_1(x, y) = (x/2, -x - y/2 + 1), f_2(x, y) = (x + y/2, y/2))$; in the *left image* the probabilities are approximately $p_1 \approx 0.51$, $p_2 \approx 0.49$; in the *right image* the probabilities are approximately $p_1 \approx 0.49$, $p_2 \approx 0.51$

It is an intriguing fact that any attractor A of $\mathcal{F}$ can be described using the chaos game algorithm applied to any disjunctive sequence, starting from any point in the basin of A (consult Chap. 7). Each fibre A_σ is visited densely by any disjunctive orbit, despite the fact that there is no point-valued coordinate map. This is surprising also in view of Theorem 2.2 which says that the coordinate map is not continuous (with respect to the Vietoris topology).

3 Fractal Transformations Between Attractors of Point-Fibred IFS

In this chapter we summarise how the fibres of pairs of point-fibred IFS attractors can be used to construct fascinating transformations between attractors. These transformations, which we dub “fractal” because they can change the fractal dimensions of sets and measures upon which they act, are of great interest when they are constructed using IFSs of classical geometrical maps. They are themselves of a geometrically elementary nature, in the sense that they can be simply and briefly described, yet their behaviour is complex and interesting, suggestive of the kinds of transformations between physical objects, such as between one cloud and another, that may be used to describe relationships between objects in the real world.

Throughout this chapter, $\mathcal{F} = (X; f_1, \dots, f_N)$ is an IFS on a compact Hausdorff space X , and A is a point-fibred attractor of $\mathcal{F}$. For simplicity, we suppose here that the basin of A is X . Let F be the Hutchinson operator for $\mathcal{F}$ and $I^\infty = \{1, \dots, N\}^\infty$ is

the code space for $\mathcal{F}$. Since A is point-fibred, the coding map $\pi_{\mathcal{F}} : I^{\infty} \rightarrow \mathcal{K}(X)$ is single-valued and continuous. Hence we can define a continuous map $\pi_{\mathcal{F}} : I^{\infty} \rightarrow X$, also called a coding map, where we use the same notation for both maps.

This coding map $\pi_{\mathcal{F}} : I^{\infty} \rightarrow X$ shows that the IFS $\mathcal{F}|_A$ is a factor of the IFS $(I^{\infty}; s_1, s_2, \dots, s_N)$ where $s_n\sigma = n\sigma$ for all $\sigma \in I^{\infty}$. We have the following commutative diagram for all $n = 1, 2, \dots, N$

$$\begin{array}{ccc} I^{\infty} & \xrightarrow{s_n} & I^{\infty} \\ \pi_{\mathcal{F}} \downarrow & & \downarrow \pi_{\mathcal{F}} \\ X & \xrightarrow{f_n} & X \end{array}$$

Following [BEtal11] we define the notion of a section associated with a point-fibred attractor.

Definition 3.1 *Let $\mathcal{F}$ be an iterated function system on a compact Hausdorff space, with point-fibred attractor A , code space I^{∞} , and coding map $\pi_{\mathcal{F}} : I^{\infty} \rightarrow A$. A subset $\Omega \subset I^{\infty}$ is called an address space for $\mathcal{F}$ if $\pi_{\mathcal{F}}(\Omega) = A$ and $\pi_{\mathcal{F}}|_{\Omega} : \Omega \rightarrow A \subset X$ is one-to-one. The corresponding map*

$$\tau : A \rightarrow \Omega, x \mapsto (\pi_{\mathcal{F}}|_{\Omega})^{-1}(x),$$

is called a section of $\pi_{\mathcal{F}}$.

Elementary properties of sections are described in [BEtal11]. A section $\tau : A \rightarrow \Omega \subset I^{\infty}$ is said to be *shift-forward invariant* if $S(\Omega) \subset \Omega$ where $S : I^{\infty} \rightarrow I^{\infty}$ is defined by $S(\sigma) = \omega$ where $\omega_i = \sigma_{i+1}$ for $i = 1, 2, \dots$. Shift-forward invariant sections are related to masks. A *mask* $\mathcal{M} = (M_1, M_2, \dots, M_N)$ is an N -tuple of subsets of A such that $\{M_n\}_{n=1}^N$ is a partition of A with the property $M_n \subset f_n(A)$ for all n . If the maps $f_n|_A : A \rightarrow A$ ($n = 1, \dots, N$) are injective, then we define a *masked dynamical system* for $\mathcal{F}$ to be

$$W_{\mathcal{M}} : A \rightarrow A, M_n \ni x \mapsto f_n^{-1}(x), (n = 1, 2, \dots, N).$$

It is proved in [BEtal11] that, given a mask $\mathcal{M}$, if the maps $f_n|_A : A \rightarrow A$ ($n = 1, 2, \dots, N$) are invertible, we can define a section for $\mathcal{F}$, called a *masked section*, $\tau_{\mathcal{M}}$ for $\mathcal{F}$, by using itineraries of $W_{\mathcal{M}}$, as follows. Let $x \in A$ and let $\{x_k\}_{k=0}^{\infty}$ be the orbit of x under $W_{\mathcal{M}}$; that is, $x_0 = x$ and $x_k = W_{\mathcal{M}}^k(x_0)$ for $k = 1, 2, \dots$. Define

$$\tau_{\mathcal{M}}(x) = \sigma_0\sigma_1\sigma_2\dots \quad (3)$$

where $\sigma_k \in I$ is the unique symbol such that $x_k \in M_{\sigma_k}$ for $k = 0, 1, \dots$

A section for $\pi_{\mathcal{F}}$ is shift-forward invariant if and only if $\tau_{\mathcal{M}}$ is a masked section for some mask $\mathcal{M}$.

Theorem 3.2 *Let the maps $f_n|_A : A \rightarrow A$ ($n = 1, 2, \dots, N$) be invertible.*

- (i) Any mask $\mathcal{M}$ for $\mathcal{F}$ defines a shift-forward invariant section, $\tau_{\mathcal{M}} : A \rightarrow I^\infty$, for $\mathcal{F}$.
- (ii) Let $\Omega_{\mathcal{M}} = \tau_{\mathcal{M}}(A)$. The following diagram commutes:

$$\begin{array}{ccc} \Omega_{\mathcal{M}} & \xrightarrow{S|_{\Omega_{\mathcal{M}}}} & \Omega_{\mathcal{M}} \\ \pi \downarrow \uparrow \tau_{\mathcal{M}} & & \pi \downarrow \uparrow \tau_{\mathcal{M}} \\ A & \xrightarrow{W_{\mathcal{M}}} & A \end{array}$$

- (iii) Any section $\tau : A \rightarrow I^\infty$ for $\mathcal{F}$ defines a mask $\mathcal{M}_\tau$ for $\mathcal{F}$.
- (iv) If the section τ in (iii) is shift-forward invariant, then $\tau = \tau_{\mathcal{M}_\tau}$.

Definition 3.3 Let $\mathcal{F}$ be an IFS on a compact Hausdorff space, with point-fibred attractor A , code space I^∞ , and coding map $\pi_{\mathcal{F}} : I^\infty \rightarrow A$. Let $\tau_{\mathcal{F}} : A_{\mathcal{F}} \rightarrow \Omega_{\mathcal{F}} \subset I^\infty$ be a section of $\pi_{\mathcal{F}}$. Let $\mathcal{G} = (Y; g_1, g_2, \dots, g_N)$ be a point-fibred iterated function system over a compact Hausdorff space Y . Let $A_{\mathcal{G}}$ be the attractor of $\mathcal{G}$. Let $\pi_{\mathcal{G}} : I^\infty \rightarrow A_{\mathcal{G}}$ be the coding map of $\mathcal{G}$. The corresponding fractal transformation is defined to be

$$T_{\mathcal{F}\mathcal{G}} : A_{\mathcal{F}} \rightarrow A_{\mathcal{G}}, x \mapsto \pi_{\mathcal{G}} \circ \tau_{\mathcal{F}}(x).$$

In Theorem 3.4 we describe some key properties of fractal transformations. These properties make fractal transformations applicable to digital imaging, see for example [BEtal11, B10, Nik07].

Theorem 3.4 [BEtal11]. Let $\mathcal{F}$ and $\mathcal{G}$ be point-fibred iterated function systems as in Definition 3.3. Let $\tau : A_{\mathcal{F}} \rightarrow I^\infty$ be a section for $\mathcal{F}$ and let $\Omega = \tau(A)$ be an address space the attractor A of $\mathcal{F}$.

- (i) If Ω is an address space for $\mathcal{G}$ then $\mathcal{T}_{\mathcal{F}\mathcal{G}} : A_{\mathcal{F}} \rightarrow A_{\mathcal{G}}$ is a bijection.
- (ii) If, whenever $\sigma, \omega \in \overline{\Omega}$, $\pi(\sigma) = \pi(\omega) \Rightarrow \pi_{\mathcal{G}}(\sigma) = \pi_{\mathcal{G}}(\omega)$, then $\mathcal{T}_{\mathcal{F}\mathcal{G}} : A_{\mathcal{F}} \rightarrow A_{\mathcal{G}}$ is continuous.
- (iii) If, whenever $\sigma, \omega \in \overline{\Omega}$, $\pi(\sigma) = \pi(\omega) \Leftrightarrow \pi_{\mathcal{G}}(\sigma) = \pi_{\mathcal{G}}(\omega)$, then $\mathcal{T}_{\mathcal{F}\mathcal{G}} : A_{\mathcal{F}} \rightarrow A_{\mathcal{G}}$ is a homeomorphism.
- (iv) If τ is a masked section of $\mathcal{F}$ such that the condition in (iii) holds then the corresponding pair of masked dynamical systems, $W_{\mathcal{M}} : A \rightarrow A$ and, say, $W_{\mathcal{M},\mathcal{G}} : A_{\mathcal{G}} \rightarrow A_{\mathcal{G}}$ are topologically conjugate.

Figure 2 illustrates a fractal transformation applied to a picture. In this example one of the IFSs is affine while the other is bi-affine, see [BVin11a].

In recent work [BEtal10] it is established that interesting fractal homeomorphisms exist between pairs of “overlapping” attractors of IFSs, each comprising a pair of maps. A simple formula for the topological entropy of the associated masked dynamical system is obtained. Current work in this direction involves the search for new families of fractal homeomorphisms and their invariants.

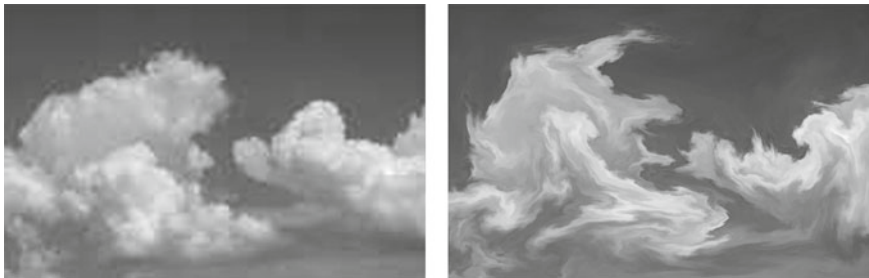


Fig. 2 Before, on *left* (a low resolution digital image of clouds) and after, on *right*, a fractal homeomorphism has been applied. In each case the attractor is a rectangle, represented by the support of the image. The fractal homeomorphism preserves the rectangular shape but modifies internal shapes

4 Progress on a Fundamental Question of Kameyama

Throughout this chapter we suppose that A is a point-fibred attractor of an IFS $\mathcal{F} = (X; f_1, f_2, \dots, f_N)$ on a complete metrizable space X , with $N > 1$. We suppose that the basin of A is X . Kameyama [Kam04] asks the following fundamental question. *Does there exist a metric on A such that $\mathcal{F}|_A$ is contractive and the topology on A induced by this metric is the same as the topology of X restricted to A ?* A more general question is obtained by replacing $\mathcal{F}|_A$ by $\mathcal{F}$; this more general question, in the case where $\mathcal{F}$ comprises one map ($N = 1$), amounts to asking if the converse of Banach’s contraction mapping theorem holds, and was answered by Janos in 1967, [Jan67], by constructing a specific metric that does the job. Janos et al. gave also, ahead of its time, a formulation of Kameyama’s question in [JanEtal79], since for $N = 1$ a strict attractor is nothing else but asymptotically stable fixed point.

Kameyama has shown in [Kam04] that the answer to his question is “No” by demonstrating the existence of an IFS on an equivalence class on code space that has a point-fibred attractor, but for which there exists no metric, compatible with the natural topology on code space, with respect to which the IFS is contractive. But there remains the interesting question of whether the property of being point-fibred implies that there exists a contractive metric. For example, what is the answer to Kameyama’s fundamental question and its generalization if (a) we require that X is a geometrical space such as $\mathbb{R}^2$, or (b) we require that $\mathcal{F}$ comprises maps that belong to a geometrical family, such as (b-i) affines, (b-ii) projectives, or (b-iii) Möbius transformations? The answer to (a) remains unknown, but in a series of recent papers [AtkEtal10, BVin12a, Vin13] the answer to (bi), (bii) and (biii) have been shown to be a gently qualified “Yes”.

In seeking to answer (a) Kameyama has also defined an interesting pseudometric (one which obeys nonnegativity and the triangle inequality, but for which $d(x, y) = 0$ does not imply $x = y$) such that, with respect to his metric $\mathcal{F}|_A$ is contractive. We refer to this pseudometric as *Kameyama’s pseudometric*, and will next describe a

recent generalization of it that yields a “contractive” pseudometric for $\mathcal{F}$ in the case that X is a compact metrizable space. This contractive pseudometric is useful because it is a metric in some interesting cases related to conformal dynamics, see [Kam04, Theorem 4.5].

Let T be the set of all metrics on X that generate the given topology on X .

Theorem 4.1 *There is a metric d in T such that $d(f_n(x), f_n(y)) \leq d(x, y)$ for $x, y \in X$ and $n = 1, 2, \dots, N$.*

The proof is given in [Bigu11]. Here we observe a definition of the metric d . Simply, let $d' \in T$ and define for given $x, y \in X$,

$$d(x, y) = \sup\{d'(f_{\sigma|k}(x), f_{\sigma|k}(y)) : \sigma \in I^\infty, k = 1, 2, \dots\},$$

where for each $\sigma = (\sigma_1, \sigma_2, \dots) \in I^\infty$, $I = \{1, \dots, N\}$, we denote $f_{\sigma|k} = f_{\sigma_1} \circ f_{\sigma_2} \circ \dots \circ f_{\sigma_k}$.

We say that $\mathcal{F}$ is a *metric contractive IFS* if there exists a metric $d \in T$ and real numbers $0 < \alpha_j < 1$ ($j = 1, 2, \dots, N$) such that $d(f_j(x), f_j(y)) \leq \alpha_j d(x, y)$ for every $x, y \in X$. Fix a polyratio $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (0, 1)^N$, i.e., a list of contraction constants. Write

$$\alpha(x, y) = \inf\left\{\prod_{j=1}^k \alpha_{\sigma_j} : x, y \in f_{\sigma|k}(X), \sigma \in I^\infty, k = 1, 2, \dots\right\}.$$

for $x, y \in X$. Now fix an nonexpansive metric d on X , as provided by Theorem 4.1. For $0 < \beta \leq 1$, α as fixed before and $x, y \in X$ we define

$$d_\alpha^\beta(x, y) = \inf \sum_{i=1}^l \alpha(x_i, x_{i+1}) (d(x_i, x_{i+1}))^\beta$$

where the infimum is taken over all finite chains of elements $x_1, x_2, \dots, x_l \in X$ such that $x = x_1$ and $x_l = y$ (there is no restriction on the length of the links in the chain).

Theorem 4.2 *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N) \in (0, 1)^N$ be a polyratio, and let $0 < \beta \leq 1$. Then d_α^β is a pseudometric on X . Furthermore, for $x, y \in X$, $d_\alpha^\beta(f_j(x), f_j(y)) \leq \alpha_j d_\alpha^\beta(x, y)$ for $j = 1, 2, \dots, N$. Moreover, if $d_\alpha^\beta(x, y) > 0$ for all $x, y \in X$ with $x \neq y$, then $d_\alpha^\beta \in T$.*

The proof is given in [Bigu11].

If $N = \beta = 1$ we have Janos’s contractive metric on X . If $\beta = 0$ we obtain Kameyama’s pseudometric on A . By means of our construction we have extended Kameyama’s metric to the whole space X from only the attractor A .

Finally let us note that the similar (in spirit) question of Hata [YaEtal97]: *Whether a continuum can be encoded by IFS?*, has been answered in the series of papers [BaNow13, KulNow12, Kwi99] (cf. [Wad03] for a practical application). Full geo-

metric/topological characterization of those continua which are (homeomorphic to) attractors of IFSs is an open problem.

5 Conley Decomposition Structure and Attractor/Repeller Pairs

Throughout this chapter, X is a compact Hausdorff space and we have an IFS $\mathcal{F} = (X; f_1, f_2, \dots, f_N)$ that comprises homeomorphisms $f_n : X \rightarrow X$ ($n = 1, 2, \dots, N$).

We define the *inverse IFS* $\mathcal{F}^{-1}$ to be $(X; f_1^{-1}, f_2^{-1}, \dots, f_N^{-1})$. We define a *strict repeller* of $\mathcal{F}$ to be a strict attractor of $\mathcal{F}^{-1}$, and vice versa. We define the *basin of a strict repeller* R of $\mathcal{F}$ to be the basin of the strict attractor R of $\mathcal{F}^{-1}$.

We say, following ideas in [McG92] (later developed in [BVin12b]), that $A \in \mathcal{K}(X) \cup \{\emptyset\}$ is a *Conley-McGehee attractor* of the IFS $\mathcal{F} = (X; f_1, \dots, f_N)$ when $F(A) = A$ and there exists an open neighbourhood $U(A) \supset A$, such that

$$\bigcap_n \overline{\bigcup_{k>n} F^k(B)} \subset A, \text{ for all } U(A) \supset B \in \mathcal{K}(X). \quad (4)$$

The *basin* of a Conley-McGehee attractor A is the union of all open neighborhoods $U(A)$ such that (4) holds. We denote the basin of a Conley-McGehee attractor A by $\mathcal{B}(A)$.

We define a *Conley-McGehee repeller* R of $\mathcal{F}$ to be a Conley-McGehee attractor of $\mathcal{F}^{-1}$ and the basin of the Conley-McGehee repeller R to be the basin of R treated as a Conley-McGehee attractor of $\mathcal{F}^{-1}$. We denote the basin of the Conley-McGehee repeller R by $\mathcal{B}(R)$.

The following theorem follows from [McG92]. We write S^C to denote the complement, with respect to X , of $S \subset X$.

Theorem 5.1 *If A is a strict attractor of $\mathcal{F}$, then A is a Conley-McGehee attractor of $\mathcal{F}$. The basin of A (treated as a strict attractor) is the same as the basin $\mathcal{B}(A)$ of the Conley-McGehee attractor A of $\mathcal{F}$. If A is a Conley-McGehee attractor of $\mathcal{F}$ then $A^* := \mathcal{B}(A)^C$ is a Conley-McGehee repeller of $\mathcal{F}$, and $\mathcal{B}(A^*) = A^C$. Moreover $(A^*)^* = A$ (where the second $*$ operation corresponds to $\mathcal{F}^{-1}$) and $\mathcal{B}(A^*) = A^C$.*

The Conley-McGehee repeller A^* provided by Theorem 5.1 is called the dual (Conley-McGehee) repeller corresponding to the (Conley-McGehee) attractor A , and (A, A^*) is called a (Conley-McGehee) *attractor/repeller pair* of $\mathcal{F}$. The set

$$\mathcal{C}(A, A^*) := (A \cup A^*)^C$$

is called the *set of connecting orbits* corresponding to the attractor/repeller pair (A, A^*) .

An attractor/repeller pair for $\mathcal{F}$ is $(X, \emptyset)$ because X is a Conley-McGehee attractor for $\mathcal{F}$, and the corresponding dual repeller is $\emptyset$. But this dual repeller is not a strict dual repeller because it is empty. However, if $\mathcal{F}$ has a strict attractor $A \subsetneq X$ then X is not a strict attractor for $\mathcal{F}$. In particular we observe that if $\mathcal{F}$ has a strict attractor $A \subsetneq X$ then $A^* \neq \emptyset$ and $\mathcal{C}(A, A^*) \neq \emptyset$.

To explain the beautiful and fundamental theorem which follows, we need to define the chain recurrent set $\mathcal{R}(\mathcal{F})$ of $\mathcal{F}$. Here we define the notion in the case that X is a metric space (X, d) , since that suffices for the examples we will mention. A point $x \in X$ is called *chain recurrent* for $\mathcal{F}$ if for every $\varepsilon > 0$ there exists a finite sequences of points $\{p_i \in X : i = 0, 1, \dots, n\}$ with $p_0 = p_n = x$, and a corresponding sequence of indices $\{n_i \in \{1, 2, \dots, N\} : i = 1, 2, \dots, n\}$ satisfying $d(p_{i+1}, f_{n_i}(p_i)) \leq \varepsilon$ for $i = 1, 2, \dots, n - 1$. (Such a sequence of points is called an ε -chain from x to x : similarly we can define an ε -chain from x to y .) The set of chain recurrent points of $\mathcal{F}$ is called the *chain recurrent set* for $\mathcal{F}$ and is denoted by $\mathcal{R}(\mathcal{F})$. More generally, the chain recurrent set can be defined for any iterated closed relation on a compact Hausdorff space, and, in fact, all of the theory that we are describing in this section goes through to iterated closed relations (and much more too: see [McGW06, Wi08]).

Theorem 5.2 (Conley decomposition). [McG92] *Let X be a compact Hausdorff space and let $\mathcal{F} = (X; f_1, f_2, \dots, f_N)$ where $f_n : X \rightarrow X$ is a homeomorphism for all $n = 1, 2, \dots, N$. Let $\mathcal{A}$ denote the set of Conley-McGehee attractors of $\mathcal{F}$. The chain-recurrent set for $\mathcal{F}$ is*

$$\mathcal{R}(\mathcal{F}) = \left(\bigcup_{A \in \mathcal{A}} \mathcal{C}(A, A^*) \right)^C.$$

In particular, if $\mathcal{F}$ possesses a unique Conley-McGehee attractor, then $\mathcal{R}(\mathcal{F}) = A \cup A^*$.

If A is a strict attractor then A is a component of $\mathcal{R}(\mathcal{F})$. The components of $\mathcal{R}(\mathcal{F})$ are defined (in the metric space case) using the equivalence relation defined on $\mathcal{R}(\mathcal{F})$ by $x \sim y$ if and only if there is an ε -chain from x to y .

A pretty family of examples of attractor/repeller pairs is provided by Möbius IFSs. A *Möbius IFS* consists of Möbius transformations on the extended complex plane, equivalently on the Riemann sphere. The simplest possible example is the following one.

Example 5.3. Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere, equivalently the complex plane together with the “point at infinity”. Let $\mathcal{F} = (\widehat{\mathbb{C}}; f_1)$ where $f_1(z) = z/2$, $f_1(\infty) = \infty$. There are two attractor/repeller pairs, $(\widehat{\mathbb{C}}, \emptyset)$ and $(\{0\}, \{\infty\})$. The only strict attractor is $\{0\}$ and its dual Conley-McGehee repeller is a strict repeller. The corresponding set of connecting orbits is $\widehat{\mathbb{C}} \setminus (\{0\} \cup \{\infty\})$ and $\mathcal{R}(\mathcal{F}) = \{0\} \cup \{\infty\}$.

The situation described in this example is, in some senses, typical, as the following result shows.

Theorem 5.4 [Vin13]. *A Möbius IFS has a strict attractor $A \neq \widehat{\mathbb{C}}$ if and only if $\mathcal{R}(\mathcal{F}) \neq \widehat{\mathbb{C}}$, in which case $\mathcal{R}(\mathcal{F}) = A \cup A^*$ where the Conley-McGehee repeller A^* is a strict repeller.*

Pictures of attractor/repeller pairs for Möbius IFSs can be found in [Vin13], as well as an interesting conjecture.

It turns out that the situation for Möbius IFSs is quite special. For affine and projective IFSs which possess a strict attractor not equal to all of the underlying space, the dual (Conley-McGehee) repeller is in general not a strict repeller. We discuss how this works in Chap. 6.

6 Projective IFSs

Here we describe recent progress concerning real projective IFSs. This topic is rich both in geometrical and topological features; including examples of attractor/repeller pairs, construction of contractive metrics, and of an interesting topological invariant. We expect that real projective IFS will be the subject of future research. The basis of this chapter is [BVin12a].

A real *projective IFS* is $\mathcal{F} = (\mathcal{P}, f_1, f_2, \dots, f_N)$ where $\mathcal{P}$ is M -dimensional real projective space and $f_n : \mathcal{P} \rightarrow \mathcal{P}$ ($n = 1, 2, \dots, N$) are real (invertible) projective transformations, where M and N finite positive integers. Notice that, since the range of each projective transformation is $\mathcal{P}$, there does not exist a metric on $\mathcal{P}$ such that f_n is a contraction.

Each function of a projective IFS $\mathcal{F}$ can be represented by an $(M+1) \times (M+1)$ real matrix with nonzero determinant. The adjoint of $\mathcal{F}$ is the real projective IFS defined by the adjoints of these matrices. Similarly, the inverse of $\mathcal{F}$ is the real projective IFS defined by the inverses of these matrices.

It is established in [BVin12a] that a projective IFS possesses at most one strict attractor. Simple examples show that $\mathcal{F}$ may have no strict attractor. (But $\mathcal{F}$ always possesses a Conley-McGehee attractor, namely the space $\mathcal{P}$.) Let $\mathcal{F}$ have a strict attractor A . Then either (i) $A = \mathcal{P}$; or (ii) $A \neq \mathcal{P}$ and $A \cap \mathcal{H} \neq \emptyset$ for all hyperplanes $\mathcal{H} \subset \mathcal{P}$; or (iii) there exists a hyperplane $\mathcal{H} \subset \mathcal{P}$ with $A \cap \mathcal{H} = \emptyset$.

The following theorem characterizes strict attractors in case (iii). Definitions of convex bodies, the notion of $\mathcal{F}$ being contractive on $\overline{U}$, as well as the proof are provided in [BVin12a].

Theorem 6.1 *If $\mathcal{F}$ is a projective IFS on $\mathbb{P}^n$, then the following statements are equivalent.*

1. $\mathcal{F}$ has a strict attractor A that avoids a hyperplane.
2. There is a nonempty open set U that avoids a hyperplane such that $\mathcal{F}(\overline{U}) \subset U$.
3. There is a nonempty finite collection of disjoint convex bodies $\{C_i\}$ such that $\mathcal{F}(\cup_i C_i) \subset \text{int}(\cup_i C_i)$.
4. There is a nonempty open set $U \subset \mathbb{P}^n$ such that $\mathcal{F}$ is contractive on $\overline{U}$.

5. *The adjoint projective IFS $\mathcal{F}^t$ has a strict attractor A^t that avoids a hyperplane.*

The classical projective duality between points and hyperplanes manifests itself in interesting ways in the theory of projective IFSs. Theorem 6.2 below, which depends on statement (5) in Theorem 6.1, is an example.

An interesting feature of projective IFS theory is that if A is a strict attractor of a projective IFS, such that A avoids a hyperplane, then the corresponding dual Conley-McGehee repeller A^* is in general not a strict repeller, but it is a *hyperplane repeller*. The point is that a projective IFS defines another IFS $\widehat{\mathcal{F}}$ that acts on the space of hyperplanes. This new IFS also comprises homeomorphisms and can have well defined attractor/repeller pairs and basins.

Theorem 6.2 (1) *A projective IFS has an attractor that avoids a hyperplane if and only if it has a hyperplane repeller that avoids a point. The basin of attraction of the attractor is the complement of the union of the hyperplanes in the repeller.*

(2) *A projective IFS has a hyperplane attractor that avoids a point if and only if it has a repeller that avoids a hyperplane. The basin of attraction of the hyperplane attractor is the set of hyperplanes that do not intersect the repeller.*

An interesting example of a projective IFS, discussed in [DeL12] with regard to Hausdorff dimensions and related matters, involves the “Cubic Gasket”; the projective IFS on $\mathbb{R}P^2$ corresponds to the three matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

This projective IFS appears to have an “attractor” equal to a “projective Sierpinski triangle”. It is “parabolic”, in the terminology used by [DeL12], because all the eigenvalues of each matrix are equal to one. In particular, the chaos game algorithm applied to this system, and to its adjoint, and to the inverse of it and to the inverse of its adjoint, appears to yield well-defined “attractor/repeller” pairs, as illustrated in Fig. 3. But in fact the only Conley-McGehee attractor in this example is $\mathbb{R}P^2$ with dual repeller equal to the empty set. That is, the chain-recurrent set in this example must be $\mathbb{R}P^2$. The apparent “attractor/repeller” pair meet, whence there is no open neighborhood of the “attractor” from within which all orbits converge to the “attractor”.

More clearly: the reason that the “projective Sierpinski triangle” is not an attractor of the IFS in this last example is the same as the reason that the chain-recurrent set for the projective IFS on $\mathbb{R}P^1$ comprising the single map represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

is equal to $\mathbb{R}P^1$. Despite this, the chaos game will yield the origin as the “attractor”; the orbit of any point will converge to the origin. Similarly for the inverse of this single



Fig. 3 The *right half* of this image illustrates the “attractor” and the “repeller” (shades of gray or colored) for a real projective IFS on $\mathbb{R}P^2$, represented using the disk model, see [BVin12a]. The *left half* illustrates the adjoint system. In both these systems the “attractor” touches the “repeller” and so neither “attractor” can be an attractor (of an IFS)

map system, the orbit of any point will converge to the origin. A related example is the projective IFS on $\mathbb{R}P^1$ defined by two maps, f_1 and f_2 given by the two matrices

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix};$$

equivalently, in the obvious notation, treated as automorphisms on $\mathbb{R} \cup \{\infty\}$,

$$f_1(x) = \frac{x}{x+1} \text{ and } f_2(x) = x+1 \text{ for } x \in \mathbb{R} \cup \{\infty\},$$

for which $[0, \infty]$ is an invariant set but not an attractor. Note however that the restricted IFS $([0, \infty], f_1, f_2)$ does have an attractor, namely $[0, \infty]$.

Theorem 6.2 enables us to associate a geometrical index with a strict attractor that avoids a hyperplane. More specifically, if an attractor A avoids a hyperplane then A lies in the complement of the union of the hyperplanes in the repeller. Since the connected components of this complement form an open cover of A and since A is compact, A is actually contained in a finite set of components of the complement. The number of components is defined to be a geometrical index of A , $\text{index}(A)$. This index is an integer associated with an attractor A , not any particular IFS that generates A . It is shown in [BVin12a] that this index is nontrivial, in the sense that it can take positive integer values other than one. Moreover, it is invariant under $PGL(n+1, \mathbb{R})$, the group of real, dimension n , projective transformations. That is, $\text{index}(A) = \text{index}(g(A))$ for all $g \in PGL(n+1, \mathbb{R})$.

7 The Chaos Game from a Topological Point of View

The path from combinatorics to probability goes through topology.

Let $\mathcal{F} = (X; f_1, \dots, f_N)$ be the IFS (consisting of continuous functions) with the strict attractor $A \in \mathcal{K}(X)$ and its basin of attraction $\mathcal{B}(A)$. According to (1) an infinite word of symbols $(\sigma_1, \sigma_2, \dots) \in I^\infty$, $I = \{1, \dots, N\}$, together with an initial point $x_0 \in B(A)$ give rise to the *orbit* :

$$\begin{cases} x_0 \in B(A), \\ x_n := f_{\sigma_n}(x_{n-1}), \quad n \geq 1. \end{cases} \quad (5)$$

Note that unlike in symbolic dynamics, here we compose functions in a forward direction: $x_n = f_{\sigma_n} \circ \dots \circ f_{\sigma_1}(x_0)$. Under appropriate conditions of probabilistic nature (whence we often speak about random orbits) such an orbit shall fill densely the attractor revealing its 'shape'. This procedure has been coined the *chaos game algorithm* [B93].

It turns out that the random iteration procedure has a universal character: if a strict attractor exists, then it can be discovered using random orbits under very mild assumptions ([BVin11], cf. [B93] and [LasMac94] Theorem 12.8.2 for earlier versions). On the other hand it was observed (often experimentally) that in some cases the algorithm does not produce full picture of the attractor. Since the algorithm is probabilistic, the quality of the random number generator has been blamed for the effect. We give below a purely deterministic combinatorial answer to these issues. Early observations and explanations of this phenomenon have been provided for example in [Goo91, HogMcF94].

From a combinatorial version of the chaos game algorithm there smoothly follows a topological (Baire category like) version and consequently a stochastic version. The direction from topology to probability might seem unexpected, if not impossible (after incompatibility of measure and category clarified in [Oxt80]), but replacing topological meagre sets with sets satisfying a metric albeit stronger porosity property makes this transfer possible, sometimes even customary.

We say that the infinite word $(\sigma_1, \sigma_2, \dots) \in I^\infty$ is *disjunctive* [CalEtal97, Sta02] if it contains all possible finite words, i.e.,

$$\forall_{m \geq 1} \forall_{w \in \{1, \dots, N\}^m} \exists_{j \geq 1} \forall_{l=1, \dots, m} \sigma_{(j-1)+l} = w_l.$$

A natural example of disjunctive sequence is known in number theory as *Champernowne normal number* : it is obtained by writing down all one-symbol words, then all two-symbol words, and so on.

Subset $\Psi \subset M$ of a metric space M is *porous* ([Zaj05], [Luc06, Chap. 11.2]) when

$$\exists_{0 < \lambda < 1} \exists_{r_0 > 0} \forall_{\psi \in \Psi} \forall_{0 < r < r_0} \exists_{v \in M} N_{\lambda r}\{v\} \subset N_r\{\psi\} \setminus \Psi.$$

A countable union of porous sets is called *σ -porous*.

We note that σ -porous sets have first Baire category and, in Euclidean space, have null Lebesgue measure. Porosity is used in optimization, e.g. [Luc06], in fractal geometry and analysis, e.g. [Cho09, MauUrb03], and elsewhere.

Theorem 7.1 (Combinatorial & Topological Chaos Game, [BL13]). *Let A be a strict attractor of an IFS $\mathcal{F}$ that is strongly-fibred. If the orbit $(x_n)_{n=0}^\infty$ is generated in (5) via a disjunctive sequence of symbols, then its tails*

$$\{x_n : n \geq p\} \xrightarrow[p \rightarrow \infty]{} A \quad (6)$$

converge to the attractor with respect to the Hausdorff distance. In particular

$$A = \bigcap_{p=1}^{\infty} \overline{\bigcup_{n=p}^{\infty} \{x_n\}} \quad (7)$$

holds.

Moreover the set of words $(\sigma_n)_{n=1}^\infty \in I^\infty$ which fail to have this property of revealing the attractor of the IFS via (6) is σ -porous (in particular it has the first Baire category) in the code space I^∞ equipped with the Baire metric ρ .

The verification that any disjunctive sequences has a combinatorial property which causes the corresponding orbit to fill densely arbitrary small neighbourhood of the attractor relies on delicate compactness and continuity conditions.

Let $Z_n : (S, \text{Pr}) \rightarrow \{1, \dots, N\}$, $n = 1, 2, \dots$, be a sequence of random variables on a probability space (S, Pr) , the time-discrete stochastic process with values in $I = \{1, \dots, N\}$. The process generates a sequence of symbols $(\sigma_1, \sigma_2, \dots) \in I^\infty$, i.e., $\sigma_n = Z_n(s)$ if event $s \in S$ happens at the n -th stage.

A stochastic process $(Z_n)_{n \geq 1}$ will be called here *disjunctive* if it generates disjunctive sequence $(\sigma_n)_{n=1}^\infty \in I^\infty$ as its outcome with probability 1.

Examples of disjunctive stochastic processes include:

1. Bernoulli schemes ([B93]);
2. homogeneous Markov chains with positive transition probabilities;
3. chain with complete connections with positively-minorized transition probabilities ([BVin11]);
4. nonhomogeneous Bernoulli scheme with probability of success in the n -th experiment not falling below $1/\log(n)$.

Establishing disjunctiveness of a process may be elementary (via Borel-Cantelli lemma) but may involve also more refined techniques (like coupling of random variables [L12]). An elegant method of transferring probabilistic results from the realm of Baire category can be based on the interplay between measure and category spelled out in [MerEtal03]. For accessibility we give a convenient formulation of this principle

Theorem 7.2 ([MerEtal03]Propositions 3.5 & 3.3). Let μ be the completion of a Borel probability measure on a separable metric space M which satisfies the doubling condition

$$\exists_{r_0, c > 0} \forall_{\psi \in M} \forall_{0 < r < r_0} \mu(N_{2r}\{\psi\}) \leq c \cdot \mu(N_r\{\psi\}).$$

If $\Psi \subset M$ is σ -porous set, then $\mu(\Psi) = 0$.

This was used in [BL13] to justify disjointness of minorized chains which appear in [BVin11]. Some early impression of this kind may be found in [CalEtal97], Chap. 4.2.

Theorem 7.3 (Stochastic Chaos Game). Let A be a strict attractor of an IFS $\mathcal{F}$ that is strongly-fibred. If the stochastic process $(Z_n)_{n \geq 1}$ with values in $\{1, \dots, N\}$ generating a random orbit (5) is disjunctive, then (6) and (7) in the statement of Theorem 7.1 hold with probability 1.

One might naively expect that the ergodicity is the driving force of the almost sure convergence of random orbit. Nevertheless it should be stressed that the ergodicity of IFS is different from ergodicity of the stochastic process underlying random orbit generation. Moreover the stochastic process, which is a strongly mixing (hence ergodic) homogeneous Markov chain, may be not disjunctive ([Lot05] Example 1.8.1). Also a sequence of independent random variables may be 'insufficiently random' to produce disjunctive outcomes (consider nonhomogeneous Bernoulli scheme with success probabilities decaying too fast).

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