Chapter 2
Fundamental Mathematics

This chapter gives a short introduction to fundamental mathematical concepts that are used in the computational methods treated in this book. These concepts are complex numbers, vectors, matrices, and graphs. Vectors and matrices belong to the field of linear algebra. For more information on linear algebra, see for example [1], which includes an appendix on complex numbers. For more information on spectral graph theory, see for example [2].

2.1 Complex Numbers

A complex number $\alpha \in \mathbb{C}$, is a number

$$\alpha = \mu + \imath\nu, \tag{2.1}$$

with $\mu, \nu \in \mathbb{R}$, and $\imath$ the imaginary unit[1] defined by $\imath^2 = -1$. The quantity $\operatorname{Re}\alpha = \mu$ is called the real part of α, whereas $\operatorname{Im}\alpha = \nu$ is called the imaginary part of the complex number. Note that any real number can be interpreted as a complex number with the imaginary part equal to 0.

Negation, addition, and multiplication are defined as

$$-(\mu + \imath\nu) = -\mu - \imath\nu, \tag{2.2}$$

$$\mu_1 + \imath\nu_1 + \mu_2 + \imath\nu_2 = (\mu_1 + \mu_2) + \imath(\nu_1 + \nu_2), \tag{2.3}$$

$$(\mu_1 + \imath\nu_1)(\mu_2 + \imath\nu_2) = (\mu_1\mu_2 - \nu_1\nu_2) + \imath(\mu_1\nu_2 + \mu_2\nu_1). \tag{2.4}$$

[1] The imaginary unit is usually denoted by i in mathematics, and by j in electrical engineering because i is reserved for the current. In this book, the imaginary unit is sometimes part of a matrix or vector equation where i and j are used as indices. To avoid ambiguity, the imaginary unit is therefore denoted by $\imath$ (iota).

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The complex conjugate is an operation that negates the imaginary part:

$$\overline{\mu + \iota\nu} = \mu - \iota\nu. \tag{2.5}$$

Complex numbers are often interpreted as points in complex plane, i.e., 2-dimensional space with a real and imaginary axis. The real and imaginary part are then the Cartesian coordinates of the complex point. That same point in the complex plane can also be described by an angle and a length. The angle of a complex number is called the argument, while the length is called the modulus:

$$\arg(\mu + \iota\nu) = \tan^{-1}\frac{\nu}{\mu}, \tag{2.6}$$

$$|\mu + \iota\nu| = \sqrt{\mu^2 + \nu^2}. \tag{2.7}$$

Using these definitions, any complex number $\alpha \in \mathbb{C}$ can be written as

$$\alpha = |\alpha| e^{\iota\varphi}, \tag{2.8}$$

where $\varphi = \arg\alpha$, and the complex exponential function is defined by

$$e^{\mu+\iota\nu} = e^{\mu}(\cos\nu + \iota\sin\nu). \tag{2.9}$$

2.2 Vectors

A vector $\mathbf{v} \in K^n$ is an element of the n-dimensional space of either real numbers ($K = \mathbb{R}$) or complex numbers ($K = \mathbb{C}$), generally denoted as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \tag{2.10}$$

where $v_1, \ldots, v_n \in K$.

Scalar multiplication and vector addition are basic operations that are performed elementwise. That is, for $\alpha \in K$ and $\mathbf{v}, \mathbf{w} \in K^n$,

$$\alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \vdots \\ \alpha v_n \end{bmatrix}, \quad \mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}. \tag{2.11}$$

The combined operation of the form $\mathbf{v} := \alpha\mathbf{v} + \beta\mathbf{w}$ is known as a vector update. Vector updates are of $O(n)$ complexity, and are naturally parallelisable.

A linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in K^n$ is an expression

$$\alpha_1 \mathbf{v}_1 + \ldots + \alpha_m \mathbf{v}_m, \tag{2.12}$$

with $\alpha_1 \ldots \alpha_m \in K$. A set of m vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in K^n$ is called linearly independent, if none of the vectors can be written as a linear combination of the other vectors.

The dot product operation is defined for real vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ as

$$\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} v_i w_i. \tag{2.13}$$

The dot product is by far the most used type of inner product. In this book, whenever we speak of an inner product, we will be referring to the dot product unless stated otherwise. The operation is of $O(n)$ complexity, but not naturally parallelisable. The dot product can be extended to complex vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}$ as $\mathbf{v} \cdot \mathbf{w} = \sum_{i=1}^{n} \overline{v}_i w_i$.

A vector norm is a function $\|.\|$ that assigns a measure of length, or size, to all vectors, such that for all $\alpha \in K$ and $\mathbf{v}, \mathbf{w} \in K^n$

$$\|\mathbf{v}\| = 0 \Leftrightarrow \mathbf{v} = \mathbf{0}, \tag{2.14}$$

$$\|\alpha \mathbf{v}\| = |\alpha| \, \|\mathbf{v}\|, \tag{2.15}$$

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|. \tag{2.16}$$

Note that these properties ensure that the norm of a vector is never negative. For real vectors $\mathbf{v} \in \mathbb{R}^n$ the Euclidean norm, or 2-norm, is defined as

$$\|\mathbf{v}\|_2 = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}. \tag{2.17}$$

In Euclidean space of dimension n, the Euclidean norm is the distance from the origin to the point $\mathbf{v}$. Note the similarity between the Euclidean norm of a 2-dimensional vector and the modulus of a complex number. In this book we omit the subscripted 2 from the notation of Euclidean norms, and simply write $\|\mathbf{v}\|$.

2.3 Matrices

A matrix $A \in K^{m \times n}$ is a rectangular array of real numbers ($K = \mathbb{R}$) or complex numbers ($K = \mathbb{C}$), i.e.,

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \tag{2.18}$$

with $a_{ij} \in K$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

A matrix of dimension $n \times 1$ is a vector, sometimes referred to as a column vector to distinguish it from a matrix of dimension $1 \times n$, which is referred to as a row vector. Note that the columns of a matrix $A \in K^{m \times n}$ can be interpreted as n (column) vectors of dimension m, and the rows as m row vectors of dimension n.

A dense matrix is a matrix that contains mostly nonzero values; all n^2 values have to be stored in memory. If most values are zeros the matrix is called sparse. For a sparse matrix A, the number of nonzero values is denoted by $\text{nnz}(A)$. With special data structures, only the $\text{nnz}(A)$ nonzero values have to be stored in memory.

The transpose of a matrix $A \in K^{m \times n}$, is the matrix $A^T \in K^{n \times m}$ with

$$\left(A^T\right)_{ij} = (A)_{ji}. \tag{2.19}$$

A square matrix that is equal to its transpose is called a symmetric matrix.

Scalar multiplication and matrix addition are elementwise operations, as with vectors. Let $\alpha \in K$ be a scalar, and $A, B \in K^{m \times n}$ matrices with columns $\mathbf{a}_i, \mathbf{b}_i \in K^m$ respectively, then scalar multiplication and matrix addition are defined as

$$\alpha \mathbf{A} = \begin{bmatrix} \alpha \mathbf{a}_1 & \dots & \alpha \mathbf{a}_n \end{bmatrix}, \tag{2.20}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} \mathbf{a}_1 + \mathbf{b}_1 & \dots & \mathbf{a}_n + \mathbf{b}_n \end{bmatrix}. \tag{2.21}$$

Matrix-vector multiplication is the product of a matrix $A \in K^{m \times n}$ and a vector $\mathbf{v} \in K^n$, defined by

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1i} v_i \\ \vdots \\ \sum_{i=1}^{n} a_{mi} v_i \end{bmatrix}. \tag{2.22}$$

Note that the result is a vector in K^m. An operation of the form $\mathbf{u} := A\mathbf{v}$ is often referred to as a matvec. A matvec with a dense matrix has $O(n^2)$ complexity, while with a sparse matrix the operation has $O(\text{nnz}(A))$ complexity. Both dense and sparse versions are naturally parallelisable.

Multiplication of matrices $A \in K^{m \times p}$ and $B \in K^{p \times n}$ can be derived as an extension of matrix-vector multiplication by writing the columns of B as vectors $\mathbf{b}_i \in K^p$. This gives

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \dots \mathbf{b}_n \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 \dots A\mathbf{b}_n \end{bmatrix}. \tag{2.23}$$

The product AB is a matrix of dimension $m \times n$.

The identity matrix I is the matrix with values $I_{ii} = 1$, and $I_{ij} = 0,\ i \neq j$. Or, in words, the identity matrix is a diagonal matrix with every diagonal element equal to 1. This matrix is such, that $IA = A$ and $AI = A$ for any matrix $A \in K^{m\times n}$ and identity matrices I of appropriate size.

Let $A \in K^{n\times n}$ be a square matrix. If there is a matrix $B \in K^{n\times n}$ such that $BA = I$, then B is called the inverse of A. If the inverse matrix does not exist, then A is called singular. If it does exist, then it is unique and denoted by A^{-1}. Calculating the inverse has $O(n^3)$ complexity, and is therefore very costly for large matrices.

The column rank of a matrix $A \in K^{m\times n}$ is the number of linearly independent column vectors in A. Similarly, the row rank is the number of linearly independent row vectors in A. For any given matrix, the row rank and column rank are equal, and can therefore simply be denoted as rank (A). A square matrix $A \in K^{n\times n}$ is invertible, or nonsingular, if and only if rank $(A) = n$.

A matrix norm is a function $\|.\|$ such that for all $\alpha \in K$ and $A, B \in K^{m\times n}$

$$\|A\| \geq 0, \tag{2.24}$$

$$\|\alpha A\| = |\alpha|\,\|A\|, \tag{2.25}$$

$$\|A + B\| \leq \|A\| + \|B\|. \tag{2.26}$$

Given a vector norm $\|.\|$, the corresponding induced matrix norm is defined for all matrices $A \in K^{m\times n}$ as

$$\|A\| = \max\left\{\|A\mathbf{v}\| : \mathbf{v} \in K^n \text{ with } \|\mathbf{v}\| = 1\right\}. \tag{2.27}$$

Every induced matrix norm is submultiplicative, meaning that

$$\|AB\| \leq \|A\|\|B\| \text{ for all } A \in K^{m\times p},\ B \in K^{p\times n}. \tag{2.28}$$

2.4 Graphs

A graph is a collection of vertices, any pair of which may be connected by an edge. Vertices are also called nodes or points, and edges are also called lines. The graph is called directed if all edges have a direction, and undirected if they do not. Graphs are often used as the abstract representation of some sort of network. For example, a power system network can be modelled as an undirected graph, with buses as vertices and branches as edges.

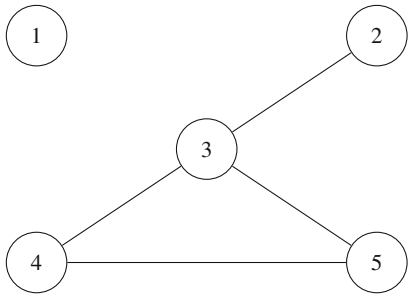


Fig. 2.1 A simple graph

Let $V = \{v_1, \ldots, v_N\}$ be a set of N vertices, and $E = \{e_1, \ldots, e_M\}$ a set of M edges, where each edge $e_k = (v_i, v_j)$ connects two vertices $v_i, v_j \in V$. The graph G of vertices V and edges E is denoted as $G = (V, E)$. Figure 2.1 shows a graph $G = (V, E)$ with vertices $V = \{1, 2, 3, 4, 5\}$ and edges $E = \{(2, 3), (3, 4), (3, 5), (4, 5)\}$.

The incidence matrix A of a graph $G = (V, E)$ is an $M \times N$ matrix in which each row i represents an edge $e_i = (p, q)$, and is defined as

$$a_{ij} = \begin{cases} -1 & \text{if } p = v_i, \\ 1 & \text{if } q = v_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

In other words, row i has value -1 at index p and value 1 at index q. Note that this matrix is unique for a directed graph. For an undirected graph, some orientation has to be chosen. For example, the matrix

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{2.30}$$

is an incidence matrix of the graph in Fig. 2.1. Such a matrix is sometimes referred to as an oriented incidence matrix, to distinguish it from the unique unoriented incidence matrix, in which all occurrences of -1 are replaced with 1. Note that some authors define the incidence matrix as the transpose of the matrix A defined here.

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