Preface

The concept of a topological manifold has been around since the middle of the nineteenth century: in his doctoral thesis "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse" of 1851, Bernhard Riemann introduced what he called "Mannigfaltigkeit," translated into English as "manifold." He discussed the concept further in his famous 1854 inaugural lecture *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*.

In those early days topological concepts were still being developed, especially around cardinality and infinite sets so such concepts as second countability would only be vague notions if thought of at all. In the early 1880s, Georg Cantor broke significant new ground in his rigorous discussion of cardinality and he surely introduced the first non-metrisable manifold, the long ray, in his 1883 paper *Ueber unendliche lineare Punktmannichfaltigkeiten*, Mathematische Annalen **21** pp 545–586.

While connected manifolds of dimension 1 and compact manifolds (surfaces) of dimension 2 were well understood by the end of the nineteenth century, manifold theory as a legitimate area of study really got off the ground with Henri Poincaré's conjectured homological characterisation of the 3-sphere in 1900. Of course Poincaré himself provided a counterexample to his original conjecture, but he tightened his conjecture by assuming a homotopy condition and kept mathematicians very busy studying compact manifolds in all dimensions for the next 100 years.

Non-metrisable manifolds did not get so much attention during the first half of the twentieth century, though new examples, such as the Prüfer manifold, were discovered. With the discovery that the Continuum Hypothesis is independent of the usual axioms of Set Theory in the early 1960s, those interested in non-metrisable manifolds began to realise that Set Theory provided not only an impediment but also another tool in the study of non-metrisable manifolds. This was cemented into place by the theorem of Mary Ellen Rudin that perfectly normal manifolds are metrisable in certain Set Theories and the counterexample to this theorem described by Rudin and Phillip Zenor; both in the 1970s.

Following on from the work of Rudin and Rudin/Zenor, Set Theoretic Topology really took off and the use of Set Theory as a further tool in the study of non-metrisable manifolds has been shown to be invaluable.

Prior to the 1970s, topologists had thought about non-metrisable manifolds, even if they were only to dismiss them. Most books on topology will at least mention the long line as a useful counterexample to a number of propositions. In his 1969 MIT Lecture Notes, found at http://www.foliations.org/surveys/FoliationLectNotes_Milnor.pdf, John W. Milnor had this to say on page seven when introducing a codimension one foliation of a 3-manifold in which there is only one leaf:

> The main object of this exercise is to imbue the reader with a suitable respect for non-paracompact[1] manifolds.

Later, in 1976, Morris W. Hirsch on page 32 of his book "Differential Topology," had this to say to justify his convention to restrict his attention to paracompact manifolds:

> Manifolds that are not paracompact are amusing, but they never occur naturally. What is perhaps worse, it is difficult to prove anything about them.

In a very brief discussion of manifolds, at Weisstein, Eric W. "Topological Manifold." From MathWorld—A Wolfram Web Resource. http://mathworld.wolfram.com/TopologicalManifold.html there is this comment about non-metrisable manifolds:

> Nonparacompact manifolds are of little use in mathematics…

In Appendix A to volume 1 of his "A comprehensive introduction to Differential Geometry," Michael Spivak devotes almost 20 pages to a study of the long line and the Prüfer manifold and some of their properties (including construction of a differential structure on the former). By 1984, sufficient theory had been developed for Peter Nyikos to write a 50-page chapter entitled "Non-metrizable manifolds" for the Handbook of Set-Theoretic Topology in which he included his structure theorem (the "Bagpipe Theorem") for a natural non-metrisable generalisation of compact surfaces, as well as many other interesting results and examples.

Perhaps what delayed the study of non-metrisable manifolds is the need for two main tools in their study. Whereas much of the success resulting in the tremendous strides in the study of metrisable manifolds was the application of Algebraic Topology, that tool on its own seems to be inadequate for non-metrisable manifolds. The second important tool (indeed, it seems currently to be of more use than Algebraic Topology) is Set Theory but the realisation of its importance did not really come until the 1970s.

In this book, like Spivak and Nyikos, we prefer to follow Milnor's philosophy. Non-metrisable manifolds **are** interesting and you **can** prove things about them. Mostly, we do not use Set Theory seriously but do make use especially of properties of the countable ordinals. This book also shows bias towards the study of non-metrisable manifolds undertaken by the author and his students at Auckland as well as collaborators in the Northern Hemisphere. It is aimed at an audience of

[1] As we shall see in Chap. 2, paracompactness and metrisability are equivalent for Hausdorff manifolds.

people who have perhaps encountered manifolds as topological objects and are curious about what happens beyond the wall of metrisability.

Chapter 1 introduces manifolds and presents some standard constructions of non-metrisable manifolds, especially of Prüfer, Moore and Nyikos. We also discuss some basic properties of the long line.

Chapter 2 explores the frontier between metrisable and non-metrisable manifolds. Not surprisingly, when we confine our attention to manifolds many topological properties which are distinct in wider contexts coincide. At one extreme, if a manifold is metrisable (and connected) then it embeds properly in some euclidean space. At the other extreme, a very weak form of paracompactness called linear ω_1-metaLindelöfness is sufficient to ensure metrisability. In the 1960s, Milnor introduced the concept of a microbundle but the theory came to a halt when James Kister showed that microbundles are fibre bundles; we include in this chapter the result that Kister's equivalence holds precisely when the underlying manifold is metrisable. We also relate metrisability to properties of function spaces on the manifold and topological games played on the manifold or its function spaces.

Chapter 3 brings together some useful geometric tools which are also of use for those working in metrisable manifolds: Morton Brown's result that a countable union of open n-cells is an open n-cell and his collaring theorem; and a brief discussion of handlebodies.

Chapter 4 looks at a large class of manifolds called Type I by Nyikos in his 1984 chapter before specialising to his Bagpipe Theorem: that every ω-bounded surface is made up of a standard compact surface with boundary (the bag) together with finitely many long pipes. The ω-bounded property is equivalent to compactness in a metric space so ω-bounded surfaces might be seen as a natural extension of compact surfaces to the non-metrisable context. While our proof follows Nyikos's proof loosely our use of handlebodies and homology theory does open up the possibility of its generalisation to higher dimensions. We complete the chapter by showing that there are $2^{\aleph_1}$ many such surfaces, which contrasts with the compact case where there are only countably many.

Chapter 5 looks at dynamics on non-metrisable manifolds, especially discrete dynamics, i.e., homeomorphisms. Emphasis is on homeomorphisms of powers of the long line where there is a significant contrast with powers of the real line. The diagonals $y = \pm x$ in the long plane form significant barriers to the behaviour of a homeomorphism of the long plane, with similar constraints being imposed in higher dimensions. Perhaps also surprising is the fact that any homeomorphism of the long plane maps arbitrarily large squares to themselves, again with similar results in higher dimensions. As a result we can classify homeomorphisms of powers of the long plane up to isotopy.

Chapter 6 addresses the question, dating back to the 1930s, whether perfectly normal manifolds need be metrisable. We give details of the construction of the Rudin-Zenor surface mentioned above: it is a perfectly normal, non-metrisable surface and requires the Continuum Hypothesis for its construction. We also present

Rudin's proof that perfectly normal manifolds are metrisable when one assumes Martin's Axiom and the negation of the Continuum Hypothesis.

Chapter 7 looks at differential structures, especially on the long line and the long plane. As already noted, Spivak explored differential structures on the long line in one of his books introducing Differential Geometry. Nyikos took this a lot further with a long paper looking at various ways of constructing differential structures on the long line. While we do not give all of the details we do discuss Nyikos's result that there are $2^{\aleph_1}$ many mutually non-diffeomorphic differential structures on the long line. The chapter also describes exotic differential structures on the long plane, and again there are $2^{\aleph_1}$ many of them. Exotic differential structures were first described by Milnor, on the 7-sphere. More recently exotic differential structures were described by Donaldson, Freedman, Kirby et al. on $\mathbb{R}^4$. Since metrisable manifolds of dimension at most three carry essentially unique differential structures there can be no exotic structures on metrisable manifolds of dimension at most three.

Chapter 8 looks at foliations in the non-metrisable context. We exhibit a 2-dimensional foliation of a 3-manifold which has only a single leaf, something which is impossible when we confine our attention to metrisable manifolds, or, as we show, when the leaves are 1-dimensional. However, most of Chap. 8 is in the context of the long plane. As for homeomorphisms of the long plane, the diagonals $y = \pm x$ form significant barriers. Whereas the real plane carries infinitely many distinct foliations, the long plane carries only two, or six if we puncture the long plane.

In Chap. 9 we relax the hypothesis that our manifolds must be Hausdorff. Here I would agree with Hirsch that it is hard to prove anything about them. Indeed, whereas in dimension 1 there are only four connected 1-manifolds, relax the Hausdorff condition and there is no limit. We discuss some possibilities. Hausdorff manifolds are homogeneous in the sense that for any two points there is a homeomorphism sending one point to the other (and even interchanging them if the dimension is at least two), but we exhibit a non-Hausdorff 1-manifold which is rigid in the sense that the only self-homeomorphism is the identity. There is a close connection between non-Hausdorff 1-manifolds and foliations of the plane. Our rigid 1-manifold leads to the description of a 1-dimensional foliation of the plane which is rigid in the sense that any homeomorphism which respects the foliation maps each leaf to itself.

The book is rounded out with two appendices, one giving an overview of the topological background assumed and the other some Set Theory.

There are many people I should thank for this work. First off must be the Set-Theoretic topologist supreme, the late Professor Mary Ellen Rudin. I was only vaguely aware of what was going on in the area until she visited Auckland during the winter of 1988 and gave very inspiring and stimulating lectures on Set Theory and manifolds. Later, she welcomed me to Madison and introduced me to my first research problem in the area. Peter Nyikos of the University of South Carolina has also been a helpful stimulus, during many visits to Auckland as well as welcoming me to Columbia. Those who have listened to my talks at conferences and seminars in various locations and have offered valuable criticism also deserve my thanks. Much of the work on foliations was undertaken in Geneva and I thank Mathieu Baillif and Alexandre Gabard for their kindness as hosts as well as the stimulating conversations. Satya Deo welcomed me to India, as well as visiting me in Auckland and I thank him for our interesting discussions. Locally, I want to thank especially my colleague and former Ph.D. student Sina Greenwood as well as my other Ph.D. students who have been involved in aspects of this work, including Abdul Mohamad, Sunanda Dikshit, and Afshin Mardani.

In addition to support from the University of Auckland I want to acknowledge the Marsden Fund of the Royal Society of New Zealand and the New Zealand Institute of Mathematics and its Applications for financial support for some parts of the work undertaken.

Last, but not least, I thank my wife Rachel for the support, both moral and intellectual (such as listening attentively to my attempts to verbalise ideas as we tramped along mountain tracks or streams), she has given me during the long drawn-out writing of this book.

Kohimarama, June 2014 David Gauld

http://www.springer.com/978-981-287-256-2

Non-metrisable Manifolds
Gauld, D.
2014, XVI, 203 p. 51 illus., 6 illus. in color., Hardcover
ISBN: 978-981-287-256-2