On Exponential Stability of Stochastic Control Systems

Fakhreddin Abedi and Wah June Leong

Abstract In this paper, we study the exponential input-to-state stability in probability of a wider class of composite stochastic control system. Our aim is to establish sufficient conditions for exponential input-to-state stability in probability of this composite system. We also give a numerical example illustrating our results.

1 Introduction

In this paper, we study the exponential input-to-state stability in probability in the r-th mean (rEISSP) of a composite stochastic control system (CSCS). We employ the stochastic version of control Lyapunov function and extend the exponential input-to-state stability in probability (EISSP) results proved by Spiliotis and Tsinias [11] for stochastic control system (SCS) to the class of CSCS driven by two independent Wiener processes. We establish sufficient conditions for rEISSP of CSCS.

Recently, Spiliotis and Tsinias [11] used the notion of control Lyapunov function and established the rEISSP of stochastic differential system (SDS). The asymptotic and exponential stability of stochastic systems have been derived by Mao [8], Liu and Raffool [7], Lan and Dang [6], Abedi et al. [1], Khasminiskii [4] and Kushner [5]. Michel [9] established asymptotic and exponential stability in probability for some classes of continuous and discrete parameters in stochastic composite system. The sufficient conditions for exponential stability in probability of stochastic system and composite stochastic system have been developed by Boulanger [3] and Rusinek [10].

In this paper, we first introduce a class of CSCS and also recall some basic definitions and results concerning rEISSP property. Finally, we state and prove the main results of the paper.

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2 Exponential Stability

Let $(\boldsymbol{\Omega}, F, P)$ be a complete probability space and $(w_t)_{t\geq 0}$ denote a standard R^m-valued Wiener process defined on this space.

We consider the SDS

$$dx = f(x)dt + h(x)dw_t, \tag{1}$$

where the following conditions hold true:

(i) x is given in R^n,
(ii) $f : R^n \to R^n$ and $h : R^n \to R^{n\times m}$ are Lipschitz functionals in R^n with $f(0) = 0, h(0) = 0$, and there exists a constant $C \geq 0$ such that the following growth condition holds true:

$$|f(x) - f(y)| + |h(x) - h(y)| \leq C|x - y|, \quad \forall x, y \in R^n. \tag{2}$$

We recall the following definition of exponential stability in mean square given by Khasminskii [4] as follows.

Definition 1 The equilibrium state $x_t = 0$ of the SDS (1) is exponentially stable in probability in mean square if, and only if, there exist constants $c_1, c_2 > 0$ such that

$$E(|x_t(t, t_0, x)|^2) \leq c_1|x|^2 e^{-c_2(t-t_0)}, \quad \forall x \in R^n, \quad t \geq t_0. \tag{3}$$

Let $(\beta_t)_{t\geq 0}$ denote a standard R^q-valued Wiener process defined on (Ω, F, P). Now, we consider the SCS

$$d\upsilon = F(\upsilon, u)dt + G(\upsilon)d\beta_t, \tag{4}$$

where
(1) u is a R^p valued measurable control law,
(2) $F : R^r \times R^p \to R^r$ and $G : R^r \to R^r \times R^q$ are Lipschitz functionals mapping with $F(0, 0) = G(0) = 0$, and there exists a constant $C \geq 0$ such that for any $\upsilon \in R^r$ and $u \in R^p$ the following growth condition holds true:

$$|F(\upsilon, u)| + |G(\upsilon)| \leq C(1 + |\upsilon| + |u|).$$

For simplicity, let $V(t) = V(t, t_0, \upsilon, u)$ and we recall the following definition of rESP derived by Spiliotis and Tsinias [11] as follows.

Definition 2 The equilibrium state $\upsilon_t = 0$ of the SCS (4) is rESP for some $r > 0$ if, and only if, there exist constants $c_1, c_2 > 0$ such that

$$E(|V(t)|^r) \leq c_1|\upsilon|^r e^{-c_2(t-t_0)}, \quad \forall \upsilon \in R^r, \quad t \geq t_0. \tag{5}$$

Definition 3 The SCS (4) is said to satisfy the exponential Lyapunov condition if there exists a Lyapunov function $\Phi : R^r \to R^+$ of class $C^2(R^r\backslash\{0\})$ and positive constants $a_i, 1 \le i \le 5$, such that

$$a_1|\upsilon|^r \le \Phi(\upsilon) \le a_2|\upsilon|^r, \tag{6}$$

$$|\nabla\Phi(\upsilon)| \le a_3|\upsilon|^{r-1}, \quad |\nabla^2\Phi(\upsilon)| \le a_4|\upsilon|^{r-2}, \tag{7}$$

$$\mathbf{Y}\Phi(\upsilon) = \sum_{i=1}^{n} F(\upsilon,u)\frac{\partial\Phi(\upsilon)}{\partial\upsilon_i} + \frac{1}{2}\sum_{i,j=1}^{n} G(\upsilon)G(\upsilon)^T\frac{\partial^2\Phi(\upsilon)}{\partial\upsilon_i\partial\upsilon_j} \le -a_5|\upsilon|^r, \tag{8}$$

where $\mathbf{Y}$ is the infinitesimal generator for the stochastic process solution of the SCS (4).
Note that Definition 2 obtained in Spiliotis and Tsinias [11] is an extension of Definition 1 established in Khasminskii [4].

Definition 4 The equilibrium state $\upsilon_t = 0$ of the SCS (4) is rEISSP for some $r > 0$, if there exist a positive definite function $\gamma : R^+ \to R^+$ and constants $c_1, c_2 > 0$ such that (5) holds true and

$$|u(t)| \le \gamma(|V(t)|), \quad \forall\upsilon \in R^r, \quad t \ge t_0. \tag{9}$$

We shall now turn the attention to a general composite stochastic system.

Let $\{\beta_t, t \in R^+\}$ be a standard R^q-valued Wiener process defined on the space (Ω, F, P) independent of the Wiener process $\{w_t, t \in R^+\}$. Consider the pair of stochastic processes solution $(x_t, \upsilon_t) \in R^n \times R^r$ of the CSCS

$$\begin{cases} dx = (f(x) + g(x,\upsilon)D\upsilon)dt + h(x)dw_t, \\ d\upsilon = F(\upsilon,u)dt + G(\upsilon)d\beta_t, \end{cases} \tag{10}$$

where
(1) $x \in R^n$, $\upsilon \in R^r$, and D is a matrix function with value in $M_{r\times r}(R)$,
(2) f and h are functionals in $C^2(R^n, R^n)$ and $C^2(R^n, R^{n\times m})$, respectively, such that $f(0) = 0$ and $h(0) = 0$,
(3) $g : R^n \times R^r \to R^{n\times r}$ is Lipschitz functional mapping such that there exists a nondecreasing scalar function $\alpha(|\upsilon|) \ge 0$ bounded for all υ such that

$$|g(x,\upsilon)| \le \alpha(|\upsilon|)|x|, \quad \forall(x,\upsilon) \in R^n \times R^r,$$

(4) u is a R^p-valued measurable control law,
(5) $F : R^r \times R^p \to R^r$ and $G : R^r \to R^{r\times q}$ are Lipschitz functionals mapping vanishing at the origin and there exists a constant $C \ge 0$ such that the following growth condition holds true:

$$|F(\upsilon,u)| + |G(\upsilon)| \le C(1 + |\upsilon| + |u|), \quad \forall(\upsilon,u) \in R^r \times R^p. \tag{11}$$

Suppose that there exist functionals $F_1 : R^r \to R^r$ and $F_2 : R^r \to R^{r\times p}$ such that

$$F(\upsilon,u) = F_1(\upsilon) + F_2(\upsilon)u,$$

and

$$d\upsilon = (F_1(\upsilon) + F_2(\upsilon)u)dt + G(\upsilon)d\beta_t, \tag{12}$$

for any $(\upsilon, u) \in R^r \times R^p$. Then, the CSCS (10) is rewritten as

$$\begin{cases} dx = (f(x) + g(x,\upsilon)D\upsilon)dt + h(x)dw_t, \\ d\upsilon = (F_1(\upsilon) + F_2(\upsilon)u)dt + G(\upsilon)d\beta_t. \end{cases} \tag{13}$$

Considering the EISSP results in [11], we can easily establish the following elementary result.

Lemma 1 *The CSCS (13) satisfies the rEISSP property if and only if* $0 \in R^n$ *is rESP for the system*

$$\begin{cases} dx = (f(x) + g(x,\upsilon)D\upsilon)dt + h(x)dw_t, \\ d\upsilon = (F_1(\upsilon) + F_2(\upsilon)\gamma(|\upsilon|)\theta)dt + G(\upsilon)d\beta_t, \end{cases} \tag{14}$$

$$\theta \in I = \{\theta \in R^p : |\theta| \le 1\}.$$

In the next section, we will derive a state feedback law that renders the satisfaction of rEISSP property for CSCS (13).

3 Main Results

In the following theorem, we suppose that the function g is bounded on $R^n \times R^p$, U is the set of admissible control, and sufficient conditions are established for rEISSP property of CSCS (13). Theorem 1 is the stochastic extension of Proposition 4.1 and Theorem 4.1 established in Spiliotis and Tsinias [11] and Boulanger [3], respectively, to the CSCS (13). In addition, we can consider the exponential stability in mean square results of Boulanger [3] as a special case of our rEISSP results (Theorem 1) where $r = 2$.

Theorem 1 *Suppose that the function* $\gamma : R^+ \to R^+$ *is positive definite with bounded first derivative* $\gamma^{(1)}$*, and for all* $|u| \le \gamma(|\upsilon|)$*, there exists a* $C^2(R^r\backslash\{0\})$ *function* Φ *satisfying the exponential Lyapunov condition. In addition, suppose that*

$$D\upsilon = F_2(\upsilon)^T \nabla \Phi_2(\upsilon). \tag{15}$$

Then, the control law

$$k(x,\upsilon) = k_1(\upsilon) - g(x,\upsilon)^T \nabla \Phi_1(x), \tag{16}$$

where Φ_1 *is a smooth Lyapunov function corresponding to the SDS (1) and* $k_1 : R^r \to R^p$ *is a control law corresponding to the closed-loop system deduced from (12), which guarantees that the CSCS (13) satisfies the rEISSP property.*

Proof In the proof of this theorem, we shall employ Lemma 1 and show that the rEISSP is satisfied for the CSCS (13), if and only if the origin is rESP for the CSCS (14). Clearly, by our hypothesis, the function $\Phi(V(t))$ satisfies the exponential Lyapunov condition with respect to the CSCS (14). We now show that the origin satisfies rESP for the CSCS (14). Since the exponential Lyapunov condition is fulfilled for the CSCS (14), there exists a Lyapunov function $\Phi(V(t))$ and positive constants $a_i, 1 \leq i \leq 5$, such that (6)–(8) hold true. Consider the composite Lyapunov function

$$\Phi(V(t)) = \Phi_1(x) + \Phi_2(\upsilon), \quad \forall(x, \upsilon) \in R^n \times R^r, \tag{17}$$

where Φ_1 and Φ_2 is the Lyapunov function corresponding to the SDS (1) and SCS (12), respectively. Substituting $\theta(x, \upsilon) = \frac{k(x,\upsilon)}{\gamma(|\upsilon|)}$ into the closed-loop system deduced from CSCS (14), we obtain

$$\begin{aligned} dx &= (f(x) + g(x, \upsilon)D\upsilon)dt + h(x)dw_t, \\ d\upsilon &= (F_1(\upsilon) + F_2(\upsilon)k_1(\upsilon) - F_2(\upsilon)g(x, \upsilon)^T \nabla\Phi_1(x) + G(\upsilon)d\beta_t. \end{aligned} \tag{18}$$

Denoting $\mathbf{D}_\upsilon$ as the infinitesimal generator of the stochastic process solution of the closed-loop system (18) yields

$$\begin{aligned} \mathbf{D}_\upsilon \Phi(V(t)) = \mathbf{D}\Phi_1(x) + \nabla\Phi_1(x)^T g(x, \upsilon)D\upsilon + \mathbf{Y}_2\Phi_2(\upsilon) \\ - \nabla\Phi_2(\upsilon)^T F_2(\upsilon)g(x, \upsilon)^T \nabla\Phi_1(x), \end{aligned} \tag{19}$$

Substituting (15) into (19) we get

$$\mathbf{D}_\upsilon \Phi(V(t)) = \mathbf{D}\Phi_1(x) + \mathbf{Y}_2\Phi_2(\upsilon). \tag{20}$$

Then, from (8) and (20) we have

$$\begin{aligned} \frac{d}{dt} E\Phi(V(t)) &= E(\mathbf{D}_\upsilon \Phi(V(t))) \\ &= E(\mathbf{D}\Phi_1(x) + \mathbf{Y}_2\Phi_2(\upsilon)) \leq -a_5 E(|x|^r) - a_5' E(|\upsilon|^r). \end{aligned} \tag{21}$$

The desired condition (5) is a direct consequence of inequality (6) and (21). Thus, CSCS (14) satisfies the rESP property at the origin. It turns out by Lemma 1 that CSCS (13) satisfies the rEISSP property.

Remark 1 (1) Theorem 1 is the stochastic extension of Proposition 4.1 and Theorem 4.1 established in Spiliotis and Tsinias [11] and Boulanger [3], respectively, to the CSCS (13). In addition, we can consider the exponential stability in mean square results of Boulanger [3] as a special case of our rEISSP results (Theorem 1) where $r = 2$.

(2) The exponential stability results obtained in [3], [10], [11] do not permit us to make a conclusion about rEISSP while the results of this paper are still valid.

Finally, we give a numerical example illustrating our results.

Example 1 Consider the multi-input composite system

$$\begin{cases} dx = (-2x + 2\upsilon)dt + x dw_t, \\ d\upsilon = (-2\upsilon + \upsilon u)dt + 2\upsilon d\beta_t, \end{cases} \tag{22}$$

where $\{w_t, t \in R^+\}$ and $\{\beta_t, t \in R^+\}$ are two independent standard real-valued Wiener processes, $x, \upsilon \in R$, u is a real-valued measurable control law, $g(x, \upsilon) = 2$, and $D = I$ is the identity matrix in $M_{r\times r}(R)$. Consider the Lyapunov function

$$\Phi(V(t)) = \Phi_1(x) + \Phi_2(\upsilon) = \frac{1}{2}(x^2 + \upsilon^2).$$

A simple calculation shows that the Lyapunov function $\Phi(V(t))$ satisfies the exponential Lyapunov condition and

$$k_1(\upsilon) = \xi(\mathbf{D}\Phi_2(\upsilon), (\mathbf{D}_z\Phi_2(\upsilon))^2)(\mathbf{D}_z\Phi_2(\upsilon)) = -\frac{1}{1+\upsilon^2}, \tag{23}$$

is a control law corresponding to the closed-loop system deduced from

$$d\upsilon = (-2\upsilon + \upsilon u)dt + 2\upsilon d\beta_t, \tag{24}$$

where

$$\mathbf{D}_z\Phi_2(\upsilon) = \sum_{i=1}^{n} K_z^i(\upsilon)\frac{\partial\Phi_2(\upsilon)}{\partial\upsilon_i} = \upsilon^2, \tag{25}$$

$$\mathbf{D}\Phi_2(\upsilon) = \sum_{i=1}^{n} H^i(\upsilon)\frac{\partial\Phi_2(\upsilon)}{\partial\upsilon_i} + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{m} M_k^i M_k^j \frac{\partial^2\Phi_2(\upsilon)}{\partial\upsilon_i\partial\upsilon_j} = 0, \tag{26}$$

$$\xi(\mathbf{D}\Phi_2, \mathbf{D}_z\Phi_2) = \begin{cases} -\frac{\mathbf{D}\Phi_2 + \sqrt{\mathbf{D}\Phi_2^2 + \mathbf{D}_z\Phi_2^2}}{\mathbf{D}_z\Phi_2(1+\sqrt{1+\mathbf{D}_z\Phi_2})} & \mathbf{D}_z\Phi_2 > 0, \\ 0 & \mathbf{D}_z\Phi_2 = 0, \end{cases} \tag{27}$$

and $H(\upsilon) = -2\upsilon, K(\upsilon) = \upsilon, M(\upsilon) = 2\upsilon$.

(For more information about the control law $k_1(\upsilon)$, we refer to Theorem 3.2 obtained by Abedi et al. [2]). Then, by Theorem 1, the control law

$$\begin{aligned} k(x, \upsilon) &= k_1(\upsilon) - g(x, \upsilon)^T \nabla\Phi_1(x) \\ &= -\frac{1}{1+\upsilon^2} - 2x, \end{aligned} \tag{28}$$

renders the composite system (22) satisfying the rEISSP property.

4 Conclusions

In this paper, we have studied the exponential input-to-state stability in probability of a class of composite stochastic system. We have employed the stochastic version of control Lyapunov function and extended the EISSP results proved by Spiliotis and Tsinias [11] for SCS to the class of composite stochastic system driven by two independent Wiener processes. We have also established sufficient conditions for the rEISSP of composite system.

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