

# Preface

Even though this domain seems to be classic, because of the last two decades of remarkable progress in computational technique, many open problems in Diophantine Analysis made a vigorous comeback with the hope that this increased power in computation will shade new light on them. The distinctive dynamic of Diophantine Analysis is expressively reflected in the well-known review journals: “Mathematical Reviews” (USA) and “Zentralblatt für Mathematik” (now zbMATH) (Germany), the following AMS Subject Classification 2000 being designated to it: 11Dxx (Diophantine equations), 11D09 (Quadratic bilinear equations), 11A55 (Continued fractions), 11J70 (Continued fractions and generalizations), 11Y65 (Continued fraction), 11B37 (Recurrences), 11B39 (Fibonacci and Lucas numbers and polynomials and generalizations), 11R27 (Units and factorization). While in 1912 a first monograph [224] dedicated to this subject synthesized the important results known by then, a variety of papers and doctoral theses in Diophantine Analysis followed. Some of them are very recent, and we mention here just the monograph [32], published in 2003 and the syntheses [111] and [226], published in 2002.

The volume has seven chapters and moves gradually from the history and motivation of quadratic Diophantine equations to more complicated equations and their applications. The accessibility has also been taken into account, in the sense that some material could be understood even by nonspecialists in the field. In many sections the history of the advances and the problems that still remain open are also given, together with comments. Throughout this work, complete proofs are given and the sign  $\square$  denotes the end of a proof.

Chapter 1, entitled “Why Quadratic Diophantine Equations?,” describes the motivation of studying quadratic Diophantine equations. It contains ten short sections, each of them illustrating an important problem whose solution reduces to solving such an equation. Section 1.1 introduces the famous theorem of A. Thue [62] that reduces a Diophantine equation of the form  $F(x, y) = m$  to a quadratic one, pointing out the importance of the quadratic case. Section 1.2 reviews Hilbert’s tenth problem, which asks, in essence, for a finite algorithm to solve a general multivariable Diophantine equation. The proof of the impossibility of determining such an algorithm was found by Yu. Matiyasevich, who used the results of M.

Davis, H. Putnam, and J. Robinson. A crucial point in Matiyasevich's proof is the consideration of the solutions to the equations  $x^2 - (a^2 - 1)y^2 = 1$ , where  $a \geq 2$ . Section 1.3 presents the connection between Euler concordant forms and elliptic curves, the determination of the ranks of Mordell–Weil groups, and Frey curves, problems that lead to studying the solutions to two simultaneous Pell's equations. Section 1.4 makes the connection with the spectrum of the Laplace–Bertrami operator whose spectrum description involves solving a general Pell's type equation. In Section 1.5, a quadrature formula in the  $s$ -dimensional unit cube is considered, where the determination of parameters that minimize the remainder also reduces to solving some special Pell's type equations. The last five sections of this chapter briefly review: reduction algorithms of “threshold” phenomena in arbitrary lattices, the study of Einstein type Riemann metrics on homogeneous manifolds generated by the action of a semisimple Lie group, the problem of counting the number of autointersection points of the geodesics closed in the modular group  $SL(2, \mathbb{Z})$ , Hecke groups and their connection with continued fractions, as well as  $(m, n)$  type sets in the projective plane, all of these representing examples of problems from various areas of mathematics, whose solutions require solving Pell's type equations. Clearly, the list of such problems can be continued.

Chapter 2, “Continued Fractions, Diophantine Approximation and Quadratic Rings,” presents two basic instruments of investigation in Diophantine Analysis. The theory of continued fractions plays an important part in pure Mathematics and has multiple applications. The main results of this theory, which are necessary in developing efficient algorithms in Diophantine Analysis, are given in Section 2.1. Among the references used, we mention here: [1, 46, 141, 159, 164, 183, 208]. The following aspects are introduced: the Euclid algorithm and its connection with continued fractions, the problem of uniqueness of a continued fraction developing, infinite continued fractions and their connection with irrational numbers, the approximation of irrational numbers by continued fractions, the problem of the best approximation and Hurwitz's Theorem, periodic continued fractions. The second important tool in studying problems in Diophantine Analysis is the theory of quadratic rings. The fundamental concepts, the units and norms defined in a natural way in this context are featured in Section 2.2. From the rich bibliography devoted to this subject, we mention [95, 171, 198].

Chapter 3, “Pell's Equation,” is divided into six sections. The first is a comprehensive historical introduction to Pell's equation. Section 3.2 considers the problem of finding the general solution to Pell's equation by using elementary methods. The proof of the main theorem is based on our papers [13–15]. The main forms of writing the general solution are given: by recursive sequences, in matrix form, explicitly, etc. In Section 3.3, the general solution to Pell's equation is obtained by using the method of continued fractions. In Section 3.4, following the papers [171] and [95], the same problem is solved by using the theory of quadratic fields. Section 3.5 contains original contributions to the study of the more general equation  $ax^2 - by^2 = 1$ . Theorem 3.5.2 shows how one can determine the general solution to this equation, in case of solvability. The proof is the one given in our papers [13–15]. We note the fact that from our explicit form we found for the general

solution to this equation (Remark 2) one gets immediately the result in [219], which is obtained there by a very complicated and unnatural method. The last section is dedicated to the negative Pell's equation  $x^2 - Dy^2 = -1$ . It is studied as a special case of the equation in the previous section and the central result concerning it is contained in Theorem 3.6.1. The formulas (3.6.3) and (3.6.4) give the general solution explicitly. The presentation follows again our papers [13–15]. Using now our paper [18], in Theorem 3.6.4 it is given a family of negative Pell's equations, solvable only for a single value of the positive integral parameter  $k$ . The section ends with the presentation of the current stage of the problem of solvability of the negative Pell's equation, an open problem that is far from being settled, and one of the most difficult in Diophantine analysis. In this respect, partial results, as well as recent conjectures, are mentioned.

The main goal of Chapter 4, “General Pell's Equation,” is to present the general theory and major algorithms regarding the equation  $x^2 - Dy^2 = N$ . This chapter contains nine sections. In Section 4.1, the theory is exposed in a personal manner; the classes of solutions are defined, and Theorems 4.1.1, 4.1.2, and 4.1.3 give classical bounds for the fundamental solutions. These bounds were recently improved in [76] (L. Panaitopol, personal communication, December 2001). The section ends with our results concerning a problem proposed in [37], problem that we solve completely (see [16]). In addition, we present a final description of the set of all rational solutions to the Pell's equation, from which one can see clearly the complexity of the problem of determining this set explicitly. Also, an interesting result proved in [72] about the solvability of some general Pell's equation is mentioned. Section 4.2 contains results about the solvability of the general Pell's equation, and it is organized into five subsections dealing with the following aspects: Pell Decision Problem and the Square Polynomial Problem, the Legendre test, Legendre unsolvability tests, modulo  $n$  unsolvability tests, extended multiplication principle. Section 4.3 contains an algorithm for determining the fundamental solutions to the general Pell's equation, based on continued fractions. This algorithm is known as the LMM method. Numerical examples that probe the efficiency of the algorithm are also included. Section 4.4 deals with the problem of solving the general Pell's equation by using the PQa algorithm, derived from the theory of continued fractions, as well. A variant of the PQa algorithm for solving the negative Pell's equation, and the special Pell's equations  $x^2 - Dy^2 = \pm 4$  are also given. Later in this section, the problem of the structure of the solutions to the general Pell's equation is taken on. By using the PQa algorithm, we study the problem of determining the fundamental solutions in the case  $N < \sqrt{D}$  and then consider several numerical examples that illustrate how this algorithm works. In some of these examples, one compares the efficiency of the algorithm PQa versus the one of the LMM's method. Section 4.5 is dedicated to the study of the solvability and unsolvability of the equation  $ax^2 - by^2 = c$ . All results here belong to us and are based on our papers [13, 14, 17]. Two general methods for solving the above-mentioned equation are presented, and a complete answer to the problem posed in the recent paper [114] is given. An original point of view is contained in Theorem 4.5.2, where the solvability of this equation is linked to

the solvability of two other quadratic equations. In Theorem 4.5.3, a large class of solvable equations  $ax^2 - by^2 = c$  is given. Section 4.6 deals with the problem of solving the general Pell's equation by using the theory of quadratic rings. One obtains a different algorithm for solving the general Pell's equation than the one described in Section 4.7. The main goal of Section 4.8 is to discuss the more general equation  $ax^2 + bxy + cy^2 = N$ . Recently, this equation captured the attention of mathematicians (see, for example, the doctoral thesis [157]). The last section in this chapter is dedicated to the connection between the Thue's theorem (Theorem 4.9.1) and the equations  $x^2 - Dy^2 = \pm N$ . We discuss here the equation of this form with  $D = 2, 3, 5, 6, 7$ , the working method being directly obtained from the Thue's theorem.

Chapter 5 is called "Equations Reducible to Pell's Type Equations." In Section 5.1, we present the equations  $x^2 - kxy^2 + y^4 = 1$  and  $x^2 - kxy^2 + y^4 = 4$ , the main results being contained in Theorems 5.1.1, 5.1.7, and 5.1.9. Section 5.2 is dedicated to the equation  $x^{2n} - Dy^2 = 1$  and the central results are given in Theorems 5.2.3 and 5.2.4. Two special equations that finally lead to Pell's type equations are studied in Sections 5.3 and 5.4. Our point of view contained in section 5.5 encompasses in a unitary class several equations dispersed in the literature, for instance the equations in the titles of subsections 5.5.1, 5.5.2, 5.5.3, and 5.5.4. Section 5.6 points out other quadratic equations with infinitely many integral solutions. In subsection 5.6.1, we rely on our paper [12]. The main result concerning the equation  $x^2 + axy + y^2 = 1$  is given in Theorem 5.6.1. In subsection 5.6.2, we study the equation (5.6.8), our results correcting the ones in [173]. Other interesting equations of this type, both solvable, are (5.6.13) and (5.6.14). The equation (5.6.15) is studied in our paper [8], where five distinct infinite families of solutions are displayed. Based on our paper [9], we find nine different infinite families of positive integral solutions to the equation (5.6.18). The main idea of the paper [9] was used in [79] to generate six infinite families of positive integral solutions to the equation (5.6.24). By using a result in our paper [10], in the last subsection of Section 5.6 we prove that equation (5.6.24) has in certain conditions infinitely many integral solutions.

In Chapter 6, "Diophantine Representations of Some Sequences," we study a first class of applications of some theoretical results presented in the previous chapters. In Section 6.1, we define the concept of Diophantine  $r$ -representability, making the connection with the papers [27, 28] and the doctoral Dissertation [47]. From Theorem 6.1.1, in Section 6.2 we then obtain as special cases some properties concerning Fibonacci's, Lucas', and Pell's sequences, given in (6.2.1), (6.2.2), and (6.2.3). In Section 6.3, we reconsider in an original manner the study of the equations  $x^2 + axy + y^2 = \pm 1$ . The central result is contained in Theorem 6.3.1 and its proof is based on solving the special Pell's equation  $u^2 - 5v^2 = -4$ . All results in Section 6.4, concerning the equation (6.4.1) and its connection with the Diophantine representation of the Fibonacci, Lucas, and Pell sequences are original. The method we employ is different, more natural, and simpler than the one in the doctoral dissertation [47]. Section 6.5 deals with the Diophantine representation of the generalized Lucas sequences, defined by us in (6.5.1). The main results, given in Theorems 6.5.1 and 6.5.2, generalize the ones in Section 6.4. Some special cases are

considered in the papers [96, 97, 134] and [61], while a particular definition of Lucas sequences is given in [91]. Important results spelling out the conditions in which the solutions to the equation (6.5.8) are linear combinations with rational coefficients of Fibonacci and Lucas classic sequences are contained in Theorems 6.5.4 and 6.5.5. These results belong to us as well and appear in our papers [19] and [20].

The last chapter, “Other Applications,” contains five sections. Based on our papers [13] and [14], we extend the results in [122] and [123] concerning the conditions in which the numbers  $an + b$  and  $cn + d$  are simultaneously perfect squares for infinitely many values of the positive integer  $n$ . The main result is given in Theorem 7.1.1 and is based on Theorem 4.5.3. Several special cases appear in [25, 199] and [40]. Section 7.2 is dedicated to the study of some special properties of the triangular numbers. In Theorem 7.2.1, one determines all such numbers that are perfect squares, while in Theorem 7.2.2 one studies an equation solvable in the set of triangular numbers. The fact that the asymptotic density of the triangular numbers is equal to 0 is proved in what follows. Theorem 7.2.3 specifies that equation (7.2.9) is solvable for infinitely many triples  $(m, n, p)$  of positive integers and unsolvable for infinitely many triples  $(m, n, p)$  of positive integers. In [170], it is shown (Theorem 7.2.4) that any positive rational number  $r$  with  $\sqrt{r} \notin \mathbb{Q}$  can be written as a ratio of two triangular numbers. The proof of this theorem uses in an essential way our result contained in Theorem 4.5.3. Also in this section, we solve completely the problem of finding all triangular numbers  $T_m, T_n$  such that  $T_m/T_n$  is the square of a positive integer. Our approach generalizes the method given in [139] and [101]. In Section 7.3, we study some properties of the polygonal numbers that generalize the ones concerning triangular numbers (Theorem 7.3.1). In Section 7.3 we present some important results about powerful numbers. Recall that a positive integer  $r$  is a powerful number if  $p^2$  divides  $r$  whenever the prime  $p$  divides  $r$ . We prove the theorem of asymptotic behavior of the function  $k(x)$ , where  $k(x)$  is the number of powerful numbers less than or equal to  $x$  and, finally, we present results concerning consecutive powerful numbers and the possible distances between two powerful numbers. These results are also based on the general theory of quadratic equations.

In the last section, we present the solution to an open problem involving matrices in the ring  $M_2(\mathbb{Z})$ . The approach uses the properties of some quadratic Diophantine equations, and it was given in the recent paper [29].

**Acknowledgements** We feel deeply honored and grateful to Professor Preda Mihăilescu for his Foreword.

Special thanks are given to Dr. Diana Savin and to the anonymous referee for the pertinent suggestions and remarks that directly contributed to the improvement of the manuscript.

Finally, we would like to thank Dr. John P. Robertson for his useful suggestions.

Richardson, TX, USA  
Cluj-Napoca, Romania  
March 2015

Titu Andreescu  
Dorin Andrica



<http://www.springer.com/978-0-387-35156-8>

Quadratic Diophantine Equations

Andreescu, T.; Andrica, D.

2015, XVIII, 211 p., Hardcover

ISBN: 978-0-387-35156-8