Chapter 2
Topological and Metric Approach Spaces

Well, I use the metric system. It's the only way to get really exact numbers.

(Catherynne M. Valente, in The Girl Who Fell Beneath Fairyland and Led the Revels There)

As every mathematician knows, nothing is more fruitful than these obscure analogies, these indistinct reflections of one theory into another, these furtive caresses, these inexplicable disagreements …

(André Weil)

Both topological and metric spaces can be viewed as special types of approach spaces. More precisely, both the categories of topological spaces and continuous maps, Top, and of (quasi)-metric spaces and non-expansive maps, $(q)\mathsf{Met}$, can actually be embedded as full and isomorphism-closed subcategories of App. In this chapter we will see various characterizations of topological and of (quasi-)metric spaces as approach spaces and we will see exactly how Top and $q\mathsf{Met}$ (respectively Met) are embedded in App. For Top the embedding will turn out to be both concretely reflective and concretely coreflective. For both Met and $q\mathsf{Met}$ the embedding will turn out to be concretely coreflective but not reflective. We will demonstrate that it is precisely the failure of Met and $q\mathsf{Met}$ to be embedded reflectively in App which makes the theory of approach spaces particularly interesting in any situation in mathematics where initial structures of (quasi-)metric or (quasi-)metrizable topological spaces occur.

2.1 Topological Approach Spaces

As far as notation is concerned, from now on, whenever we say that X is a topological space, $\mathscr{T}$ will stand for the collection of open sets. Structures derived from $\mathscr{T}$, such as the associated closure operator, will be denoted, for example, by $\mathrm{cl}_{\mathscr{T}}$. If no confusion can arise we may also drop reference to $\mathscr{T}$ altogether. We put usc (respectively lsc) for upper semicontinuous (respectively lower semicontinuous) and $(\mathscr{V}_{\mathscr{T}}(x))_x$ or shortly $(\mathscr{V}(x))_x$ for the neighbourhood filters of a topological space. Given a quasi-metric d,

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we let $\mathscr{T}_d$ stand for the underlying topology. Given a filter $\mathscr{F}$ in a topological space, the set of adherence points of $\mathscr{F}$ is denoted by adh$\mathscr{F}$ and the set of limit points by $\lim \mathscr{F}$. That a filter $\mathscr{F}$ converges to a point x is written as $\mathscr{F} \to x$ and that it adheres to x is written as $\mathscr{F} \rightsquigarrow x$.

Given a topological space $(X, \mathscr{T})$ we associate with it a natural approach space in the following way. Let

$$\delta_{\mathscr{T}} : X \times 2^X \longrightarrow \mathbb{P} : (x, A) \mapsto \begin{cases} 0 & x \in \mathrm{cl}_{\mathscr{T}}(A), \\ \infty & x \notin \mathrm{cl}_{\mathscr{T}}(A). \end{cases}$$

2.1.1 Proposition *If $(X, \mathscr{T})$ is a topological space, then the function*

$$\delta_{\mathscr{T}} : X \times 2^X \longrightarrow \mathbb{P}$$

is a distance on X and the associated structures are given as follows.

1. *For any filter $\mathscr{F}$ on X: $\alpha_{\mathscr{T}}\mathscr{F} = \theta_{\mathrm{adh}\,\mathscr{F}}$ and $\lambda_{\mathscr{T}}\mathscr{F} = \theta_{\lim \mathscr{F}}$.*
2. *For any $x \in X$: $\mathscr{A}_{\mathscr{T}}(x) := \left\{\varphi \in \mathbb{P}^X \mid \varphi(x) = 0, \varphi \text{ usc in } x\right\}$ and a basis is given by $\mathscr{B}_{\mathscr{T}}(x) := \{\theta_V \mid V \in \mathscr{V}_{\mathscr{T}}(x)\}$.*
3. *$\mathscr{G}_{\mathscr{T}} := \{d \in q\mathsf{Met}(X) \mid \mathscr{T}_d \subseteq \mathscr{T}\}$.*
4. *The tower is given by the family $(\mathfrak{t}_\varepsilon^{\mathscr{T}})_{\varepsilon\in\mathbb{R}^+}$ where for each $\varepsilon \in \mathbb{R}^+$, $\mathfrak{t}_\varepsilon^{\mathscr{T}}$ coincides with $\mathrm{cl}_{\mathscr{T}}$.*
5. *For any $\mu \in \mathbb{P}^X$ and $x \in X$: $\mathfrak{l}_{\mathscr{T}}(\mu)(x) := \sup\limits_{V\in\mathscr{V}_{\mathscr{T}}(x)} \inf\limits_{y\in V} \mu(y)$ i.e. $\mathfrak{l}_{\mathscr{T}}(\mu)$ is the largest lower semicontinuous function smaller than μ.*
6. *$\mathfrak{L}_{\mathscr{T}} := \{\mu \in \mathbb{P}^X \mid \mu \text{ lsc}\}$.*
7. *For any $\mu \in \mathbb{P}_b^X$ and $x \in X$: $\mathfrak{u}_{\mathscr{T}}(\mu)(x) := \inf\limits_{V\in\mathscr{V}_{\mathscr{T}}(x)} \sup\limits_{y\in V} \mu(y)$ i.e. $\mathfrak{u}_{\mathscr{T}}(\mu)$ is the smallest upper semicontinuous function larger than μ.*
8. *$\mathfrak{U}_{\mathscr{T}} := \{\mu \in \mathbb{P}_b^X \mid \mu \text{ usc}\}$.*
9. *For any functional ideal $\mathfrak{I}$ on X: $\mathfrak{I} \rightarrowtail x$ if and only if $\mathfrak{f}_\alpha(\mathfrak{I}) \longrightarrow x$ for all $\alpha \in [c(\mathfrak{I}), \infty[$.*

Proof (D1) and (D2) are immediate. (D3) follows from the fact that, for any $A, B \subseteq X$, we have $\mathrm{cl}_{\mathscr{T}}(A \cup B) = \mathrm{cl}_{\mathscr{T}}(A) \cup \mathrm{cl}_{\mathscr{T}}(B)$. (D4) follows from the fact that, for all $\varepsilon < \infty$, $A^{(\varepsilon)} = \mathrm{cl}_{\mathscr{T}}(A)$ and $A^{(\infty)} = X$.

1. We give the proof for the adherence operator; the one for the limit operator is precisely the same. It follows from 1.2.64 that,

$$\alpha_{\mathscr{T}}\mathscr{F} = \sup_{F\in\mathscr{F}} (\delta_{\mathscr{T}})_F = \sup_{F\in\mathscr{F}} \theta_{\mathrm{cl}_{\mathscr{T}}(F)} = \theta_{\bigcap\limits_{F\in\mathscr{F}} \mathrm{cl}_{\mathscr{T}}(F)} = \theta_{\mathrm{adh}\,\mathscr{F}}.$$

2. From 1.2.33 it follows that, for all $x \in X$,

$$\mathscr{A}_{\mathscr{T}}(x) = \left\{ \varphi \in \mathbb{P}^X \mid \forall A \subseteq X : x \in \mathrm{cl}_{\mathscr{T}}(A) \Rightarrow \inf_{a \in A} \varphi(a) = 0 \right\}.$$

If $\varphi \in \mathscr{A}_{\mathscr{T}}(x)$, then, since $x \in \mathrm{cl}_{\mathscr{T}}(\{x\})$, it follows that $\varphi(x) = 0$. Now let $\alpha > 0$. If $x \in \mathrm{cl}_{\mathscr{T}}(\{\varphi \geq \alpha\})$, then

$$\inf_{a \in \{\varphi \geq \alpha\}} \varphi(a) = 0,$$

which is absurd. Hence, for all $\alpha > 0$, $x \in \mathrm{int}_{\mathscr{T}}(\{\varphi < \alpha\})$, which proves that φ is upper semicontinuous in x. The converse follows from the fact that if φ is upper semicontinuous in x and $\varphi(x) = 0$, then, for any $\alpha > 0$, $\{\varphi < \alpha\}$ is a neighbourhood of x. That $\mathscr{B}_{\mathscr{T}}(x)$ is a basis for $\mathscr{A}_{\mathscr{T}}(x)$ follows easily from the definitions.

3. From 1.2.4 we obtain that

$$\mathscr{G} = \left\{ d \in q\mathsf{Met}(X) \mid \ x \in \mathrm{cl}_{\mathscr{T}}(A) \Rightarrow \delta_d(x, A) = 0 \right\}$$

from which the result immediately follows.

4. For any $\varepsilon \in \mathbb{R}^+$ and $A \subseteq X$, we have

$$\begin{aligned} \mathrm{t}_{\varepsilon}^{\mathscr{T}}(A) &= \left\{ x \in X \mid \ \delta_{\mathscr{T}}(x, A) \leq \varepsilon \right\} \\ &= \left\{ x \in X \mid \ \delta_{\mathscr{T}}(x, A) = 0 \right\} \\ &= \mathrm{cl}_{\mathscr{T}}(A). \end{aligned}$$

5. The formula follows from 1.2.38. The alternative description is well known and can be found for instance in Bourbaki (1960).

6. This follows from 1.2.24.

7. This is analogous to 5.

8. This follows from 1.2.29.

9. This follows from 1.2.58. □

An approach space of type $(X, \delta_{\mathscr{T}})$ for some topology $\mathscr{T}$ on X will be called a *topological approach space*. Analogously all associated structures will be referred to as being topological and will be denoted in a similar way with a subscript or superscript referring to the original topology. Note that in particular 5 and 7 are the well-known lower semicontinuous and upper semicontinuous regularization of functions, see e.g. Bourbaki (1960).

The next result gives an internal characterization of these spaces.

2.1.2 Proposition *An approach space (X, δ) is topological if and only if any of the following equivalent properties holds.*

1. $\delta(X \times 2^X) \subseteq \{0, \infty\}$.
2. *For any filter $\mathscr{F} \in \mathsf{F}(X)$ we have that $\lambda\mathscr{F}(X) \subseteq \{0, \infty\}$.*
3. *For any lower regular function μ we have that also $\theta_{\{\mu=0\}}$ is lower regular.*
4. *For any lower regular function μ and for any $\varepsilon \in \mathbb{R}^+$ also $\theta_{\{\mu \leq \varepsilon\}}$ is lower regular.*

5. *There exists a subbasis $\mathscr{M}$ for the lower regular function frame such that for all $\mu \in \mathscr{M}$ also $\theta_{\{\mu=0\}}$ is lower regular.*
6. *For any upper regular function ν and for any $\alpha, \beta \in]0, \infty[$ also $\theta_{\{\nu<\alpha\}} \wedge \beta$ is upper regular.*
7. *There exists a subbasis $\mathscr{M}$ for the upper regular function frame such that for all $\nu \in \mathscr{M}$ and for any $\alpha, \beta \in]0, \infty[$ also $\theta_{\{\nu<\alpha\}} \wedge \beta$ is upper regular.*
8. *For any $d \in \mathscr{G}$ and $\alpha \in \mathbb{R}^+$ also $\alpha d \in \mathscr{G}$.*

Proof The only-if part of 1 follows from the definition of a topological approach space. To show the if part it suffices to note that if, for all $A \subseteq X$, we put $\mathrm{cl}(A) := \{x \in X \mid \delta(x, A) = 0\}$, then cl is a topological closure operator and δ is the associated distance. Characterization 2 is an immediate consequence. To prove 3 it suffices to look at the functions $\mu = \delta_A$ for $A \subseteq X$ and 4 is clearly equivalent to 3 because of the translation-invariance of $\mathfrak{L}$. Claims 5 to 7 follow by analogous reasoning. That 8 is necessary follows from 2.1.1 and that it is sufficient follows from 1.2.6 and the first claim. □

2.2 Embedding Top in App

In the foregoing section we have seen that a topological space can be viewed as a special type of approach space. That, moreover, Top is concretely embedded in App is a consequence of the fact that given topological spaces $(X, \mathscr{T})$ and $(X', \mathscr{T}')$ a function $f : X \longrightarrow X'$ will be continuous as a map between the topological spaces if and only if it is a contraction as a map between the associated approach spaces, as follows at once e.g. from the observation that if $A \subseteq X$, then

$$f(\mathrm{cl}_{\mathscr{T}} A) \subseteq \mathrm{cl}_{\mathscr{T}'}(f(A)) \Leftrightarrow \forall \varepsilon \in \mathbb{R}^+ : f(A^{(\varepsilon)}) \subseteq (f(A))^{(\varepsilon)'},$$

which by 1.3.3 proves our claim. Hence the concrete functor from Top to App which takes $(X, \mathscr{T})$ to $(X, \delta_{\mathscr{T}})$ is a full embedding of Top into App.

We will now prove that Top is actually very nicely embedded in App. In contrast to most known topological categories which do not have subcategories which are simultaneously reflectively and coreflectively embedded, such as for instance Top itself, we will prove that Top is simultaneously concretely reflectively and concretely coreflectively embedded in App. This is what, in 12.1 we call a *stable subcategory*.

The fact that the embeddings, reflections and coreflections are concrete implies in particular that the reflection and coreflection morphisms are carried by the identity map of the underlying set. Hence, throughout this work, when referring to a reflection or coreflection, we will only mention the objects and never the reflection or coreflection morphisms.

We recall that two important aspects of the fact that Top is reflectively embedded in App are: (1) for each approach structure, there exists a finest coarser topological structure on the same underlying set and (2) initial structures of topological spaces are the same whether they are taken in Top or in App. We refer to the seminal work of Herrlich on these matters (Herrlich 1968, 1983).

2.2.1 Proposition *For any approach space X, the operation defined by*

$$\mathrm{cl}(A) := \left\{x \in X \mid \delta(x, A) < \infty\right\}$$

is a pretopological closure operator.

Proof This follows from (D1), (D2), and (D3). □

We recall that the category PrTop of pretopological spaces and continuous maps is a topological category in which Top is reflectively embedded. A *pretopological space* is a set equipped with a closure operation which satisfies all the usual axioms with the possible exception of idempotency. For more information on PrTop we refer the reader to Choquet (1947) and Colebunders (1989).

2.2.2 Theorem Top *is embedded as a concretely reflective subcategory of* App. *For any approach space (X, δ), its* Top*-reflection is determined by the distance δ^{tr} associated with the topological reflection of the pretopological closure operator* cl.

Proof The topological reflection of this closure operator is obtained by a standard transfinite process which produces a topological closure operator, cl_{tr}. To verify that $1_X : (X, \delta) \longrightarrow (X, \delta^{tr})$ is a contraction it suffices to note that if $\delta(x, A) < \infty$, for some $x \in X$ and $A \subseteq X$, then $\delta^{tr}(x, A) = 0$. Now suppose that $(Y, \mathscr{T})$ is a topological space and that

$$f : (X, \delta) \longrightarrow (Y, \delta_{\mathscr{T}})$$

is a contraction. Then, for any $x \in X$ and $A \subseteq X$, we have

$$\begin{aligned} x \in \mathrm{cl}(A) &\Rightarrow \delta(x, A) < \infty \\ &\Rightarrow \delta_{\mathscr{T}}(f(x), f(A)) < \infty \\ &\Rightarrow f(x) \in \mathrm{cl}_{\mathscr{T}}(f(A)). \end{aligned}$$

Hence $f : (X, \mathrm{cl}) \longrightarrow (Y, \mathrm{cl}_{\mathscr{T}})$ is continuous as a function between pretopological spaces. It then follows that also $f : (X, \mathrm{cl}_{tr}) \longrightarrow (Y, \mathrm{cl}_{\mathscr{T}})$ is continuous as a function between topological spaces, which in turn means that $f : (X, \delta^{tr}) \longrightarrow (Y, \delta_{\mathscr{T}})$ is a contraction. □

2.2.3 Corollary Top *is closed under the formation of limits and initial structures in* App. *In particular, a product in* App *of a family of topological approach spaces is a topological approach space and, likewise, a subspace in* App *of a topological approach space is a topological approach space.*

Although, as the previous results show, it is important to know from a structural point of view that Top is reflectively embedded in App, we will not often have recourse to considering the Top-reflection of an approach space. It is easily seen that if the distance is finite, then the Top-reflection is indiscrete. This shows that it will usually not be a very interesting topology to consider. The situation becomes totally different for the dual property, coreflectivity, as we will now see. We also recall that two important aspects of the fact that Top is coreflectively embedded in App are: (1) for each approach structure there exists a coarsest finer topological structure on the same underlying set and (2) final structures of topological spaces are the same whether they are taken in Top or in App.

2.2.4 Theorem Top *is embedded as a concretely coreflective subcategory of* App. *For any approach space* (X, δ), *its* Top-*coreflection is determined by the distance* δ^{tc} *associated with the topological closure operator given by*

$$\mathrm{cl}_{\delta}(A) := \left\{x \in X \mid \ \delta(x, A) = 0\right\}.$$

Proof It is easily verified that cl_{δ} is indeed a topological closure operator and that $1_X : (X, \delta^{tc}) \longrightarrow (X, \delta)$ is a contraction. Now suppose that $(Y, \mathscr{T})$ is a topological space and that

$$f : (Y, \delta_{\mathscr{T}}) \longrightarrow (X, \delta)$$

is a contraction. Then, for any $x \in Y$ and $A \subseteq Y$ such that $x \in \mathrm{cl}_{\mathscr{T}}(A)$, we have

$$\delta(f(x), f(A)) \leq \delta_{\mathscr{T}}(x, A) = 0,$$

which proves that

$$f : (Y, \delta_{\mathscr{T}}) \longrightarrow (X, \delta^{tc})$$

is also a contraction. $\square$

2.2.5 Corollary Top *is closed under the formation of colimits and final structures in* App. *In particular, a coproduct in* App *of a family of topological approach spaces is a topological approach space and, likewise, a quotient in* App *of a topological approach space is a topological approach space.*

In the sequel, in order not to overload notation and terminology, we will call the topological spaces associated with the topological reflection and coreflection also simply the topological reflection and coreflection of a given approach space. In other words, unless required for technical reasons, we do not differentiate between a topological space and the associated approach space.

At the end of this chapter we will discuss the importance of the Top-coreflection of an approach space. Anticipating this, we will now describe this coreflection by means of the most important other basic structures. Given an approach space (X, δ), we will denote the topology underlying the topological coreflection by $\mathscr{T}_\delta$, or with any other index referring to the original structure.

2.2.6 Proposition *If X is an approach space with $(\mathscr{B}(x))_{x\in X}$ a basis for the approach system and $\mathscr{H}$ a basis for the gauge, then the following properties hold.*

1. *Convergence in the topological coreflection is characterized by*

$$\mathscr{F} \to x \Leftrightarrow \lambda\mathscr{F}(x) = 0 \text{ and } \mathscr{F} \rightsquigarrow x \Leftrightarrow \alpha\mathscr{F}(x) = 0.$$

2. *The neighbourhoods in the topological coreflection are characterized by*

$$\begin{aligned}\mathscr{V}(x) &= \left\{V \in 2^X \mid \exists\varepsilon > 0, \exists\varphi \in \mathscr{B}(x) : \{\varphi < \varepsilon\} \subseteq V\right\} \\ &= \left\{V \in 2^X \mid \exists\varepsilon > 0, \exists d \in \mathscr{H} : B_d(x, \varepsilon) \subseteq V\right\}.\end{aligned}$$

Proof 1. This follows from

$$\begin{aligned}\mathscr{F} \to x \Leftrightarrow x &\in \bigcap_{A\in\sec(\mathscr{F})} \mathrm{cl}_\delta(A) \\ &\Leftrightarrow \sup_{A\in\sec(\mathscr{F})} \delta(x, A) = 0 \\ &\Leftrightarrow \lambda\mathscr{F}(x) = 0,\end{aligned}$$

and analogously for the adherence.

2. For the approach system this follows from

$$\begin{aligned}V \in \mathscr{V}(x) &\Leftrightarrow x \notin \mathrm{cl}_\delta(X \setminus V) \\ &\Leftrightarrow \exists\varepsilon > 0 : \sup_{\varphi\in\mathscr{B}(x)} \inf_{y\in X\setminus V} \varphi(y) > \varepsilon \\ &\Leftrightarrow \exists\varepsilon > 0, \exists\varphi \in \mathscr{B}(x) : \{\varphi < \varepsilon\} \subseteq V\end{aligned}$$

and for the gauge it is entirely similar. □

2.2.7 Example We refer to the two examples which we considered in 1.2.62.

The distance of the first example is $\delta_{\mathbb{E}} : \mathbb{P}\times 2^{\mathbb{P}} \longrightarrow \mathbb{P}$. The topological coreflection of this space is $(\mathbb{P}, \mathscr{T}_{\mathbb{E}})$, where $\mathscr{T}_E$ is the topology of the Alexandroff compactification of $[0, \infty[$ with the usual (Euclidean) topology.

The distance of the second example is $\delta_{\mathbb{P}} : \mathbb{P} \times 2^{\mathbb{P}} \longrightarrow \mathbb{P}$. The topological coreflection of this space is $(\mathbb{P}, \mathscr{T}_{\mathbb{P}})$, where

$$\mathscr{T}_{\mathbb{P}} := \{]a, \infty] | a \in \mathbb{P}\} \cup \{\mathbb{P}\}.$$

2.2.8 Proposition *If X is an approach space, then any lower regular function $\xi \in \mathfrak{L}$ considered as a map $\xi : (X, \mathscr{T}_\delta) \longrightarrow (\mathbb{P}, \mathscr{T}_{\mathbb{P}})$ is continuous, in particular the distance functionals, and limits and adherences of filters are continuous maps.*

Proof This follows from 1.3.5, 2.2.4, and 2.2.7. □

If, in the foregoing result, we replace the topology $\mathcal{T}_{\mathbb{P}}$ by the Euclidean topology $\mathcal{T}_{\mathbb{E}}$, then all maps are lower semicontinuous.

In the foregoing chapter we introduced three types of maps, namely, open and closed expansions and proper contractions. The following result tells us that the terminology for closed and open was appropriately chosen.

2.2.9 Proposition (Top) *If X and X' are topological spaces and $f : X \longrightarrow X'$ is a map then the following properties hold.*

1. *f is closed as a map between the topological spaces if and only if it is a closed expansion between the associated approach spaces.*
2. *f is open as a map between the topological spaces if and only if it is an open expansion between the associated approach spaces.*

Proof (1) This follows from 2.1.1 (6), 1.4.2 (5) and the observation that if f is closed and μ is lsc then $f(\mu)$ is lsc.

(2) Analogously, this follows from 2.1.1 (8), 1.4.6 (5) and the observation that if f is open and μ is usc then $f(\mu)$ is usc. □

We will treat the case of proper contractions later when we have a more appropriate characterization at our disposal (see 4.3.30).

2.3 (Quasi-)Metric Approach Spaces

Given a quasi-metric space (X, d), we associate with it a natural approach space by defining in the usual way the function

$$\delta_d : X \times 2^X \longrightarrow \mathbb{P} : (x, A) \mapsto \inf_{a \in A} d(x, a).$$

2.3.1 Proposition *If (X, d) is a quasi-metric space, then the function*

$$\delta_d : X \times 2^X \longrightarrow \mathbb{P}$$

is a distance on X and the associated structures are given as follows.

1. *For all $\mathcal{F} \in \mathsf{F}(X)$ and $x \in X$:*

   a. $\alpha_d \mathcal{F}(x) = \sup\limits_{F \in \mathcal{F}} \inf\limits_{y \in F} d(x, y) = \sup_{F \in \mathcal{F}} \delta_d(x, F)$,

   b. $\lambda_d \mathcal{F}(x) = \inf\limits_{F \in \mathcal{F}} \sup\limits_{y \in F} d(x, y)$.

2. *For any $x \in X$: $\mathcal{A}_d(x) := \left\{\varphi \in \mathbb{P}^X \mid \varphi \leq d(x, \cdot)\right\}$ and a basis is given by the singleton $\{d(x, \cdot)\}$.*

3. $\mathcal{G}_d := \{e \in q\mathsf{Met}(X) \mid e \leq d\}$ *and a basis is given by the singleton* $\{d\}$.
4. *The tower is given by the family* $(\mathfrak{t}^d_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ *where*

$$\mathfrak{t}^d_\varepsilon : 2^X \longrightarrow 2^X : A \mapsto \left\{x \in X \mid \delta_d(x, A) \leq \varepsilon\right\}.$$

5. *For any* $\mu \in \mathbb{P}^X$ *and* $x \in X$: $\mathfrak{l}_d(\mu)(x) := \inf_{y \in X}(\mu(y) + d(x, y))$ *i.e.* $\mathfrak{l}_d(\mu)$ *is the largest non-expansive map smaller than* μ.
6. $\mathfrak{L}_d := \left\{\mu \in \mathbb{P}^X \mid \mu \text{ non-expansive}\right\}$.
7. *For any* $\mu \in \mathbb{P}^X_b$ *and* $x \in X$: $\mathfrak{u}_d(\mu)(x) := \sup_{y \in X}(\mu(y) - d(x, y))$ *i.e.* $\mathfrak{u}_d(\mu)$ *is the smallest non-expansive map larger than* μ.
8. $\mathfrak{U}_d := \left\{\mu \in \mathbb{P}^X_b \mid \mu \text{ non-expansive}\right\} = \mathfrak{L}_d \cap \mathbb{P}^X_b$.
9. *For any functional ideal* $\mathfrak{I}$ *on* X: $\mathfrak{I} \rightarrowtail x$ *if and only if* $d(x, \cdot) \wedge \omega \in \mathfrak{I}$ *for each* $\omega \in \mathbb{R}^+$.

Proof That δ_d is a distance is merely the special case of 1.2.6 where we take for the gauge basis $\mathcal{H} := \{d\}$.

1. This is an immediate consequence of the definition of the adherence operator 1.2.64 and of 1.2.44.

2. By 1.2.33 we have that, for all $x \in X$,

$$\mathcal{A}_d(x) = \left\{\varphi \in \mathbb{P}^X \mid \forall A \in 2^X : \inf_{a \in A} \varphi(a) \leq \inf_{a \in A} d(x, a)\right\}.$$

Clearly, $\inf_{a \in A} \varphi(a) \leq \inf_{a \in A} d(x, a)$ holds, for all $A \subseteq X$, if and only if $\varphi(a) \leq d(x, a)$ holds, for all $a \in X$.

3. Referring to 1.2.4 instead of to 1.2.33, this is precisely the same as the proof of the foregoing result.

4. This follows from 1.2.21.

5. This follows from 1.2.16.

6. This follows from 1.2.24.

7. This follows from 1.2.25.

8. This follows from 1.2.29 and 6.

9. This follows from 1.2.31 and 2. □

The above expression for δ_d is of course well known and notationally often no distinction is made between d and δ_d. We emphasize, however, that for our purposes it is important to use different notations for a quasi-metric and for the distance derived from it in the sense of the foregoing definition. In the first place, the two functions have different domains and in the second place, they determine categorically different structures. We have seen that Top is simultaneously reflectively and coreflectively embedded in App, and hence for topological approach spaces, it makes no difference whether we perform constructions, such as the making of limits, colimits, initial, and final structures, in Top or in App. However, for quasi-metric approach spaces, as we will see later in this chapter when we study the precise way in which $q\mathsf{Met}$ is embedded in App, it does make a difference whether we make initial structures of quasi-metric approach spaces in qMet or in App. Hence it is important to make the distinction.

For a sequence $(x_n)_n$ we denote the generated filter by $\langle (x_n)_n \rangle$.

2.3.2 Corollary *If (X, d) is a quasi-metric space, $(x_n)_n$ is a sequence in X and $x \in X$, then the following formulas hold.*

1. $\alpha_d \langle (x_n)_n \rangle (x) = \liminf_{n\to\infty} d(x, x_n)$.
2. $\lambda_d \langle (x_n)_n \rangle (x) = \limsup_{n\to\infty} d(x, x_n)$.

An approach space of type (X, δ_d), for some quasi-metric d on X, will be called a *(quasi-)metric approach space*. Analogously all associated structures will be referred to as being (quasi-)metric.

2.3.3 Proposition *An approach space is quasi-metric if and only if it has one, and hence all of the following equivalent properties.*

1. *For all $x \in X$ and $\mathscr{A} \subseteq 2^X$, we have $\delta(x, \cup\mathscr{A}) = \inf_{A\in\mathscr{A}} \delta(x, A)$.*
2. *For all $x \in X$ and $A \subseteq X$, we have $\delta(x, A) = \inf_{a\in A} \delta(x, \{a\})$.*
3. *The lower regular function frame is closed under the formation of arbitrary infima.*
4. *The upper regular function frame is closed under the formation of arbitrary bounded suprema.*
5. *The gauge is a principal gauge generated by a unique function, which necessarily is a quasi-metric.*

Proof The equivalence of the first and second property with being quasi-metric follows at once from the definition. That a quasi-metric space fulfils the third property follows from the description of the lower regular function frame in 2.3.1. Conversely it follows from 3 that given $x \in X$ and $A \subseteq X$ the function

$$\eta : X \longrightarrow \mathbb{P} : y \mapsto \inf_{a\in A} \delta(y, \{a\})$$

belongs to $\mathfrak{L}$ and vanishes on A. Consequently it follows from 1.2.45 that

$$\begin{aligned}\delta(x, A) &= \sup\{\mu(x) \mid \mu \in \mathfrak{L}, \mu_{|A} = 0\}\\ &\geq \eta(x)\\ &= \inf_{a\in A} \delta(x, \{a\}).\end{aligned}$$

Since the other inequality always holds this proves 3. The third and fourth properties are obviously equivalent. Property 5 finally is evident. □

2.3.4 Proposition *An approach space is metric if and only if for all $A, B \in 2^X$*

$$\inf_{a\in A} \delta(a, B) = \inf_{b\in B} \delta(b, A).$$

Proof This follows from the foregoing result and the symmetry of metrics. □

The above results clarify our terminology of local distances. In a quasi-metric approach space the basic local distance at a point x is simply the quasi-metric "localized" at x.

The pretopological closures $\mathfrak{t}_\varepsilon^d$, $\varepsilon \in \mathbb{R}^+$, are referred to, in Beer and Luchetti (1993), as (ε)-enlargement operators.

Making use of the results of this section it is possible to formulate some of the transitions in a more concise way.

2.3.5 Proposition *The following formulas for various transitions hold.*

1. *The transition from distances to gauges:* $\mathcal{G} = \{d \in q\mathsf{Met}(X) \mid \delta_d \leq \delta\}$.
2. *The transition from gauges to distances:* $\delta = \sup_{d \in \mathcal{G}} \delta_d$.
3. *The transition from gauges to lower hull operators:* $\mathfrak{l} = \sup_{d \in \mathcal{G}} \mathfrak{l}_d$.
4. *The transition from gauges to upper hull operators:* $\mathfrak{u} = \inf_{d \in \mathcal{G}} \mathfrak{u}_d$.
5. *The transition from gauges to limit operators:* $\lambda = \sup_{d \in \mathcal{G}} \lambda_d$.
6. *The transition from gauges to adherence operators:* $\alpha = \sup_{d \in \mathcal{G}} \alpha_d$.

Proof This follows from 2.3.1 and all the respective transition formulas proved in the first chapter. □

2.3.6 Proposition *If (X, d) is a quasi-metric space, $\mathcal{F}$ is a filter on X and $x \in X$, then the following properties hold.*

1. *$\mathcal{F} \rightsquigarrow x$ in $(X, \mathcal{T}_d)$ if and only if $\alpha_d \mathcal{F}(x) = 0$.*
2. *$\mathcal{F} \rightarrow x$ in $(X, \mathcal{T}_d)$ if and only if $\lambda_d \mathcal{F}(x) = 0$.*

Proof This follows from 2.3.1. □

2.4 Embedding qMet in App

In 2.3 we have seen that quasi-metric spaces can be viewed as special types of approach spaces. That qMet is concretely embedded in App is a consequence of the fact that given quasi-metric spaces (X, d) and (X', d') a function $f : X \longrightarrow X'$ will be nonexpansive between the quasi-metric spaces if and only if it is a contraction between the associated approach spaces. Hence the concrete functor from qMet to App which takes (X, d) to (X, δ_d) is a full embedding of qMet into App.

In 2.2 we were able to show both concrete reflectivity and coreflectivity of the embedding of Top in App. For Met and qMet only concrete coreflectivity of the embedding holds. However, as we will see later, it is precisely the fact that neither Met nor qMet is embedded reflectively in App which makes the theory of approach spaces especially interesting.

2.4.1 Theorem *qMet is embedded as a concretely coreflective subcategory of App. For any approach space (X, δ), its qMet-coreflection is determined by the distance δ^{qm} associated with the quasi-metric*

$$d_\delta : X \times X \longrightarrow \mathbb{P} : (x, y) \mapsto \delta(x, \{y\}).$$

Proof To show that $1_X : (X, \delta^{qm}) \longrightarrow (X, \delta)$ is a contraction let $x \in X$ and let $A \subseteq X$. Then we have

$$\delta(x, A) \leq \inf_{a \in A} \delta(x, \{a\}) = \delta^{qm}(x, A).$$

Now suppose that (Y, d) is a quasi-metric space and that

$$f : (Y, \delta_d) \longrightarrow (X, \delta)$$

is a contraction. Then, for any $x \in Y$ and $A \subseteq Y$, we have

$$\begin{aligned}\delta^{qm}(f(x), f(A)) &= \inf_{a \in A} \delta(f(x), \{f(a)\}) \\ &\leq \inf_{a \in A} \delta_d(x, \{a\}) \\ &= \delta_d(x, A),\end{aligned}$$

which proves that

$$f : (Y, \delta_d) \longrightarrow (X, \delta^{qm})$$

is also a contraction. □

2.4.2 Corollary *qMet is closed under the formation of colimits and final structures in App. In particular, a coproduct in App of a family of quasi-metric approach spaces is a quasi-metric approach space and, likewise, a quotient in App of a quasi-metric approach space is a quasi-metric approach space.*

In the following result we describe the qMet-coreflection of an approach space by means of approach systems and gauges. This result is the counterpart of 2.2.6.

2.4.3 Proposition *If (X, δ) is an approach space with $(\mathcal{B}(x))_{x \in X}$ a basis for the approach system and $\mathcal{H}$ a basis for the gauge, then for any $x, y \in X$: $d_\delta(x, y) = \sup_{\varphi \in \mathcal{B}(x)} \varphi(y) = \sup_{d \in \mathcal{H}} d(x, y)$.*

Proof The first equality follows from 1.2.34 and 2.4.1 while the second one follows from 1.2.6 and 2.4.1. □

Given a quasi-metric d on a set X we call d^-, defined by

$$d^-(x, y) := d(y, x)$$

the *adjoint quasi-metric*. Further we put $d^* := d \vee d^-$.

2.4.4 Theorem Met *is embedded as a concretely coreflective subcategory of* App. *For any approach space* (X, δ)*, its* Met*-coreflection is determined by the distance* δ^m *associated with the metric*

$$d^*_\delta : X \times X \longrightarrow \mathbb{P} : (x, y) \mapsto d_\delta(x, y) \vee d^-_\delta(x, y).$$

Proof This is analogous to 2.4.1 and we leave this to the reader. □

2.4.5 Corollary Met *is closed under the formation of colimits and final structures in* App. *In particular, a coproduct in* App *of a family of metric approach spaces is a metric approach space and, likewise, a quotient in* App *of a metric approach space is a metric approach space.*

The description of the Met-coreflection of an approach space by means of a basis for the approach system or a basis for the gauge is easily deduced from 2.4.3 and 2.4.4. For instance, if $\mathscr{H}$ is a basis for the gauge associated with δ, then $d^*_\delta(x, y) = \sup_{d \in \mathscr{H}} d(x, y) \vee d(y, x)$, for all $x, y \in X$.

2.4.6 Example Again we refer to the examples which we considered in 1.2.62.

The distance of the first example is $\delta_{\mathbb{E}} : \mathbb{P} \times 2^{\mathbb{P}} \longrightarrow \mathbb{P}$, and both the qMet-coreflection and the Met-coreflection of this space are given by $(\mathbb{P}, d_{\mathbb{E}})$ where $d_{\mathbb{E}}$ is the "Euclidean" metric on $\mathbb{P}$, i.e. for all $x, y \in \mathbb{P}$

$$d_{\mathbb{E}}(x, y) := |x - y| .$$

The distance of the second example is $\delta_{\mathbb{P}} : \mathbb{P} \times 2^{\mathbb{P}} \longrightarrow \mathbb{P}$, and the qMet-coreflection of this space is $(\mathbb{P}, d_{\mathbb{P}})$, where for all $x, y \in \mathbb{P}$

$$d_{\mathbb{P}}(x, y) := (x - y) \vee 0$$

and the Met-coreflection is $(\mathbb{P}, d_{\mathbb{E}})$.

2.4.7 Proposition *If* (X, δ) *is an approach space, then for any* $\xi \in \mathfrak{L}$,

$$\xi : (X, d_\delta) \longrightarrow (\mathbb{P}, d_{\mathbb{P}})$$

is a nonexpansive map. In particular distance functionals, limits and adherences are nonexpansive maps.

Proof This follows from 2.4.1, and 2.4.6. □

From 2.3.1 it follows at once that, given a quasi-metric space (X, d), $\xi \in \mathfrak{L}_d$ if and only if the function $\xi : (X, d) \longrightarrow (\mathbb{P}, d_{\mathbb{P}})$ is nonexpansive, which means that for quasi-metric approach spaces the condition given in 2.4.7 is both necessary and sufficient. This also implies that, for any $\mu \in \mathbb{P}^X$, the lower hull $\mathfrak{l}_d(\mu)$, as given in 2.3.1, can be described as the largest nonexpansive function smaller than μ. This alternative characterization is well known in the classical situation (i.e. when considering only real-valued functions), and can for instance be found in Singer (1986).

2.4.8 Proposition *If (X, d) is a quasi-metric space, then the following properties hold.*

1. *The* Top*-coreflection of (X, δ_d) is $(X, \delta_{\mathscr{T}_d})$, where $\mathscr{T}_d$ is the topology generated by d.*
2. *The* Met*-coreflection of (X, δ_d) is (X, δ_{d^*}).*

Proof 1. This follows from the second property in 2.2.6. This result indeed implies that the Top-coreflection of (X, δ_d) has as neighbourhood system

$$\mathscr{V}(x) = \left\{V \in 2^X \mid \exists \varepsilon > 0 : B_d(x, \varepsilon) \subseteq V\right\}.$$

2. This follows from 2.4.4. □

2.4.9 Proposition *(q*Met*) If X and X' are quasi-metric approach spaces and $f : X \longrightarrow X'$ is a map, then the following properties hold.*

1. *f is open expansive if and only if for all $x \in X$ and $y \in X'$:*

$$\inf_{z \in f^{-1}(y)} d(x, z) \leq d'(f(x), y).$$

2. *f is closed expansive if and only if for all $x \in X$ and $y \in X'$:*

$$\inf_{z \in f^{-1}(y)} d(z, x) \leq d'(y, f(x)).$$

Proof This follow from 1.4.5 and the definition of δ_d for a quasi-metric d. □

2.4.10 Corollary (Met) *If X and X' are metric approach spaces and $f : X \longrightarrow X'$ is a map, then f is closed-expansive if and only if it is open-expansive.*

2.4.11 Example 1. The situation with closed- and open-expansiveness is quite different in the metric case when compared to the topological case. For instance, whereas a projection $\mathbb{R}^2 \longrightarrow \mathbb{R}$ is open but not closed in the topological sense when both spaces are equipped with their usual Euclidean topologies, it is both closed-expansive and open-expansive when both spaces are equipped with their usual Euclidean metrics.

2. Let $X := \{a, b_1, b_2\}$ and $X' := \{c_1, c_2\}$ with quasi-metrics defined by

$$d(b_1, a) = d(b_1, b_2) = d(b_2, b_1) = 1, d(b_2, a) = 2, d(a, b_1) = d(a, b_2) = 0$$

and

$$d(c_1, c_2) = 1, d(c_2, c_1) = 0$$

and let $f : X \longrightarrow X'$ be the function defined as $f(a) = c_2, f(b_1) = f(b_2) = c_1$. Then it is easily verified that f is closed-expansive but not open-expansive. Replacing both quasi-metrics by their adjoints (see 3.1, $d^-(x, y) := d(y, x)$) gives an example of a function which is open-expansive but not closed-expansive.

2.4.12 Example 1. If $(X, \delta_{\mathscr{T}})$ is a topological approach space then the qMet-coreflection is given by (X, δ_{d_1}), where d_1 is the quasi-metric

$$d_1(x, y) := \begin{cases} 0 & x \in \mathrm{cl}_{\mathscr{T}}\{y\}, \\ \infty & x \notin \mathrm{cl}_{\mathscr{T}}\{y\}, \end{cases}$$

and the Met-coreflection is given by (X, δ_{d_0}), where d_0 is the metric

$$d_0(x, y) := \begin{cases} 0 & x \in \mathrm{cl}_{\mathscr{T}}\{y\} \text{ and } y \in \mathrm{cl}_{\mathscr{T}}\{x\}, \\ \infty & x \notin \mathrm{cl}_{\mathscr{T}}\{y\} \text{ or } y \notin \mathrm{cl}_{\mathscr{T}}\{x\}. \end{cases}$$

Hence we can deduce that $(X, \mathscr{T})$ is T_1 if and only if d_1 is a separated metric and that it is T_0 if and only if d_0 is a separated metric.

2. An object in App is at the same time topological and quasi-metric if and only if it is a finitely generated topological space. (Recall that a topological space is said to be finitely generated if the closure is entirely determined by the closures of the singletons in the sense that a point will be in the closure of a set if and only if it is in the closure of a point of the set.)

3. An object in App is at the same time topological and metric if and only if it is a coproduct of indiscrete topological spaces.

4. An object in App is at the same time topological and separated metric if and only if it is discrete.

We now return to the initially dense objects which where found by Claes in (2009). In that paper the research was performed in the setting of metrically generated theories, and the results are far more general that what we require. Hence we will give short proofs restricted to our case (see also Colebunders et al. 2011). We consider the same underlying set as $\mathbb{P}$ but now equipped with the following distances

$$\mathbb{P} \times 2^{\mathbb{P}} \longrightarrow \mathbb{P} : (x, A) \mapsto \inf_{a \in A} a \ominus x,$$

and

$$\mathbb{P} \times 2^{\mathbb{P}} \longrightarrow \mathbb{P} : (x, A) \mapsto \inf_{a \in A} x \ominus a.$$

Note that both distances are quasi-metric, precisely, the first distance is nothing else but $\delta_{d_{\mathbb{P}}^-}$ and the second distance is $\delta_{d_{\mathbb{P}}}$.

2.4.13 Theorem *Both* $(\mathbb{P}, \delta_{d_{\mathbb{P}}^-})$ *and* $(\mathbb{P}, \delta_{d_{\mathbb{P}}})$ *are initially dense objects in* App.

Proof Since we already know one initially dense object, namely $(\mathbb{P}, \delta_{\mathbb{P}})$, it suffices to show that we can obtain that object via initial sources from either of the two objects above. We recall (see 1.2.62) that a gauge basis for $(\mathbb{P}, \delta_{\mathbb{P}})$ is given by the family $\{d_\alpha \mid \alpha \in \mathbb{R}^+\}$ where $d_\alpha(x, y) = (x \wedge \alpha) \ominus (y \wedge \alpha)$.

For the first we consider the following source:

$$(f_\alpha : (\mathbb{P}, \delta_{\mathbb{P}}) \longrightarrow (\mathbb{P}, \delta_{d_{\mathbb{P}}^-}) : x \mapsto \alpha \ominus x)_{\alpha \in \mathbb{R}^+}$$

then the equality $f_\alpha(y) \ominus f_\alpha(x) = d_\alpha(x, y)$ holds because if $x \leq y$ and $\alpha < y < x$ both sides are zero, if $y < x$ and $y \leq \alpha \leq x$ both sides are equal to $\alpha - y$ and if $y < x \leq \alpha$ both sides are equal to $x - y$. Hence for any $\alpha \in \mathbb{R}^+$ we have $d_{\mathbb{P}}^- \circ (f_\alpha \times f_\alpha) = d_\alpha$ which shows that this first source is initial.

For the second we consider the source:

$$(g_\alpha : (\mathbb{P}, \delta_{\mathbb{P}}) \longrightarrow (\mathbb{P}, \delta_{d_{\mathbb{P}}}) : x \mapsto x \wedge \alpha)_{\alpha \in \mathbb{R}^+}$$

then here too it follows that for any $\alpha \in \mathbb{R}^+$ we have $d_{\mathbb{P}} \circ (g_\alpha \times g_\alpha) = d_\alpha$, which shows that this source too is initial. □

2.4.14 Corollary App *is the epireflective hull of* $q\mathsf{Met}$ *in* App.

2.4.15 Theorem *If* (X, δ) *is an approach space and we put* $J := \mathbb{R}^+ \times 2^X$, *then*

$$((X, \delta) \longrightarrow (\mathbb{P}, \delta_{d_{\mathbb{P}}}) : x \mapsto \delta(x, A) \wedge \alpha)_{(\alpha, A) \in J}$$

and

$$((X, \delta) \longrightarrow (\mathbb{P}, \delta_{d_{\mathbb{P}}^-}) : x \mapsto \alpha \ominus \delta(x, A))_{(\alpha, A) \in J}$$

are initial sources.

Proof This follows from the combination of the initial sources in 1.3.19 and in 2.4.13. □

We know that Top consists precisely of subspaces of products (in Top) of quasi-metrizable topological spaces, Herrlich (1968). This is strengthened in the approach case.

In the following theorem we use the notation of 1.2.5. This means that, given an approach space (X, δ), $Z \subseteq X$, and $\zeta \in \mathbb{R}^+$, we consider the quasi-metric d_Z^ζ which, for all $x, y \in X$, is given by

$$d_Z^\zeta(x, y) = (\delta(x, Z) \wedge \zeta) \ominus (\delta(y, Z) \wedge \zeta).$$

We will prove the following result by a straightforward calculation of the distances involved.

2.4.16 Theorem *If X is an approach space and we put $J := \mathbb{R}^+ \times 2^X$, then*

$$\psi : (X, \delta) \longrightarrow (X^J, \prod_{(\zeta,Z)\in J} \delta_{d_Z^\zeta}) : x \mapsto (x_{(\zeta,Z)} := x)_{(\zeta,Z)\in J}$$

is an embedding.

Proof Let us put $\delta^* := \prod_{(\zeta,Z)\in J} \delta_{d_Z^\zeta}$, the product distance on X^J. Let $x \in X$ and $A \subseteq X$. Then, making use of 1.3.11, on the one hand we have

$$\begin{aligned}
\delta^*(\psi(x), \psi(A)) &= \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \inf_{a\in A} \sup_{Z\in\mathscr{Z}} d_Z^\zeta(x, a) \\
&= \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \sup_{\varphi\in\mathscr{Z}^A} \inf_{a\in A} d_{\varphi(a)}^\zeta(x, a) \\
&= \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \sup_{\varphi\in\mathscr{Z}^A} \inf_{Z\in\mathscr{Z}} \inf_{a\in\varphi^{-1}(Z)} d_Z^\zeta(x, a) \\
&= \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \sup_{\varphi\in\mathscr{Z}^A} \inf_{Z\in\mathscr{Z}} (\delta(x, Z) \wedge \zeta) \ominus (\sup_{a\in\varphi^{-1}(Z)} \delta(a, Z) \wedge \zeta) \\
&\le \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \sup_{\varphi\in\mathscr{Z}^A} \inf_{Z\in\mathscr{Z}} \delta(x, \varphi^{-1}(Z)) \\
&= \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \sup_{\varphi\in\mathscr{Z}^A} \delta(x, A) = \delta(x, A).
\end{aligned}$$

On the other hand it follows from 1.2.8 that we have

$$\begin{aligned}
\delta(x, A) &= \sup_{Z\in 2^X} \sup_{\zeta\in\mathbb{R}^+} \inf_{a\in A} d_Z^\zeta(x, a) \\
&\le \sup_{\mathscr{Z}\in 2^{(2^X)}} \sup_{\zeta\in\mathbb{R}^+} \inf_{a\in A} \sup_{Z\in\mathscr{Z}} d_Z^\zeta(x, a) \\
&= \delta^*(\psi(x), \psi(A)).
\end{aligned}$$

This proves that (X, δ) is indeed embedded in (X^J, δ^*). □

The foregoing results are important for the sequel. A fundamental relationship among the different types of structures which we are considering in this work is that of a topology generated by a metric. As we argued in the introduction, it is the failure of this relation to be well behaved with respect to products in particular and initial structures in general which is one of the motivations for considering approach spaces.

What the foregoing results tell us is that the operation of taking the topology underlying a (quasi-)metric is recaptured in App as a canonical functor, namely the

Top-coreflector restricted to qMet. In the case of a quasi-metric space the Top-coreflector gives us precisely the underlying topological space. It is natural therefore to extend this interpretation to the whole of App and given an arbitrary approach space (X, δ), we will speak of (X, δ^{tc}) or $(X, \mathscr{T}_\delta)$ (the Top-coreflection of (X, δ)) as the *underlying topological approach space* and of the topology $\mathscr{T}_\delta$ as the *topology underlying* δ or the topology *generated by* δ. The situation is clarified in the following commutative diagram.

$$\begin{tikzcd} q\mathsf{Met} \arrow[r, "F_1"] \arrow[d, "E"'] & \mathsf{Top} \\ \mathsf{App} \arrow[ur, "F_2"'] & \end{tikzcd}$$

The functor E is the embedding of qMet in App, F_1 is the forgetful functor associating with each quasi-metric space its underlying topological space, and F_2 is the Top-coreflector. The diagram commutes and F_2 thus is an extension of F_1.

Although it is a fundamental aspect of the theory of approach spaces that qMet and, especially also, Met are not epireflectively embedded in App, for the restricted case of subspaces we do have the following result.

2.4.17 Theorem qMet *and* Met *are closed under the formation of subspaces in* App.

Proof This follows from the definitions. □

Referring to the foregoing diagram we can now further point out that the problem of the non-(quasi-)metrizability, in general, of initial topologies of (quasi-)metric topological spaces gets completely resolved in the setting of approach spaces. Consider the source in Top

$$(f_i : X \longrightarrow (X_i, \mathscr{T}_{d_i}))_{i \in I}$$

where d_i is a (quasi-)metric on X_i for each $i \in I$. The initial topology of this source is in general not (quasi-)metrizable. However, it is sufficient to embed the (quasi-) metric spaces (X_i, d_i) in App, there to consider the source

$$(f_i : X \longrightarrow (X_i, \delta_{d_i}))_{i \in I}$$

and then to take the initial approach structure on X and finally to apply the Top-coreflection to this initial structure. As coreflections preserve initial structures, that topological coreflection will be exactly the initial topology, which is generated, not by a (quasi-)metric but by an approach structure.

In the following diagram we give an overview of the categorical situation.

$$\begin{tikzcd}
 & \mathsf{App} & & \\
\mathsf{Top} \arrow[ur, "c", "r"'] & & q\mathsf{Met} \arrow[ul, "c"'] & \mathsf{Met} \arrow[ll, "c"', "r"] \arrow[ull, bend right, "c"'] \\
 & \mathsf{Fin} \arrow[ul, "c"] \arrow[ur, "r", "c"'] & &
\end{tikzcd}$$

Both qMet and Met are concretely coreflectively embedded in App. Initial structures can be taken in App rather than in either qMet or Met (both of which do indeed have initial structures, neither being compatible with the initial structures of the underlying topologies) and then the coreflection to Top can be applied. The concept of a metric simply is too restricted, what is required to resolve the incompatibility is precisely the category of approach spaces.

Now further note that in the diagram Fin stands for the category of finitely generated topological spaces. This category is not only a coreflective subcategory of Top but also of qMet. Given a finitely generated space it suffices to define a quasi-metric by

$$d(x, y) := \begin{cases} 0 & x \in \mathrm{cl}\{y\}, \\ \infty & x \notin \mathrm{cl}\{y\}. \end{cases}$$

Thus, within App it turns out that Fin is precisely the intersection of Top and qMet. In a certain sense, with regard to Top, Fin therefore plays the role that qMet plays with regard to App. A metric distance from x to A is completely determined by the distances between x and the points of A, and analogously for a finitely generated topology, whether x is in the closure of A or not, is entirely determined by whether x is in the closure of any of the points of A. Referring to the diagram in the introduction we see that App fills in the place of the first question mark (Fig. 2.1).

$$\begin{tikzcd}
\mathsf{Met} \arrow[r] & \boxed{?_2} \arrow[r] & \boxed{\mathsf{App}} & q\mathsf{Met} \arrow[l] \\
m\mathsf{Top} \arrow[r] & \mathsf{CReg} \arrow[r] \arrow[u] & \mathsf{Top} \arrow[u] & qm\mathsf{Top} \arrow[l]
\end{tikzcd}$$

Fig. 2.1 The categorical position of App with regard to Top and (q)Met

2.5 Comments

1. **Comparison of structures**

In the table below we compare various approach concepts in Top and qMet. The purpose is not to describe in precise mathematical terms what these different approach concepts are, but rather to indicate conceptually what are the basic ideas behind these concepts. In the table, d stands for a (quasi-)metric, $\mathscr{F}$ stands for a filter, and x is a point in the underlying set.

Concept in App	**Basic** Top-**analog**	**Basic** qMet-**analog**
Distance	Closure operator	Point-set distance
Adherence of $\mathscr{F}$	Adherence points of $\mathscr{F}$	liminf $d(\mathscr{F}, \cdot)$
Limit of $\mathscr{F}$	Limit points of $\mathscr{F}$	lim sup $d(\mathscr{F}, \cdot)$
Approach system	Neighbourhoods	Localized quasi-metrics $d(x, \cdot)$
Gauge	Quasi-metrics determining coarser topologies	Quasi-metrics smaller than d
Tower	Closure operator	Enlargement operators
Lower hull operator	Lower semicontinuous regularization	Nonexpansive regularization
Lower regular function frame	Lower semicontinuous $\mathbb{P}$-valued functions	Nonexpansive $\mathbb{P}$-valued functions
Upper hull operator	Upper semicontinuous regularization	Nonexpansive regularization
Upper regular function frame	Upper semicontinuous $\mathbb{P}$-valued functions	Nonexpansive $\mathbb{P}$-valued functions
Contraction	Continuous map	Nonexpansive map

2. **Supercategories of** Top

There are many topological categories wherein Top is embedded as a full subcategory in a more or less nice way. Some of these categories are intended for their better categorical properties such as the category PrTop of pretopological spaces which is extensional (see Herrlich 1987, 1988a, b) and the category PsTop of pseudotopological spaces which is a quasi-topos or topological universe (see Herrlich et al. 1991). We also refer to Antoine (1966a, b, c), Bentley et al. (1991), Bourdaud (1975, 1976), Choquet (1947), Colebunders and Verbeeck (2000), Day and Kelley (1970), Lowen-Colebunders and Sonck (1993, 1996) and Machado (1973). These categories are basically smallest possible extensions of Top with certain better properties, and they do not, and were not meant to contain, embeddings of other interesting categories.

Some categories however are specifically intended, as the category of approach spaces, to merge two familiar and somewhat related categories in one supercategory.

Another typical such example is the category of *nearness spaces* as introduced by Herrlich (1974a). Nearness spaces constitute a supercategory of the categories Unif of uniform spaces and R_0Top of R_0-topological spaces. For further information we refer to Bentley et al. (1998), Császár (1963) (for related concepts), Herrlich (1974b), Herrlich et al. (1991) and Hušek (1964a, b).

A historical overview can be found in "Handbook of the History of General Topology" Volume 3 (edit: Aull and Lowen 2001) in the articles "Supercategories of Top and the inevitable emergence of topological constructs" by Colebunders and Lowen and "The historical development of uniform, proximal and nearness concepts in topology" by Bentley, Herrlich and Hušek.

3. **Limit operators and approximation theory**

The formula given in 2.3.2 for the limit operator of a sequence in a metric space is well known in approximation theory. It was introduced in this field in 1972 by Edelstein in (1972) and was later, in 1980, generalized for nets by Lim (1980) (see also Amir et al. 1982; Benyamini 1985; Lami Dozo 1981; Liu 2001). The setting there was mainly restricted to bounded sequences or nets in closed convex subsets of a Banach space E and the main interest was to find a point where the limit operator would be minimal. Such a point is called an *asymptotic center*, i.e. a point x where

$$\lambda_d \langle (x_n)_n \rangle (x) = \min\{\lambda_d \langle (x_n)_n \rangle (y) \mid y \in E\}$$

and the value of the limit operator at such an asymptotic center is called the *asymptotic radius*, i.e. $\inf_{x \in X} \lambda_d \langle (x_n)_n \rangle (x)$. Since an asymptotic center, if it exists, need not be unique, the term asymptotic center is also used for the set of all asymptotic centers in the sense of the first definition, i.e.

$$\{x \mid \lambda_d \langle (x_n)_n \rangle (x) = \min_{y \in E} \lambda_d \langle (x_n)_n \rangle (y)\}.$$

As an example, consider the real line $\mathbb{R}$ with the usual Euclidean metric and topology, and consider the sequences $(x_n)_n$ and $(y_n)_n$ where

$$x_n := \begin{cases} \varepsilon & n \text{ even} \\ -\varepsilon & n \text{ odd,} \end{cases} \quad \text{and } y_n := \begin{cases} n & n \text{ even} \\ -n & n \text{ odd.} \end{cases}$$

Neither of these sequences converges. The first one, however, has two main convergent subsequences, and from the point of view of numerical analysis or approximation theory, for a "sufficiently small" ε, the sequence itself might actually be considered "sufficiently" convergent, e.g. to 0. The second sequence on the other hand has no convergent subsequences, and could not even remotely be considered to be "approximately convergent" to any point of $\mathbb{R}$.

A more striking example is obtained as follows. Let $\varphi : \mathbb{R} \longrightarrow \ ]-\varepsilon, \varepsilon[$ be a homeomorphism, and let $(r_n)_n$ be an enumeration of the rationals. The sequence $(r_n)_n$ is not remotely "approximately convergent" to any point of $\mathbb{R}$. The sequence $(\varphi(r_n))_n$ on the other hand, for a "sufficiently small" ε, might again be considered "sufficiently" convergent.

By means of the topology of $\mathbb{R}$, not only can we not detect the different behaviour of these sequences, we must conclude that they are "identical". In order to "see" the difference we require the metric and the concepts of limit and adherence operator.

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