Chapter 2
Robust Performance Analysis of LTI Systems

2.1 Introduction

In this chapter, we study simple control system analysis problems that embrace many, if not most, engineering issues with respect to performances of dynamical systems close to an equilibrium point. Systems are modeled in state-space and assumed linear time-invariant. Inevitable modeling errors are considered in the simplest affine polytopic form. Having chosen this rather well-established and highly studied framework, the aim of the chapter is to give a comprehensive, step by step understanding of key characteristics of the S-variable approach. Meanwhile, the chapter recalls some fundamental proofs and reasonings that structure the robust control field. One fundamental understanding is that results should be formalized in terms of convex optimization (linear matrix inequalities) for which polynomial-time solvers exist. It is shown that the S-variable approach fits perfectly this specification and has a numerical burden that remains reasonable compared to prior existing results. The other fundamental understanding is conservatism, that is the ability of an approach to merge the sometimes inevitable gap between necessary and sufficient conditions. With respect to conservatism it is shown that the S-variable approach is much less conservative than the prior existing methods as long as robustness issues are considered. This feature, combined with the very easy understanding of mathematical manipulations, explains its rapid success in the control literature.

2.2 Robust Stability Analysis of Uncertain LTI Systems

Let us consider the linear time-invariant (LTI) system described by

$$\dot{x}(t) = Ax(t). \tag{2.1}$$

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Here, $x \in \mathbb{R}^n$ is the state variable and $A \in \mathbb{R}^{n \times n}$ is the coefficient matrix. This system is said to be *stable* if

$$\lim_{t \to \infty} x(t) \to 0 \quad (\forall x(0) \in \mathbb{R}^n). \tag{2.2}$$

Stability is of course the premier requirement in any engineering system. Therefore, analyzing stability of a given system, and moreover, designing a controller ensuring stability of closed-loop systems, appear to be important issues in control theory.

If a practical system of interest can be modeled accurately as (2.1), it is straightforward to determine its stability. It is well known that the system (2.1) is stable if and only if the matrix A is Hurwitz stable, i.e., every eigenvalue of A is located in $\mathbb{C}_-$. Here, $\mathbb{C}_-$ stands for the open left-half plane in $\mathbb{C}$. Therefore, we can easily check the stability of (2.1) by computing the eigenvalues of A.

However, due to inevitable uncertainties, it is impossible to model a given practical system accurately as (2.1) even if it is LTI. In particular, if the system of interest includes physical parameters whose exact values are hardly available, it is often natural to employ a "model set" described by

$$\dot{x}(t) = A(\theta)x(t), \quad \theta \in \mathbb{E}^L \subset \mathbb{R}^L. \tag{2.3}$$

Here, the matrix-valued function $A(\cdot) : \mathbb{R}^L \to \mathbb{R}^{n \times n}$ is linear and given by

$$A(\theta) := \sum_{j=1}^{L} \theta_j A_j \tag{2.4}$$

where $A_j \in \mathbb{R}^{n \times n}$ $(j \in \mathscr{I}_L)$ are known matrices. On the other hand, the set $\mathbb{E}^L$ is a standard simplex in $\mathbb{R}^L$ defined by

$$\mathbb{E}^L := \left\{ \theta \in \mathbb{R}^L : \theta \geq 0,\ \mathbf{1}^T \theta = 1 \right\}.$$

We emphasize that the parameter θ is time-invariant, but whose exact value is unknown. That is, θ denotes *uncertain parameters* in the system (2.3), and the only available information is $\theta \in \mathbb{E}^L$. The model of the form (2.3) is often called *polytopic uncertain model*, and A_j $(j \in \mathscr{I}_L)$ in (2.4) are *vertex matrices* of $A(\theta)$.

For a concrete example, let us consider a quarter-car suspension model depicted in Fig. 2.1. This model is frequently used for designing active suspensions.

The model is composed of a chassis of weight M and a wheel of weight m, connected by a spring of constant k_1 and a damper of constant c. The spring of constant k_2 stands for the wheel stiffness. The control input is the force to chassis and wheel and denoted by f. The vertical positions of chassis and wheel (from their equilibrium points) are represented by z_c and z_w, respectively.

In this model, the parameters M and k_2 should be treated uncertain, since in practice they depend on the number of passengers and wheel conditions, etc. Therefore,

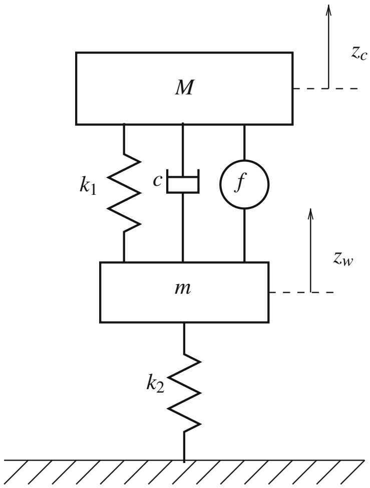


Fig. 2.1 Quarter-car suspension model

we assume that these two parameters are unknown but bounded with known (or say, reasonably estimated) minimum and maximal values as follows:

$$M \in [M_{\min}, M_{\max}], \quad k_2 \in [k_{2,\min}, k_{2,\max}]. \tag{2.5}$$

On the other hand, we assume that other parameters are exactly known.

The equations of motion for the quarter-car suspension model are given by

$$\begin{aligned} M\ddot{z}_c &= -k_1(z_c - z_w) - c(\dot{z}_c - \dot{z}_w) + f, \\ m\ddot{z}_w &= k_1(z_c - z_w) + c(\dot{z}_c - \dot{z}_w) - k_2 z_w - f. \end{aligned} \tag{2.6}$$

If we define $x := [z_c \; \dot{z}_c \; z_w \; \dot{z}_w]^T$ and $u := f$, the above equations can be converted to the state-space form as follows:

$$\begin{aligned} &\dot{x}(t) = A(M, k_2)x(t) + B_2(M)u(t), \quad M \in [M_{\min}, M_{\max}], \quad k_2 \in [k_{2,\min}, k_{2,\max}], \\ &A(M, k_2) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{M} & -\frac{c}{M} & \frac{k_1}{M} & \frac{c}{M} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m} & \frac{c}{m} & -\frac{k_1+k_2}{m} & -\frac{c}{m} \end{bmatrix}, \quad B_2(M) := \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{m} \end{bmatrix}. \end{aligned} \tag{2.7}$$

For this system, suppose a static state-feedback control $u = Kx$ is designed to achieve typical requirements for active suspensions such as satisfactory ride comfort and road-holding ability, etc. If we apply this state-feedback, then the closed-loop system can be written as

$$\dot{x}(t) = A_{\mathrm{cl}}(M, k_2)x(t), \quad M \in [M_{\min}, M_{\max}], \quad k_2 \in [k_{2,\min}, k_{2,\max}],$$
$$A_{\mathrm{cl}}(M, k_2) := A(M, k_2) + B_2(M)K.$$

It is now elementary to see that this can be rewritten, equivalently, to a polytopic model of the form (2.3) and (2.4) as follows:

$$\dot{x}(t) = A_{\mathrm{cl}}(\theta)x(t) \quad \theta \in \mathbb{E}^4, \quad A_{\mathrm{cl}}(\theta) = \sum_{j=1}^{4} \theta_j A_{cl,j},$$
$$A_{\mathrm{cl},1} = A_{\mathrm{cl}}(M_{\max}, k_{2,\max}), \quad A_{\mathrm{cl},2} = A_{\mathrm{cl}}(M_{\max}, k_{2,\min}),$$
$$A_{\mathrm{cl},3} = A_{\mathrm{cl}}(M_{\min}, k_{2,\max}), \quad A_{\mathrm{cl},4} = A_{\mathrm{cl}}(M_{\min}, k_{2,\min}).$$

In this example, we emphasize that the open-loop system (without feedback $u = Kx$) is, of course, stable irrespective of the parameters $M(>0)$ and $k_2(>0)$. However, such physically ensured stability property cannot retain in general if we apply feedback control to achieve desired control objectives. Therefore, analyzing robust stability of designed controllers, or moreover, designing feedback controllers ensuring robust stability against parameter variations, are definitely important issues in control engineering.

We now go back to the discussion on the general uncertain system description (2.3). Since the parameter θ is assumed to be time-invariant, it is clear that the uncertain system (2.3) is *robustly stable* (i.e., it is stable for all $\theta \in \mathbb{E}^L$) if and only if all the eigenvalues of $A(\theta)$ remain to be $\mathbb{C}_-$. However, it is hard to check the variations of the eigenvalues of $A(\theta)$ over $\theta \in \mathbb{E}^L$ rigorously. One of the effective ways to get around this difficulty is relying on the Lyapunov's stability theorem described below.

Theorem 2.1 *The system* (2.1) *is stable, or equivalently, the coefficient matrix in* (2.1) *is Hurwitz stable if and only if there exists* $P \in \mathbb{S}^n$ *such that*

$$P \succ 0, \quad PA + A^T P \prec 0. \tag{2.8}$$

The inequality above is known as *Lyapunov inequality*, which is an LMI for the matrix variable P to be determined. The matrix P is often called *Lyapunov (matrix) variable*. In the following, we give brief remarks on the Lyapunov inequality (2.8) since it forms an important basis for stability analysis of (uncertain) LTI systems. First, the inequality is strongly related to the existence of quadratic-in-the-state Lyapunov function $V_P(x) := x^T Px$ that certificates the stability of (2.1). Indeed, if we define $V_P(x)$ by means of a feasible solution P for (2.8), we see that the time-derivative of $V_P(x)$ along the trajectory of (2.1) satisfies

$$\frac{\mathrm{d}V_P(x)}{\mathrm{d}t} = x^T(PA + A^T P)x < 0 \quad (\forall x \neq 0).$$

On the other hand, since $P \succ 0$, it is obvious that $V_P(x) > 0$ $(\forall x \neq 0)$. With this fact and the above inequality, we have $V_P(x) \to 0$ $(t \to \infty)$ for any $x(0) \in \mathbb{R}^n$. Again, since $P \succ 0$, this also ensures $x \to 0$ $(t \to \infty)$, that is, the system (2.1) is stable.

It is also possible to prove the sufficiency of (2.8) in a purely algebraic manner (without invoking the dynamical behavior of (2.1)). Indeed, if (2.8) holds, then we have

$$\xi^*(PA + A^T P)\xi < 0 \quad \forall \xi \in \mathbb{C}^n \setminus \{0\}.$$

In particular, if we let ξ as an eigenvector of A corresponding to an eigenvalue $\lambda \in \mathbb{C}$ (i.e., $A\xi = \lambda\xi$), the above inequality implies

$$(\lambda + \lambda^*)\xi^* P\xi < 0.$$

Since $P \succ 0$, we have $\xi^* P\xi > 0$ and hence we can conclude

$$\lambda + \lambda^* < 0.$$

This clearly shows that every eigenvalue of A is located in $\mathbb{C}_-$. The exposition above proves the sufficiency of (2.8) for stability. On the other hand, its necessity also follows from elementary arguments on Lyapunov equality. See [1] for details.

We next apply Theorem 2.1 to the robust stability analysis of the uncertain system (2.3). Since $A(\theta)$ in (2.4) is linear on θ, the following theorem readily follows:

Theorem 2.2 *The uncertain LTI system* (2.3) *is stable for all* $\theta \in \mathbb{E}^L$ *if there exists* $P \in \mathbb{S}^n$ *such that*

$$P \succ 0, \quad PA_j + A_j^T P \prec 0 \ (j \in \mathscr{I}_L). \tag{2.9}$$

Proof Suppose (2.9) holds. Then, for any $\theta \in \mathbb{E}^L$, we have

$$P \succ 0, \quad \sum_{j=1}^{L} \theta_j (PA_j + A_j^T P) \prec 0$$

or equivalently,

$$P \succ 0, \quad PA(\theta) + A(\theta)^T P \prec 0. \tag{2.10}$$

From this inequality and Theorem 2.1, we can conclude that the uncertain LTI system (2.3) is stable for all $\theta \in \mathbb{E}^L$. □

The condition (2.9) is numerically tractable since it is finite-dimensional finite-number of LMI feasibility problems. This is the main advantage in working with the

LMI condition (2.8) instead of directly examining the variations of the eigenvalues of $A(\theta)$. The condition (2.9) is known as *quadratic stability condition* for the uncertain system (2.3), since, as clearly shown in (2.10), the condition (2.9) is necessary and sufficient for the existence of a single quadratic-in-the-state Lyapunov function $V_P(x)$ that certificates the stability of (2.3) for all $\theta \in \mathbb{E}^L$. Finally, we emphasize that the condition (2.9) is *conservative*, i.e., it is only sufficient for the robust stability of (2.3) and far from necessary in general. The conservatism stems from the fact we seek for a single, or say, common Lyapunov function that uniformly ensures the stability of the uncertain system (2.3). Therefore, we need to derive a condition that allows us to narrow the "gap" toward the exact analysis, but at the same time we need to keep the condition still being computationally tractable. The S-variable approach was conceived to achieve these objectives.

2.3 Robust Stability Analysis Using S-Variable Approach

We now state how the stability of the LTI system (2.1) is characterized by an LMI with S-variables.

Theorem 2.3 *The system* (2.1) *is stable, or equivalently, the coefficient matrix in* (2.1) *is Hurwitz stable if and only if there exist* $P \in \mathbb{S}^n$ *and* $F_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$ *such that*

$$P \succ 0, \quad \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} I & -A \end{bmatrix} \right\} \prec 0. \tag{2.11}$$

Proof We will establish the validity by showing the equivalence of (2.8) and (2.11). To this end, let us note that the following choice is valid:

$$\begin{bmatrix} -I \\ A^T \end{bmatrix}^{\perp} = \begin{bmatrix} A^T & I \end{bmatrix}.$$

It also reads as

$$\begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} -I \\ A^T \end{bmatrix}^{\perp T} = \begin{bmatrix} I & -A \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} = 0$$

and hence

$$\begin{aligned}
&\begin{bmatrix} -I \\ A^T \end{bmatrix}^{\perp} \left(\begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} I & -A \end{bmatrix} \right\} \right) \begin{bmatrix} -I \\ A^T \end{bmatrix}^{\perp T} \\
&= \begin{bmatrix} A^T & I \end{bmatrix} \begin{bmatrix} 0 & P \\ P & 0 \end{bmatrix} \begin{bmatrix} A \\ I \end{bmatrix} \\
&= PA + A^T P.
\end{aligned}$$

Therefore, the equivalence of (2.8) and (2.11) readily follows from Lemma 1.2, i.e., Elimination Lemma [2, 3]. □

As in the proof, the equivalence of (2.8) and (2.11) can be proved by eliminating the variables F_1 and F_2 from (2.11) via Elimination lemma. Since (2.8) is originally given, this elimination procedure can be restated conversely that the condition (2.11) is derived from (2.8) by introducing (or creating) S-variables F_1 and F_2.

The main advantage in working with (2.11) instead of (2.8) lies in the fact that in (2.11) the Lyapunov variable P is free from multiplication with A, whereas F_1 and F_2 are. This *decoupling* property is the main feature of SV-LMIs in general. In control system analysis and synthesis, we often encounter matrix inequality formulations that include undesirable multiplications among data matrices, Lyapunov variables, scalings, and controller variables, etc. Such multiplications can be decoupled by introducing SVs, and this decoupling property indeed brings great advantage when we deal with wide range of "difficult" problems for which definitive approaches are hardly available. On the other hand, a drawback of the SV-LMI (2.11) is of course obvious; it is computationally more demanding than the original one (2.8) due to the increase of number of variables and size of LMIs. This is, again, a typical drawback of SV-LMIs in general.

As noted, the SV-LMI (2.11) can be obtained by dilating (2.8) with S-variables F_1 and F_2. However, we see that the proof based on Elimination lemma does not provide any concrete ways to create F_1 and F_2 satisfying (2.11) from the original LMI (2.8). Indeed, deriving explicit expressions of SVs satisfying SV-LMIs is an important issue in the study of SV-LMIs and therefore this issue will be discussed repeatedly in this book. In the following, we derive such an explicit expression for (2.8) and (2.11). To this end, suppose (2.8) holds with $P = \Pi(\succ 0)$. Then, there exists $\overline{\varepsilon} > 0$, which depends on Π, such that for any $\varepsilon \in (0, \overline{\varepsilon})$ the following inequality holds:

$$\Pi A + A^T \Pi + \frac{1}{2}\varepsilon A^T \Pi A \prec 0. \tag{2.12}$$

By Schur complement, we see that this is equivalent to

$$\begin{bmatrix} -2\varepsilon\Pi & \varepsilon\Pi A \\ \varepsilon A^T \Pi & \Pi A + A^T \Pi \end{bmatrix} \prec 0.$$

Furthermore, we can rewrite this matrix inequality as

$$\begin{bmatrix} 0 & \Pi \\ \Pi & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} -\varepsilon\Pi \\ -\Pi \end{bmatrix}\begin{bmatrix} I & -A \end{bmatrix}\right\} \prec 0.$$

This clearly shows that (2.11) holds with $P = \Pi$, $F_1 = -\varepsilon\Pi$ and $F_2 = -\Pi$. This result is summarized in the next lemma.

Lemma 2.1 *Suppose* (2.8) *holds with* $P = \Pi$. *Then, there exists* $\overline{\varepsilon} > 0$ *such that the LMI* (2.11) *holds with* $(P, F_1, F_2) = (\Pi, -\varepsilon\Pi, -\Pi)$ *for all* $\varepsilon \in (0, \overline{\varepsilon})$.

We are now ready to apply the SV-LMI (2.11) to the robust stability analysis of the uncertain system (2.3). The usefulness of the the SV-LMI will be clarified immediately.

Theorem 2.4 *The uncertain LTI system* (2.3) *is stable for all* $\theta \in \mathbb{E}^L$ *if there exist* $P_j \in \mathbb{S}^n$ $(j \in \mathscr{I}_L)$, $F_1, F_2 \in \mathbb{R}^{n\times n}$ *such that*

$$P_j \succ 0,\quad \begin{bmatrix} 0 & P_j \\ P_j & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\begin{bmatrix} I & -A_j \end{bmatrix}\right\} \prec 0 \; (j \in \mathscr{I}_L). \tag{2.13}$$

Proof Suppose (2.13) holds. Then, for any $\theta \in \mathbb{E}^L$, we have

$$\sum_{j=1}^{L} \theta_j P_j \succ 0,\quad \sum_{j=1}^{L} \theta_j \left(\begin{bmatrix} 0 & P_j \\ P_j & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\begin{bmatrix} I & -A_j \end{bmatrix}\right\}\right) \prec 0$$

or equivalently,

$$\begin{aligned} &P(\theta) \succ 0,\quad \begin{bmatrix} 0 & P(\theta) \\ P(\theta) & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\begin{bmatrix} I & -A(\theta) \end{bmatrix}\right\} \prec 0, \\ &P(\theta) := \sum_{j=1}^{L} \theta_j P_j. \end{aligned} \tag{2.14}$$

From the first two inequalities above and Theorem 2.3, we can conclude that the uncertain LTI system (2.3) is stable for all $\theta \in \mathbb{E}^L$. □

This theorem provides an SV-LMI-based condition for the robust stability of (2.3). Unfortunately, the condition (2.13) is still sufficient to conclude robust stability in general. Since (2.9) is also sufficient, we are naturally lead to the following question: Is the condition (2.13) better (or more precisely, no worse) than (2.9)? The answer is "yes," as formally stated in the following theorem.

Theorem 2.5 *Suppose* (2.9) *holds with* $P = \Pi$. *Then, there exists* $\varepsilon > 0$ *such that* (2.13) *holds with* $P = \Pi$, $F_1 = -\varepsilon\Pi$ *and* $F_2 = -\Pi$.

Proof Suppose (2.9) holds with $P = \Pi$. Then, from Lemma 2.1, there exist $\overline{\varepsilon}_j$ $(j \in \mathscr{I}_L)$ such that the jth LMI in (2.13) holds with $P_j = \Pi$, $F_1 = -\varepsilon_j\Pi$ and $F_2 = -\Pi$ for any $\varepsilon_j \in (0, \overline{\varepsilon}_j)$. Therefore, by letting $\varepsilon \in (0, \min_j \overline{\varepsilon}_j)$, the assertion is verified. □

Theorem 2.5 says that if we can conclude robust stability of (2.3) by the quadratic stability condition (2.9), then we can always conclude the same by the SV-LMI condition (2.13) as well. The converse is not true in general. As illustrated by numerical examples in Sect. 2.6, the effectiveness of (2.13) is remarkable, and we frequently experience that (2.13) works fine to conclude robust stability even for those problem instances where (2.9) fails.

As we have seen in the proof of Theorem 2.5, Lemma 2.1 plays a key role in ensuring an explicit advantage of (2.13) over (2.9). However, qualitatively speaking, the effectiveness of (2.13) can be explained from (2.14), which implies that the condition (2.13) ensures the robust stability of (2.3) via *parameter-dependent* quadratic-in-the state Lyapunov function $V_P(\theta, x) := x^T P(\theta)x$. Namely, the restriction to a common Lyapunov function in the quadratic stability approach (2.9) is relaxed to a Lyapunov function that can depend linearly on the parameter θ. In the literature, there are several ways to introduce parameter-dependent Lyapunov functions so that less-conservative analysis/synthesis results can be achieved. Among them, the SV approach can be regarded as a powerful tool, and the most attractive feature is that it allows us to derive LMI conditions for (robust) controller synthesis as well. See Chaps. 4, 7, 8 for detailed discussions on this point.

Anticipating slightly the discussions of these chapters, we can state some issues for deriving the control design results. The first one is related to the fact that state-feedback design method have classically nice formulations when LMI conditions are derived not for the original system $\dot{x} = Ax$ but for the *dual system* $\dot{x}_d = A^T x_d$. Related to Theorems 2.1 and 2.3, the following corollaries readily follow for the dual system.

Corollary 2.1 *The system* (2.1) *is stable, or equivalently, the coefficient matrix in* (2.1) *is Hurwitz stable if and only if there exists* $X \in \mathbb{S}^n$ *such that*

$$X \succ 0, \quad AX + XA^T \prec 0. \tag{2.15}$$

Corollary 2.2 *The system* (2.1) *is stable, or equivalently, the coefficient in* (2.1) *is Hurwitz stable if and only if there exist* $X \in \mathbb{S}^n$ *and* $F_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$ *such that*

$$X \succ 0, \quad \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} I \\ -A \end{bmatrix}\begin{bmatrix} F_1 & F_2 \end{bmatrix}\right\} \prec 0. \tag{2.16}$$

Corollary 2.3 *Suppose* (2.15) *holds with* $X = \Xi$. *Then, there exists* $\overline{\varepsilon} > 0$ *such that the LMI* (2.16) *holds with* $(X, F_1, F_2) = (\Xi, -\varepsilon\Xi, -\Xi)$ *for all* $\varepsilon \in (0, \overline{\varepsilon})$.

Corollary 2.4 *The uncertain LTI system* (2.3) *is stable for all* $\theta \in \mathbb{E}^L$ *if there exist* $X_j \in \mathbb{S}^n$ $(j \in \mathscr{I}_L)$ *and* $F_i \in \mathbb{R}^{n\times n}$ $(i = 1, 2)$ *such that*

$$X_j \succ 0, \quad \begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} I \\ -A_j \end{bmatrix}\begin{bmatrix} F_1 & F_2 \end{bmatrix}\right\} \prec 0. \tag{2.17}$$

In the LMIs (2.15), (2.16) and (2.17), the matrix A has multiplication with variables from the right-hand side and this is different from (2.8), (2.11) and (2.13). Due to this reason, the LMIs (2.15) and (2.16) are effective for feedback controller synthesis as well (see related discussions in Chaps. 4 and 8).

Theorem 2.4 and Corollary 2.4 are two versions of a same result but applied to either the original system $\dot{x} = A(\theta)x$ or to its dual $\dot{x}_d = A^T(\theta)x_d$. Both conclude on the robust stability of $\dot{x} = A(\theta)x$ and one could therefore expect the two results to be equivalent. This is not the case.

Theorem 2.6 *The LMI condition* (2.13) *may be feasible and* (2.17) *not, and the converse is true as well.*

The proof of this result needs only to give one such example. This is done later on in the Sect. 2.6 devoted to numerical examples. To understand the reasons for this result recall that if $x_d^T X x_d$ is a Lyapunov function proving the stability of the dual system, then $x^T X^{-1} x$ is a Lyapunov function for the original system. Now, from the proof of Theorem 2.4 it can be noticed that the LMI conditions (2.13) imply that $x^T P(\theta) x$ is a Lyapunov function for the system, with $P(\theta) = \sum_{j=1}^{L} \theta_j P_j$ linear with respect to θ. Similarly, the solution of (2.17) implies that $x_d^T X(\theta) x_d$ is a Lyapunov function for the dual system, with $X(\theta) = \sum_{j=1}^{L} \theta_j X_j$ linear with respect to θ. Hence, Theorem 2.4 is related to the existence of a linear parameter-dependent Lyapunov matrix $P(\theta)$ while Corollary 2.4 is related to the existence of a Lyapunov matrix $X^{-1}(\theta)$ which is not linear. These two properties are trivially not equivalent.

The second issue for deriving design conditions that will be discussed in the next chapters is that F_1 and F_2 give too many degrees of freedom. To make the control design conditions convex it is needed to structure the S-variables. To this end, Theorem 2.5 can give the impression that the choice $F_1 = -\varepsilon \Pi$ and $F_2 = -\Pi$ for some sufficiently small ε is appropriate when solving (2.13). Unfortunately, it is not quite the case for the following reason.

Theorem 2.7 *For the choice $F_1 = -\varepsilon \Pi$ and $F_2 = -\Pi$ with $\varepsilon > 0$ and $\Pi \in \mathbb{R}^{n \times n}$, if ε tends to zero then* (2.13) *converges to the quadratic stability conditions of* (2.9).

Proof Assume a solution to (2.13) with S-variables structured as $F_1 = -\varepsilon \Pi$ and $F_2 = -\Pi$ where $\varepsilon > 0$:

$$\begin{bmatrix} 0 & P_j \\ P_j & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} -\varepsilon \Pi \\ -\Pi \end{bmatrix} \begin{bmatrix} I & -A_j \end{bmatrix} \right\}$$
$$= \begin{bmatrix} -\varepsilon \Pi - \varepsilon \Pi^T & P_j - \Pi^T + \varepsilon \Pi A_j \\ P_j - \Pi + \varepsilon A_j^T \Pi^T & \Pi A_j + A_j^T \Pi^T \end{bmatrix} \prec 0.$$

After a Schur complement argument applied to the (1,1) block, this LMI also reads as

$$\mathrm{He}\left\{ \left((P_j - \Pi)(\Pi + \Pi^T)^{-1} + I \right) \Pi A_j \right\}$$
$$\prec -\varepsilon A_j^T \Pi^T (\Pi + \Pi^T)^{-1} \Pi A_j - \varepsilon^{-1} (P_j - \Pi)(\Pi + \Pi^T)^{-1} (P_j - \Pi^T).$$

Assume now that $\varepsilon > 0$ is indeed small, i.e., assume $\varepsilon \to 0$. Then the right-hand side of the inequality is a matrix with eigenvalues going to $-\infty$ except if $\Pi = P_j$ $(j \in \mathscr{I}_L)$. Since the left-hand side is independent of ε, the only possible finite solution is indeed that $(P_j - \Pi) \to 0$ as ε goes to zero. Hence we conclude that for small values of ε the SV-LMI with structured S-variables $F_1 = -\varepsilon \Pi$ and $F_2 = -\Pi$

implies asymptotically

$$\Pi \succ 0, \quad \Pi A_j + A_j^T \Pi \prec 0 \, (j \in \mathcal{I}_L)$$

which is the condition of the quadratic stability Theorem 2.2. □

Theorem 2.7 is a negative result in the sense that the structure $F_1 = -\varepsilon\Pi$ and $F_2 = -\Pi$ with ε small brings no advantage compared to quadratic stability conditions. The question of structuring the S-variables is left open at this point of the book. The discussion is continued in the next chapters.

2.4 Lemmas for SV-LMI Derivation

In the previous section, SV results are presented for the case of stability analysis of continuous-time LTI systems. Some properties of the result are discussed for this specific criterion. These properties happen to be valid for many more cases some of which are discussed in the following section. Before that, we expose the technical lemmas that formalize the properties.

Lemma 2.2 *For given* $\mathsf{M}_{11} \in \mathbb{H}_+^n$, $\mathsf{M}_{12} \in \mathbb{C}^{n\times m}$, $\mathsf{M}_{22} \in \mathbb{H}^m$, $\mathsf{A} \in \mathbb{C}^{n\times m}$, $\mathsf{E} \in \mathbb{C}^{n\times n}$ *the following two conditions are equivalent.*

(i) The matrix E *is invertible and*

$$\begin{bmatrix} \mathsf{E}^{-1}\mathsf{A} \\ I \end{bmatrix}^* \begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathsf{E}^{-1}\mathsf{A} \\ I \end{bmatrix} \prec 0. \tag{2.18}$$

(ii) There exist matrices $\mathsf{F}_1 \in \mathbb{C}^{n\times n}$ *and* $\mathsf{F}_2 \in \mathbb{C}^{m\times n}$ *such that*

$$\begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{M}_{22} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathsf{F}_1 \\ \mathsf{F}_2 \end{bmatrix} \begin{bmatrix} \mathsf{E} & -\mathsf{A} \end{bmatrix} \right\} \prec 0. \tag{2.19}$$

Moreover,

(a) If A *and* E *are real valued, then* F_1 *and* F_2 *can be chosen real valued.*
(b) If (2.18) *holds, then whatever* $\mathsf{W} \in \mathbb{H}_{++}^n$, (2.19) *holds as well with* $\mathsf{F}_1 = -(\mathsf{M}_{11} + \varepsilon\mathsf{W})\mathsf{E}^{-1}$ *and* $\mathsf{F}_2 = -\mathsf{M}_{12}^*\mathsf{E}^{-1}$ *and a sufficiently small* $\varepsilon > 0$.
(c) If (2.18) *holds and* $M_{11} \succ 0$, *then* (2.19) *holds as well with* $\mathsf{F}_1 = -\mathsf{M}_{11}\mathsf{E}^{-1}$ *and* $\mathsf{F}_2 = -\mathsf{M}_{12}^*\mathsf{E}^{-1}$.

Proof The equivalence of (i) and (ii) is the direct application of the elimination lemma [2, 3] due to the fact that for invertible E:

$$\begin{bmatrix} \mathsf{E}^{-1}\mathsf{A} \\ I \end{bmatrix}^* = \begin{bmatrix} \mathsf{E}^* \\ -\mathsf{A}^* \end{bmatrix}^\perp.$$

Note that $\mathsf{M}_{11} \in \mathbb{H}^n_+$ ensures the invertibility of E in (2.19). Property (a) comes from Finsler's lemma [3]. That lemma states that if (2.18) holds then there exists some positive scalar $\tau \geq 0$ such that

$$\begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{M}_{22} \end{bmatrix} \prec \tau \begin{bmatrix} \mathsf{E}^T \\ -\mathsf{A}^T \end{bmatrix} \begin{bmatrix} \mathsf{E} & -\mathsf{A} \end{bmatrix}.$$

This inequality provides a possible choice of S-variables is $\mathsf{F}_1 = \frac{\tau}{2}\mathsf{E}^T$ and $\mathsf{F}_2 = -\frac{\tau}{2}\mathsf{A}^T$ which are real valued.

We shall now prove property (b). Assume (2.18) holds, then for any positive definite matrix $\mathsf{W} \succ 0$ and a sufficiently small $\varepsilon > 0$ the following inequality holds as well:

$$\begin{bmatrix} \mathsf{E}^{-1}\mathsf{A} \\ I \end{bmatrix}^* \begin{bmatrix} \mathsf{M}_{11} + \varepsilon\mathsf{W} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{M}_{22} \end{bmatrix} \begin{bmatrix} \mathsf{E}^{-1}\mathsf{A} \\ I \end{bmatrix}$$
$$= \mathsf{A}^*\mathsf{E}^{-*}(\mathsf{M}_{11} + \varepsilon\mathsf{W})\mathsf{E}^{-1}\mathsf{A} + \mathsf{A}^*\mathsf{E}^{-*}\mathsf{M}_{12} + \mathsf{M}_{12}^*\mathsf{E}^{-1}\mathsf{A} + \mathsf{M}_{22} \prec 0.$$

The matrix $\mathsf{M}_{11} + \varepsilon\mathsf{W}$ is positive definite, one can therefore apply the Schur complement result and get

$$\begin{bmatrix} -\mathsf{M}_{11} - \varepsilon\mathsf{W} & (\mathsf{M}_{11} + \varepsilon\mathsf{W})\mathsf{E}^{-1}\mathsf{A} \\ \mathsf{A}^*\mathsf{E}^{-*}(\mathsf{M}_{11} + \varepsilon\mathsf{W}) & \mathsf{A}^*\mathsf{E}^{-*}\mathsf{M}_{12} + \mathsf{M}_{12}^*\mathsf{E}^{-1}\mathsf{A} + \mathsf{M}_{22} \end{bmatrix} \prec 0,$$

which implies

$$\begin{bmatrix} -\mathsf{M}_{11} - 2\varepsilon\mathsf{W} & (\mathsf{M}_{11} + \varepsilon\mathsf{W})\mathsf{E}^{-1}\mathsf{A} \\ \mathsf{A}^*\mathsf{E}^{-*}(\mathsf{M}_{11} + \varepsilon\mathsf{W}) & \mathsf{A}^*\mathsf{E}^{-*}\mathsf{M}_{12} + \mathsf{M}_{12}^*\mathsf{E}^{-1}\mathsf{A} + \mathsf{M}_{22} \end{bmatrix} \prec 0,$$

This inequality is exactly (2.19) for the choice $\mathsf{F}_1 = -(\mathsf{M}_{11} + \varepsilon\mathsf{W})\mathsf{E}^{-1}$ and $\mathsf{F}_2 = -\mathsf{M}_{12}^*\mathsf{E}^{-1}$.

Property (c) is a special case of (b) for the choice $\varepsilon = 0$. This choice can be done only if M_{11} is positive definite. □

To illustrate how Lemma 2.2 works fine, let us derive (2.11) from (2.8) along the line of the lemma. We first note that the second inequality (2.8) can be identified with (2.18) by the correspondences

$$\mathsf{M}_{11} = 0, \quad \mathsf{M}_{12} = P, \quad \mathsf{M}_{22} = 0, \quad \mathsf{E} = I, \quad \mathsf{A} = A. \tag{2.20}$$

Then, we obtain directly (2.11) from (2.19) by letting $F_i = \mathsf{F}_i$ $(i = 1, 2)$. The results in Lemma 2.1 follow from property *(b)* in Lemma 2.2 for the choice $\mathsf{W} = P$.

As illustrated above, we can use Lemma 2.2 as follows for the derivation of SV-LMIs. First, we identify a known LMI condition with (2.18) by finding out appropriate correspondences as in (2.20). Once this step is done, it is straightforward to derive a corresponding SV-LMI of the form (2.19).

In comparison with (2.18), we see that the SV-LMI (2.19) has the following preferable properties:

(i) *Decoupling*: In (2.19), variables that may be contained in M_{11}, M_{12}, and M_{22} have no multiplication relation with E and A, while F_1 and F_2 do.
(ii) *Eliminating inverse*: In (2.19), the inverse of the matrix E is eliminated.

As partly illustrated in the preceding section, the property (i) is effective when we deal with robust performance analysis problems for polytopic uncertain systems. Indeed, it enables us to employ parameter-dependent Lyapunov functions so that less-conservative analysis/synthesis conditions can be achieved.

Before formulating the central technical results of this chapter, we first give an intermediate technical lemma. It states in which case testing an LMI over a polytope or over its vertices is equivalent.

Lemma 2.3 *Assume* G *and* M *are given where* $\mathsf{G} \succeq 0$. *Then the following two conditions are equivalent*

$$\mathsf{H}_j + \mathrm{He}\{\mathsf{K}_j^*\mathsf{M}\} + \mathsf{J}_j^*\mathsf{G}\mathsf{J}_j \prec 0, \quad \forall j \in \mathscr{I}_L \tag{2.21}$$

$$\mathsf{H}(\theta) + \mathrm{He}\{\mathsf{K}^*(\theta)\mathsf{M}\} + \mathsf{J}^*(\theta)\mathsf{G}\mathsf{J}(\theta) \prec 0, \quad \forall \theta \in \mathbb{E}_L \tag{2.22}$$

where $\mathsf{H}(\theta) = \sum_{j=1}^L \theta_j \mathsf{H}_j$, $\mathsf{K}(\theta) = \sum_{j=1}^L \theta_j \mathsf{K}_j$, $\mathsf{J}(\theta) = \sum_{j=1}^L \theta_j \mathsf{J}_j$.

Proof The implication (2.22)⇒(2.21) is obtained by taking the choices $\theta_j = 1$, $\theta_{i\neq j} = 0$. The implication (2.21)⇒(2.22) is obtained by convexity. Since G is positive semi-definite it has a factorization $\mathsf{G} = \mathsf{L}^*\mathsf{L}$. Applying a Schur complement argument (2.21) implies that

$$\begin{bmatrix} \mathsf{H}_j + \mathrm{He}\{\mathsf{K}_j^*\mathsf{M}\} & \mathsf{J}_j^*\mathsf{L}^* \\ \mathsf{L}\mathsf{J}_j & -I \end{bmatrix} \prec 0, \quad \forall j \in \mathscr{I}_L$$

Summing these conditions with weights θ_j gives

$$\begin{aligned} &\sum_{j=1}^L \theta_j \begin{bmatrix} \mathsf{H}_j + \mathrm{He}\{\mathsf{K}_j^*\mathsf{M}\} & \mathsf{J}_j^*\mathsf{L}^* \\ \mathsf{L}\mathsf{J}_j & -I \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{H}(\theta) + \mathrm{He}\{\mathsf{K}^*(\theta)\mathsf{M}\} & \mathsf{J}^*(\theta)\mathsf{L}^* \\ \mathsf{L}\mathsf{J}(\theta) & -I \end{bmatrix} \prec 0, \quad \forall \theta \in \mathbb{E}^L. \end{aligned}$$

A converse Schur complement argument allows to conclude with (2.22). □

We now give the two central results which explicate the advantage of the SV-LMIs in case of polytopic systems. For the greatest generality of the result, the matrix M_{22} from Lemma 2.2 is decomposed as $\mathsf{M}_{22} = \mathsf{N} + \mathsf{C}^*\mathsf{V}\mathsf{C}$ where $\mathsf{V} \succeq 0$ is assumed to be positive semidefinite.

Lemma 2.4 *For given* $\mathsf{M}_{11j} \in \mathbb{H}^n$, $\mathsf{M}_{12j} \in \mathbb{C}^{n\times m}$, $\mathsf{M}_{22j} \in \mathbb{H}^m$ *with* $\mathsf{M}_{22j} = \mathsf{N}_j + \mathsf{C}_j^*\mathsf{V}\mathsf{C}_j$ $(j \in \mathscr{I}_L)$, $\mathsf{A}_j \in \mathbb{C}^{n\times m}$, *and* $\mathsf{E}_j \in \mathbb{C}^{n\times n}$, *suppose* $\mathsf{V} \succeq 0$, $\mathsf{E}(\theta) = \sum_{j=1}^{L} \theta_j \mathsf{E}_j$ *is invertible for all* $\theta \in \mathbb{E}^L$ *and there exist* $\mathsf{F}_1 \in \mathbb{C}^{n\times n}$ *and* $\mathsf{F}_2 \in \mathbb{C}^{m\times n}$ *such that*

$$\begin{bmatrix} \mathsf{M}_{11j} & \mathsf{M}_{12j} \\ \mathsf{M}_{12j}^* & \mathsf{N}_j + \mathsf{C}_j^*\mathsf{V}\mathsf{C}_j \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} \mathsf{F}_1 \\ \mathsf{F}_2 \end{bmatrix}\begin{bmatrix} \mathsf{E}_j & -\mathsf{A}_j \end{bmatrix}\right\} \prec 0 \quad (\forall j \in \mathscr{I}_L). \tag{2.23}$$

Then, the following condition hold for all $\theta \in \mathbb{E}^L$*:*

$$\begin{bmatrix} \mathsf{E}^{-1}(\theta)\mathsf{A}(\theta) \\ I \end{bmatrix}^* \begin{bmatrix} \mathsf{M}_{11}(\theta) & \mathsf{M}_{12}(\theta) \\ \mathsf{M}_{12}^*(\theta) & \mathsf{N}(\theta) + \mathsf{C}^*(\theta)\mathsf{V}\mathsf{C}(\theta) \end{bmatrix} \begin{bmatrix} \mathsf{E}^{-1}(\theta)\mathsf{A}(\theta) \\ I \end{bmatrix} \prec 0. \tag{2.24}$$

where $\mathsf{M}_{11}(\theta) = \sum_{j=1}^{L} \theta_j \mathsf{M}_{11j}$, $\mathsf{M}_{12}(\theta) = \sum_{j=1}^{L} \theta_j \mathsf{M}_{12j}$, $\mathsf{N}(\theta) = \sum_{j=1}^{L} \theta_j \mathsf{N}_j$, $\mathsf{A}(\theta) = \sum_{j=1}^{L} \theta_j \mathsf{A}_j$ *and* $\mathsf{C}(\theta) = \sum_{j=1}^{L} \theta_j \mathsf{C}_j$.

This lemma can be seen as the extension of Theorem 2.4 to a general formulation. The proof follows the same lines as that of Theorem 2.4.

Proof The first step is, thanks to Lemma 2.3, to notice that (2.23) is equivalent to assessing that for all $\theta \in \mathbb{E}^L$ one has

$$\begin{bmatrix} \mathsf{M}_{11}(\theta) & \mathsf{M}_{12}(\theta) \\ \mathsf{M}_{12}^*(\theta) & \mathsf{N}(\theta) + \mathsf{C}^*(\theta)\mathsf{V}\mathsf{C}(\theta) \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} \mathsf{F}_1 \\ \mathsf{F}_2 \end{bmatrix}\begin{bmatrix} \mathsf{E}(\theta) & -\mathsf{A}(\theta) \end{bmatrix}\right\} \prec 0$$

Finally (2.24) is obtained by post- and premultiplying by $\begin{bmatrix} \mathsf{E}^{-1}(\theta)\mathsf{A}(\theta) \\ I \end{bmatrix}$ and its transpose, respectively. □

By analogy to Theorem 2.4 one expects some results such as Theorem 2.5. It indeed holds and is based on the properties (b) and (c) of Lemma 2.2.

Lemma 2.5 *Under the same setting as Lemma 2.4 except for* $\mathsf{M}_{11j} = \mathsf{M}_{11}$, $\mathsf{M}_{12j} = \mathsf{M}_{12}$, $\mathsf{N}_j = \mathsf{N}$, *and* $\mathsf{E}_j = \mathsf{E}$ $(j \in \mathscr{I}_L)$, *suppose* $\mathsf{V} \succeq 0$, E *is invertible, and* $\mathsf{M}_{11} \succeq 0$. *Then the following three conditions are equivalent:*

(i) The following LMI holds for all $\theta \in \mathbb{E}^L$

$$\begin{bmatrix} \mathsf{E}^{-1}\mathsf{A}(\theta) \\ I \end{bmatrix}^* \begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{N} + \mathsf{C}^*(\theta)\mathsf{V}\mathsf{C}(\theta) \end{bmatrix} \begin{bmatrix} \mathsf{E}^{-1}\mathsf{A}(\theta) \\ I \end{bmatrix} \prec 0. \tag{2.25}$$

(ii) The following LMI holds for all $j \in \mathscr{I}_L$

$$\begin{bmatrix} \mathsf{E}^{-1}\mathsf{A}_j \\ I \end{bmatrix}^* \begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{N} + \mathsf{C}_j^*\mathsf{V}\mathsf{C}_j \end{bmatrix} \begin{bmatrix} \mathsf{E}^{-1}\mathsf{A}_j \\ I \end{bmatrix} \prec 0. \tag{2.26}$$

(iii) There exists F_1 *and* F_2 *such that for all* $j \in \mathscr{I}_L$

$$\begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ \mathsf{M}_{12}^* & \mathsf{N} + \mathsf{C}_j^* \mathsf{V} \mathsf{C}_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \mathsf{F}_1 \\ \mathsf{F}_2 \end{bmatrix} \begin{bmatrix} \mathsf{E} & -\mathsf{A}_j \end{bmatrix} \right\} \prec 0 \tag{2.27}$$

Moreover, if these equivalent inequality conditions hold, then the S-variables in (2.27) *can be chosen as* $\mathsf{F}_1 = -(\mathsf{M}_{11} + \varepsilon \mathsf{W})\mathsf{E}^{-1}$ *and* $\mathsf{F}_2 = -\mathsf{M}_{12}^* \mathsf{E}^{-1}$ *where* W *is a free-positive definite matrix* $\mathsf{W} \succ 0$ *and* $\varepsilon > 0$ *is sufficiently small. If* $M_{11} \succ 0$ *one can chose* $\varepsilon = 0$.

Proof Equivalence between (i) and (ii) is due to Lemma 2.3. Equivalence between (ii) and (iii) is direct by the properties (b) and (c) of Lemma 2.2. □

Notice that Lemma 2.4 provides a finite dimensional LMI test for rationally dependent matrices $\mathsf{E}^{-1}(\theta)\mathsf{A}(\theta)$. This feature is important for the analysis of more elaborated uncertain systems than LTI polytopic systems. This topic is also closely related to the analysis of uncertain descriptor systems. These issues are treated in detail in Chap. 3.

2.5 SV-LMI Results for Robust Performance Analysis Problems

Let us consider the LTI system G described by

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t),\ x(0) = 0, \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{2.28}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^{n_w}$ the disturbance input, $z \in \mathbb{R}^{n_z}$ the performance output, and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_w}$, $C \in \mathbb{R}^{n_z \times n}$, $D \in \mathbb{R}^{n_z \times n_w}$. In this section, we derive various SV-LMIs characterizing time- and frequency-domain performances of the system (2.28). The performances of interest includes pole location, H_2 performance, H_∞ performance, and impulse-to-peak performances. Then, we apply those SV-LMIs to robust performance analysis of polytopic uncertain LTI system G_θ described by

$$\begin{aligned} &G_\theta : \begin{cases} \dot{x}(t) = A(\theta)x(t) + B(\theta)w(t),\ x(0) = 0, \\ z(t) = C(\theta)x(t) + D(\theta)w(t) \end{cases}, \\ &\theta \in \mathbb{E}^L,\quad \begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} = \sum_{j=1}^{L} \theta_j \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}. \end{aligned} \tag{2.29}$$

Here, A_j ($j \in \mathscr{I}_L$), etc., are known matrices.

Some of the results are derived for the dual system described by

$$G_\theta^* : \begin{cases} \dot{x}_d(t) = A^T(\theta)x_d(t) + C^T(\theta)w_d(t),\ x_d(0) = 0, \\ z_d(t) = B^T(\theta)x_d(t) + D^T(\theta)w_d(t) \end{cases}. \tag{2.30}$$

2.5.1 Robust Regional Pole Location Analysis

In this subsection, we analyze the location of eigenvalues of a given matrix A in the complex plane. Namely, we will discuss whether $\lambda(A) \subset \mathbb{O}$ holds for given $\mathbb{O} \subset \mathbb{C}$. This analysis could be considered as a preliminary step for regional pole placement by feedback control.

2.5.1.1 QMI Regions

For the analysis of regional pole location, we focus on the following Quadratic Matrix Inequality (QMI) regions defined by $R \in \mathbb{H}^{2d}$

$$\mathbb{O}(R) := \left\{ \lambda \in \mathbb{C} : \begin{bmatrix} \lambda^* I_d & I_d \end{bmatrix} R \begin{bmatrix} \lambda I_d \\ I_d \end{bmatrix} = R_{11}\lambda^*\lambda + R_{12}^*\lambda + R_{12}\lambda^* + R_{22} \prec 0 \right\}. \tag{2.31}$$

This definition of regions of the complex plane has been introduced in [4]. Such regions are convex if $R_{11} \succeq 0$. In that case, take L such that $R_{11} = L^*L$. A Schur complement argument indicates that the region is also described by the LMI

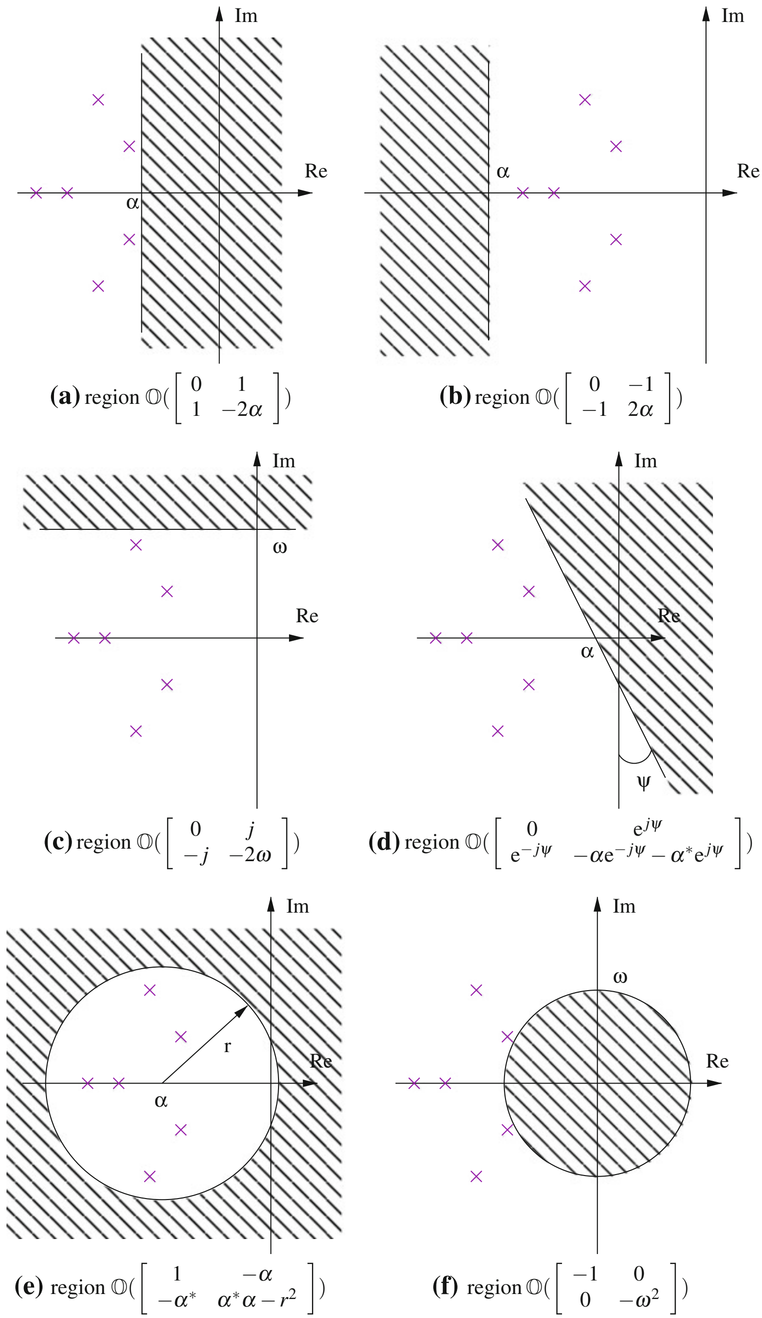
$$\begin{aligned} &\begin{bmatrix} R_{12}^*\lambda + R_{12}\lambda^* + R_{22} & \lambda^* L^* \\ \lambda L & -I \end{bmatrix} = \\ &\begin{bmatrix} R_{12}^* & 0 \\ L & 0 \end{bmatrix} \lambda + \begin{bmatrix} R_{12} & L^* \\ 0 & 0 \end{bmatrix} \lambda^* + \begin{bmatrix} R_{22} & 0 \\ 0 & -I \end{bmatrix} \prec 0. \end{aligned} \tag{2.32}$$

This fact illustrates that QMI regions are more general than LMI regions defined in [5]. The subclass of QMI regions with R_{11} positive semi-definite is exactly identical to the LMI regions.

Before giving examples of QMI regions, recall that for real-valued LTI system (2.28), if $\lambda_\pm = -\zeta\omega_n \pm j\omega_d \subset \lambda(A)$ is a pair of poles, then for the associated mode $0 < \zeta < 1$ is the damping ratio, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped natural frequency, $\omega_n (= |\lambda_\pm|)$ is the undamped natural frequency, and $\tau = \frac{1}{\zeta\omega_n}$ is the time constant.

For $d = 1$, there are two types of regions: half planes (if $R_{11} = 0$) and discs (otherwise). Examples of such regions follow.

- $R_{11} = 0$, $R_{12} = 1$, $R_{22} = 0$ then $\mathbb{O}(R) = \mathbb{C}_-$ the left-hand side of the complex plane. Proving that poles in this region are equivalent to Hurwitz stability.
- $R_{11} = 0$, $R_{12} = -1$, $R_{22} = 0$ then $\mathbb{O}(R) = \mathbb{C}_+$ the right-hand side of the complex plane. A system having its poles in such region is said to be anti-stable.
- $R_{11} = 0$, $R_{12} = 1$, $R_{22} = -2\alpha$ then $\mathbb{O}(R) = \{\lambda \in \mathbb{C} : \ \mathrm{Re}(\lambda) < \alpha\}$. If poles are in such region the modes of the system have time constants smaller than $-\frac{1}{\alpha}$ for $\alpha < 0$. This property is also called α-stability. It is illustrated in Fig. 2.2a.



(a) region $\mathbb{O}(\begin{bmatrix} 0 & 1 \\ 1 & -2\alpha \end{bmatrix})$

(b) region $\mathbb{O}(\begin{bmatrix} 0 & -1 \\ -1 & 2\alpha \end{bmatrix})$

(c) region $\mathbb{O}(\begin{bmatrix} 0 & j \\ -j & -2\omega \end{bmatrix})$

(d) region $\mathbb{O}(\begin{bmatrix} 0 & \mathrm{e}^{j\psi} \\ \mathrm{e}^{-j\psi} & -\alpha \mathrm{e}^{-j\psi} - \alpha^* \mathrm{e}^{j\psi} \end{bmatrix})$

(e) region $\mathbb{O}(\begin{bmatrix} 1 & -\alpha \\ -\alpha^* & \alpha^*\alpha - r^2 \end{bmatrix})$

(f) region $\mathbb{O}(\begin{bmatrix} -1 & 0 \\ 0 & -\omega^2 \end{bmatrix})$

Fig. 2.2 Examples of pole location regions. The striped region is the exterior of the QMI region

- $R_{11} = 0$, $R_{12} = -1$, $R_{22} = 2\alpha$ then $\mathbb{O}(R) = \{\lambda \in \mathbb{C} : \quad \mathrm{Re}(\lambda) > \alpha\}$. If poles are in such region the modes of the system have time constants greater than $-\frac{1}{\alpha}$ for $\alpha < 0$. Such region is illustrated in Fig. 2.2b.
- $R_{11} = 0$, $R_{12} = j$, $R_{22} = -2\omega$ then $\mathbb{O}(R) = \{\lambda \in \mathbb{C} : \quad \mathrm{Im}(\lambda) < \omega\}$. If poles are in such region the modes of a real-valued system have damped natural frequencies smaller than ω: $\mathrm{Im}(-\zeta\omega_n \pm j\omega_d) \leq \omega$ if and only if $|\omega_d| \leq \omega$. Such region is illustrated in Fig. 2.2c.
- $R_{11} = 0$, $R_{12} = \mathrm{e}^{j\psi}$, $R_{22} = -\alpha\mathrm{e}^{-j\psi} - \alpha^*\mathrm{e}^{j\psi}$ then $\mathbb{O}(R)$ is the half plane limited by a line that crosses the values α and orthogonal to $\mathrm{e}^{j\psi}$. Such region is illustrated in Fig. 2.2d. For $\alpha = 0$ and $\psi = \arcsin\zeta$, if poles are in such region the modes of a real-valued system have a damping ratio greater than ζ. If $\psi = 0$ and α is real it corresponds to the set $\mathbb{O}(R) = \{\lambda \in \mathbb{C} : \quad \mathrm{Re}(\lambda) < \alpha\}$. If $\psi = \pi/2$ and $\alpha = j\omega$ it corresponds to the set $\mathbb{O}(R) = \{\lambda \in \mathbb{C} : \quad \mathrm{Im}(\lambda) < \omega\}$.
- $R_{11} = 1$, $R_{12} = 0$, $R_{22} = -1$ then $\mathbb{O}(R) = \mathbb{D}$ is the open unit disc. Proving that poles are in this region is equivalent to Schur stability of discrete-time systems.
- $R_{11} = -1$, $R_{12} = 0$, $R_{22} = 1$ then $\mathbb{O}(R)$ is the exterior of the unit disc. A discrete-time system having its poles in such region is said to be anti-stable.
- $R_{11} = 1$, $R_{12} = -\alpha$, $R_{22} = \alpha^*\alpha - r^2$ then $\mathbb{O}(R)$ is the open disc centered at $\alpha \in \mathbb{C}$ with radius r. Such region is illustrated in Fig. 2.2e. For $\alpha = 0$ and $r = \omega$, if poles are in such region the modes of the system have undamped natural frequencies ω_n smaller than ω.
- $R_{11} = -1$, $R_{12} = \alpha$, $R_{22} = r^2 - \alpha^*\alpha$ then $\mathbb{O}(R)$ is the exterior of the disc centered at $\alpha \in \mathbb{C}$ with radius r. For $\alpha = 0$ and $r = \omega$, if poles are in such region the modes of the system have undamped natural frequencies ω_n greater than ω. Such region is illustrated in Fig. 2.2f.

Pole location in QMI regions can therefore without difficulty specify characteristics on the modes of linear systems. These are frequently used in the regional pole placement designs [5, 6].

In the definition of QMI regions, (2.31) the matrix R has $2d$ rows and $2d$ columns. But, the regions that have been described above are those with $d = 1$ only. Known examples from the literature of regions with $d > 1$ are intersections of regions of smaller dimensions. It is indeed quite easy to see that

$$\mathbb{O}\left(\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix}\right) \cap \mathbb{O}\left(\begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^* & \tilde{R}_{22} \end{bmatrix}\right) = \mathbb{O}\left(\left[\begin{array}{cc|cc} R_{11} & 0 & R_{12} & 0 \\ 0 & \tilde{R}_{11} & 0 & \tilde{R}_{12} \\ \hline R_{12}^* & 0 & R_{22} & 0 \\ 0 & \tilde{R}_{12}^* & 0 & \tilde{R}_{22} \end{array}\right]\right).$$

This feature is most useful for design problems. In analysis it is not. Indeed, an alternative is to prove first that the poles belong to one region and, separately to prove they belong to the other region. It amounts to solving two separate LMI problems of smaller size than the LMI problem defined for the intersection of regions. Yet, a useful application of this property is to convert a region defined by a complex valued R into a region defined by a real-valued matrix.

To illustrate this feature consider the region defined by $R_{11} = 0$, $R_{12} = \mathrm{e}^{j\psi}$, $R_{22} = 0$ that amounts to poles located below a line crossing the origin and that makes an angle ψ with the imaginary axis. Eigenvalues of a real matrix are symmetric with respect to the real axis. Hence, proving that a real-valued matrix has its eigenvalues in $\mathbb{O}(R)$ implies that the poles are as well in the conjugate of that region. Thus, the eigenvalues are in the intersection of the two regions defined by:

$$\mathbb{O}\left(\left[\begin{array}{cc|cc} 0 & 0 & \mathrm{e}^{j\psi} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{-j\psi} \\ \hline \mathrm{e}^{-j\psi} & 0 & 0 & 0 \\ 0 & \mathrm{e}^{j\psi} & 0 & 0 \end{array}\right]\right) \tag{2.33}$$

Since any change of basis applied simultaneously to all blocs of R does not modify the feasibility of the inequality in (2.31), this conic region is equivalently defined by:

$$\mathbb{O}\left(\left[\begin{array}{cc|cc} 0 & 0 & \cos(\psi) & \sin(\psi) \\ 0 & 0 & -\sin(\psi) & \cos(\psi) \\ \hline \cos(\psi) & -\sin(\psi) & 0 & 0 \\ \sin(\psi) & \cos(\psi) & 0 & 0 \end{array}\right]\right). \tag{2.34}$$

This is how conic regions are defined in [5].

2.5.1.2 LMI Tests for Pole Location in QMI Regions

Let us recall that how the condition $\lambda(A) \subset \mathbb{O}(R)$ can be written in terms of LMIs.

Lemma 2.6 *For given matrices $A \in \mathbb{C}^{n\times n}$, $R \in \mathbb{H}^{2d}$, the following conditions are equivalent:*

(i) $\lambda(A) \subset \mathbb{O}(R)$.
(ii) There exists $P \in \mathbb{H}^n$ such that

$$\begin{bmatrix} I_d \otimes A^* & I \end{bmatrix} \begin{bmatrix} R_{11} \otimes P & R_{12} \otimes P \\ R_{12}^* \otimes P & R_{22} \otimes P \end{bmatrix} \begin{bmatrix} I_d \otimes A \\ I \end{bmatrix} \prec 0. \tag{2.35}$$

By convention, we call P the Lyapunov (matrix) variable in (2.35).

Proof In this lemma, the implication (ii) $\Rightarrow$ (i) is straightforward to see. Indeed, if (2.35) holds, then for any eigenvalue-eigenvector pair $(\lambda, \xi) \in \mathbb{C} \times \mathbb{C}^n$ of A we have

$$\begin{aligned} &(I_d \otimes \xi)^* \begin{bmatrix} I_d \otimes A^* & I \end{bmatrix} \begin{bmatrix} R_{11} \otimes P & R_{12} \otimes P \\ R_{12}^* \otimes P & R_{22} \otimes P \end{bmatrix} \begin{bmatrix} I_d \otimes A \\ I \end{bmatrix} (I_d \otimes \xi) \\ &= \begin{bmatrix} I_d \otimes (\lambda\xi)^* & I_d \otimes \xi^* \end{bmatrix} \begin{bmatrix} R_{11} \otimes P & R_{12} \otimes P \\ R_{12}^* \otimes P & R_{22} \otimes P \end{bmatrix} \begin{bmatrix} I_d \otimes (\lambda\xi) \\ I_d \otimes \xi \end{bmatrix} \\ &= (\xi^* P \xi) \begin{bmatrix} \lambda^* I_d & I_d \end{bmatrix} R \begin{bmatrix} \lambda I_d \\ I_d \end{bmatrix} \prec 0. \end{aligned}$$

Since $P \succ 0$, we have $\xi^* P \xi > 0$, and hence the above inequality implies

$$\begin{bmatrix} \lambda^* I_d & I_d \end{bmatrix} R \begin{bmatrix} \lambda I_d \\ I_d \end{bmatrix} \prec 0.$$

It follows that $\lambda \in \mathbb{O}(R)$.

The implication (i) $\Rightarrow$ (ii) is more technical. It follows exactly the lines of the proof given in appendix of [5]. □

The LMI condition (2.35) is complex valued in general and P is complex as well. The LMI should be interpreted as follows: a complex Hermitian matrix is negative definite $X \prec 0$ if and only if

$$\begin{bmatrix} \mathrm{Re}(X) & \mathrm{Im}(X) \\ -\mathrm{Im}(X) & \mathrm{Re}(X) \end{bmatrix} \prec 0. \tag{2.36}$$

A special case of the LMI condition (2.35) is when A is real valued. If R is real valued as well, then one can without conservatism look for real-valued P. If A is real valued and R is not, P should be considered complex valued. An alternative is to double the size of the matrix by including the conjugate region (as it is done for the cone in (2.33)) and, by a change of basis to convert it to a region defined by a real-valued matrix (as it is done for the cone in (2.34)).

2.5.1.3 LMI Tests for Robust Pole Location Analysis

Based on Lemma 2.6 one gets the following robust pole location result based on S-variables.

Theorem 2.8 *The uncertain LTI system* (2.29) *has all its poles located in* $\mathbb{O}(R)$ *for all* $\theta \in \mathbb{E}^L$ *if there exist* $P_j \in \mathbb{H}^n$ $(j \in \mathscr{I}_L)$ *and* $F_i \in \mathbb{C}^{dn \times dn}$ $(i = 1, 2)$ *such that*

$$\begin{aligned} & P_j \succ 0, \\ & \begin{bmatrix} R_{11} \otimes P_j & R_{12} \otimes P_j \\ R_{12}^* \otimes P_j & R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} I & -I_d \otimes A_j \end{bmatrix} \right\} \prec 0 \ (\forall j \in \mathscr{I}_L). \end{aligned} \tag{2.37}$$

Proof The result is a direct application of Lemma 2.4. □

By analogy to Corollary 2.4, an SV-LMI-based pole location result can be produced for the dual system $\dot{x}_d = A^T(\theta) x_d$.

Corollary 2.5 *The uncertain LTI system* (2.29) *has all its poles located in* $\mathbb{O}(R)$ *for all* $\theta \in \mathbb{E}^L$ *if there exist* $X_j \in \mathbb{H}^n$ $(j \in \mathscr{I}_L)$ *and* $F_i \in \mathbb{C}^{dn \times dn}$ $(i = 1, 2)$ *such that*

$$\begin{aligned} & X_j \succ 0, \\ & \begin{bmatrix} R_{11} \otimes X_j & R_{12} \otimes X_j \\ R_{12}^* \otimes X_j & R_{22} \otimes X_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} I \\ -I_d \otimes A_j \end{bmatrix} \begin{bmatrix} F_1 & F_2 \end{bmatrix} \right\} \prec 0 \ (\forall j \in \mathscr{I}_L). \end{aligned} \tag{2.38}$$

It is shown on examples that these two results are not equivalent. The reason for it, as for Hurwitz stability, is that the underlying parameter-dependent Lyapunov matrices are not of the same type (see Theorem 2.6).

The two S-variable results are valid for all regions, even for nonconvex regions for which R_{11} is not positive semi-definite. If such assumption is added, then one gets the following result.

Theorem 2.9 *If $R_{11} \succeq 0$ the uncertain LTI system (2.29) has all its poles located in $\mathbb{O}(R)$ for all $\theta \in \mathbb{E}^L$ if there exist $\Pi \in \mathbb{H}^n$ such that*

$$\Pi \succ 0, \quad \begin{bmatrix} I_d \otimes A_j^* & I \end{bmatrix} \begin{bmatrix} R_{11} \otimes \Pi & R_{12} \otimes \Pi \\ R_{12}^* \otimes \Pi & R_{22} \otimes \Pi \end{bmatrix} \begin{bmatrix} I_d \otimes A_j \\ I \end{bmatrix} \prec 0. \tag{2.39}$$

Moreover, if (2.39) holds, then (2.37) also holds. A possible choice of variables is $P_j = \Pi$, $F_1 = -(R_{11} + \varepsilon I) \otimes \Pi$, and $F_2 = -R_{12}^* \otimes \Pi$ where $\varepsilon > 0$ is small enough. In case $R_{11} \succ 0$ the choice $\varepsilon = 0$ is admissible.

Proof The results readily follow from Lemma 2.5 with the choice $\mathsf{M}_{11} = R_{11} \otimes \Pi$, $\mathsf{M}_{12} = R_{12} \otimes \Pi$, $\mathsf{M}_{22} = R_{22} \otimes \Pi$, $\mathsf{E} = I$, $\mathsf{A}_j = I_d \otimes A_j$. The S-variables can be obtained if we let $W = \varepsilon I \otimes \Pi \succ 0$ in Lemma 2.5. □

Notice that for $R_{11} = 0$, $R_{12} = 1$ and $R_{22} = 0$ one has $\mathbb{O}(R) = \mathbb{C}_-$ and one recovers the LMI conditions obtained for Hurwitz stability. Theorem 2.9 is the pole location extension of Theorem 2.2. For that reason it is called a "quadratic stability" result. Theorem 2.8 is the pole location extension of Theorem 2.4. Corollary 2.5 is the pole location extension of Corollary 2.4. As for the case of Hurwitz stability, it is guaranteed that Theorem 2.8 and Corollary 2.5 are less conservative than Theorem 2.9.

On page 26, it is shown that identical QMI regions may be described by different choices of the R matrix. Under some conditions, one can have $\mathbb{O}(R) = \mathbb{O}(\tilde{R})$ with $R \neq \tilde{R}$. Moreover, these matrices may be of different dimensions. For results of Lemma 2.6, modifying the matrix that defines the regions has no effect since the LMI conditions are necessary and sufficient. For the S-variables tests of Theorem 2.8 and Corollary 2.5, describing regions with matrices R of enlarged dimensions increases the dimensions of the S-variables, thus the number of decision variables and therefore potentially reduces the conservatism. However, in several systematic ways for modifying the matrix defining the pole location region, such modification does not bring any improvement as illustrated below:

- For all square nonsingular $\breve{R}$ one has

$$\mathbb{O}\left(\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix}\right) = \mathbb{O}\left(\begin{bmatrix} \breve{R}^* R_{11} \breve{R} & \breve{R}^* R_{12} \breve{R} \\ \breve{R}^* R_{12}^* \breve{R} & \breve{R}^* R_{22} \breve{R} \end{bmatrix}\right).$$

This modification is exactly what is applied when going from (2.33) to (2.34) (for the choice $\breve{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -j \\ j & -1 \end{bmatrix}$). Modifying in this way the matrix that defines the region has no effect on the conservatism of Theorem 2.8. To prove this fact, write (2.37) for the modified matrix:

$$\begin{bmatrix} \breve{R}^* R_{11} \breve{R} \otimes P_j & \breve{R}^* R_{12} \breve{R} \otimes P_j \\ \breve{R}^* R_{12}^* \breve{R} \otimes P_j & \breve{R}^* R_{22} \breve{R} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} I_{dn} & -I_d \otimes A_j \end{bmatrix} \right\} \prec 0. \tag{2.40}$$

Pre- and postmultiply this inequality by $\begin{bmatrix} \breve{R}^{-*} \otimes I_n & 0 \\ 0 & \breve{R}^{-*} \otimes I_n \end{bmatrix}$ and its transpose conjugate, respectively, due to properties of the Krönecker product it gives

$$\begin{bmatrix} R_{11} \otimes P_j & R_{12} \otimes P_j \\ R_{12}^* \otimes P_j & R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \breve{F}_1 \\ \breve{F}_2 \end{bmatrix} \begin{bmatrix} I_{dn} & -I_d \otimes A_j \end{bmatrix} \right\} \prec 0 \tag{2.41}$$

with $\breve{F}_i = (\breve{R}^{-*} \otimes I_n) F_i (\breve{R}^{-1} \otimes I_n)$, for $i = 1, 2$. It is exactly the same as the conditions in Theorem 2.8.

- For all positive definite $\hat{R} \succ 0$ one has

$$\mathbb{O}\left(\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix} \right) = \mathbb{O}\left(\begin{bmatrix} \hat{R} \otimes R_{11} & \hat{R} \otimes R_{12} \\ \hat{R} \otimes R_{12}^* & \hat{R} \otimes R_{22} \end{bmatrix} \right). \tag{2.42}$$

The choice of $\hat{R}$ may seem a critical issue. Indeed, one cannot search simultaneously for the P_j matrices and $\hat{R}$. Fortunately, one can always choose $\hat{R} = I_r$ equal to the identity matrix without conservatism. To prove this fact write (2.37) for the increased dimension region with $\hat{R} \in \mathbb{H}^r$:

$$\begin{bmatrix} \hat{R} \otimes R_{11} \otimes P_j & \hat{R} \otimes R_{12} \otimes P_j \\ \hat{R} \otimes R_{12}^* \otimes P_j & \hat{R} \otimes R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} I_{rdn} & -I_{rd} \otimes A_j \end{bmatrix} \right\} \prec 0. \tag{2.43}$$

Pre- and postmultiply this inequality by $\begin{bmatrix} \hat{R}^{-1/2} \otimes I_{dn} & 0 \\ 0 & \hat{R}^{-1/2} \otimes I_{dn} \end{bmatrix}$, due to properties of the Krönecker product it gives:

$$\begin{bmatrix} I_r \otimes R_{11} \otimes P_j & I_r \otimes R_{12} \otimes P_j \\ I_r \otimes R_{12}^* \otimes P_j & I_r \otimes R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \hat{F}_1 \\ \hat{F}_2 \end{bmatrix} \begin{bmatrix} I_{rdn} & -I_{rd} \otimes A_j \end{bmatrix} \right\} \prec 0 \tag{2.44}$$

where $\hat{F}_i = (R^{-1/2} \otimes I_{dn}) F_i (R^{-1/2} \otimes I_{dn})$ for $i = 1, 2$. The LMIs hence hold for $\hat{R}$ replaced by I_r. There is no conservatism reduction brought by the degrees of freedom in $\hat{R}$.

When increasing r it is trivial to see that conditions are at least no more conservative (if the LMIs for a value $\hat{R} = I_{r_1}$ then they imply that these also hold for $\hat{R} = I_{r_2}$ if $r_2 \geq r_1$). Conservatism reduction may be expected thanks to the increased number of decision variables in the S-variables $\hat{F}_i$. Unfortunately it is not. The inequality (2.44) may as well be written (when reordering the rows and columns) as

$$I_r \otimes \begin{bmatrix} R_{11} \otimes P_j & R_{12} \otimes P_j \\ R_{12}^* \otimes P_j & R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \check{F}\left(I_r \otimes \begin{bmatrix} I_{dn} & -I_d \otimes A_j \end{bmatrix} \right) \right\} \prec 0 \quad (2.45)$$

where $\check{F}$ contains the same coefficients as $\begin{bmatrix} \hat{F}_1^T & \hat{F}_2^T \end{bmatrix}$ but reordered accordingly. The upper left block of size $2n \times 2n$ of inequality (2.45) needs to be negative-definite which reads as

$$\begin{bmatrix} R_{11} \otimes P_j & R_{12} \otimes P_j \\ R_{12}^* \otimes P_j & R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \check{F}_{11} \\ \check{F}_{21} \end{bmatrix} \begin{bmatrix} I_{dn} & -I_d \otimes A_j \end{bmatrix} \right\} \prec 0.$$

It is exactly the same as the conditions in Theorem 2.8. There is thus no improvement to be expected when enlarging the dimensions of the matrix defining the region in this way.

- Suppose R_{11} contains one or more positive eigenvalue. It can then be decomposed in $R_{11} = \acute{R}_{11} + L^* L$ and by Schur complement argument one gets

$$\mathbb{O}\left(\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^* & R_{22} \end{bmatrix} \right) = \mathbb{O}\left(\left[\begin{array}{cc|cc} \acute{R}_{11} & 0 & R_{12} & L^* \\ 0 & 0 & 0 & 0 \\ \hline R_{12}^* & 0 & R_{22} & 0 \\ L & 0 & 0 & -I \end{array} \right] \right). \quad (2.46)$$

This way of modifying the matrix defining the region is exactly what is applied to get the LMI description of convex QMI regions in (2.32). Again, this way of modifying the matrix defining the region does not bring any reduction of conservatism. To observe this fact, consider the condition (2.37) for the modified the matrix defining the region:

$$\begin{gathered} \left[\begin{array}{cc|cc} \acute{R}_{11} \otimes P_j & 0 & R_{12} \otimes P_j & L^* \otimes P_j \\ 0 & 0 & 0 & 0 \\ \hline R_{12}^* \otimes P_j & 0 & R_{22} \otimes P_j & 0 \\ L \otimes P_j & 0 & 0 & -I \otimes P_j \end{array} \right] \\ + \mathrm{He}\left\{ \left[\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \\ \hline F_{31} & F_{32} \\ F_{41} & F_{42} \end{array} \right] \left[\begin{array}{cc|cc} I_{dn} & 0 & -I_d \otimes A_j & 0 \\ 0 & I_{dn} & 0 & -I_d \otimes A_j \end{array} \right] \right\} \prec 0. \end{gathered} \quad (2.47)$$

Define the following matrices

$$\begin{array}{l} Z_1^* = (F_{12} + (L^* \otimes I) F_{42}) F_{22}^{-1} \\ Z_2^* = F_{32} F_{22}^{-1} \end{array}, \quad Z^* = \begin{bmatrix} I_{dn} & -Z_1^* & 0 & L^* \otimes I_n \\ 0 & -Z_2^* & I_{dn} & 0 \end{bmatrix}.$$

and notice the following facts

$$Z^* \left[\begin{array}{cc|cc} \acute{R}_{11} \otimes P_j & 0 & R_{12} \otimes P_j & L^* \otimes P_j \\ 0 & 0 & 0 & 0 \\ \hline R_{12}^* \otimes P_j & 0 & R_{22} \otimes P_j & 0 \\ L \otimes P_j & 0 & 0 & -I \otimes P_j \end{array}\right] Z = \begin{bmatrix} R_{11} \otimes P_j & R_{12} \otimes P_j \\ R_{12}^* \otimes P_j & R_{22} \otimes P_j \end{bmatrix}$$

$$Z^* \left[\begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \\ \hline F_{31} & F_{32} \\ F_{41} & F_{42} \end{array}\right] = \begin{bmatrix} F_{11} - Z_1^* F_{21} + (L^* \otimes I_n) F_{41} & 0 \\ -Z_2^* F_{21} + F_{31} & 0 \end{bmatrix} = \begin{bmatrix} \acute{F}_1 & 0 \\ \acute{F}_2 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} I_{dn} & 0 & -I_d \otimes A_j & 0 \\ 0 & I_{dn} & 0 & -I_d \otimes A_j \end{array}\right] Z = \begin{bmatrix} I_{dn} & -I_d \otimes A_j \\ -Z_1 - L \otimes A_j & -Z_2 \end{bmatrix}.$$

Hence pre- and postmultiplying (2.47) by Z^* and Z, respectively, implies the existence of $\acute{F}_{11}$ and $\acute{F}_{21}$ such that

$$\begin{bmatrix} R_{11} \otimes P_j & R_{12} \otimes P_j \\ R_{12}^* \otimes P_j & R_{22} \otimes P_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \acute{F}_1 \\ \acute{F}_2 \end{bmatrix} \begin{bmatrix} I_{dn} & -I_d \otimes A_j \end{bmatrix} \right\} \prec 0.$$

It is exactly the same as the conditions in Theorem 2.8. There is again no improvement in terms of conservatism brought by increasing artificially the dimensions of the matrix defining the region in this way.

2.5.2 Robust H_2 Performance Analysis

Let us consider the LTI system G described by (2.28). Suppose G is stable and strictly proper, i.e., the matrix A is Hurwitz stable and $D = 0$. Then, the H_2 norm of the system G is defined by

$$\|G\|_2 := \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{trace}\{G(j\omega)^* G(j\omega)\} \mathrm{d}\omega}. \tag{2.48}$$

From the Parseval relation, this can be rewritten as

$$\|G\|_2 = \sqrt{\int_{-\infty}^{\infty} \mathrm{trace}\{g(t)^T g(t)\} \mathrm{d}t} \tag{2.49}$$

where $g(\cdot) : \mathbb{R} \to \mathbb{R}^{n_z \times n_w}$ is the impulse response matrix of the system G defined by

$$g(t) := \begin{cases} C\exp(At)B & t \geq 0, \\ 0 & t < 0. \end{cases} \tag{2.50}$$

It is well known that the H_2 norm is useful if we evaluate the size of the system G in terms of stochastic disturbance input w. Indeed, if the input signal w is a standard white noise, then the squared H_2 norm $\|G\|_2^2$ coincides with the asymptotic mean of the squared norm of the output z [7, pp.237–239].

From (2.48) and (2.50), we can derive an algebraic characterization of the H_2 norm $\|G\|_2$. Indeed, from (2.49) and (2.50), we have

$$\|G\|_2^2 = \int_0^\infty \mathrm{trace}\left\{B^T\exp(A^Tt)C^TC\exp(At)B\right\} = \mathrm{trace}(B^TP_oB)$$

where P_o is the observability gramian of the pair (C, A) defined by

$$P_o := \int_0^\infty \exp(A^Tt)C^TC\exp(At)\mathrm{d}t.$$

As is well known, the observability gramian P_o can also be characterized as a unique solution for the Lyapunov equation

$$P_oA + A^TP_o + C^TC = 0.$$

It follows that the next lemma holds.

Lemma 2.7 *For the LTI system G described by (2.28), suppose A is Hurwitz stable and $D = 0$. Then, we have*

$$\|G\|_2^2 = \mathrm{trace}(B^TP_oB) \tag{2.51}$$

where P_o is a unique solution for the Lyapunov equation

$$P_oA + A^TP_o + C^TC = 0. \tag{2.52}$$

We next characterize the H_2 norm in terms of LMIs.

Theorem 2.10 *For the LTI system G described by (2.28), suppose $D = 0$. Then, for a given scalar $\overline{\gamma}_2 > 0$, the following conditions are equivalent:*

(i) The matrix A is Hurwitz stable and $\|G\|_2 < \overline{\gamma}_2$.
(ii) There exists $P \in \mathbb{S}_{++}^n$ and $Z \in \mathbb{S}_{++}^{n_w}$ such that

$$PA + A^T P + C^T C \prec 0, \tag{2.53a}$$

$$B^T PB - Z \prec 0 \tag{2.53b}$$

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z). \tag{2.53c}$$

Proof of Theorem 2.10 (i) $\Rightarrow$ (ii) Suppose (i) holds. Then, we have

$$\|G\|_2^2 = \mathrm{trace}(B^T P_o B) < \overline{\gamma}_2^2$$

where P_o is the observability gramian obtained from (2.52). To complete the proof, let us define W as a unique solution for the Lyapunov equation

$$WA + A^T W + I = 0.$$

Since A is Hurwitz, we see that $W \succ 0$. Moreover, let us define a scalar $\beta > 0$ by

$$\beta = \frac{\overline{\gamma}_2^2 - \|G\|_2^2}{2\mathrm{trace}(B^T WB)}.$$

Then, we can confirm

$$\begin{aligned} &(P_o + \beta W)A + A^T(P_o + \beta W) + C^T C = -\beta I, \\ &\mathrm{trace}\left(B^T(P_o + \beta W)B\right) = \|G\|_2^2 + \tfrac{\overline{\gamma}_2^2 - \|G\|_2^2}{2} = \tfrac{\overline{\gamma}_2^2 + \|G\|_2}{2} < \overline{\gamma}_2^2. \end{aligned} \tag{2.54}$$

It follows that (2.53a)–(2.53c) holds with $P = P_o + \beta W$ and $Z = B^T(P_o + \beta W)B + \varepsilon I$ for $\varepsilon \in (0, \frac{\overline{\gamma}_2^2 - \|G\|_2^2}{2})$.
(ii) $\Rightarrow$ (i) Suppose (2.53a) holds. Then, it is clear that A is Hurwitz stable form Theorem 2.1. Moreover, if we define

$$Q := -(PA + A^T P + C^T C) \succ 0,$$

it is obvious that

$$PA + A^T P + C^T C + Q = 0.$$

Subtracting (2.52) from the above equality, we have

$$(P - P_o)A + A^T(P - P_o) + Q = 0.$$

Since A is stable as noted, the above Lyapunov equality implies $P \succ P_o$. Therefore, we readily conclude

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z) > \mathrm{trace}(B^T P B) \geq \mathrm{trace}(B^T P_o B) = \|G\|_2^2.$$

This completes the proof. □

Based on (2.53a)–(2.53c), one gets the following result on SV-LMI-based robust H_2 performance analysis.

Theorem 2.11 *The uncertain LTI system* (2.29) *is robustly stable and satisfies the robust H_2 performance $\|G_\theta\|_2 < \overline{\gamma}_2$ $(\forall\theta \in \mathbb{E}^L)$ if there exist $P_j \in \mathbb{S}^n_{++}$, $Z_j \in \mathbb{S}^{n_w}_{++}$ $(j \in \mathscr{I}_L)$ and F_i $(i = 1, 2, 3, 4)$ such that*

$$\begin{bmatrix} 0 & P_j \\ P_j & C_j^T C_j \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}\begin{bmatrix} I & -A_j \end{bmatrix}\right\} \prec 0 \ (j \in \mathscr{I}_L), \tag{2.55a}$$

$$\begin{bmatrix} P_j & 0 \\ 0 & -Z_j \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} F_3 \\ F_4 \end{bmatrix}\begin{bmatrix} I & -B_j \end{bmatrix}\right\} \prec 0 \ (j \in \mathscr{I}_L), \tag{2.55b}$$

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z_j) \ (j \in \mathscr{I}_L). \tag{2.55c}$$

Proof The result is a direct application of Lemma 2.4. Indeed, if we identify (2.55a) and (2.55b) with (2.23), respectively, the LMIs corresponding to (2.24) read as

$$P(\theta)A(\theta) + A(\theta)^T P(\theta) + C(\theta)^T C(\theta) \prec 0, \tag{2.56a}$$

$$B(\theta)^T P(\theta) B(\theta) - Z(\theta) \prec 0, \tag{2.56b}$$

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z(\theta)), \quad \forall\theta \in \mathbb{E}^L \tag{2.56c}$$

where $P(\theta) := \sum_{j=1}^L \theta_j P_j$, $Z(\theta) := \sum_{j=1}^L \theta_j Z_j$. From Theorem 2.10, the above condition (2.56a)–(2.56b) clearly shows that the uncertain LTI system (2.29) is robustly stable and satisfies the robust H_2 performance $\|G_\theta\|_2 < \overline{\gamma}_2$ $(\forall\theta \in \mathbb{E}^L)$. □

By analogy to Corollary 2.4 and noticing that $\|G_\theta\|_2 = \|G_\theta^*\|_2$, a second robust S-variable H_2 performance result can be produced for the dual system.

Corollary 2.6 *The uncertain LTI system* (2.29) *is robustly stable and satisfies the robust H_2 performance $\|G_\theta\|_2 < \overline{\gamma}_2$ $(\forall\theta \in \mathbb{E}^L)$ if there exist $X_j \in \mathbb{S}^n$ $(j \in \mathscr{I}_L)$ and F_i $(i = 1, 2, 3, 4)$ such that*

$$\begin{bmatrix} 0 & X_j \\ X_j & B_j B_j^T \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} I \\ -A_j \end{bmatrix}\begin{bmatrix} F_1 & F_2 \end{bmatrix}\right\} \prec 0 \ (j \in \mathscr{I}_L), \tag{2.57a}$$

$$\begin{bmatrix} X_j & 0 \\ 0 & -Z_j \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} I \\ -C_j \end{bmatrix} \begin{bmatrix} F_3 & F_4 \end{bmatrix} \right\} \prec 0 \ (j \in \mathscr{I}_L), \tag{2.57b}$$

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z_j) \ (j \in \mathscr{I}_L). \tag{2.57c}$$

It is shown on examples that these two results are not equivalent. One reason for it is, similarly to the Hurwitz stability cases, that the underlying parameter-dependent Lyapunov matrices are not of the same type (see Theorem 2.6). Another reason comes from the fact that the "quadratic stability" counterparts of these results are not equivalent either.

Theorem 2.12 *The uncertain LTI system* (2.29) *is robustly stable and satisfies the robust* H_2 *performance* $\|G_\theta\|_2 < \overline{\gamma}_2$ $(\forall \theta \in \mathbb{E}^L)$ *if there exist* $\Pi \in \mathbb{S}^n$ *such that*

$$\Pi A_j + A_j^T \Pi + C_j^T C_j \prec 0 \ (j \in \mathscr{I}_L), \tag{2.58a}$$

$$B_j^T \Pi B_j - Z_j \prec 0 \ (j \in \mathscr{I}_L), \tag{2.58b}$$

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z_j) \ (j \in \mathscr{I}_L). \tag{2.58c}$$

Moreover, if (2.58a)–(2.58c) *holds, then* (2.55a)–(2.55c) *also holds. A possible choice of variables is* $P_j = \Pi$, $F_1 = -\varepsilon \Pi$, $F_2 = -\Pi$, $F_3 = -\Pi$, *and* $F_4 = 0$ *where* $\varepsilon > 0$ *is small enough.*

Corollary 2.7 *The uncertain LTI system* (2.29) *is robustly stable and satisfies the robust* H_2 *performance* $\|G_\theta\|_2 < \overline{\gamma}_2$ $(\forall \theta \in \mathbb{E}^L)$ *if there exist* $\Xi \in \mathbb{S}^n$ *such that*

$$\Xi A_j^T + A_j \Xi + B_j B_j^T \prec 0 \ (j \in \mathscr{I}_L), \tag{2.59a}$$

$$C_j \Xi C_j^T - Z_j \prec 0 \ (j \in \mathscr{I}_L), \tag{2.59b}$$

$$\overline{\gamma}_2^2 > \mathrm{trace}(Z_j) \ (j \in \mathscr{I}_L). \tag{2.59c}$$

Moreover, if (2.59a)–(2.59c) *holds, then* (2.57a)–(2.57c) *also holds. A possible choice of variables is* $X_j = \Xi$, $F_1 = -\varepsilon \Xi$, $F_2 = -\Xi$, $F_3 = -\Xi$, *and* $F_4 = 0$ *where* $\varepsilon > 0$ *is small enough.*

The LMI in Theorem 2.12 readily follows from Theorem 2.10 with simple convexity arguments. The latter assertion of this theorem is a direct consequence of Lemma 2.5. Note that the S-variable LMI in Theorem 2.11 is no more conservative than the LMI in Theorem 2.12 due to this assertion. Similar comments apply also to the LMIs in Corollary 2.7–2.6.

2.5.3 Robust H_∞ Performance Analysis

Suppose that the LTI system G described by (2.28) is stable (i.e., the matrix A is Hurwitz stable). Then, the H_∞ norm of the system G is defined by

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \|G(j\omega)\|. \tag{2.60}$$

The H_∞ norm plays an important role in robust control theory. The cerebrated small gain theorem ensures that the feedback connection with stable LTI systems G_1 and G_2 is stable if $\|G_1\|_\infty \|G_2\|_\infty < 1$ holds. It follows that if $\|G\|_\infty < \gamma_\infty$, then the feedback connection with G and an uncertain stable LTI system Δ is stable whenever $\|\Delta\|_\infty < 1/\gamma_\infty$. In this way, the H_∞ norm serves as a reasonable measure for robust stability of feedback systems against unstructured uncertainties. Another important interpretation of the H_∞ norm is that it coincides with the L_2 induced norm of the system G. Namely, we have

$$\|G\|_\infty = \sup_{w \in L_2, \|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2}. \tag{2.61}$$

Usually, when evaluating the H_∞ norm of the system G, the disturbance input w and the performance output z are chosen so that the good performance of the system G can be translated to the insensitiveness of z with respect to w. It follows that we can ensure good performance if $\|G\|_\infty$ is small enough.

From the definition (2.60), we see that $\|G\|_\infty < \overline{\gamma}_\infty$ holds for given $\overline{\gamma}_\infty > 0$ if and only if

$$G(j\omega)^* G(j\omega) \prec \overline{\gamma}_\infty^2 I \ (\forall \omega \in (\mathbb{R} \cup \{\infty\}))$$

or equivalently,

$$\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \hat{\Theta} \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \prec 0 \ (\forall \omega \in (\mathbb{R} \cup \{\infty\})), \tag{2.62}$$

$$\hat{\Theta} = \begin{bmatrix} C & D \\ 0 & I_{n_w} \end{bmatrix}^T \Theta_{\mathrm{BD}} \begin{bmatrix} C & D \\ 0 & I_{n_w} \end{bmatrix}, \quad \Theta_{\mathrm{BD}} := \begin{bmatrix} I & 0 \\ 0 & -\overline{\gamma}_\infty^2 I \end{bmatrix} \tag{2.63}$$

It is well recognized that numerous performances of LTI systems defined in the frequency domain can be characterized in the form of (2.62). For example, if we set

$$\hat{\Theta} := \begin{bmatrix} C & D \\ 0 & I_{n_w} \end{bmatrix}^T \Theta_{\mathrm{PR}} \begin{bmatrix} C & D \\ 0 & I_{n_w} \end{bmatrix}, \quad \Theta_{\mathrm{PR}} := \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}$$

then the corresponding condition becomes

$$G(j\omega) + G(j\omega)^* \succ 0 \ (\forall \omega \in (\mathbb{R} \cup \{\infty\})).$$

This is nothing but the positive realness condition for the system G.

The condition (2.62) is often called *frequency domain inequality (FDI)*. The FDI condition is apparently hard to check since it involves semi-infinite constraint (i.e., we have to take care of all ω belonging to $\mathbb{R} \cup \{\infty\}$). The celebrated Kalman–Yakubovich–Popov (KYP) lemma successfully remove this semi-infinite constraint and translate the FDI into a finite-dimensional LMI.

Lemma 2.8 (KYP Lemma)[8] *For given* $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_w}$ *and*

$$\hat{\Theta} = \begin{bmatrix} \hat{\Theta}_{11} & \hat{\Theta}_{12} \\ \hat{\Theta}_{12}^T & \hat{\Theta}_{22} \end{bmatrix} \in \mathbb{S}_{n+n_w}, \quad \hat{\Theta}_{11} \in \mathbb{S}_n, \quad \hat{\Theta}_{22} \in \mathbb{S}_{n_w},$$

suppose $\hat{\Theta}_{11} \succeq 0$. *Then, the following conditions are equivalent.*

(i) The matrix A is Hurwitz stable and the FDI (2.62) *holds.*
(ii) There exists $P \in \mathbb{S}^n_{++}$ such that

$$\begin{bmatrix} PA + A^T P & PB \\ B^T P & 0 \end{bmatrix} + \hat{\Theta} \prec 0. \tag{2.64}$$

The condition (ii) is easy to check numerically since (2.64) is a finite-dimensional LMI. For the proof of (i) $\Rightarrow$ (ii), we need some mathematical preliminaries [8]. However, the proof of (ii) $\Rightarrow$ (i) is fairly easy. Indeed, if (2.64) holds, then we have $PA + A^T P \prec 0$ since $\hat{\Theta}_{11} \succeq 0$. Since $P \in \mathbb{S}^n_{++}$, this implies that A is Hurwitz stable. On the other hand, (2.64) can be rewritten equivalently as

$$\mathrm{He}\left\{\begin{bmatrix} P \\ 0 \end{bmatrix}\begin{bmatrix} (A - j\omega I) & B \end{bmatrix}\right\} + \hat{\Theta} \prec 0.$$

where $\omega \in \mathbb{R}$ is arbitrary. Since A is Hurwitz stable, we see $j\omega I - A$ is nonsingular, and hence we have

$$\begin{aligned} &\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \left(\mathrm{He}\left\{\begin{bmatrix} P \\ 0 \end{bmatrix}\begin{bmatrix} (A - j\omega I) & B \end{bmatrix}\right\} + \hat{\Theta}\right)\begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \\ &= \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix}^* \hat{\Theta} \begin{bmatrix} (j\omega I - A)^{-1} B \\ I \end{bmatrix} \prec 0 \ (\forall \omega \in \mathbb{R}). \end{aligned}$$

Finally, if (2.64) holds, then it is obvious that $\hat{\Theta}_{22} \prec 0$ and hence the FDI (2.62) holds for $\omega = \infty$.

We next provides a robust SV-LMI corresponding to (2.64) for the case when $\hat{\Theta}$ is structured as follows:

$$\hat{\Theta} = \begin{bmatrix} C & D \\ 0 & I_{n_w} \end{bmatrix}^T \Theta \begin{bmatrix} C & D \\ 0 & I_{n_w} \end{bmatrix}, \quad \Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^T & \Theta_{22} \end{bmatrix}, \quad \Theta_{11} \succeq 0. \tag{2.65}$$

It trivially includes the case when $\hat{\Theta}$ has no structure since it corresponds to $C = I$ and $D = 0$.

Theorem 2.13 *The uncertain LTI system* (2.29) *is robustly stable and the FDI* (2.62) *holds for all* $\theta \in \mathbb{E}^L$ *if there exist* $P_j \in \mathbb{S}^n$ $(j \in \mathscr{I}_L)$ *and* F_i $(i = 1, 2, 3)$ *such that*

$$\begin{bmatrix} 0 & P_j & 0 \\ P_j & \hat{\Theta}_{11j} & \hat{\Theta}_{12j} \\ 0 & \hat{\Theta}_{12j}^T & \hat{\Theta}_{22j} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \begin{bmatrix} I & -A_j & -B_j \end{bmatrix} \right\} \prec 0 \; (j \in \mathscr{I}_L) \tag{2.66}$$

where $\hat{\Theta}_j := \begin{bmatrix} \hat{\Theta}_{11j} & \hat{\Theta}_{12j} \\ \hat{\Theta}_{12j}^T & \hat{\Theta}_{22j} \end{bmatrix} = \begin{bmatrix} C_j & D_j \\ 0 & I \end{bmatrix}^T \Theta \begin{bmatrix} C_j & D_j \\ 0 & I \end{bmatrix}$.

Proof The result is a direct application of Lemma 2.4. □

The "quadratic stability" counterpart of this result can be given as follows.

Theorem 2.14 *The uncertain LTI system* (2.29) *is robustly stable and the FDI* (2.62) *holds for all* $\theta \in \mathbb{E}^L$ *if there exist* $\Pi \in \mathbb{S}^n$ *such that for all* $j \in \mathscr{I}_L$

$$\begin{bmatrix} \Pi A_j + A_j^T \Pi & \Pi B_j \\ B_j^T \Pi & 0 \end{bmatrix} + \hat{\Theta}_j \prec 0. \tag{2.67}$$

Moreover, if (2.67) *holds, then* (2.66) *also holds. A possible choice of variables is* $P_j = \Pi$, $F_1 = -\varepsilon\Pi$, $F_2 = -\Pi$, *and* $F_3 = 0$ *where* $\varepsilon > 0$ *is small enough.*

Proof The LMI (2.67) readily follows from (2.64) with simple convexity arguments. The latter result is a direct application of Lemma 2.5. □

Dual versions of the above two theorems are accessible thanks to the following technical result. That applies assuming that diagonal blocks of Θ satisfy certain conditions.

Lemma 2.9 *The two following conditions on* G *are equivalent*

$$\begin{bmatrix} G \\ I \end{bmatrix}^* \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{22} \end{bmatrix} \begin{bmatrix} G \\ I \end{bmatrix} \prec 0 \tag{2.68}$$

$$\begin{bmatrix} G^* \\ I \end{bmatrix}^* \begin{bmatrix} \Theta_{11d} & \Theta_{12d} \\ \Theta_{12d}^* & \Theta_{22d} \end{bmatrix} \begin{bmatrix} G^* \\ I \end{bmatrix} \prec 0 \tag{2.69}$$

where

(i) If $\Theta_{11} \succ 0$

$$\begin{aligned}\Theta_{11d} &= (\Theta_{12}^* \Theta_{11}^{-1} \Theta_{12} - \Theta_{22})^{-1} \\ \Theta_d &= \begin{bmatrix} \Theta_{11d} & \Theta_{12d} \\ \Theta_{12d}^* & \Theta_{22d} \end{bmatrix} = \begin{bmatrix} \Theta_{11d} & \Theta_{11d} \Theta_{12}^* \Theta_{11}^{-1} \\ \Theta_{11}^{-1} \Theta_{12} \Theta_{11d} & \Theta_{11}^{-1} \Theta_{12} \Theta_{11d} \Theta_{12}^* \Theta_{11}^{-1} - \Theta_{11}^{-1} \end{bmatrix}\end{aligned}$$

(ii) If $\Theta_{11} = 0$ *and* Θ_{12} *is square non singular:*

$$\Theta_d = \begin{bmatrix} \Theta_{11d} & \Theta_{12d} \\ \Theta_{12d}^* & \Theta_{22d} \end{bmatrix} = \begin{bmatrix} 0 & \Theta_{12}^{-1} \\ \Theta_{12}^{-*} & \Theta_{12}^{-1} \Theta_{22} \Theta_{12}^{-*} \end{bmatrix}$$

Proof Let us start with the simplest case (ii). Condition (2.68) also reads as

$$G^* \Theta_{12} + \Theta_{12}^* G + \Theta_{22} \prec 0.$$

Pre- and postmultiply by the invertible matrix Θ_{12}^{-*} and its transpose, respectively. It gives exactly (2.69) with the given Θ_d matrix.

Now let us consider the case (i). Condition (2.68) reads as

$$G^* \Theta_{11} G + G^* \Theta_{12} + \Theta_{12}^* G + \Theta_{22} \prec 0.$$

Since Θ_{11} is assumed to positive definite, it is quite simple to show that this inequality can also be written as

$$(G^* \Theta_{11} + \Theta_{12}^*) \Theta_{11}^{-1} (\Theta_{11} G + \Theta_{12}) + \Theta_{22} - \Theta_{12}^* \Theta_{11}^{-1} \Theta_{12} \prec 0.$$

Since again Θ_{11} is positive definite, we can apply Schur complement argument to the above inequality to get

$$\begin{bmatrix} \Theta_{22} - \Theta_{12}^* \Theta_{11}^{-1} \Theta_{12} & G^* \Theta_{11} + \Theta_{12}^* \\ \Theta_{11} G + \Theta_{12} & -\Theta_{11} \end{bmatrix} \prec 0.$$

A converse Schur complement argument applied to the upper left block gives

$$-\Theta_{11} - (\Theta_{11} G + \Theta_{12})(\Theta_{22} - \Theta_{12}^* \Theta_{11}^{-1} \Theta_{12})^{-1} (G^* \Theta_{11} + \Theta_{12}^*) \prec 0.$$

Pre- and postmultiply this inequality by Θ_{11}^{-1} and develop the expression to get (2.69) with the given Θ_d matrix. □

In case one of the two conditions in Lemma 2.9 is applicable, then the following dual versions of Theorem 2.13 and Theorem 2.14 are obtained.

Corollary 2.8 *The uncertain LTI system* (2.29) *is robustly stable and the FDI* (2.62) *holds for all* $\theta \in \mathbb{E}^L$ *if there exist* $X_j \in \mathbb{S}^n$ $(j \in \mathcal{I}_L)$ *and* F_i $(i = 1, 2, 3)$ *such that*

$$\begin{bmatrix} 0 & X_j & 0 \\ X_j & \hat{\Theta}_{11j} & \hat{\Theta}_{12j} \\ 0 & \hat{\Theta}_{12j}^T & \hat{\Theta}_{22j} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} I \\ -A_j \\ -C_j \end{bmatrix} \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix} \right\} \prec 0 \; (j \in \mathscr{I}_L) \tag{2.70}$$

where $\begin{bmatrix} \hat{\Theta}_{11j} & \hat{\Theta}_{12j} \\ \hat{\Theta}_{12j}^T & \hat{\Theta}_{22j} \end{bmatrix} = \begin{bmatrix} B_j & 0 \\ D_j & I \end{bmatrix} \Theta_d \begin{bmatrix} B_j & 0 \\ D_j & I \end{bmatrix}^T.$

Corollary 2.9 *The uncertain LTI system* (2.29) *is robustly stable and the FDI* (2.62) *holds for all* $\theta \in \mathbb{E}^L$ *if there exist* $\Xi \in \mathbb{S}^n$ *such that for all* $j \in \mathscr{I}_L$

$$\begin{bmatrix} \Xi A_j^T + A_j \Xi & \Xi C_j^T \\ C_j \Xi & 0 \end{bmatrix} + \hat{\Theta}_j \prec 0, \quad \hat{\Theta}_j = \begin{bmatrix} B_j & 0 \\ D_j & I \end{bmatrix} \Theta_d \begin{bmatrix} B_j & 0 \\ D_j & I \end{bmatrix}^T. \tag{2.71}$$

Moreover, if (2.71) *holds, then* (2.70) *also holds. A possible choice of variables is* $X_j = \Xi$, $F_1 = -\varepsilon \Xi$, $F_2 = -\Xi$, *and* $F_3 = 0$ *where* $\varepsilon > 0$ *is small enough.*

2.5.4 Robust Impulse-To-Peak Performance Analysis

Suppose the LTI system G described by (2.28) is stable (i.e., the matrix A is Hurwitz stable) and $D = 0$. In this subsection, we focus on the value

$$\gamma_{IP} := \sup_{t \geq 0, \|\alpha\|=1} \|z(t)\| \tag{2.72}$$

where $z(t)$ is the response of (2.28) to an impulse input $w = \delta(t)\alpha$, $\alpha \in \mathbb{R}^{n_w}$. The value γ_{IP} defined by (2.72) is the peak (supremum) of the performance output measured in Euclidean norm with respect to all unitary impulse inputs. It should be noted that z can also be characterized as the output of the input-free system described by

$$\begin{cases} \dot{x}(t) = Ax(t), \; x(0) = x_0 = B\alpha, \\ z(t) = Cx(t). \end{cases} \tag{2.73}$$

For given $\overline{\gamma}_{IP} > 0$, a sufficient condition to guarantee $\gamma_{IP} < \overline{\gamma}_{IP}$ can be given in terms of LMIs [2].

Theorem 2.15 *For the LTI system G described by* (2.28)*, suppose* $D = 0$. *The matrix A is stable and* $\gamma_{IP} < \overline{\gamma}_{IP}$ *holds if there exists* $P \in \mathbb{S}_{++}^n$ *such that*

$$PA + A^T P \prec 0, \tag{2.74a}$$

$$B^T PB - \overline{\gamma}_{IP}^2 I_{n_w} \prec 0, \tag{2.74b}$$

$$C^T C - P \prec 0. \tag{2.74c}$$

Proof Suppose (2.74a)–(2.74c) holds with $P \in \mathbb{S}^n_{++}$. From (2.74a), the matrix A is Hurwitz stable. Moreover, if define $g_P(t) := P^{\frac{1}{2}} \exp(At)B\alpha$ for given $\alpha \in \mathbb{R}^{n_w}$ with $\|\alpha\| = 1$, we can readily see from (2.74c) that

$$\|z(t)\| \le \|g_P(t)\| \quad (\forall t \ge 0). \tag{2.75}$$

With this in mind, let us focus on g_P. From (2.74b), we first obtain

$$\|g_P(0)\| = \|P^{\frac{1}{2}} B\alpha\| < \overline{\gamma}_{IP}\|\alpha\| = \overline{\gamma}_{IP}. \tag{2.76}$$

Moreover, for all $t > 0$, we have

$$\begin{aligned} \frac{\mathrm{d}\|g_P(t)\|^2}{\mathrm{d}t} &= \frac{\mathrm{d}(\alpha^T B^T \exp(A^T t)P \exp(At)B\alpha)}{\mathrm{d}t} \\ &= \alpha^T B^T \exp(A^T t)(PA + A^T P)\exp(At)B\alpha \\ &\le 0 \end{aligned} \tag{2.77}$$

since (2.74a) holds. From (2.76) and (2.77), we have $\|g_P(t)\| < \overline{\gamma}_{IP}$ $(\forall t \ge 0)$. Since (2.75) holds, we can conclude that $\|z(t)\| < \overline{\gamma}_{IP}$ $(\forall t \ge 0)$ holds as well. □

In contrast with other performances discussed in the preceding subsections, the LMI characterization in Theorem 2.15 is sufficient for $\gamma_{IP} < \overline{\gamma}_{IP}$. The LMI condition (2.74a)–(2.74c) is somewhat complicated, but the corresponding SV-LMIs can be derived without any further ado. This is done in the following. Before that we illustrate the usefulness of impulse-to-peak performance in characterizing time responses for bounded initial conditions.

Ensuring adequate impulse-to-peak performance (i.e., keeping γ_{IP} small enough) is often very important. In some practical applications, the state vector must be kept "small" to avoid undesirable phenomena such as saturation, etc. Since the magnitude of the state vector of course depends on the initial condition, it is reasonable to analyze whether

$$z(t)^T W_z z(t) \le 1 \; \forall (t, x_0) \in (\mathbb{R}_+, \mathbb{X}_0), \quad \mathbb{X}_0 := \{x_0 \in \mathbb{R}^n : x_0^T W_0 x_0 \le 1\} \tag{2.78}$$

holds for (2.73), where $W_z \in \mathbb{S}^{n_z}_{++}$ and $W_0 \in \mathbb{S}^n_{++}$ are appropriately chosen weighting matrices. A sufficient condition for (2.78) readily follows from Theorem 2.15.

Corollary 2.10 *Suppose $W_z \in \mathbb{S}^{n_z}_{++}$ and $W_0 \in \mathbb{S}^n_{++}$ are given. Then, for the LTI system (2.73), the matrix A is stable and the condition (2.78) holds if there exists $P \in \mathbb{S}^n_{++}$ such that*

$$PA + A^T P \prec 0, \tag{2.79a}$$

$$P - W_0 \prec 0, \tag{2.79b}$$

$$C^T W_z C - P \prec 0. \tag{2.79c}$$

Proof of Corollary 2.10 Again, the condition (2.79a) implies the Hurwitz stability of A. As for the condition (2.78), we first note that $x_0 \in \mathbb{X}_0$ can always be represented as $x_0 = W_0^{-\frac{1}{2}} v$ for some $\|v\| \le 1$. If $x_0 = 0$, the condition (2.78) is obviously satisfied, and therefore we consider the case where $x_0 \neq 0$ (and hence $v \neq 0$). From (2.79b) we readily obtain

$$x_0^T W_0 x_0 > x_0^T P x_0 \;\; \Leftrightarrow \;\; v^T v > x_0^T P x_0 \;\; \Rightarrow \;\; 1 > x_0^T P x_0 \; (\forall x_0 \in \mathbb{X}_0).$$

From (2.79a), the above last inequality and (2.79c), the assertion of the corollary readily follows by Theorem 2.15. □

We next consider briefly the robust impulse-to-peak performance analysis for the polytopic uncertain LTI system (2.29). The issue is to determine whether the following two robustness properties hold or not:

(A) $\lambda(A(\theta)) \subset \mathbb{C}_- \; (\forall \theta \in \mathbb{E}^L)$,
(B) $\|z_\theta(t)\| < \overline{\gamma}_{\mathrm{IP}} \; (\forall (t, \theta) \in \mathbb{R}_+ \times \mathbb{E}^L)$.

Here, z_θ is the response matrix to an impulse input $w = \delta(t)\alpha, \alpha \in \mathbb{R}^{n_w}$ with $\|\alpha\| = 1$ of the plant (2.29).

Theorem 2.16 *The uncertain LTI system* (2.29) *is robustly stable and the robust impulse-to-peak performance* (B) *holds if there exist* $P_j \in \mathbb{S}^n$ $(j \in \mathscr{I}_L)$ *and* F_i $(i = 1, 2, 3, 4)$ *such that*

$$\begin{bmatrix} 0 & P_j \\ P_j & 0 \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \begin{bmatrix} I & -A_j \end{bmatrix} \right\} \prec 0 \; (j \in \mathscr{I}_L). \tag{2.80a}$$

$$\begin{bmatrix} P_j & 0 \\ 0 & -\overline{\gamma}_{IP}^2 I_{n_w} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} F_3 \\ F_4 \end{bmatrix} \begin{bmatrix} I & -B_j \end{bmatrix} \right\} \prec 0 \; (j \in \mathscr{I}_L) \tag{2.80b}$$

$$C_j^T C_j - P_j \prec 0 \; (j \in \mathscr{I}_L). \tag{2.80c}$$

Proof The result is a direct application of Lemma 2.4. □

The "quadratic stability" counterpart of this result can be given in the following. The latter assertion of the next theorem follows from Lemma 2.5.

Theorem 2.17 *The uncertain LTI system* (2.29) *is robustly stable and the robust impulse-to-peak performance* (B) *holds if there exist* $\Pi \in \mathbb{S}^n$ *such that for all* $j \in \mathscr{I}_L$

$$\Pi A_j + A_j^T \Pi \prec 0, \tag{2.81a}$$

$$B_j^T \Pi B_j - \overline{\gamma}_{IP}^2 I_{n_w} \prec 0, \tag{2.81b}$$

$$C_j^T C_j - \Pi \prec 0. \tag{2.81c}$$

Moreover, if (2.81a)–(2.81c) *holds, then* (2.80a)–(2.80c) *also holds. A possible choice of variables is* $P_j = \Pi$, $F_1 = -\varepsilon\Pi$, $F_2 = -\Pi$, $F_3 = -\Pi$, *and* $F_4 = 0$ *where* $\varepsilon > 0$ *is small enough.*

We shall next consider the results for the dual system.

Lemma 2.10 *For the LTI system G described by* (2.28)*, suppose* $D = 0$*. The matrix A is stable and* $\gamma_{IP} < \overline{\gamma}_{IP}$ *holds if there exists* $X \in \mathbb{S}^n_{++}$ *such that*

$$XA^T + AX \prec 0, \tag{2.82a}$$

$$CXC^T - \overline{\gamma}_{IP}^2 I_{n_z} \prec 0, \tag{2.82b}$$

$$BB^T - X \prec 0. \tag{2.82c}$$

Moreover, if (2.74a)–(2.74c) *hold for some matrix P, then* (2.82a)–(2.82c) *also hold and* $X = \overline{\gamma}_{IP}^2 P^{-1}$ *is a possible choice.*

Proof We shall prove that (2.82a)–(2.82c) with $X = \overline{\gamma}_{IP}^2 P^{-1}$ follows immediately from (2.74a)–(2.74c). Pre- and postmultiply (2.74a) by $X = \overline{\gamma}_{IP}^2 P^{-1}$ gives exactly (2.82a). Divide (2.74b) and (2.74c) by $\overline{\gamma}_{IP}^{-2}$, it gives

$$B^T X^{-1} B - I_{n_w} \prec 0, \qquad C^T(\overline{\gamma}_{IP}^{-2} I_{n_z})^{-1} C - X^{-1} \prec 0.$$

Apply a Schur complement argument to these inequalities to get

$$\begin{bmatrix} -X & B \\ B^T & -I_{n_w} \end{bmatrix} \prec 0, \qquad \begin{bmatrix} -X^{-1} & C^T \\ C & -\overline{\gamma}_{IP}^2 I_{n_z} \end{bmatrix} \prec 0$$

Converse Schur complement argument concludes that (2.82c) and (2.82b) hold. □

Based on Lemma 2.10 the dual counterparts of Theorem 2.16 and Theorem 2.17 are derived.

Theorem 2.18 *The uncertain LTI system* (2.29) *is robustly stable and the robust impulse-to-peak performance* (B) *holds if there exist* $X_j \in \mathbb{S}^n$ $(j \in \mathscr{I}_L)$ *and* F_i $(i = 1, 2, 3, 4)$ *such that*

$$\begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} I \\ -A_j \end{bmatrix}\begin{bmatrix} F_1 & F_2 \end{bmatrix}\right\} \prec 0 \; (j \in \mathscr{I}_L). \tag{2.83a}$$

$$\begin{bmatrix} X_j & 0 \\ 0 & -\overline{\gamma}_{IP}^2 I_{n_z} \end{bmatrix} + \mathrm{He}\left\{\begin{bmatrix} I \\ -C_j \end{bmatrix}\begin{bmatrix} F_3 & F_4 \end{bmatrix}\right\} \prec 0 \; (j \in \mathscr{I}_L) \tag{2.83b}$$

$$B_j B_j^T - X_j \prec 0 \; (j \in \mathscr{I}_L). \tag{2.83c}$$

Theorem 2.19 *The uncertain LTI system* (2.29) *is robustly stable and the robust impulse-to-peak performance* (B) *holds if there exist* $\Pi \in \mathbb{S}^n$ *such that for all* $j \in \mathscr{I}_L$

$$\Xi A_j^T + A_j \Xi \prec 0, \tag{2.84a}$$

$$C_j \Xi C_j - \overline{\gamma}_{IP}^2 I_{n_z} \prec 0, \tag{2.84b}$$

$$B_j B_j^T - \Xi \prec 0. \tag{2.84c}$$

Moreover, if (2.84a)–(2.84c) *holds, then* (2.83a)–(2.83c) *also holds. A possible choice of variables is* $X_j = \Xi$, $F_1 = -\varepsilon\Xi$, $F_2 = -\Xi$, $F_3 = -\Xi$, $F_4 = 0$ *where* $\varepsilon > 0$ *is small enough.*

Once again, there are no implications in either direction between Theorem 2.16, Theorem 2.17 and their dual counterparts, Theorem 2.18 and Theorem 2.19. The only implications are that the S-variable results of Theorem 2.16 and Theorem 2.18 are less conservative (no more conservative) than the "quadratic stability" counterparts of Theorem 2.17 and Theorem 2.19, respectively.

2.6 Numerical Examples

In this section, we illustrate the effectiveness of SV-LMIs for robust performance analysis problems for polytopic uncertain LTI systems.

2.6.1 Quarter-Car Suspension Example

2.6.1.1 Definition of the Uncertain System

Let us revisit the quarter-car suspension model discussed in Sect. 2.2. By following [9], we let the following nominal values of the parameters

$$\begin{aligned} &M_{\text{nom}} = 320\,\text{Kg},\ m = 40\,\text{Kg}, \\ &k_1 = 180\,\text{KN/m},\ k_{2,\text{nom}} = 200\,\text{KN/m},\ c = 1000\,\text{Ns/m} \end{aligned}$$

The main objectives for the suspension system design are to achieve (a) good ride comfort, (b) good road-holding ability, and (c) small suspension deflection. To fulfill these specifications, the following generalized plant is defined:

$$\begin{cases} \dot{x}(t) = A(M, k_2)x(t) + B_1(k_2)w(t) + B_2(M)u(t), \\ z(t) = C_1(M)x(t) \qquad\qquad\qquad\quad + D_{12}(M)u(t). \end{cases}$$

Here, $A(M, k_2)$ and $B_2(M)$ are as in (2.7)

$$A(M,k_2) := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1}{M} & -\frac{c}{M} & \frac{k_1}{M} & \frac{c}{M} \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m} & \frac{c}{m} & -\frac{k_1+k_2}{m} & -\frac{c}{m} \end{bmatrix}, \quad B_2(M) := \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{m} \end{bmatrix}$$

and input/output performance signals are defined by

$$B_1(k_2) := \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{k_2}{m} \end{bmatrix}, \; C_1(M) := \begin{bmatrix} -\frac{k_1}{M} & -\frac{c}{M} & \frac{k_1}{M} & \frac{c}{M} \\ W_{z_c} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; D_{12}(M) := \begin{bmatrix} \frac{1}{M} \\ 0 \\ W_u \end{bmatrix}.$$

In this generalized plant, the disturbance input w corresponds to the deviation of the ground height, whereas the performance output z is nothing but $z = [\ddot{z}_c \; W_{z_c} z_c \; W_u u]^T$. Here, $W_{z_c} = 2.5 > 0$ and $W_u = 0.15 > 0$ are weightings for the position z_c and the input u. Therefore, the H_∞ norm of the transfer $w \to z$ characterizes a tradeoff between the three required specifications (a), (b), and (c).

The considered uncertainties are as follows. The weight of the chassis can be increased by 20 % from its nominal value. The wheel stiffness can have up to 10 % discrepancy around the nominal value.

$$\begin{aligned} M_{\min} &= M_{\text{nom}} = 320, & M_{\max} &= 1.2 \cdot M_{\text{nom}} = 384, \\ k_{2,\min} &= 0.9 \cdot k_{2,\text{nom}} = 180, & k_{2,\max} &= 1.1 \cdot k_{2,\text{nom}} = 220. \end{aligned}$$

2.6.1.2 H_∞ Performance

First, we study the open-loop system. The nominal system is stable and its H_∞ norm is 13.080. The H_∞ norm computed on the vertices of the polytope (the four extremal combinations of the uncertainties) gives the following values 12.433, 13.814, 11.172, and 12.464. The value 13.814 obtained for $M = M_{\min}$ and $k_2 = k_{2,\min}$ may be a worst-case value of the H_∞ norm of the open-loop system, but some combination of uncertainties may as well give some larger value of the H_∞ norm, or even destabilize the system. To guarantee robustly some upper-bound on the H_∞ norm we apply results of Theorem 2.14, that is, we seek for a unique quadratic Lyapunov function for all uncertainties. The obtained optimum of this LMI problem is 88.978. It is an upper bound on the robust H_∞ performance. Still, there is clearly a huge gap between this upper bound and the values computed on vertices. In order to reduce the gap, we apply SV-LMI results of Theorem 2.13 which are guaranteed to be no more conservative. The obtained optimum of the SV-LMI problem is 13.814. Since it coincides with the H_∞ norm computed for $M = M_{\min}$ and $k_2 = k_{2,\min}$, we can

Table 2.1 H_∞ performance of the quarter-car suspension

	Nominal	Worst Vertex	SV-LMI (2.66)	Dual-SV-LMI (2.70)	Quad-LMI (2.67)	Dual-quad-LMI (2.71)
Open-loop	13.080	13.814	13.814	13.814	88.978	88.990
Closed-loop	9.710	9.859	9.859	9.859	18.497	18.497

conclude that

- 13.814 is the worst-case H_∞ norm and
- the SV-LMIs of Theorem 2.13 are nonconservative on this example.

The same study is now performed for the closed-loop system with the following state-feedback control

$$u = \begin{bmatrix} 7 & 21 & -8 & 3 \end{bmatrix} x.$$

For the nominal values of the parameters the H_∞ norm is 9.710. This state-feedback control improves the performance of the system. Computed on the vertices the H_∞ norm takes the following values 9.027, 9.539, 9.162, 9.859. The value 9.859 obtained for $M = M_{\min}$ and $k_2 = k_{2,\max}$ may be a worst-case value of the H_∞ norm of the closed-loop system. However, as above, this cannot be ensured at this stage. The LMIs of Theorem 2.14 are solved and give the guaranteed upper bound on the worst-case H_∞ norm 18.497. The SV-LMIs of Theorem 2.13 are solved; at the optimum, these give the improved upper-bound 9.859. It follows that the same conclusions apply as for the open-loop case. The SV-LMIs are nonconservative for this example.

Recall that results of Theorem 2.14 and Theorem 2.13 have dual counterparts which are, respectively, Corollary 2.9 and Corollary 2.8. The LMIs of these corollaries happen to produce the same values as for the SV-LMIs. All results for the open-loop and closed-loop cases are summarized in Table 2.1.

2.6.1.3 H_2 Performance

The same tests are now performed but assuming that the performance is measured according to an H_2 norm. Results are summarized in Table 2.2. The state-feedback gain is tuned in order to attenuate the H_∞ norm. It happens to have a (small) negative effect on the H_2 norm. Again the results show that, for this example, worst-case values of the uncertainties are at vertices and the SV-LMIs are sometimes exact (nonconservative). More precisely, notice that primal and dual LMIs produce different results both in the SV-LMI case or in the standard "quadratic stability" case. For this example, the dual versions are less conservative and the dual SV-LMIs are exact. The primal SV-LMIs are conservative with a small conservatism gap while the classical "quadratic stability" conditions are all highly conservative.

Table 2.2 H_2 performance of the quarter-car suspension

	Nominal	Worst vertex	Dual-SV-LMI (2.57a)–(2.57c)	SV-LMI (2.55a)–(2.55c)	Dual-quad-LMI (2.59a)–(2.59c)	Quad-LMI (2.58a)–(2.58c)
Open-loop	4.136	4.251	4.251	4.401	9.972	16.568
Closed-loop	4.143	4.388	4.388	4.396	5.469	8.058

Table 2.3 Impulse-to-peak performance of the quarter-car suspension

	Nominal	Worst vertex	Dual-SV-LMI (2.83a)–(2.83c)	SV-LMI (2.80a)–(2.80c)	Dual-quad-LMI (2.84a)–(2.84c)	Quad-LMI (2.81a)–(2.81c)
Open-loop	15.625	17.187	17.330	17.331	17.523	17.522
Closed-loop	15.578	17.136	17.339	17.339	17.382	17.382

2.6.1.4 Impulse-To-Peak Performance

Similar tests are done assuming performance is measured according to impulse-to-peak properties. Since the system has scalar perturbation input $w \in \mathbb{R}$, computation of the impulse-to-peak performance of a given system, let say the nominal one for example, is possible by simulation. In practice, we applied the `impulse` Matlab function. Results are summarized in Table 2.3.

2.6.2 Stability of Randomly Generated Examples

The tests that follow are done on artificial randomly generated uncertain models. Generating models of increasing sizes (order of the model, number of vertices of the polytope) allows to test the conservatism and the numerical complexity of the exposed LMI results.

2.6.2.1 Procedure for Random Generation of Polytopes

Relevance of randomly generated examples is often debatable. Indeed, although mathematically correct, the generated examples may not be illustrative for engineering issues. In order to have randomly generated models that fit some of the features of models issued from modeling of engineering problems, we have tried to fulfill the following criteria:

- Sparsity: Usual state-space models issued from real-life problems are sparse and the sparse structure is preserved by the uncertainties. A zero coefficient usually means that there is no information transiting between components. It cannot be considered to be uncertain.

- Few uncertain coefficients: The number of uncertain coefficients should be of the same order of magnitude as the number of vertices. Typically, for real-life models, the uncertainties are related to coefficients that occur a few times in the matrices of the model, all other coefficients being assumed to be known.
- Not all critical vertices: In order to test conservatism with respect to stability, one should generate examples in which worst-case poles are close to be unstable. Since only vertices are generated and worst cases are not at vertices, this issue cannot be handled exactly. A heuristic approach is to generate vertices close to be unstable. But, to be realistic, not all vertices should have this feature.

In order to comply with these specifications, we have developed a heuristic method combining randomized and line search features.

- Inputs: The inputs are the sizes of the system to be generated, that is, n the order and L the number of vertices; a positive coefficient d specifying the density of the sparse matrices; a positive coefficient α specifying the distance to instability; a positive coefficient p specifying the maximal amplitude of uncertainties.
- Initialization: The initialization step aims at generating a nominal A matrix with specified density d and with poles such that the distance to instability is clearly satisfied. Namely the matrix should be such that for all eigenvalues λ of A one has $\mathrm{Re}(\lambda) < -1.3\alpha$. The way this initialization is done is as follows:

  0.1. Generate a random matrix A_0 with density d and a second random matrix $\tilde{A}_0$ with the same sparsity pattern (should have same zero coefficients as A_0).
  0.2. Perform a line search over the scalar decision variable x, starting with $x = 0$, with the constraints

  $$x^* = \max x : \quad \max(\mathrm{Re}(\lambda(A_0 + x\tilde{A}_0))) \leq -1.3\alpha \ , \ 0 \leq x \leq \mathsf{p}.$$

  If p is sufficiently large and $\tilde{A}_0$ is a destabilizing direction, the matrix $A_0 + x^*\tilde{A}_0$ has one (or more) eigenvalue with real part equal to -1.3α, that is, at the specified distance to instability. If p is small or if $A_0 + x\tilde{A}_0$ has all eigenvalues with real part smaller than -1.3α whatever positive x then the matrix $A_0 + x^*\tilde{A}_0$ has all eigenvalues with real part smaller than -1.3α. Both these cases are satisfactory.
  0.3. If all real parts of eigenvalues of $A_0 + x^*\tilde{A}_0$ are smaller than -1.3α take $A = A_0 + x^*\tilde{A}_0$. If not start again at step 0.1.

- Generate vertices: Almost the same procedure is applied to generate the vertices but this time with perturbation matrices with smaller density and by requiring the strict distance to instability α. For each vertex $j \in \mathscr{I}_L$ the exact procedure is as follows:

  j.1 Generate a random matrix $\tilde{A}_j$ with low density (typically one or two non zero coefficients) and fitting the sparsity pattern of A (no non zero coefficients at row-column positions where the coefficients of A are zero).

j.2 Perform a line search over the scalar decision variable x_j, starting with $x_j = 0$, with the constraints

$$x_j^* = \max x_j : \quad \max(\mathrm{Re}(\lambda(A + x_j \tilde{A}_j))) \leq -\alpha \ , \ 0 \leq x_j \leq \mathsf{p}.$$

If p is sufficiently large and $\tilde{A}_j$ is a destabilizing direction, the matrix $A + x_j^* \tilde{A}_j$ has one (or more) eigenvalue with real part equal to $-\alpha$, that is, at the specified distance to instability. If p is small or if $A + x_j \tilde{A}_j$ has all eigenvalues with real part smaller than $-\alpha$ whatever positive x_j then the matrix $A + x_j^* \tilde{A}_j$ has all eigenvalues with real part smaller than $-\alpha$. Since A is chosen to have poles with real part strictly smaller than -1.3α and by continuity arguments one of these two cases holds.

j.3 Take $A_j = A + x_j^* \tilde{A}_j$ to be a vertex of the polytope and proceed to the next one.

For illustration, the procedure is applied twice with $n = 4$, $L = 2$, $\mathsf{d} = 0.7$, $\alpha = 0.1$,s and $\mathsf{p} = 1$. One (random) result gives the following two vertices

$$A_1 = \begin{bmatrix} 0 & 1.0851 & 0 & -1.3040 \\ 0 & -0.0883 & -0.5298 & 0 \\ 0.6548 & 0 & -1.0984 & 1.2803 \\ 1.0987 & 0 & 0 & -1.5144 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1.0851 & 0 & -1.3040 \\ 0 & -0.0883 & -0.5943 & 0 \\ 0.6548 & 0 & -1.2043 & 1.2725 \\ 1.0987 & 0 & 0 & -1.4806 \end{bmatrix}$$

with the following eigenvalues

$$\lambda(A_1) = \begin{matrix} -0.1000 \pm 0.8430j \\ -1.2505 \pm 0.7363j \end{matrix}, \quad \lambda(A_2) = \begin{matrix} -0.1000 \pm 0.8758j \\ -1.2865 \pm 0.7112j \end{matrix}$$

In this example, both vertices have poles at the specified distance to instability. Both matrices have the same sparsity pattern and only four coefficients out of nine are different.

Another (random) result of the procedure gives the following two vertices

$$A_1 = \begin{bmatrix} 0.8117 & 0 & 0.1146 & -1.3448 \\ 0.9845 & -0.6881 & 1.1752 & 0 \\ 0 & 0 & -0.2259 & 0 \\ 0.6236 & 0 & 0 & -1.0199 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.1851 & 0 & 0.1146 & -1.3448 \\ 0.9845 & -2.3668 & 1.1752 & 0 \\ 0 & 0 & -0.2259 & 0 \\ 0.6236 & 0 & 0 & -1.0199 \end{bmatrix}$$

with the following eigenvalues

$$\lambda(A_1) = \begin{matrix} -0.6881 \\ -0.1041 \pm 0.0045j \\ -0.2259 \end{matrix}, \quad \lambda(A_2) = \begin{matrix} -2.3668 \\ -0.6025 \pm 0.8152j \\ -0.2259 \end{matrix}$$

In this second example, none of the vertices are exactly at the specified distance to instability (although one is quite close). Sparsity and low number of uncertain elements criteria are satisfied.

Note that the procedure for generating randomly these systems has a numerical cost that is not negligible due to the possibly many trials for having a good initial guess of the matrix A and due to the many line search optimizations. The computation time for this random generation is large compared to the LMI tests to be evaluated as illustrated in the following. This is a price to pay to have interesting, reality-like, illustrative tests.

2.6.2.2 Robust Stability Testing

Series of tests are done for different values of the order n and the number of vertices L. The generated random models are according to the above-described procedure with parameters $\alpha = 0.05$, $\mathsf{d} = 0.6$, and $\mathsf{p} = 10$. The results are given in Table 2.4. The values are the number of samples that comply the tests and the values in curly braces are the mean computation times.

Computation time of LMI conditions that is given in the table includes both the parser YALMIP time (the time used to build the LMI problem and to convert it into the appropriate SDP-like format acceptable for the solver) and the SDP solver computation time (SDPT3 was used for these tests). Both these computations times grow with the size of the problem but not quite in the same way. The parser time mainly depends on the number of LMIs to declare, that is the number of vertices L, while the solver time grows polynomially with respect to the overall size of the problem characterized by n^2L. This difference explains why for small size problems the quad-LMI tests and SV-tests have almost identical computation time and the difference increases rapidly with the order of the system n.

When reading the table it is clear that SV results are much less conservative than classical quadratic stability type results. Moreover, the reduction of conservatism increases as the size of the problems grow (in n and L). The improvement brought by SV results is of 40 % for $n = 3$, $L = 3$. It is of 1,000 % for $n = 5$, $L = 6$. This reduction of conservatism is not at the expense of large increase in computation time.

Table 2.4 Robust stability tests for 100 random systems (for each row of the table), generated with parameters $\alpha = 0.05$, $d = 0.6$ and $p = 10$

n	L	rand gen	quad-LMI (2.9)	SV-LMI (2.13)	Dual-SV-LMI (2.17)	SV different (Theorem 2.6)	Unstable	Unknown
3	3	{ 2.47s}	70 {0.40s}	98 {0.47s}	96 {0.50s}	4	0	1
4	3	{ 3.18s}	48 {0.44s}	97 {0.54s}	98 {0.51s}	1	1	1
5	3	{10.90s}	32 {0.49s}	92 {0.65s}	94 {0.63s}	2	2	4
3	4	{ 2.11s}	53 {0.45s}	96 {0.52s}	96 {0.52s}	4	0	2
4	4	{ 4.28s}	32 {0.54s}	91 {0.69s}	95 {0.64s}	6	0	4
5	4	{10.57s}	28 {0.53s}	90 {0.79s}	90 {0.79s}	0	3	7
6	4	{32.36s}	27 {0.58s}	90 {0.95s}	91 {0.92s}	1	2	7
3	5	{ 2.45s}	54 {0.46s}	92 {0.62s}	91 {0.62s}	5	0	6
4	5	{ 3.77s}	22 {0.51s}	85 {0.77s}	84 {0.78s}	3	2	12
5	5	{10.78s}	23 {0.64s}	87 {1.05s}	87 {1.02s}	0	4	9
3	6	{ 2.50s}	42 {0.49s}	87 {0.72s}	87 {0.73s}	12	1	6
4	6	{ 3.67s}	17 {0.55s}	76 {0.91s}	76 {0.95s}	8	6	14
5	6	{11.02s}	7 {0.58s}	79 {1.18s}	78 {1.18s}	5	4	15
3	7	{ 2.32s}	47 {0.48s}	88 {0.77s}	91 {0.76s}	7	0	7
4	7	{ 4.38s}	16 {0.59s}	72 {1.16s}	78 {1.12s}	10	1	19
5	7	{11.40s}	8 {0.60s}	68 {1.40s}	66 {1.47s}	8	2	27
3	8	{ 2.93s}	32 {0.58s}	78 {0.93s}	83 {0.87s}	12	1	12
4	8	{ 4.65s}	15 {0.70s}	63 {1.37s}	65 {1.35s}	6	5	28
5	8	{10.06s}	6 {0.66s}	67 {1.59s}	67 {1.61s}	6	2	28
6	8	{32.62s}	5 {0.69s}	66 {2.10s}	66 {2.08s}	10	1	28

The value given in curly braces is the mean cpu time for generating the examples and for solving the LMIs

As mentioned in Theorem 2.6, the two primal and dual SV-LMI results may not be feasible simultaneously. This fact is illustrated by the value in the column "SV different." Considering, for example, the row corresponding to $n = 3$, $L = 6$, 87 systems are proved robustly stable by each SV LMI conditions. Among the 87 cases proved robustly stable with the SV-LMI condition (2.13), six give unfeasible dual SV-LMIs (2.17). Conversely, among the 87 cases proved robustly stable with the dual SV-LMI condition (2.17), six give unfeasible primal SV-LMIs (2.13). Overall, combining the two results, $87 + 6 = 93$ of the 100 tested systems are proved to be robustly stable by either one of the SV-LMI results. Among the remaining seven cases, one is proved to correspond to a nonrobustly stable system. This fact is obtained by finding a "worst case" value of the uncertainties for which the system is unstable. The way to find such value of the uncertainties is described in Chap. 3. Since all vertices of the generated polytopes are stable, these "worst case" values are not at vertices.

The last column of Table 2.4 indicates the number of generated random examples for which we were not able to conclude with neither robust stability nor instability.

The fact that these are not equal to zero illustrates the conservatism of the SV results (and the imperfectness of the worst-case uncertainty extraction scheme). Although much less conservative than the quadratic stability results, these results do have some conservatism. Further reduction of this conservatism is considered in Chap. 3.

2.6.2.3 Robust Discrete-Time Stability Testing

A same series of tests is performed for discrete-time systems. Stability corresponds to pole location in the unit disc. The system generation is identical except for the maximal real part of eigenvalues test replaced by maximal modulus test. Results are given in Table 2.5. The values are the number of samples that comply the tests and the values in curly braces are the mean computation times.

Conclusions that can be drawn from these results are quite similar to those in the continuous-time case. The main difference is in the very low number of samples for

Table 2.5 Robust stability tests for 100 random discrete-time systems (for each row of the table), generated with parameters $\alpha = 0.05$, d = 0.6 and p = 10

n	L	Rand gen	Quad-LMI (2.9)	SV-LMI (2.37)	Dual-SV-LMI (2.38)	SV different (see Theorem 2.6)	Unstable	Unknown
3	3	{0.70s}	83 {0.41s}	97 {0.49s}	96 {0.44s}	1	4	0
4	3	{0.86s}	61 {0.48s}	96 {0.51s}	96 {0.50s}	0	4	0
5	3	{0.89s}	49 {0.50s}	92 {0.67s}	92 {0.61s}	0	8	0
3	4	{0.84s}	58 {0.45s}	90 {0.57s}	88 {0.53s}	2	12	0
4	4	{0.92s}	42 {0.53s}	92 {0.68s}	90 {0.64s}	2	10	0
5	4	{1.13s}	39 {0.61s}	87 {0.82s}	87 {0.73s}	0	13	0
6	4	{1.62s}	28 {0.58s}	97 {0.91s}	96 {0.90s}	1	4	0
3	5	{0.90s}	67 {0.45s}	89 {0.57s}	87 {0.54s}	2	13	0
4	5	{0.95s}	34 {0.48s}	78 {0.80s}	77 {0.66s}	1	23	0
5	5	{1.11s}	33 {0.53s}	81 {0.97s}	80 {0.82s}	1	20	0
3	6	{1.04s}	54 {0.50s}	83 {0.68s}	81 {0.61s}	2	18	1
4	6	{1.12s}	38 {0.52s}	83 {0.94s}	78 {0.85s}	5	22	0
5	6	{1.41s}	22 {0.58s}	77 {1.26s}	77 {1.04s}	0	22	1
3	7	{1.14s}	62 {0.51s}	82 {0.75s}	82 {0.65s}	0	18	0
4	7	{1.23s}	34 {0.52s}	86 {0.98s}	83 {0.90s}	3	17	0
5	7	{1.38s}	13 {0.57s}	69 {1.36s}	67 {1.11s}	2	33	0
3	8	{1.40s}	58 {0.60s}	80 {0.90s}	78 {0.71s}	2	22	0
4	8	{1.45s}	26 {0.56s}	69 {1.15s}	68 {0.96s}	1	32	0
5	8	{1.69s}	6 {0.68s}	66 {1.64s}	26 {1.39s}	4	38	0
6	8	{2.10s}	4 {0.68s}	73 {1.98s}	70 {1.78s}	3	30	0

The value given in curly braces is the mean cpu time for generating the examples and for solving the LMIs which we are not able to state whether these are robustly stable systems or not. It gives the feeling that these SV-LMI results are close to be nonconservative. Compared to the continuous-time case this is rather surprising but is maybe due to the random system generation.

2.6.2.4 Airbus Provided Models

In 2009–2012 a study took place involving the Directorate General for Civil Aviation, LAAS-CNRS and the aircraft manufacturer Airbus through convention NGCI F/20 334/DA PPUJ. Among many studied topics, one of them was the evaluation of LMI methods for robust performance analysis of closed-loop longitudinal dynamics of a civil aircraft. The results that are described in the following are results of joint work on this topic involving Denis Arzelier, Guilherme De Calazans Chevarria, Dimitri Peaucelle at LAAS-CNRS and Guilhem Puyou at Airbus. These are also published in [10]. The numerical results of the book are recomputed values based on updated codes.

In the study, the certification is lead for a control law of the aircraft motion in the vertical plane (longitudinal). All the closed-loop components have been modeled. It includes:

- the actuators, namely the elevators, that have been modeled by a first order transfer function;
- the flight mechanics which has been restricted to only short period dynamics (angle of attack x_α and pitch rate x_q);
- the sensors delay and filtering have been reduced to first order transfer functions;
- the flight control law is mostly static but includes a scheduling part with respect to the speed (V_c), the Mach number (M_a) and the center of gravity (c_g).

The closed-loop model has been linearized on 633 flight points that map the parametric domain with respect to the weight (3 values [min medium max]), balance (center of gravity position [max forward, max aft]), speed V_c and Mach number (see Fig. 2.3). A regular spacing is used for V_c points generation. Nevertheless, minimum speed depends on the weight parameter value so that original points which are below the minimal bound are always brought back to the closest limit (see Fig. 2.3 in the lower part of the V_c range). A non regular spacing is used for Mach points generation. It comes from the aerodynamic coefficient study that highlights nonlinearity in the high Mach range. A tightest grid is used for high Mach number values.

For each of the $\mathrm{j} = 1, \ldots, 633$ points, the linearized model is given by:

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} A_\mathrm{j} & B_\mathrm{j} \\ C_\mathrm{j} & D_\mathrm{j} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \tag{2.85}$$

where $x \in \mathbb{R}^9$ is the state vector, w the vector of exogenous inputs, z the vector of controlled outputs. The channels w and z will be defined in the following for several performance criteria.

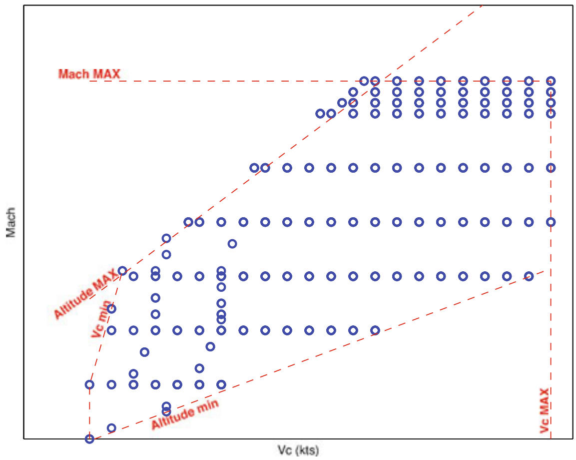


Fig. 2.3 Flight Domain in the (V_c, M_a) plane

Based on the knowledge of the 633 linearized models and of the true model dependency upon the flight conditions, it could be possible to derive an uncertain model with respect to flight condition parameters (load factor, Mach number etc.). Such uncertain model would depend on the 6 parameters and should fit exactly the given linearized models on the 633 points. In between the 633 it would correspond to one possible interpolation, possibly not fitting the real values. But the major drawback of such model would be its size. Some attempts, handling the interpolation issue and based on the physics of the plant are presented in [11]. The LFT models provided have uncertainty blocks of size 250–300. These are not for the longitudinal axis problem but based on these results one can expect an uncertainty block of size 150 for the problem we consider and maybe only for a part of the flight envelope. Such dimension are incompatible with the tools we aim at testing.

It is therefore proposed here to build several small size uncertain models, each of which are valid in the neighborhood of a given fight condition point. The collection of these uncertain models for the 633 flight conditions therefore cover the flight envelope. Analysis of each of these gives for each individual fight condition the expected level of robust performances valid in the close neighborhood.

A heuristic algorithm has been designed for choosing neighboring flight points. Given a selected flight point j, it selects among the 632 others the set $N(\mathrm{j}) \subset \{1 \ldots 633\}$ of neighboring points by using a criteria that combines the Euclidian distance in the 6-dimensional space of flight condition parameters and the search along parametric directions. Define $L(\mathrm{j})$ the number of elements of $N(\mathrm{j})$. The algorithm was tuned specifically for the given problem to generate sets $N(\mathrm{j})$ with acceptable number of elements. Namely, $L(\mathrm{j})$ goes from 8 in the corners of the flight envelope to 13 and has a mean value of 12.2.

As indicated above, the chosen uncertain modeling process leads to describing the system at a point j taking into account the neighboring points $N(\mathrm{j})$. We have chosen to do so in the natural affine polytopic way considered in this book. It needs no assumption on the physics of the problem. Each uncertain model at point j is defined as the linear combination of the $L(\mathrm{j})$ neighboring models with indexes in $N(\mathrm{j})$:

$$\begin{bmatrix} A_{\mathrm{j}}(\theta_{\mathrm{j}}) & B_{\mathrm{j}}(\theta_{\mathrm{j}}) \\ C_{\mathrm{j}}(\theta_{\mathrm{j}}) & D_{\mathrm{j}}(\theta_{\mathrm{j}}) \end{bmatrix} = \sum_{j \in N(\mathrm{j})} \theta_{\mathrm{j},j} \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \qquad \theta_{\mathrm{j}} \in \mathbb{E}^{L(\mathrm{j})} \tag{2.86}$$

2.6.2.5 Robust Performance Tests

The first main issue in analyzing feedback control systems is closed-loop robust stability of the models (2.86). For those flight conditions at which robust stability is assessed, 5 robust criteria are evaluated to assess different performances of the flight control system. Three of these criteria are input–output type performances. The inputs w and the outputs z for each of these depend on the criteria and are explained below.

- Robust pole placement in the left-half plane such that $\mathrm{Re}(\lambda) \leq \alpha$ to evaluate settling time.
- Robust pole placement in conic sectors with an angle ψ with respect to the imaginary axis to evaluate damping of modes.
- H_2 norm usually helps to evaluate the structural loads in turbulence and can also be used as an indirect measure of consumption. In our study, we evaluate robust upper bounds on the H_2-norm assuming the performance input w is a noise on the measured signals: pitch rate x_q and load factor x_{nz}, while the performance output $z = u \in \mathbb{R}$ is the control signal. The H_2 norm thus gives an information about the effect of measurement noise on the control effort.
- H_∞ norm may be used for measuring several features: comfort against turbulence, actuator activity or stability margins. In our study z is composed of the following three signals: the angle of attack x_α, the pitch rate x_q and the load factor x_{nz}, while $w \in \mathbb{R}$ is an additional disturbance signal applied on the control inputs. The H_∞-norm thus measures a stability margin.
- The impulse-to-peak performance is a way to evaluate the peak of the time response of the system to nonzero initial conditions. The issue of saturation of actuators signals can be analyzed using this criteria. Here, it is used to assess the behavior of the control signals to nonzero initial conditions in the aircraft. Namely, $w \in \mathbb{R}^2$ is chosen to define a unit ball of initial conditions and $z \in \mathbb{R}$ measures the maximal deflection of the elevator actuators.

We have done tests for all the 633 flight points. For each flight point $\mathrm{j} = 1 \ldots 633$ the steps were the following:

1. Generate the polytope of neighbors.
2. Test if all vertices $j \in N(\mathrm{j})$ are stable. If not, the polytope is proved not to be robustly stable. If all vertices are stable the quad-LMI test (2.9) is performed. In case it fails to prove robust stability, SV-LMI test (2.13) is performed. Robust stability is undetermined if all vertices are stable and SV-LMI test fails.

3. For robustly stable systems, upper bounds on the 5 robust criteria are evaluated with available methods and compared.

Robust stability results.

For robust stability assessment, it turned out that 3 of the 633 flight points are unstable. These are involved as neighbors in 15 other polytopes. A total of 18 flight points were thus proved not to be robustly stable. Of the 615 remaining polytopes, 612 were proved to be robustly stable using the quad-LMI test (2.9). The remaining 3 are proved to be robustly stable using SV-LMI test (2.13).

The size of the LMIs (number of variables and number of rows of the constraints) depends on the number of vertices. Since the number of vertices is not the same for all flight points, Table 2.6 gives the mean values of these dimensions over the 633 points. The table also gives the mean computation time for testing feasibility of the LMIs.

Settling time.

Finding an upper bound on the real part of the poles of the uncertain system cannot be done by LMI optimization. A bisection algorithm is applied in which LMI tests are run at each step to test a possible upper bound α. The initial interval of the bisection algorithm is $[\alpha_m(\mathrm{j})\ 0]$ where $\alpha_m(\mathrm{j})$ is the largest real part of all poles at the vertices of the robustly stable polytope of neighbors around flight point j:

$$\alpha_m(\mathrm{j}) = \max_{j \in N(\mathrm{j})} \max \mathrm{Real}(\lambda(A_j)).$$

Bisection stops when 1 % tolerance is reached. The optimal upper bound is denoted $\alpha^*(\mathrm{j})$. For comparison of the methods over all flight points, the following ratio is computed $\alpha_{\%}(\mathrm{j}) = (\alpha^*(\mathrm{j}) - \alpha_m(\mathrm{j}))/|\alpha_m(\mathrm{j})|$. Its mean value in percentage is given in Table 2.7 for the different methods.

The SV-LMI methods happen to be much less conservative than the quad-LMI method. Because of that, the number of iterations in the bisection is much lower for SV-LMI methods, thus making these not only less conservative but also of comparable bisection convergence time.

Damping.

Exactly the same procedure as above is applied to compare methods for the damping criterion. Results are given in Table 2.8.

Again, the SV-LMI method overcomes the quad-LMI results in terms of conservatism. It is at the expense of an increase in computation time. Compared to the tests

Table 2.6 LMI sizes and times for stability tests

	Mean nb of variables	Mean nb of rows	Mean time (s)
Quad-LMI (2.9)	45.0	127.7	0.49
SV-LMI (2.13)	683.3	204.3	1.53

Table 2.7 Results for settling time criterion

	$\alpha_{\%}$	Mean time per LMIs (s)	Mean number of bisection iterates	Mean bisection time (s)
Quad-LMI (2.39)	11.69 %	0.85	8.0	6.82
SV-LMI (2.37)	2.09 %	2.21	4.33	9.57

Table 2.8 Results for damping criterion

	$\psi_{\%}$	Mean time per LMIs (s)	Mean number of bisection iterates	Mean bisection time (s)
Quad-LMI (2.39)	9.45 %	1.78	8.0	14.27
SV-LMI (2.37)	1.69 %	7.78	4.43	34.48

Table 2.9 Results for robust H_2 cost

	$\gamma_{2\,\%}$	Mean nb of variables	Mean nb of rows	Mean time (s)
Quad-LMI (2.58a)–(2.58a)	30.82 %	46.0	136.7	0.89
SV-LMI (2.55a)–(2.55c)	0.01 %	728.3	257.1	3.16

Table 2.10 Results for robust H_∞ cost

	$\gamma_{\infty\%}$	Mean nb of variables	Mean nb of rows	Mean time (s)
Quad-LMI (2.67)	31.21 %	46.0	154.1	0.89
SV-LMI (2.66)	0.14 %	694.1	231.3	2.83

for settling-time criterion, the size of the LMIs happen to be double because pole location regions such as sectors used to prove damping are complex valued regions. These need double-sized LMI to be coded, see equation (2.36).

Input–Output performances.

Computation of the upper bounds on the three input–output criteria are done by direct LMI optimization. Again, a ratio $\gamma_{\%}(\mathrm{j})$ between the worst performance over all vertices of the polytope at point j and the obtained upper bound is computed. For the impulse-to-peak performance, $\gamma_{IP,m}$ is computed by simulation.

Results for the H_2, H_∞ and impulse-to-peak performances are given, respectively, in Tables 2.9, 2.10 and 2.11.

The reduction of conservatism due to the SV-LMI methods is significant and the comparisons with vertices indicate that SV-LMI results are close to being nonconservative. Moreover, globally the computation time does not increase dramatically.

Table 2.11 Results for robust impulse-to-peak performance

	$\gamma_{IP}\%$	Mean nb of variables	Mean nb of rows	Mean time (s)
Quad-LMI (2.81a)–(2.81c)	42.27 %	46.0	281.8	1.23
SV-LMI (2.80a)–(2.80c)	27.91 %	688.7	375.8	2.87

2.7 Conclusions

This chapter exposes the SV-LMI technique applied for a set of classical robust control system analysis problems. The considered problems are stability, pole location in convex sets of the complex plane, H_∞, H_2 and impulse-to-peak performances. For all these problems, classical "quadratic stability" based methods are recalled and compared in terms of conservatism and numerical complexity. SV-LMIs are proved to be always less conservative and as illustrated on examples the conservatism reduction is most significant. Noticeably, the conservatism reduction is not at the expense of huge increase of the computation time. Nevertheless, there is still some conservatism in SV-LMIs. The reasons for this conservatism are precisely explained and noticed on examples. The aim of the chapter that follows is to build with the same SV-LMI technique conditions with further reduced conservatism.

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