

Chapter 1

Introduction

This book is about monotone complete C^* -algebras, their properties and their classification. We also give a basic account of generic dynamics because of its useful connections to these algebras.

1.1 Monotone Complete Algebras of Operators

Fundamental to analysis is the completeness of the real numbers. Each bounded, monotone increasing sequence of real numbers has a least upper bound. Monotone complete algebras of operators have a similar property.

Let A be a C^* -algebra. Its self-adjoint part, A_{sa} , is a partially ordered, real Banach space whose positive cone is $\{zz^* : z \in A\}$. If each upward directed, norm-bounded subset of A_{sa} , has a least upper bound then A is said to be *monotone complete*. Every von Neumann algebra is monotone complete but the converse is false.

Recently there have been major advances in the theory of monotone complete C^* -algebras; for example the construction of classification semigroups [144]. This followed an important breakthrough in [66], which introduced huge numbers of new examples. But much remains to be discovered. The purpose of this book is to expound the new theory. We want to take readers from the basics to the frontiers of the subject. We hope they will be stimulated to work on the many fascinating open problems. Our intention is to strive for clarity rather than maximal generality. Our intended reader has a grounding in elementary functional analysis and point set topology and some exposure to the fundamentals of C^* -algebras, say, the first chapters of [161]. But prior knowledge of von Neumann algebras or operator systems is not essential. However, in this introduction, we may use terminology with which some readers are unfamiliar. If so, we apologise and reassure them that all necessary technicalities will be discussed later in the text.

Algebras of operators on Hilbert space, including C^* -algebras, von Neumann algebras and their generalisations, are the focus of intense research activity world-wide. They are fundamental to non-commutative geometry and intimately related to work on operator systems and operator spaces and have connections to many other fields of mathematics and quantum physics. But the first to be investigated (with a different name and a more “spatial” viewpoint) were the von Neumann algebras.

Monotone complete C^* -algebras arise in several different areas. There are close connections with operator systems, with operator spaces and with generic dynamics. In the category of operator systems, with completely positive maps as morphisms, each injective object can be given the structure of a monotone complete C^* -algebra in a canonical way. Injective operator spaces can be embedded as “corners” of monotone complete C^* -algebras, see Theorem 6.1.3 and Theorem 6.1.6 [38] and [25, 59, 60]. When a monotone complete C^* -algebra is commutative, its lattice of projections is a complete Boolean algebra. Up to isomorphism, every complete Boolean algebra arises in this way.

Let A be a monotone complete C^* -algebra then A is a von Neumann algebra precisely when it has a separating family of normal states. If a monotone complete C^* -algebra does not possess any normal states it is called *wild*.

The best known commutative example of a wild monotone complete C^* -algebra is straightforward to construct. Let $B(\mathbb{R})$ be the commutative C^* -algebra of all bounded, complex valued Borel measurable functions on \mathbb{R} . Let $M(\mathbb{R})$ be the ideal of all functions h in $B(\mathbb{R})$ such that $\{t \in \mathbb{R} : h(t) \neq 0\}$ is a meagre subset of \mathbb{R} . (Let us recall that a set is meagre if it is contained in the union of countably many nowhere dense sets; a set is nowhere dense if its closure has empty interior.) Then the quotient algebra $B(\mathbb{R})/M(\mathbb{R})$ is a commutative monotone complete C^* -algebra which has no normal states and so is not a von Neumann algebra. It turns out that if we replace \mathbb{R} by any complete separable metric space, without isolated points, and perform the same construction then we end up with the same commutative monotone complete C^* -algebra.

A monotone complete C^* -algebra, like a von Neumann algebra, is said to be a *factor* if its centre is trivial. In other words, factors are as far as possible from being commutative. Just as for von Neumann algebras, monotone complete C^* -factors can be divided into Type I, Type II₁, Type II_∞ and Type III. It turns out that all Type I factors are von Neumann algebras. So it is natural to ask: are *all* monotone complete C^* -factors, in fact, von Neumann algebras? The answer is “no” in general but to clarify the situation, we need some extra notions. Let H be a separable Hilbert space and $L(H)$ the bounded operators on H . A C^* -algebra A is said to be *separably representable* if there exists a $*$ -isomorphism π from A into $L(H)$. It is known that if A is a monotone complete C^* -factor which is also a separably representable C^* -algebra then A must be a von Neumann algebra [179]. So where are the wild factors?

A (unital) C^* -algebra B is said to be *small* if there exists a unital complete isometry from B onto an operator system in $L(H)$, where H is separable. When an algebra is separably representable then it is small but the converse is false. In other words, there exist C^* -algebras which can be regarded as operator systems

on a separable Hilbert space but which can only be represented as $*$ -algebras of bounded operators on a larger Hilbert space. There do exist small Type III monotone complete C^* -factors which are not von Neumann algebras. In fact they exist in huge abundance. There are 2^c , where c is the cardinality of the real numbers. By contrast, there are only c small von Neumann algebras. (Each small von Neumann algebra is isomorphic to a $*$ -subalgebra of $L(H)$ where the subalgebra is closed in the weak operator topology. In particular, each small von Neumann algebra is separably representable. This follows from [1].) Incidentally, if a small C^* -algebra is a wild factor then it is always of Type III.

One way to find a wild monotone complete C^* -factor is to start with a separable, simple, unital C^* -algebra and use a kind of “Dedekind cut” completion [173]. This approach will be discussed later. Another method is to associate a monotone complete C^* -algebra with a dynamical system. This “generic dynamics” approach is outlined below.

Monotone complete C^* -algebras are a generalisation of von Neumann algebras. The theory of the latter is now very well advanced. But it took many years before it was demonstrated that there were continuum many von Neumann factors of Type III [126], Type II_1 [100] and Type II_∞ [148]. Then the pioneering work of Connes, Takesaki and other giants of the subject transformed our knowledge of von Neumann algebras, see [8, 30, 96, 162]. By comparison, the theory of monotone complete C^* -algebras is in its infancy with many fundamental questions unanswered. But great progress has been made in recent years. In the early study of monotone complete C^* -algebras the emphasis was on showing how similar they were to von Neumann algebras. Nowadays we realise how different they can be.

In 2001 Hamana [66] made a major breakthrough which implied that there are 2^c small monotone complete C^* -factors. In 2007 [144] we found a way to classify monotone complete C^* -algebras. This is set out in Chap. 3.

In [144] we introduced a quasi-ordering between monotone complete C^* -algebras. From this quasi-ordering we defined an equivalence relation and used this to construct, in particular, a classification semigroup \mathcal{W} for small monotone complete C^* -algebras. This semigroup is abelian, partially ordered, and has the Riesz decomposition property. For each small monotone complete C^* -algebra A we assign a “normality weight”, $w(A) \in \mathcal{W}$. If A and B are algebras then $w(A) = w(B)$, precisely when these algebras are equivalent. It turns out that algebras which are very different can be equivalent. In particular, the von Neumann algebras are equivalent to each other and correspond to the zero element of the semigroup. It might have turned out that \mathcal{W} is very small and fails to distinguish between more than a few algebras. This is not so; the cardinality of \mathcal{W} is 2^c , where $c = 2^{\aleph_0}$.

A natural reaction by anyone familiar with K -theory, is to construct the Grothendieck group of \mathcal{W} . But this group is trivial because each element of the semigroup is idempotent. However this implies that \mathcal{W} has a natural structure as a semi-lattice. Furthermore, the Riesz Decomposition Property for \mathcal{W} ensures that the semi-lattice is distributive.

As we shall see later, one of the useful properties of \mathcal{W} is that it can sometimes be used to replace problems about factors by problems about commutative algebras [144].

To each monotone complete C^* -algebra we can associate a *spectroid* invariant, ∂A [144]. Just as a spectrum is a set which encodes information about an operator, a spectroid encodes information about a monotone complete C^* -algebra. It turns out that if $wA = wB$ then A and B have the same spectroid. So the spectroid may be used as a tool for classifying elements of \mathcal{W} .

Kaplansky wished to capture the algebraic essence of von Neumann algebras and to do it, introduced AW^* -algebras [90–92]. An AW^* -algebra may be defined as a unital C^* -algebra in which every maximal abelian $*$ -subalgebra is monotone complete [146]. Every monotone complete C^* -algebra is easily seen to be an AW^* -algebra. Nobody has ever seen an AW^* -algebra which is *NOT* monotone complete. It is strongly suspected that *EVERY* AW^* -algebra is monotone complete. But in full generality this is a difficult open problem. But many positive results are known. In particular, all “small” AW^* -factors are known to be monotone complete. Since our interest is strongly focused on small C^* -algebras we shall postpone a discussion of AW^* -algebras until Chap. 8. (But this can be read now, without working through all the earlier chapters.) They will appear on our list of open problems, some of which have been unsolved for over 60 years. For a scholarly account of the classical theory of AW^* -algebras the reader may consult [13].

Generic dynamics is used in an essential way in this book but we shall not introduce this tool until Chap. 6. So some readers may prefer to turn immediately to Chap. 2 and postpone reading the introduction to generic dynamics.

1.2 Generic Dynamics

An elegant account of generic dynamics was given by Weiss [165]; the term occurred earlier in [157]. In these articles, the underlying framework is a countable group of homeomorphisms acting on a complete separable metric space with no isolated points (a perfect Polish space). The key result of [157] was a strong uniqueness theorem. As a consequence, the wild factor discovered by Dyer [36] and the factor found by Takenouchi [159] were shown to be isomorphic.

We devote a chapter to aspects of generic dynamics useful for monotone complete C^* -algebra theory, including some recent discoveries [145]. This is an elementary exposition. In this book, generic dynamics is only developed as far as we need it for applications to C^* -algebras. But this does require us to consider generic dynamics on compact non-metrisable separable spaces; a topic which has been little explored and gives rise to interesting open questions.

Let G be a countable group. Unless we specify otherwise, G will always be assumed to be infinite. Let X be a Hausdorff topological space with no isolated points. Further suppose that X is a Baire space i.e. such that the only meagre open set is the empty set. In other words, the Baire Category Theorem holds for X . We

shall also suppose that X is completely regular. (These conditions are satisfied if X is compact or homeomorphic to a complete separable metric space or, more generally, a G_δ -subset of a compact Hausdorff space or is the extreme boundary of a compact convex set in a locally convex Hausdorff topological vector space.) A subset Y of X is said to be *generic* if $X \setminus Y$ is meagre.

Let ε be an action of G on X as homeomorphisms of X .

In classical dynamics we would require the existence of a Borel measure on X which was G -invariant or quasi-invariant, and discard null sets. In topological dynamics, no measure is required and no sets are discarded. In generic dynamics, we discard meagre Borel sets.

We shall concentrate on the situation where, for some $x_0 \in X$, the orbit $\{\varepsilon_g(x_0) : g \in G\}$ is dense in X . Of course this cannot happen unless X is separable. (A topological space is *separable* if it has a countable dense subset. This is a weaker property than having a countable base.) Let S be the Stone space of the (complete) Boolean algebra of regular open sets of X . Then, see below, the action ε of G on X induces an action $\hat{\varepsilon}$ of G as homeomorphisms of S ; which will also have a dense orbit.

When, as in [165] and [157], X is a perfect Polish space, then S is unique; it can be identified with the Stone space of the regular open sets of \mathbb{R} . But if we let X range over all separable compact subspaces of the separable space, $2^{\mathbb{R}}$, then we obtain 2^c essentially different S ; where S is compact, separable and extremally disconnected. For each such S , $C(S)$ is a subalgebra of ℓ^∞ .

Let E be the relation of orbit equivalence on S . That is, sEt , if, for some group element g , $\hat{\varepsilon}_g(s) = t$. Then we can construct a monotone complete C^* -algebra M_E from the orbit equivalence relation. When there is a free dense orbit, the algebra will be a factor with a maximal abelian $*$ -subalgebra, A , which is isomorphic to $C(S)$. There is always a faithful, normal, conditional expectation from M_E onto A . It can be shown that $wM_E = wA$. So some classification questions about factors can be replaced by questions about commutative algebras. *When E and F are orbit equivalence relations which coincide on a dense G_δ -subset of S then M_E is isomorphic to M_F .*

For $f \in C(S)$, let $\gamma^g(f) = f \circ \hat{\varepsilon}_{g^{-1}}$. Then $g \mapsto \gamma^g$ is an action of G as automorphisms of $C(S)$. Then we can associate a monotone complete C^* -algebra $M(C(S), G)$, the *monotone cross-product* with this action (see Chap. 7). When the action $\hat{\varepsilon}$ is free, then $M(C(S), G)$ is naturally isomorphic to M_E . In other words, the monotone cross-product does not depend on the group, only on the orbit equivalence relation.

In this book we shall consider 2^c algebras $C(S)$. Each is a subalgebra of ℓ^∞ and each takes different values in the weight semigroup \mathcal{W} . (Here $c = 2^{\aleph_0}$, the cardinality of \mathbb{R} .)

For general S there is no uniqueness theorem but we do show the following. Let G be a countably infinite group. Let α be an action of G as homeomorphisms of S and suppose this action has at least one orbit which is dense and free. Then, modulo meagre sets, the orbit equivalence relation obtained can also be obtained by an action of $\bigoplus \mathbb{Z}_2$ as homeomorphisms of S . This should be contrasted with the situation in

classical dynamics. (e.g. It is shown in [31] that any action by an amenable group is orbit equivalent to an action of \mathbb{Z} . But, in general, non-amenable groups give rise to orbit equivalence relations which do not come from actions of \mathbb{Z} .)

On each of 2^c , essentially different, compact extremally disconnected spaces we construct a natural action of $\bigoplus \mathbb{Z}_2$ with a free, dense orbit. Let Λ be a set of cardinality 2^c , where $c = 2^{\aleph_0}$. Then by applying generic dynamics, as in [144], we can find a family of monotone complete C^* -algebras $\{B_\lambda : \lambda \in \Lambda\}$ with the following properties. Each B_λ is a monotone complete factor of Type III, and also a small C^* -algebra. For $\lambda \neq \mu$, B_λ and B_μ have different spectroids and so $wB_\lambda \neq wB_\mu$ and, in particular, B_λ is not isomorphic to B_μ . Furthermore each B_λ is generated by an increasing sequence of full matrix algebras.

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