Chapter 2
Fourier Series Expansion

In Chap. 1, it was shown, mostly by using graphics, that various waves can be expressed by a summation of sine and cosine functions, i.e., by the Fourier series (see Eq. 1.5). In this chapter, first, a method of determining coefficients of Fourier series will be given. A key idea is the integral of the products of sine and cosine functions. It was shown that an addition of sine and cosine functions with the same frequency can be combined into one sine or cosine function by introducing a phase term (see Eq. 1.5). It is also possible to express an arbitrary function by a combination of even and odd functions. The former and the latter can be expressed by cosine and sine functions, respectively. The next step is the expression of a Fourier series by complex exponential functions. The coefficients in this case are also complex, but since the mathematical manipulations are simpler, this method will be used most of the time hereafter.

The steps will be given one by one in this chapter and the reader should understand that the same thing is dealt with from different angles. A derivation of the Fourier transform pair as the extreme case of the Fourier series is the last subject in this chapter.

2.1 Integrals of Sine and Cosine Functions

As the starting point of this chapter, Eq. (1.2) is shown here again. It states that a waveform $x(t)$ with period T can be expressed by sine and cosine functions with frequencies $kf_0 = k/T\,(k = 0,\ 1,\ 2, \ldots)$.

$$\begin{aligned} x(t) = A_0 &+ A_1 \cos 2\pi \frac{1}{T} t + A_2 \cos 2\pi \frac{2}{T} t + A_3 \cos 2\pi \frac{3}{T} t + \cdots \\ &+ B_1 \sin 2\pi \frac{1}{T} t + B_2 \sin 2\pi \frac{2}{T} t + B_3 \sin 2\pi \frac{3}{T} t + \cdots \end{aligned} \tag{2.1}$$

The first task is to find a way of determining the Fourier coefficients A_k and B_k assuming that a periodic function can be represented in the form of Eq. (2.1). In

K. Kido, *Digital Fourier Analysis: Fundamentals*,
Undergraduate Lecture Notes in Physics, DOI: 10.1007/978-1-4614-9260-3_2,

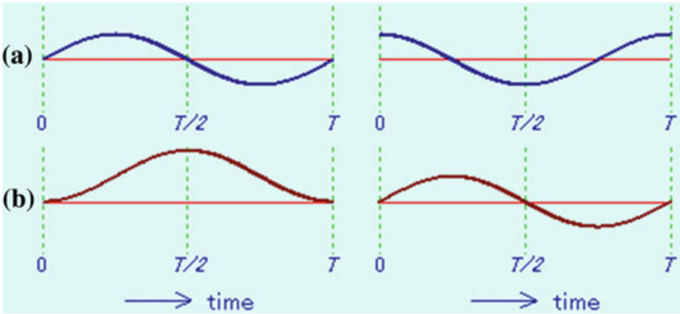

Fig. 2.1 Sine and cosine functions (*top*) and their integrations (*bottom*) starting from $t = 0$ and ending at $t = T$. Animation available in supplementary files under filename E2-01_SinCos.exe

order to determine the Fourier coefficients, both sides of Eq. (2.1) are multiplied by $\cos\{2\pi(k/T)t\}$ or $\sin\{2\pi(k/T)t\}$ and integrated over one period T. Since the integration can be carried out term by term, the basic problem is how to integrate the sine, cosine, and their products over one period (T).

With regard to the first term A_0, we need to integrate $1 \cdot \cos\{2\pi(k/T)t\}$ or $1 \cdot \sin\{2\pi(k/T)t\}$. For $k = 0$, $\cos\{2\pi(k/T)t\} = 1$ and $\sin\{2\pi(k/T)t\} = 0$. Therefore, their integration over period T is T and zero, respectively. It is obvious that the integrations of sine and cosine functions over multiples of their fundamental period are zero if $k \geq 1$. Figure 2.1a shows $\sin\{2\pi(1/T)t\}$ and $\cos\{2\pi(1/T)t\}$, and Fig. 2.1b shows how the integrations vary as the integration time is increased from $\mathrm{t} = 0$ to T. Since the positive and negative areas are the same, the integration over one period becomes zero for both cases. It should be clear that the results are the same for higher orders ($k \geq 2$).

With regard to the higher order terms A_n ($n \geq 1$), it is necessary to investigate the integration of $\cos\{2\pi(k/T)t\}$ and $\sin\{2\pi(k/T)t\}$ multiplied by either $\cos\{2\pi(m/T)t\}$ or $\sin\{2\pi(m/T)t\}$. It should be remembered that both k and m are integers. The situation varies depending on the two cases: $k = m$ and $k \neq m$. Since equations of integration of products of sine and cosine functions are shown in many books, an emphasis will be put on gaining a physical image of these integrations.

Figure 2.2 shows integrations of cosine (a) and sine (b) functions multiplied by themselves. In this figure (a), $x_1(t)$ and $x_2(t)$ are the same cosine functions with the period T. The product of these two functions is shown by the curve $W_{12}(t)$ (thin line). This function never takes negative values and it has the period of $T/2$ (twice of the original frequency). The integration of $W_{12}(t)$ is shown by $E_{12}(t)$ (thick line), which is a monotonically increasing function. The integration of $W_{12}(t)$ over period T takes a finite value. A formula (equation) is needed to obtain this value, which will be discussed in Sect. 2.2. In the case of the sine function, the integration takes a different path but the final value at $t = T$ is the same as that of the cosine function.

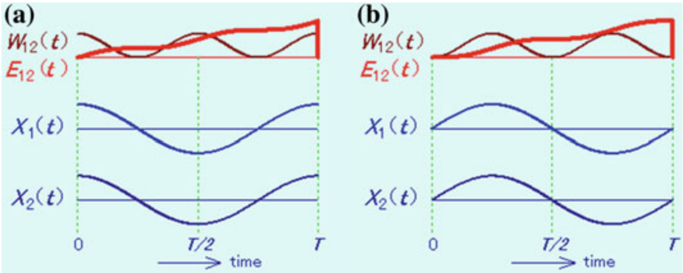

Fig. 2.2 Integration of $\cos^2\{2\pi(1/T)t\}$ and $\sin^2\{2\pi(1/T)t\}$. Animation available in supplementary files under filename E2-02_SCIntegral.exe

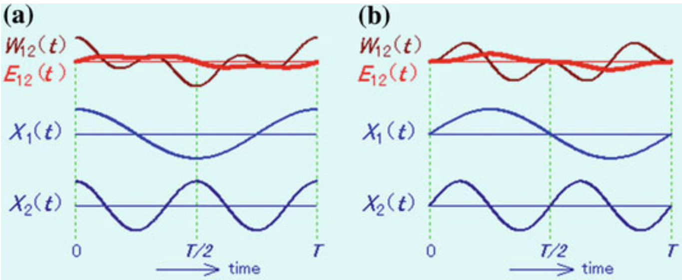

Fig. 2.3 Integration of **a** $\cos\{2\pi(1/T)t\} \times \cos\{2\pi(2/T)t\}$ and **b** $\sin\{2\pi(1/T)t\} \times \sin s\{2\pi(2/T)t\}$. Animation available in supplementary files under filename E2-03_SCIntegral.exe

Even if the functions $x_1(t)$ and $x_2(t)$ are both sine or cosine functions, if $k \neq m$, the integration becomes zero. This is shown in Fig. 2.3 for the case with the frequency ratio of 1:2 (Fig. 2.3a: cosine functions, Fig. 2.3b: sine functions). The validity can be checked for any ratio of two integers (k and m) by running the program.

If $x_1(t)$ is a cosine function and $x_2(t)$ is a sine function, the integration becomes zero regardless of the values of k and m. Fig. 2.4 shows these cases.

In summary, it is concluded that the integrals of products of two sine or two cosine functions are zero except for the cases when the frequencies of the two sine or two cosine functions are identical (i.e., $k = m$). This statement is also valid for the cases of integration of $\cos\{2\pi(k/T)t\}$ and $\sin\{2\pi(k/T)t\}$. Each of these functions is considered as the product of itself with the cosine function of zero frequency ($m = 0$) since $\cos\{2\pi(0/T)t\} = 1$. A set of functions with the property that the integration of a product of any two of its functions over the same fixed range is zero unless the functions are identical is called an "orthogonal system." A set of sine and cosine functions that have integer multiples of a fundamental frequency has this "orthogonality" property.

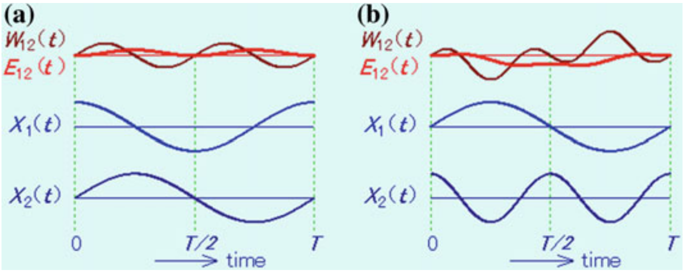

Fig. 2.4 Integration of **a** $\cos\{2\pi(1/T)t\} \times \sin\{2\pi(1/T)t\}$ and **b** $\sin\{2\pi(1/T)t\} \times \cos\{2\pi(2/T)t\}$. Animation available in supplementary files under filename E2-04_SCIntegral.exe

It has been made clear from the above discussion that, if the right-hand side is multiplied by $\cos\{2\pi(k/T)t\}$ or $\sin\{2\pi(k/T)t\}$ and integrated from $t = 0$ to T, only the k-th term of the cosine or sine series remains. This is the means by which the coefficients A_k and B_k can be determined. Since we cannot determine the values of integration from the charts, we must use equations, which will be developed in the next section.

2.2 Calculations of Fourier Coefficients

In order to derive formulae to determine the coefficients of the Fourier series, explanations given in Sect. 2.1 will be followed using equations.

First, what kinds of results are obtained if we integrate both sides of Eq. (2.1)?

$$\int_0^T x(t)dt = \int_0^T [A_0 + A_1 \cos 2\pi\frac{1}{T}t + A_2 \cos 2\pi\frac{2}{T}t + A_3 \cos 2\pi\frac{3}{T}t + \cdots + B_1 \sin 2\pi\frac{1}{T}t + B_2 \sin 2\pi\frac{2}{T}t + B_3 \sin 2\pi\frac{3}{T}t + \cdots]dt$$

The integration of the right-hand side will be done term by term. The first term with A_0, is given by

$$\int_0^T A_0 dt = TA_0 \tag{2.2}$$

The integrations of the following terms with coefficients A_k, $k = 1, 2,\ldots$ are all equal to zero.

$$\int_0^T A_k \cos 2\pi\frac{k}{T}tdt = A_k\frac{T}{2\pi}\sin 2\pi\frac{k}{T}t\Big|_0^T = 0 \tag{2.3}$$

It is the same for the coefficients B_k, $k = 1, 2, \ldots$

$$\int_0^T B_k \sin 2\pi \frac{k}{T} t dt = -B_n \frac{T}{2\pi} \cos 2\pi \frac{k}{T} t \bigg|_0^T = 0 \tag{2.4}$$

Equation (2.2) is rewritten, giving an equation to determine A_0.

$$A_0 = \frac{1}{T} \int_0^T x(t) dt \tag{2.5}$$

Next, we will multiply $\cos\{2\pi(k/T)t\}$ on both sides of Eq. (2.1).

$$\begin{aligned}\int_0^T x(t) \cos\left(2\pi \frac{k}{T} t\right) dt = \int_0^T & [A_0 \cos\left(2\pi \frac{k}{T} t\right) + A_1 \cos\left(2\pi \frac{1}{T} k\right) \cos\left(2\pi \frac{k}{T} t\right) \\ & + A_2 \cos\left(2\pi \frac{2}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) + \cdots + A_n \cos^2\left(2\pi \frac{k}{T} t\right) + \cdots \\ & + B_1 \sin\left(2\pi \frac{1}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) + \cdots + B_k \sin\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{k}{T} t\right) + \cdots] dt\end{aligned} \tag{2.6}$$

Let's check this integration term by term. The first term becomes zero as shown by Eq. (2.3). The second and higher order terms are the integrations of products of cosine functions with different frequencies or the products of sine and cosine functions. The discussion in Sect. 2.1 showed that there is only one nonzero term, which is the product of cosine functions with the same frequency.

The product of $\cos\{2\pi(k/T)t\}$ and $\sin\{2\pi(k/T)t\}$ is rewritten as

$$\cos\left(2\pi \frac{k}{T} t\right) \cos\left(2\pi \frac{m}{T} t\right) = \frac{1}{2}\left\{\cos\left(2\pi \frac{k+m}{T} t\right) + \cos\left(2\pi \frac{k-m}{T} t\right)\right\}.$$

The integration of this product from $t = 0$ to T is zero if $k \neq m$. If $k = m$, the second term of the right-hand side is $T/2$ since $\cos\{2\pi(0/T)t\} = 1$. Similarly, the product of $\cos\{2\pi(k/T)t\}$ and $\sin\{2\pi(m/T)t\}$ is rewritten as

$$\cos\left(2\pi \frac{k}{T} t\right) \sin\left(2\pi \frac{m}{T} t\right) = \frac{1}{2}\left\{\sin\left(2\pi \frac{k+m}{T} t\right) - \sin\left(2\pi \frac{k-m}{T} t\right)\right\}.$$

The integration of this product from $t = 0$ to T becomes zero for both cases: $k \neq m$ and $k = m$.

Finally, only one nonzero term remains, which is

$$\int_0^T x(t) \cos\left(2\pi \frac{k}{T} t\right) dt = A_k \frac{T}{2}.$$

This gives the equation to determine A_k,

$$A_k = \frac{2}{T}\int_0^T x(t)\cos\left(2\pi\frac{k}{T}t\right)dt. \tag{2.7}$$

Similarly, the equation to determine B_k is given by

$$B_k = \frac{2}{T}\int_0^T x(t)\sin\left(2\pi\frac{k}{T}t\right)dt. \tag{2.8}$$

The range of integration need not be from $t = 0$ to T. It can be over any range from $t = T_1$ to T_2 as long as $T_2 - T_1 = T$. Then the equations to determine Fourier coefficients are given as

$$A_0 = \frac{1}{T}\int_{T_1}^{T_1+T} x(t)dt \tag{2.9}$$

$$A_k = \frac{2}{T}\int_{T_1}^{T_1+T} x(t)\cos\left(2\pi\frac{k}{T}t\right)dt \tag{2.10}$$

$$B_k = \frac{2}{T}\int_{T_1}^{T_1+T} x(t)\sin\left(2\pi\frac{k}{T}t\right)dt. \tag{2.11}$$

If a symmetrical expression is preferred, the equations will be

$$A_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt \tag{2.12}$$

$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(2\pi\frac{k}{T}t\right)dt \tag{2.13}$$

$$B_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(2\pi\frac{k}{T}t\right)dt. \tag{2.14}$$

The three sets of equations to calculate Fourier coefficients have been presented (Eqs. (2.5), (2.7)–(2.14)). The most general expression is the second set. Each set will give different values of A_k and B_k. This is due to the phase difference caused by the change of starting point of each harmonic of the waveform. However, the magnitude of each harmonic $\sqrt{A_k^2 + B_k^2}$ is independent of the starting point.

A reason why a waveform is expanded into a series of sine and cosine functions is because each coefficient can be determined using the orthogonality property of sine and cosine functions. There are other types of orthogonal functions that could be used to expand the same waveform. However, there is no reason here to seek other orthogonal functions since the sine and cosine functions constitute one of the most elegant sets of orthogonal systems.

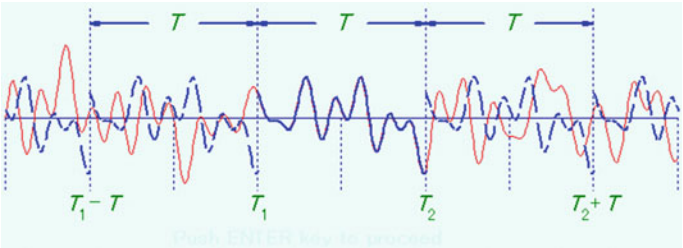

Fig. 2.5 An infinitely long waveform (*thin line*) and its Fourier expansion in the range $T_1 \le t < T_2$ (*thick line*). The Fourier series expansion repeats the same (extracted) waveform outside the range $T_1 \le t < T_2$, as shown by the dotted line. Animation available in supplementary files under filename E2-05_AnalysisRange.exe

Until now, we have not paid much attention to the region outside of the range $0 \le t < T$, $T_1 \le t < T_2$, or $-T/2 \le t < T/2$, within which the function is defined. However, since the fundamental component of the Fourier series has the periodicity T and its k-th harmonic has the periodicity T/k, any waveform expressed by use of the Fourier series will exhibit the periodicity T. Therefore, as shown in Fig. 2.5, if the portion of a waveform between $T_1 \le t < T_2$ is expressed by a Fourier series, the Fourier series expansion of that waveform repeats the waveform with period T (thick line) within the range $T_1 \le t < T_2$. Note that the original waveform (thin line) and the Fourier series expansion extended outside the range $T_1 \le t < T_2$ (dotted line) may be different.

Since selecting a portion of a continuous waveform is analogous to looking at the waveform through a window, it is called *time-windowing* in the field of signal processing. The period T or $T_2 \sim T_1$ is the length of the *time window*. The waveform expressed by the Fourier series repeats the extracted waveform within that time window, with the period of the window length.

As shown by Eq. (2.1), the Fourier series is expressed by the series of cosine functions with coefficients A_k and the series of sine functions with coefficients B_k. Once A_k and B_k are obtained, each cosine and sine combination can be combined into one cosine or one sine function as shown by Eqs. (1.23) and (1.25). Only the results will be shown here.

The expression using only cosine functions is given by

$$x(t) = C_0 + C_1 \cos\left(2\pi \frac{1}{T} t - \phi_1\right) + C_2 \cos\left(2\pi \frac{2}{T} t - \phi_2\right) + \cdots \tag{2.15}$$

where

$$C_0 = A_0,\ C_k = \sqrt{A_k^2 + B_k^2},\ \phi_k = \arctan(B_k/A_k) \tag{2.16}$$

A similar expression using only sine functions is given by

$$x(t) = C_0 + C_1 \sin\left(2\pi\frac{1}{T}t + \theta_1\right) + C_2 \sin\left(2\pi\frac{2}{T}t + \theta_2\right) + \cdots \tag{2.17}$$

where C_0, C_k are given by Eq. (2.16) and θ_k is given by

$$\theta_k = \arctan(A_k/B_k) \tag{2.18}$$

The coefficient $C_k = \sqrt{A_k^2 + B_k^2}$, which is common for Eqs. (2.15) and (2.17), is the amplitude of the combined component of the cosine and sine functions with frequency k/T. Therefore, the set of amplitudes C_k is called the *amplitude spectrum*, which is expressed as a function of k (order) or the real frequency (k/T). On the other hand, a set of squares C_k^2 is called the *power spectrum*, since, for example, the square of the amplitude of the voltage or the current is proportional to the (electrical) power. The frequencies of the spectra given as components of the Fourier series are integer multiples of $1/T$. The spectra are distributed at discrete points on the frequency axis and their magnitudes are expressed by vertical thin lines. Therefore, they are called *line spectra*.

2.3 Expressing Waveforms by Even Functions

The Fourier coefficients, or Fourier spectra, obtained using an extracted waveform with length T have cosine and sine components as shown in Fig. 2.6. The time windowed (extracted) waveform repeats itself with period T, and the spacing between the adjacent spectra is $1/T$. Each spectrum is actually composed of two components, the cosine ("real") and sine ("imaginary") components, A_k and B_k, respectively. These two can be combined into single cosine or sine components while introducing the phase terms as shown by Eq. (2.15) or (2.17). The introduction of phase may make the situation more complex, rather than making it simpler. Then, a question arises, "Is there a way of expressing a waveform by cosine or sine functions without using phase terms?"

As we have learned in Chap. 1, an even function, which is symmetric with respect to the origin of time $t = 0$, is expressed only by cosine functions, and an odd function, which is anti-symmetric with respect to the origin of time $t = 0$, is expressed only by sine functions. Then, if a waveform that is symmetric with $x(t)$ is introduced into the time range, $-T \leq t < 0$, as shown in Fig. 2.7, and if the combined waveform (A_{2TE}) is expressed by the Fourier series with period $2T$, the series will contain only the cosine terms.

Figure 2.7 shows exactly what we expect. Figures 2.6 and 2.7 are the spectra of waveform constructed from the same portion of another waveform. Figure 2.6 shows Fourier coefficients of a periodic waveform A_T with a single period T and Fig. 2.7 shows Fourier coefficients of a periodic waveform (A_{2TE}) with period $2T$. Therefore, the frequency spacing (resolution) of the former figure is $1/T$ and

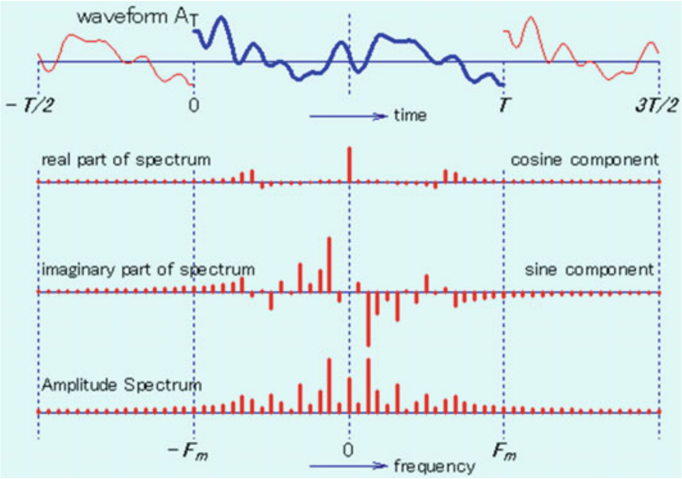

Fig. 2.6 Fourier coefficients (real and imaginary parts of the spectrum) and amplitude spectra obtained when the extracted waveform A_T is assumed to be periodic. Animation available in supplementary files under filename E2-06_F-Coeff_A.exe

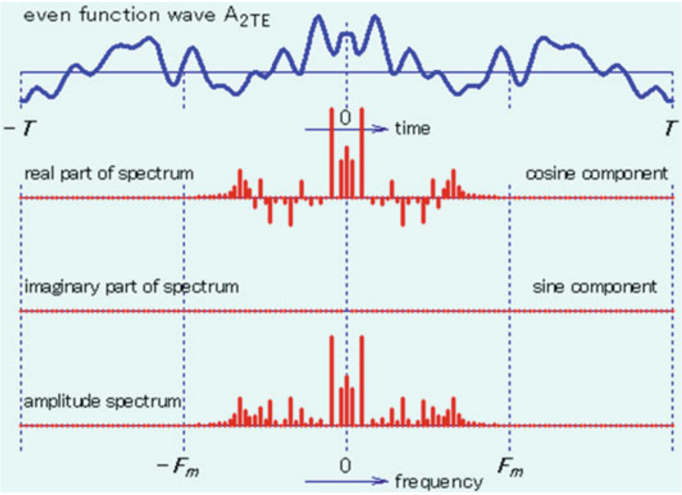

Fig. 2.7 Even waveform A_{2TE} with period $2T$ and its spectra obtained by adding a symmetric waveform of A_T (see Fig. 2.6) in the region $-T \le t < 0$. Animation available in supplementary files under filename E2-07_EvenF_A.exe

that of the latter figure is $1/2T$ (one half of the former). In order to make this clear, the horizontal axis is scaled by frequency (instead of integer k) and vertical dotted lines are inserted at every F_m frequency. Following charts are shown using the same format so that the charts may be compared more easily.

The waveform shown in Fig. 2.6 has many spectral components outside of $\pm F_m$. A reason why so many spectral lines are necessary is that the constructed waveform has a large discontinuity at the connections when the portion of the waveform with the period T is repeated. The constructed waveform in Fig. 2.7, has a smaller degree of discontinuity and, therefore, more of the harmonics are kept within $\pm F_{m.}$ As shown in Fig. 2.7, if a symmetric waveform is introduced into the time range, $-T \leq t < 0$, the constructed waveform becomes an even function and the whole range from $-T$ to T must be taken into account when applying the Fourier series expansion. In this case, the waveform in the range from 0 to T is completely recovered and the series has only cosine terms. This may seem to be an advantage to be able to avoid the use of the phase terms. However, since the period is doubled (in other words, the frequency spacing is one half), two times the number of spectra are necessary to cover the same frequency range. The number of spectral lines of the waveforms shown in Figs 2.6 and 2.7 are the same since the latter needs only the cosine terms even though the spectral density is twice.

For later use, what was explained above will be described using equations. In order to express the waveform $x(t)$ in $0 \leq t < T$ using only cosine terms, the symmetric waveform $x(-t)$ is added to the range $-T \leq t < 0$. The combined waveform is of course symmetric (even), which will be named $z(t)$. The k-th order Fourier coefficients defined in the range $-T \leq t < T$, is given by

$$A_k = \frac{1}{T}\int_{-T}^{T} z(t)\cos\left(2\pi\frac{k}{2T}t\right)dt$$

Since $z(t)$ and cosine functions are both even functions, the integration from $t = -T$ to 0 is the same with the integration from 0 to T. Therefore, the above integration can be given by

$$A_k = \frac{2}{T}\int_{0}^{T} z(t)\cos\left(2\pi\frac{k}{2T}t\right)dt.$$

Since $x(t) = z(t)$ in the range from 0 to T, it is given by

$$A_k = \frac{2}{T}\int_{0}^{T} x(t)\cos\left(2\pi\frac{k}{2T}t\right)dt. \tag{2.19}$$

The coefficient A_0 can be obtained by the same way.

$$A_0 = \frac{1}{T}\int_{0}^{T} x(t)dt \tag{2.20}$$

With these coefficients, the waveform $x(t)$ in $0 \leq t < T$ can be expressed by

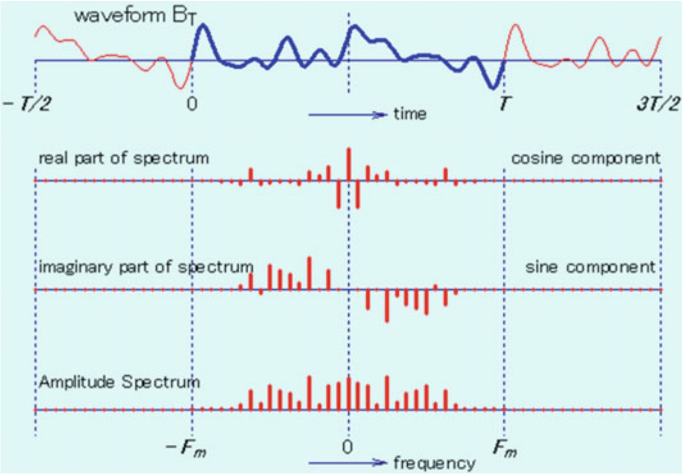

Fig. 2.8 Fourier series obtained when the extracted waveform B_T is assumed to be periodic. Animation available in supplementary files under filename E2-08_F-Coeff_B.exe

$$x(t) = A_0 + A_1 \cos\left(2\pi \frac{1}{2T} t\right) + A_2 \cos\left(2\pi \frac{2}{2T} t\right) + A_3 \cos\left(2\pi \frac{3}{2T} t\right) + \cdots \tag{2.21}$$

This is called the "cosine Fourier series." By expressing the waveform in Fig 2.6 using the cosine Fourier series, the frequency components higher than F_m are not visible and it seems possible to approximate $x(t)$ with relatively low frequency components. At first glance, this method seems to be a good idea, but it is necessary to make sure if this is always the case.

Let's take an example ("waveform B_T") shown in Fig. 2.8. As shown in the figure, there are very few spectral lines in the range outside of $\pm F_m$. One reason is that the original waveform itself contains small levels of high frequency components. Another reason is that the waveform in $0 \leq t < T$ is connected smoothly at the joint to the preceding and following repetitive waveforms, which are shown by the dotted lines in the figure. The reader can check this by running the program Fig. 2.8.

Figure 2.9 shows an even function, "waveform B_{2TE}" with period $2T$ produced by adding the time-reversed waveform of B_T in the region $-T \leq t < 0$ and its spectra. Since waveform B_T increases sharply at $t = 0$, the symmetric waveform B_{2TE} has a large discontinuity at $t = 0$. Since it is an even function, the Fourier series is composed of cosine functions only. A comparison of Fig. 2.9 with Fig. 2.8 shows that waveform B_{2TE} has a larger distribution of spectra in the high frequency region. A reason for this is that there is a large trough (discontinuity) around $t = 0$. Since, the more abrupt the waveform change is, the larger the high frequency components are, Fig. 2.9 has larger high frequency components. This is necessary not to produce the waveform in $0 \leq t < T$, but to produce the symmetric

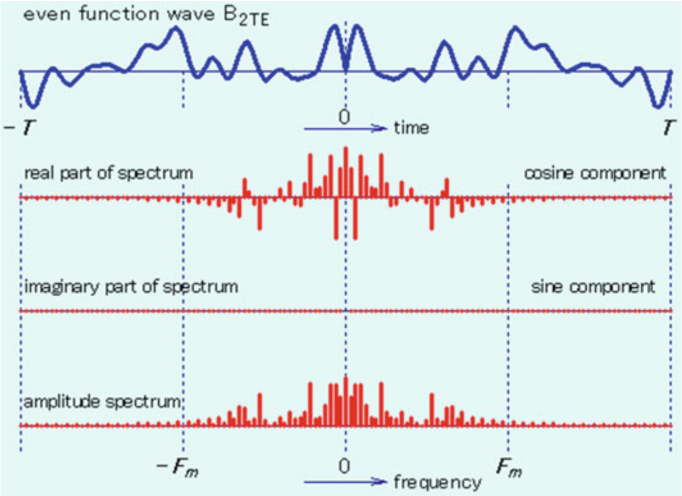

Fig. 2.9 Even waveform B_{2TE} with period $2T$ obtained by adding a symmetric waveform of B_T in the region $-T \le t < 0$ and its spectra. Animation available in supplementary files under filename E2-09_EvenF_B.exe

waveform in the negative time range. If more components are necessary, the benefit of expressing the waveform only by the cosine series is lost. What can be done to avoid this?

2.4 Expressing Waveforms by Odd Functions

The even function B_{2TE} with period $2T$ produced by adding the time-reversed waveform of B_T into the region $-T \le t < 0$ has more higher frequency components than B_T. Another way of connecting B_T in the region $-T \le t < 0$ is to reverse B_T in time and also to reverse its sign, resulting in an *odd function* as shown in Fig. 2.10.

Figure 2.10 clearly shows that the connection at $t = 0$ is now smooth and the spectral distribution is narrower than Fig. 2.9. Since the waveform is an odd function, its Fourier series contains only sine components. It became possible to express the series only by sine functions with lower levels of the high frequency components. The reason why the levels of the high frequency components are kept low is that the connected waveform has no abrupt change at $t = 0$ and at $\pm T$. This assures smooth connections at $t = 0$, and $\pm T$. On the other hand, if an odd waveform A_{2TO} is made from A_T, levels of the high frequency components are increased. This can be checked by running the program in the CD

Let's write down the Fourier coefficients for the case of odd functions. Since the waveform is made anti-symmetric, the coefficients are all sine waves. If a time-

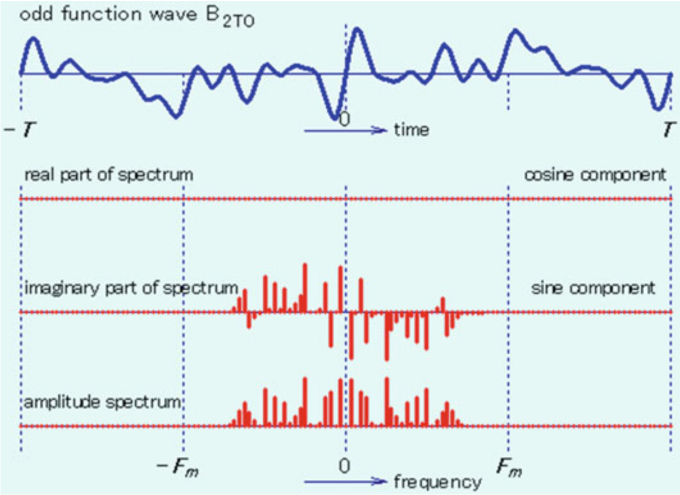

Fig. 2.10 Odd waveform B_{2TO} with period $2T$ produced by adding the time-reversed and sign-reversed waveform of B_T in the region $-T \leq t < 0$, and its spectra. Animation available in supplementary files under filename E2-10_OddF_B.exe

reversed and sign-reversed signal $-x(-t)$ of the original waveform $x(t)$ $(0 \leq t < T)$ is introduced into the time range $-T \leq t < 0$, an odd function with period $2T$ is produced, which will be referred to as $z(t)$. The Fourier coefficients of $z(t)$ is given by changing the region of integration as

$$B_k = \frac{1}{T}\int_{-T}^{T} z(t)\sin\left(2\pi\frac{k}{2T}t\right)dt.$$

Since $z(t)$ and the sine functions are both odd functions, their products are even functions, and the above integral can be obtained by doubling the integration in the region $0 \leq t < T$.

$$B_k = \frac{2}{T}\int_{0}^{T} x(t)\sin\left(2\pi\frac{k}{2T}t\right)dt \tag{2.22}$$

By the use of these coefficients, the waveform $x(t)$ $(0 \leq t < T)$ is given only by sine terms.

$$x(t) = B_1\sin\left(2\pi\frac{1}{2T}t\right) + B_2\sin\left(2\pi\frac{2}{2T}t\right) + B_3\sin\left(2\pi\frac{3}{2T}t\right) + \cdots \tag{2.23}$$

This is called the *sine Fourier series*.

In the Fourier series expansion of a portion of a waveform, both sine and cosine terms are necessary. In the discussion in Sects. 2.3 and 2.4, it was made clear that, depending on the way of combining the same waveform and making a new waveform with period $2T$, it can be expressed solely by sine or cosine terms. Which one to choose may depend on the property of the waveform as well as signal processing needed later. However, it is clear that the choice of methods is dependent on the waveform.

2.5 Expressing Waveforms by Complex Exponential Functions

In Sect. 1.7, it was shown that the sine and cosine waves can be replaced by a complex exponential function, of which the real part is the cosine function and the imaginary part is the sine function. A geometrical expression of the complex exponential function by a rotating vector on the complex plane led to the idea of phase "lead" or phase "delay", which corresponds to the positive or negative angle of the rotating vector at $t = 0$ measured counter-clockwise from the positive real axis. More benefits, such as the simpler expression of the series and convenient mathematical handling tools, are gained by the introduction of the complex exponential functions into the Fourier series expansion.

In order to introduce the Fourier series expansion expressed by complex exponential functions, some of the equations that have been shown before will be listed here.

$$a_0 = A_0 T = \int_{-T/2}^{T/2} x(t)dt \tag{2.24}$$

$$a_k = \frac{A_k}{2} T = \int_{-T/2}^{T/2} x(t)\cos\left(2\pi\frac{k}{T}t\right)dt \tag{2.25}$$

$$b_k = \frac{B_k}{2} T = \int_{-T/2}^{T/2} x(t)\sin\left(2\pi\frac{k}{T}t\right)dt \tag{2.26}$$

Using the above equations, Eq. (2.1) can be rewritten as Eq. (2.27)

$$\begin{aligned} x(t) =& \frac{1}{T}[a_0 + 2a_1\cos\left(2\pi\frac{1}{T}t\right) + 2a_2\cos\left(2\pi\frac{2}{T}t\right) + 2a_3\cos\left(2\pi\frac{3}{T}t\right) + \cdots \\ & + 2b_1\sin\left(2\pi\frac{1}{T}t\right) + 2b_2\sin\left(2\pi\frac{2}{T}t\right) + 2b_3\sin\left(2\pi\frac{3}{T}t\right) + \cdots] \\ =& \frac{1}{T}a_0 + \frac{2}{T}\sum_{k=1}^{\infty}\left[a_k\cos\left(2\pi\frac{k}{T}t\right) + b_k\sin\left(2\pi\frac{k}{T}t\right)\right]. \end{aligned} \tag{2.27}$$

The use of Euler's formula (Eq. 1.3) enables us to express the cosine and sine functions using complex exponential functions.

$$x(t) = \frac{1}{T}a_0 + \frac{2}{T}\sum_{k=1}^{\infty}\frac{1}{2}[(a_k - jb_k)\exp(j2\pi\frac{k}{T}t)$$
$$+ (a_k + jb_k)\exp(-j2\pi\frac{k}{T}t)]$$

By introducing new coefficients X_k, and defining $X_{-k} = X_k^*$, Eq. (2.28) can be derived.

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T}t\right) \tag{2.28}$$

Now let's get an expression for the coefficient X_k. Multiplying both sides of Eq. (2.28) by $\exp\{-j2\pi(m/T)\}$ and integrating from $-T/2$ to $T/2$:

$$\int_{-T/2}^{T/2} x(t)\exp\left(-j2\pi\frac{m}{T}t\right)dt = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k \int_{-T/2}^{T/2}\exp\left(j2\pi\frac{k}{T}t\right)\exp\left(-j2\pi\frac{m}{T}t\right)dt$$

The integration on the right-hand side becomes

$$\int_{-T/2}^{T/2}\exp\left(j2\pi\frac{m}{T}t\right)\exp\left(-j2\pi\frac{m}{T}t\right)dt = \int_{-T/2}^{T/2} dt = T$$

for $k = m$, and

$$\int_{-T/2}^{T/2}\exp\left(j2\pi\frac{k-m}{T}t\right)dt = 0$$

for $k \neq m$. Then Eq. (2.29) is obtained.

$$X_k = \int_{-T/2}^{T/2} x(t)\exp\left(-j2\pi\frac{k}{T}t\right)dt \tag{2.29}$$

Equation (2.28) is the Fourier series expressed by the use of complex exponential functions and Eq. (2.29) is the equation used to obtain the coefficients. The coefficient X_k is referred to as the "amplitude" of the complex wave componentexp$\{-j2\pi(k/T)\}$, in the same way that A_k and B_k are the amplitudes of cosine and sine waves, respectively. However, since X_k is complex, it is referred to as the *complex amplitude*". The situation may seem to be more complicated but it is not. There is the same number of coefficients A_k's for positive and negative k's and the real and imaginary parts are even and odd, respectively. Since $\exp\{-j2\pi(k/T)\}$

has a real (even) part and an imaginary (odd) part, the expression of the Fourier series becomes simpler than using sine and cosine functions. In following sections, describing digital processing carried out using computers, the complex exponential functions will be used most of the time.

Equation (2.28) is the equation that gives a waveform from the complex coefficients when the Fourier series is expressed by the complex exponential functions. On the other hand, Eq. (2.29) gives a complex amplitude of the coefficients from the waveform. These are the pairs of expressions that exist between the waveform and the complex amplitude of each harmonic. If a waveform is given, its complex coefficients are obtained by Eq. (2.29); if the complex coefficients are given, the waveform is recovered by Eq. (2.28).

The waveform $x(t)$ is the time-dependent function and the Fourier coefficient X_k is dependent on the order k of each frequency component. Therefore, X_k is considered a frequency-dependent function. In this sense, $x(t)$ and X_k are referred to as *time domain* and *frequency domain functions*, respectively. The terms X_k are referred to as *complex Fourier coefficients* or *complex spectra*.

Let's check the similarities and differences between Eqs. (2.28) and (2.29). Equation (2.28) gives a waveform of time domain function obtained by multiplying the Fourier coefficients, i.e., the complex spectra, by complex exponential functions whose exponents are purely imaginary with positive sign. Equation (2.29) gives Fourier coefficients (complex spectra) of the frequency domain function, which is obtained by multiplying the waveform in the time domain by the complex exponential functions, whose exponents are purely imaginary with negative sign. There is much similarity between the two equations except that the exponents have opposite signs and Eq. (2.28) is composed of summations and Eq. (2.29) is composed of integrations. The latter may seem to be a major difference. This is caused by the fact that the frequency spectra exist at discrete points on the frequency axis (because the waveform is periodic). In this case, the integration on the frequency domain becomes a summation (with multiplication) at its extremity. The opposite sign of the exponent will be discussed in the next section. Thus far, the region of integration has been from $-T/2$ to $T/2$. As has been discussed before, this is not a necessary condition. It can be from T_1 to T_2 as long as $T = T_2 - T_1$. Then Eq. (2.29) becomes

$$X_k = \int_{T_1}^{T_1+T} x(t) \exp\left(-j2\pi \frac{k}{T} t\right) dt. \tag{2.30}$$

The equation for obtaining the time function is still the same (Eq. 2.28). The difference is that the integration interval is from T_1 to T_2. The property that the cosine function is even and the sine function is odd is inherent in the property of X_k in that the real and imaginary parts are the even and odd functions of k.

The property of periodicity remains the same when Eq. (2.28) is applied to the region outside of $T_1 \leq t < T_2$. This can be checked by substituting t by $t + pT$ (p: integer).

$$x(t+pT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k \exp\left\{j2\pi\frac{k}{T}(t+pT)\right\}$$
$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T}t\right)\exp(j2\pi kp)$$

Since k and p are both integers,

$$\exp(j2\pi kp) = 1.$$

Therefore

$$x(t+pT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi\frac{k}{T}t\right) = x(t). \tag{2.31}$$

The time function $x(t + pT)$ repeats itself with period T.

2.6 Fourier Transform

We have discussed the method of expressing a waveform with length T by the sine and cosine functions with period T. The reader may have noticed that there is no restriction on the period T. Then, what happens if the period T is made infinite?

Let's make the region of integration from $-T$ to T by keeping the waveform unchanged in Eq. (2.29), that is, by using the same time period $-T/2 \leq t < T/2$ to extract the portion of the waveform, and assigning zeros to the regions $-T \leq t < -T/2$ and $T/2 \leq t < T$. Equation (2.29) will be rewritten using m instead of k,

$$U_m = U\left(\frac{m}{2T}\right) = \int_{-T}^{T} x(t)\exp(-j2\pi\frac{m}{2T}t)dt \tag{2.32}$$

The integration length is doubled to $2T$, and the fundamental frequency is halved to $1/2T$ but no other changes are made. The length of the extracted waveform is T but the spacing between the spectral lines is $1/2T$. Since zeros have been added to the regions $-T \leq t < -T/2$ and $T/2 \leq t < T$, the region of integration of Eq. (2.32) can be reduced to $-T/2 \leq t < T/2$.

$$U_m = U\left(\frac{m}{2T}\right) = \int_{-T/2}^{T/2} x(t)\exp(-j2\pi\frac{m}{2T}t)dt \tag{2.33}$$

Comparing Eqs. (2.29)–(2.32), the reader will find that they are equal if $m = 2\,k$. Since m and k are both integers, the value U_m is equal to X_k when m is even.

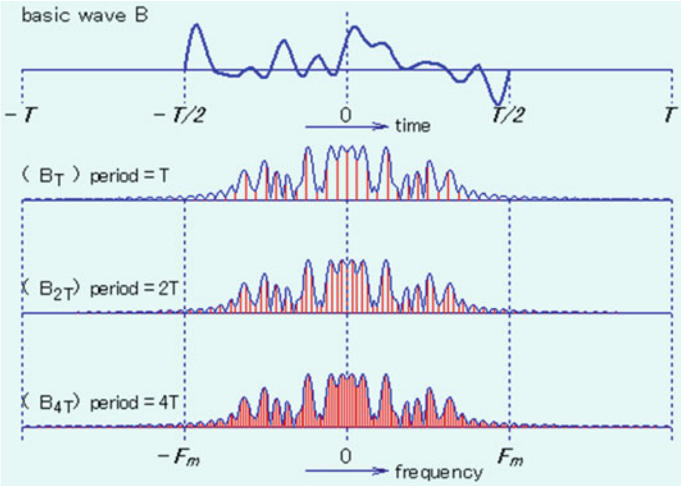

Fig. 2.11 Change in the power spectrum due to the change in the period for the computation of Fourier coefficients of waves B_T, B_{2T} and B_{4T}. Animation available in supplementary files under filename E2-11_VariP.exe

$$U_{2k} = X_k \tag{2.34}$$

What is described above is shown in Fig. 2.11. The top chart is the waveform B_T. The spectrum obtained with the integration T is shown by the vertical lines in the second chart (S_T). If zeros are added to the region $-T \leq t < -T/2$ and $T/2 \leq t < T$, and if the integration over the period $2T$ is carried out, the spectrum shown in the next chart (B_{2T}) is obtained. The shape of the spectrum distribution is not changed but the spacing between the spectral lines is halved.

In the case of the waveform that has zeros except for the region $-T/2 \leq t < T/2$, after integration over the region $-mT \leq t < mT$, the fundamental frequency becomes $1/(2mT)$. The following relation exists (m : integer).

$$U_{2mk} = X_k \tag{2.35}$$

The bottom chart in Fig. 2.11 shows the spectrum of B_{4T} for the case $m = 2$ (integration over period $-2T \leq t < 2T$). The spectral spacing is 1/2 of B_{2T} and 1/4 of B_T. The continuous thin line in each chart is the spectrum when m is made infinite, i.e., when the period of the periodic function is made infinite.

In Fig. 2.11, the same lengths of zeros are added to both sides of the extracted waveform B_T. It is also possible to add zeros to one side of the extracted waveform, but in this case, the origin of the time axis is changed and the equivalent phase shifts will be produced. However, the power spectrum is kept unchanged. This can be confirmed by running the program.

Now, let's consider the case when the extracted waveform is unchanged and the integration range is made infinite (i.e., the integration region is $-\infty \leq t < \infty$). In this case the spectral spacing becomes infinitely small and the distribution of Fourier coefficients becomes a continuous function of frequency. Then, Eq. (2.29) becomes:

$$X(f) = \lim_{m \to \infty} X\left(\frac{k}{2mT}\right) = \lim_{m \to \infty} \int_{-mT}^{mT} x(t) \exp\left(-j2\pi \frac{k}{2mT} t\right) dt \tag{2.36}$$

If we use $f = \lim_{m \to \infty} \{k/2mT\}$, the frequency f can take continuous values, and the integration region becomes $-\infty \leq t < \infty$.

$$X(f) = \int_{-\infty}^{+\infty} x(t) \exp(-j2\pi ft) dt \tag{2.37}$$

This is the equation known as the *Fourier transform* using complex exponential functions . The Fourier coefficients X_k of the periodic function are line spectra, but $X(f)$ defined by Eq. (2.37) is a continuous function of frequency. The continuous curve shown in each of the spectral charts in Fig. 2.11 is $X(f)$ obtained by letting m be infinite. As shown by Fig. 2.11, the envelope of the line spectra of a periodic waveform is the continuous spectrum of the waveform that is made from only one extracted waveform in the infinite time domain. The k-th Fourier coefficient of the periodic waveform is equal to the value of the continuous spectrum at $f = k/T$. These will be understood by considering the process starting from Eqs. (2.32) and (2.33) and reaching Eq. (2.37), which is obtained by letting the integration region be $-\infty \leq t < \infty$.

Equation (2.27) is the Fourier transform that calculates the spectrum $X(f)$ from a waveform $x(t)$ which is a function of time. It is necessary to have an inverse Fourier transform as the counterpart of the Fourier transform. This will be derived by replacing k/T in Eq. (2.28) by f and replacing the summation by the integration. Since $1/T$ in Eq. (2.28) is an inverse of time, it has the dimension of frequency, and as time T becomes infinitely large, its inverse $1/T$ should be represented as df. Then, Eq. (2.28) is rewritten as

$$x(t) = \int_{-\infty}^{+\infty} X(f) \exp(j2\pi ft) df \tag{2.38}$$

Equations (2.37) and (2.38) are the Fourier transform and inverse Fourier transform using complex exponential functions, respectively, and is known as the *Fourier transform pair*.

Let's check what happens when the waveform $x(t)$ is delayed by τ, which is represented by $x(t-\tau)$. Replacing $x(t)$ in Eq. (2.37) by $x(t-\tau)$:

$$X'(f) = \int_{-\infty}^{+\infty} x(t-\tau)\exp(-j2\pi ft)dt$$

An introduction of a new parameter $u = (t-\tau)$, since $t = u + \tau$ and $dt = du$, makes it possible to replace the parameter t by $u + \tau$.

$$\begin{aligned} X'(f) &= \int_{-\infty}^{+\infty} x(u)\exp(-j2\pi fu)\exp(-j2\pi f\tau)du \\ &= \int_{-\infty}^{+\infty} x(u)\exp(-j2\pi fu)du \cdot \exp(-j2\pi f\tau) = X(f)\exp(-j2\pi f\tau) \end{aligned} \tag{2.39}$$

The spectrum of the waveform with time delay τ is given by the product of the original Fourier spectrum $X(f)$ andexp$(-j2\pi f\tau)$.

Conversely, the inverse Fourier transform of the product of $X(f)$ andexp$(-j2\pi ft)$ gives the original waveform but with time delay τ. This is shown as follows. The substitution of $X(f)$ by $X(f)\exp(-j2\pi f\tau)$ gives

$$\begin{aligned} x'(t) &= \int_{-\infty}^{+\infty} X(f)\exp(-j2\pi f\tau)\exp(j2\pi ft)df \\ &= \int_{-\infty}^{+\infty} X(f)\exp\{j2\pi f(t-\tau)\}df = x(t-\tau). \end{aligned} \tag{2.40}$$

In preparation for later digital processing, let's check the Fourier transform of the unit impulse (Dirac's delta function in the time domain)$\delta(t)$. The unit impulse satisfies

$$\int_{-\infty}^{+\infty} \delta(t)dt = 1.$$

The spectrum of the unit impulse is given by

$$X(f) = \int_{-\infty}^{+\infty} \delta(t)\exp(-j2\pi ft)dt.$$

Since $\delta(t) = 0$ for $t \neq 0$ and, $\exp(-j2\pi ft) = 1$ for $t = 0$,

$$X(f) = \int_{-\infty}^{+\infty} \delta(t)\exp(-j2\pi ft)|_{t=0}dt = \int_{-\infty}^{\infty} \delta(t)dt = 1. \tag{2.41}$$

The Fourier transform of the unit impulse is equal to 1 at all frequencies.

The Fourier transform (FT) of the impulse $\delta(t-\tau)$ that exists at $t = \tau$, is obtained directly from Eq. (2.37). By letting $x(t) = \delta(t-\tau)$, we have

$$\mathrm{FT}\{\delta(t-\tau)\} = \int_{-\infty}^{+\infty} \delta(t-\tau)\exp(-j2\pi ft)dt$$

Since the integrand is nonzero at $t = \tau$, the above equation is rewritten as

$$\begin{aligned}\mathrm{FT}\{\delta(t-\tau)\} &= \int_{-\infty}^{+\infty} \delta(t-\tau)\exp(-j2\pi f\tau)dt \\ &= \exp(-j2\pi f\tau)\int_{-\infty}^{\infty} \delta(t-\tau)dt \\ &= \exp(-j2\pi f\tau)\end{aligned} \tag{2.42}$$

Equation (2.42) shows that the spectrum of an impulse that exists at $t = \tau$ is given by $\exp(-j2\pi f\tau)$. This means that the absolute value of the spectrum of an impulse is 1 for all frequencies and the phase delay at frequency f is equal to the product of the time delay and$2\pi f$. The statement above may seem superfluous since the same result has been already shown in Fig. 1.16, and is obvious from Eqs. (2.39) and (2.42). However, this result will play an important role in Chap. 4.

Let's introduce another important theorem of the Fourier transform: the energy of the time function $x(t)$, is given by the integral of the absolute Fourier transform squared over the range $-\infty \le f < \infty$.

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt = \int_{-\infty}^{+\infty} x(t)\left[\int_{-\infty}^{+\infty} X^*(f)\exp(-j2\pi ft)df\right]dt$$

where $x^*(t)$ is the complex conjugate of $x(t)$ (the imaginary part of it would have the opposite sign of $x(t)$). By changing the order of integration,

$$\begin{aligned}\int_{-\infty}^{+\infty} |x(t)|^2 dt &= \int_{-\infty}^{+\infty} X^*(f)\left[\int_{-\infty}^{+\infty} x(t)\exp(-j2\pi ft)dt\right]df \\ &= \int_{-\infty}^{+\infty} X^*(f)X(f)df\end{aligned}$$

which gives,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df \tag{2.43}$$

Equation (2.43) affirms that "the energy of the Fourier spectrum $X(f)$ obtained from the Fourier transform of $x(t)$ has the same energy as $x(t)$." This is known as *Parseval's formula*.

The same idea is applicable to the Fourier series expansion. Let's calculate the energy in one period of $x(t)$ using Eqs. (2.28) and (2.29).

$$\int_{-T/2}^{+T/2} |x(t)|^2 dt = \int_{-T/2}^{+T/2} x(t)x^*(t)dt = \int_{-T/2}^{+T/2} x(t)\left[\frac{1}{T}\sum_{k=-\infty}^{\infty} X_k^* \exp(-j2\pi\frac{k}{T}t)\right]dt$$

Reversing the order of integration,

$$\begin{aligned}\int_{-T/2}^{+T/2} |x(t)|^2 d &= \sum_{k=-\infty}^{\infty} X_k^*\left[\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\exp(-j2\pi\frac{k}{T}t)dt\right] \\ &= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_k^* X_k\end{aligned}.$$

Then, the following equation is obtained.

$$\int_{-T/2}^{+T/2} |x(t)|^2 dt = \frac{1}{T}\sum_{k=-\infty}^{\infty} |X_k|^2 = F\sum_{k=-\infty}^{\infty} |X_k|^2 \tag{2.44}$$

where $F = 1/T$ is the spacing of the Fourier spectrum. This is Parseval's relation in the case of a Fourier series expansion. The energy of one period of a periodic function is equal to the summation of squares of Fourier coefficients multiplied by the frequency spacing.

Appendixes 2A–C are supplements of Chap. 2. They will be useful for a better understanding of the following chapters.

2.7 Gibbs' Phenomenon

By the end of the previous section, it was shown that any periodic waveform can be expanded by a Fourier series, which requires an infinite number of harmonics. As can be seen in Fig. 2.12, the Fourier series expansions of a rectangular waveform up to the 5-th, 10-th, and 50-th harmonic gradually approach the rectangle, but overshoots are observed near the edges of the waveform. The height of the overshoot seems to approach a fixed value as the number of harmonics increases. This phenomenon was named *Gibbs Phenomenon* after the discoverer, Gibbs. Let's check how the height of the Gibbs Phenomenon is determined.

The Fourier series expansion of a periodic rectangular waveform $x(t)$ with the period T, $x(t) = -1$ for $-T/2 < t < 0$, and $x(t) = 1$ for $0 < t < T/2$, is given by

$$x(t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\sin\left(2\pi\frac{2k+1}{T}t\right). \tag{2.45}$$

The partial sum up to the $(K-1)$-th harmonic is given by

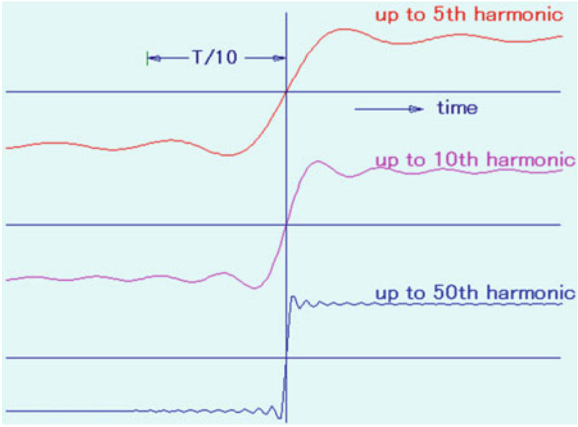

Fig. 2.12 Waveform of step wave synthesized by finite number of harmonics. Animation available in supplementary files under filename E2-12_GIBBS.exe

$$\begin{aligned} x_K(t) &= \frac{4}{\pi T}\sum_{k=0}^{K-1}\frac{T}{2k+1}\sin\left(2\pi\frac{2k+1}{T}t\right) \\ &= \frac{8}{T}\sum_{k=0}^{K-1}\int_0^t \cos\left(2\pi\frac{2k+1}{T}u\right)du. \end{aligned} \tag{2.46}$$

By exchanging the order of the sum and the integration, the following is obtained.

$$x_K(t) = \frac{8}{T}\int_0^t \sum_{k=0}^{K-1}\cos\left(2\pi\frac{2k+1}{T}u\right)du \tag{2.47}$$

Let the integrand be

$$S_K = \sum_{k=0}^{K-1}\cos\left(2\pi\frac{2k+1}{T}u\right). \tag{2.48}$$

The multiplication of S_k with $\sin(2\pi u/T)$ becomes

$$\begin{aligned} S_K\sin\left(2\pi\frac{u}{T}\right) &= \sum_{k=0}^{K-1}\left[\sin\left(2\pi\frac{u}{T}\right)\cos\left(2\pi\frac{2k+1}{T}u\right)\right] \\ &= \sin\left(2\pi\frac{u}{T}\right)\cos\left(2\pi\frac{u}{T}\right)+\sin\left(2\pi\frac{u}{T}\right)\cos\left(2\pi\frac{3u}{T}\right)+\cdots \\ &\cdots+\sin\left(2\pi\frac{u}{T}\right)\cos\left(2\pi\frac{2K-1}{T}u\right). \end{aligned}$$

Since

$$\sin\left(2\pi\frac{u}{T}\right)\cos\left(2\pi\frac{ru}{T}\right)=\frac{1}{2}\left[\sin\left(2\pi\frac{r+1}{T}u\right)-\sin\left(2\pi\frac{r-1}{T}u\right)\right]$$

it is rewritten as

$$\begin{aligned}2S_K\sin\left(2\pi\frac{u}{T}\right)&=\sin\left(2\pi\frac{2}{T}u\right)+\sin\left(2\pi\frac{4}{T}u\right)-\sin\left(2\pi\frac{u}{T}\right)+\cdots\\&\quad\cdots+\sin\left(2\pi\frac{2K}{T}u\right)-\sin\left(2\pi\frac{2K-2}{T}u\right)\\&=\sin\left(2\pi\frac{2K}{T}u\right).\end{aligned}$$

Therefore

$$S_K=\frac{1}{2}\frac{\sin\left(2\pi\frac{2K}{T}u\right)}{\sin\left(2\pi\frac{1}{T}u\right)}. \tag{2.49}$$

Then, Eq. (2.47) becomes

$$x_K(t)=\frac{4}{T}\int_0^t\frac{\sin\left(2\pi\frac{2K}{T}u\right)}{\sin\left(2\pi\frac{1}{T}u\right)}du \tag{2.50}$$

The function $x_K(t)$. takes maxima or minima at times when its time-derivative equals 0.

$$\frac{dx_K(t)}{dt}=\frac{4}{T}\frac{\sin\left(2\pi\frac{2K}{T}t\right)}{\sin\left(2\pi\frac{1}{T}t\right)}=0 \tag{2.51}$$

Those are given by

$$t=\frac{mT}{4K}\ (m=1,2,3,\cdots,K). \tag{2.52}$$

The function $x_K(t)$ takes maxima or minima when m is odd or even, respectively. The largest of the maxima is given when m is 1.

$$x_K\left(\frac{T}{4K}\right)=\frac{4}{T}\int_0^{\frac{T}{4K}}\frac{\sin\left(2\pi\frac{2K}{T}u\right)}{\sin\left(2\pi\frac{1}{T}u\right)}du$$

By the variable transformation

$$v = 2\pi \frac{2K}{T} u.$$

$x_K(t)$ is given by

$$x_K(\frac{T}{4K}) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin(v)}{v} \frac{\frac{v}{2K}}{\sin\left(\frac{v}{2K}\right)} dv. \tag{2.53}$$

The limit given when K approaches infinity is

$$\lim_{K \to \infty} x_K\left(\frac{T}{2K}\right) = \frac{2}{\pi} \int_0^{\pi} \frac{\sin v}{v} dv \cong 1.17898.$$

The overshoot is approximately 18 %.

The time width of the overshoot becomes infinitely small as K approaches infinity, and therefore the power of the overshoot becomes zero. Similar phenomena are observed in functions other than rectangular, which take different values when t approaches a discontinuity from the negative and the positive directions. These are also referred to as *Gibbs' Phenomena*.

2.8 Exercises

1. What is the orthogonal property of sine and cosine functions?
2. Derive Eq. (2.8) used to obtain Fourier coefficients B_k.
3. If a function $x(t)$ defined in the region $0 \le t < T$ is represented by a summation of an even function $x_e(t)$ and an odd function $x_o(t)$, both defined in the region $-T \le t < T$, derive equations for $x_e(t)$ and $x_o(t)$.
4. Describe the properties of $x_e(t)$ and $x_o(t)$ defined in Problem 3.
5. When the Fourier expansion is applied to a function $x(t)$ defined in the region $0 \le t < T$, what are the frequencies of the individual frequency components?
6. What is the complex exponential function?
7. What is the relationship between the even and odd functions in $x(t)$ and the real and imaginary parts of its complex Fourier transform $X(f)$?
8. Express a cosine wave with the phase lead of 45° using a complex exponential function.
9. Express a cosine wave with the phase lead of θ using a complex exponential function.
10. A function $x(t)$ defined in the region $-\infty < t < \infty$ as zero everywhere except in the region $0 \le t < T$ has Fourier transform $X(f)$. When a Fourier series expansion is applied to this same function in the region $0 \le t < T$, what would you expect to get as the Fourier series?
11. When a Fourier series expansion is applied to $x(t)$ in Problem 10 over the region $-nT \le t < nT$, what would you expect to get as the Fourier series?

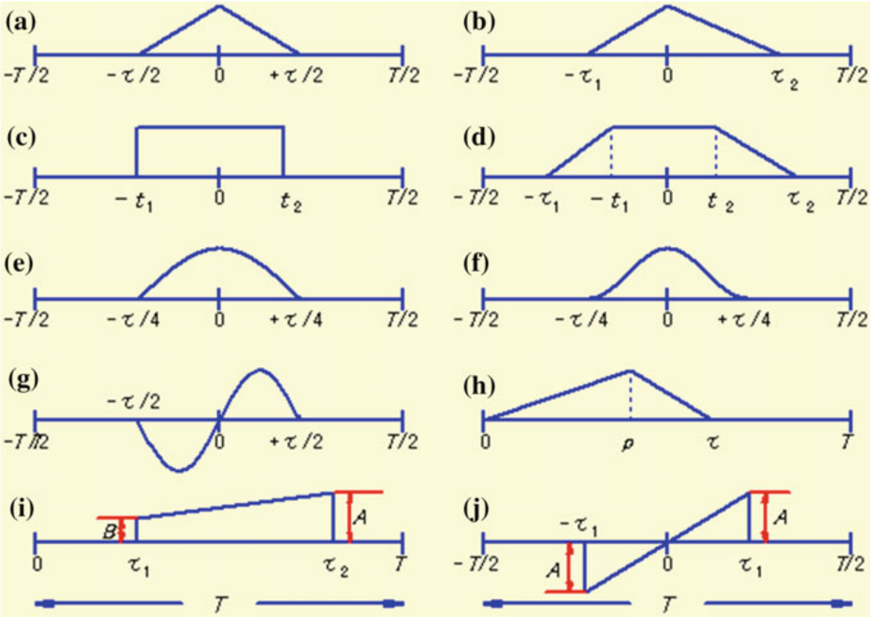

Fig. 2.13 Waveforms with period T

Give answers for $n = 1$, $n =$ arbitrary integer, and $n \cong$ an infinitely large integer.

12. Obtain Fourier coefficients of the functions (a)–(j) in Fig. 2.13. Assume the same peak value A for each waveform and the starting value in (i) be B.

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