

Chapter 2

DEA Cross Efficiency

Wade D. Cook and Joe Zhu

Abstract Data envelopment analysis (DEA) provides a relative efficiency measure for peer decision making units (DMUs) with multiple inputs and outputs. While DEA has been proven an effective approach in identifying the best practice frontiers, its flexibility in weighting multiple inputs and outputs and its nature of self-evaluation have been criticized. The cross efficiency method was developed as a DEA extension to rank DMUs with the main idea being to use DEA to do peer evaluation, rather than in pure self-evaluation mode. However, cross efficiency scores obtained from the original DEA model are generally not unique, and depend on which of the alternate optimal solutions to the DEA linear programs is used. The current chapter discusses various cross efficiency approaches in dealing with non-unique solutions from DEA

Keywords Data Envelopment Analysis (DEA) · Cross efficiency · Multiplicative · Cobb-Douglas

2.1 Introduction

While DEA has been proven an effective approach in identifying best practice frontiers, its flexibility in weighting multiple inputs and outputs and its nature of self-evaluation have been criticized. The cross efficiency method was developed as a DEA extension to rank DMUs (Sexton et al. 1986), with the main idea being to use DEA to do peerevaluation, rather than to have it operate in a pure self-evaluation mode. Cross efficiency has been further investigated by Doyle and Green (1994). There are mainly two advantages of the cross-evaluation method. It provides an ordering among DMUs, and it eliminates unrealistic weight schemes without requiring the elicitation of weight restrictions from application area experts (e.g., Anderson et al. 2002).

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Cross efficiency evaluation has been used in various applications, e.g., efficiency evaluations of nursing homes (Sexton et al. 1986), R&D project selection (Oral et al. 1991), preference voting (Green et al. 1996), and others. However, as noted in Doyle and Green (1994), the non-uniqueness of the DEA optimal weights/multipliers possibly reduces the usefulness of cross efficiency. Specifically, cross efficiency scores obtained from the original DEA methodology are generally not unique. Thus, depending on which of the alternate optimal solutions to the DEA linear programs is used, it may be possible to improve a DMU's (cross efficiency) performance rating, but generally only by worsening the ratings of others. With that in mind, Sexton et al. (1986) and Doyle and Green (1994) propose the use of a secondary goal to deal with the non-unique DEA solutions. They developed aggressive (benevolent) model formulations to identify optimal weights that not only maximize the efficiency of a particular DMU under evaluation, but also minimize (maximize) the average efficiency of other DMUs

In the current chapter, we present the standard DEA cross efficiency method, and discuss several approaches that have been developed to address the non-uniqueness issue discussed above. These approaches include the game cross efficiency methodology of Liang et al. (2008a) and the maximum cross efficiency concept (Cook and Zhu 2014) based upon a set of log-linear DEA models.

2.2 Cross Efficiency

Suppose we have a set of n DMUs and each DMU_j have s different outputs and m different inputs. We denote the i th input and r th output of DMU_j ($j = 1, 2, \dots, n$) as x_{ij} ($i = 1, \dots, m$) and y_{rj} ($r = 1, \dots, s$), respectively. Cross efficiency is generally presented as a two-phase process. Specifically, phase 1 is the self-evaluation phase where DEA scores are calculated using the constant returns-to-scale (CRS) DEA model of Charnes et al. (1978). In the second phase, the multipliers arising from phase 1 are applied to all peer DMUs to arrive at the so-called cross evaluation score for each of those DMUs.

Phase 1: Suppose DMU_d is under evaluation by the CRS model (Charnes et al. 1978). Then that DMU's (self-evaluation) efficiency score is determined by the following DEA model

$$\begin{aligned}
 \text{Max} \quad E_{dd} &= \frac{\sum_{r=1}^s u_{rd} y_{rd}}{\sum_{i=1}^m v_{id} x_{id}} \\
 \text{s.t.} \quad E_{dj} &= \frac{\sum_{r=1}^s u_{rd} y_{rj}}{\sum_{i=1}^m v_{id} x_{ij}} \leq 1, j = 1, 2, \dots, n. \\
 u_{rd} &\geq 0, r = 1, \dots, s. \\
 v_{id} &\geq 0, i = 1, \dots, m.
 \end{aligned} \tag{2.1}$$

where v_{id} and u_{rd} represent i th input and r th output weights for DMU_d .

Phase 2: The cross efficiency of DMU_j , using the weights that DMU_d has chosen in model (2.1), is given by

$$E_{dj} = \frac{\sum_{r=1}^s u_{rd}^* y_{rj}}{\sum_{i=1}^m v_{id}^* x_{ij}}, d, j = 1, 2, \dots, n \quad (2.2)$$

where (*) denotes optimal values in model (2.1). For $DMU_j (j = 1, 2, \dots, n)$, an average of all $E_{dj} (d = 1, 2, \dots, n)$,

$$\bar{E}_j = \frac{1}{n} \sum_{d=1}^n E_{dj}, \quad (2.3)$$

is referred to as the *cross efficiency score for DMU_j* .

We should point out that each individual E_{dj} is called cross efficiency and the average defined in (2.3) is also called cross efficiency in the DEA literature. In general, “cross efficiency” refers to the average defined in (2.3), not the individual scores defined in (2.2).

While the DEA model (2.1) is a non-linear model, model (2.1) is usually solved in its equivalent multiplier model,

$$\max \quad E_{dd} = \sum_{r=1}^s u_{rd} y_{rd}$$

Subject to

$$\begin{aligned} \sum_{i=1}^m v_{id} x_{id} &= 1 \\ \sum_{r=1}^s u_{rd} y_{rj} - \sum_{i=1}^m v_{id} x_{ij} &\leq 0 \quad j = 1, \dots, n \\ u_{rd}, v_{id} &\geq 0 \end{aligned} \quad (2.4)$$

Due to the fact that the above cross efficiency is based upon input-oriented models, cross efficiency scores are not greater than one.

We here briefly illustrate the concept of cross efficiency by adopting the cross efficiency matrix from Doyle and Green (1994). In Fig. 2.1, we have six DMUs. E_{dj} is the (cross) efficiency of DMU_j based upon a set of DEA weights calculated for DMU_d . This set of DMU weights gives the best efficiency score for DMU_d under evaluation by a DEA model, and E_{dd} (in the leading diagonal) is the DEA efficiency for DMU_k . The cross efficiency for a given DMU_j is defined as the arithmetic average down column j , given by \bar{E}_j . (We point out that in Doyle and Green (1994), the efficiency score for DMU k is not included as part of the average.)

Obviously, $E_{dj} (d \neq j)$ and \bar{E}_j are not unique due to the often-present multiple optimal DEA weights in model (2.4), for example. As a result of this non uniqueness, the cross efficiency concept has been criticized as unreliable.

Rating DMU	Rated DMU						Averaged appraisal of peers
	1	2	3	4	5	6	
1	E₁₁	E ₁₂	E ₁₃	E ₁₄	E ₁₅	E ₁₆	A ₁
2	E ₂₁	E₂₂	E ₂₃	E ₂₄	E ₂₅	E ₂₆	A ₂
3	E ₃₁	E ₃₂	E₃₃	E ₃₄	E ₃₅	E ₃₆	A ₃
4	E ₄₁	E ₄₂	E ₄₃	E₄₄	E ₄₅	E ₄₆	A ₄
5	E ₅₁	E ₅₂	E ₅₃	E ₅₄	E₅₅	E ₅₆	A ₅
6	E ₆₁	E ₆₂	E ₆₃	E ₆₄	E ₆₅	E₆₆	A ₆
	\bar{E}_1	\bar{E}_2	\bar{E}_3	\bar{E}_4	\bar{E}_5	\bar{E}_6	

Averaged appraisal by peers (peer appraisal)

Fig. 2.1 Cross efficiency matrix. (Doyle and Green 1994)

Note that the above discussion is based upon input-orientation. Similarly, we can use output-oriented models to calculate cross efficiency. In this case, E_{dj} in (2.2) becomes

$$E_{dj} = \frac{\sum_{i=1}^m v_{id}^* x_{ij}}{\sum_{r=1}^s u_{rd}^* y_{rj}} \quad (2.5)$$

where v_{id}^* and u_{rd}^* are optimal values in the following output-oriented model when DMU_d is under evaluation

$$\begin{aligned}
 \min \quad & E_{dd} = \frac{\sum_{i=1}^m v_{id} x_{id}}{\sum_{r=1}^s u_{rd} y_{rd}} \\
 s.t. \quad & E_{dj} = \frac{\sum_{i=1}^m v_{id} x_{ij}}{\sum_{r=1}^s u_{rd} y_{rj}} \geq 1, j = 1, 2, \dots, n \\
 & v_{id} \geq 0, i = 1, \dots, m \\
 & u_{rd} \geq 0, r = 1, \dots, s.
 \end{aligned} \quad (2.6)$$

The above model (2.6) is equivalent to the following output-oriented CRS multiplier model:

$$\text{Min} \quad \sum_{r=1}^s v_{id} x_{id}$$

subject to

$$\begin{aligned}
 & \sum_{i=1}^m v_{id} x_{ij} - \sum_{r=1}^s u_{rd} y_{rj} \geq 0, j = 1, 2, \dots, n \\
 & \sum_{i=1}^m u_{rd} y_{rd} = 1 \\
 & v_{id} \geq 0, i = 1, \dots, m \\
 & u_{rd} \geq 0, r = 1, \dots, s
 \end{aligned} \quad (2.7)$$

Table 2.1 Numerical example

DMU	Input 1	Input 2	Input 3	Output 1	Output 2
1	7	7	7	4	4
2	5	9	7	7	7
3	4	6	5	5	7
4	5	9	8	6	2
5	6	8	5	3	6

Table 2.2 Input-oriented CRS efficiency and optimal multipliers

DMU	CRS efficiency	x1	x2	x3	y1	y2
1	0.68571	0.00000	0.14286	0.00000	0.17143	0.00000
2	1.00000	0.07143	0.07143	0.00000	0.14286	0.00000
3	1.00000	0.00000	0.16667	0.00000	0.00000	0.14286
4	0.85714	0.07143	0.07143	0.00000	0.14286	0.00000
5	0.85714	0.00000	0.00000	0.20000	0.00000	0.14286

Note that under the output-oriented case, all cross efficiency scores are not less than one, as the output-oriented CRS efficiency score is not less than one. Then, the output-oriented DEA cross efficiency score can be defined in a similar manner as in (4.3).

Finally, the above discussion is based upon CRS. Similar developments can be obtained under non-CRS situations. We, however, point out that negative cross efficiency scores can be obtained under non-CRS conditions, for example, variable returns-to-scale (VRS) (see Lim and Zhu ([in press](#)) or Chap. 3).

2.3 Numerical Example

Throughout the chapter, we use the numerical example shown in Table 2.1 to illustrate various cross efficiency approaches. This example is from Liang et al. ([2008a](#)) and has five DMUs with three inputs and two outputs.

Table 2.2 reports the CRS efficiency scores obtained from model (2.4) along with a set of optimal multipliers. Based upon this set of multipliers, an input-oriented cross efficiency matrix is provided in Table 2.3. Tables 2.4 and 2.5 report cross efficiency results based upon the output-oriented model (2.7). As we can see, unlike the standard CRS efficiency scores, the input-oriented cross efficiency score is not (always) the reciprocal of the associated output-oriented cross efficiency score.

Table 2.3 Input-oriented standard cross efficiency matrix

	Cross efficiency matrix				
Rating DMU	DMU1	DMU2	DMU3	DMU4	DMU5
1	0.68571	0.93333	1.00000	0.80000	0.45000
2	0.57143	1.00000	1.00000	0.85714	0.42857
3	0.48980	0.66667	1.00000	0.19048	0.64286
4	0.57143	1.00000	1.00000	0.85714	0.42857
5	0.40816	0.71429	1.00000	0.17857	0.85714
Cross efficiency	0.54531	0.86286	1.00000	0.57667	0.56143

Table 2.4 Output-oriented CRS efficiency and optimal multipliers

DMU	CRS efficiency	x1	x2	x3	y1	y2
1	1.45833	0.00000	0.20833	0.00000	0.25000	0.00000
2	1.00000	0.07143	0.07143	0.00000	0.14286	0.00000
3	1.00000	0.00000	0.16667	0.00000	0.00000	0.14286
4	1.16667	0.08333	0.08333	0.00000	0.16667	0.00000
5	1.16667	0.00000	0.00000	0.23333	0.00000	0.16667

Table 2.5 Output-oriented standard cross efficiency matrix

	Cross efficiency matrix				
Rating DMU	DMU1	DMU2	DMU3	DMU4	DMU5
1	1.45833	1.07143	1.00000	1.25000	2.22222
2	1.75000	1.00000	1.00000	1.16667	2.33333
3	2.04167	1.50000	1.00000	5.25000	1.55556
4	1.75000	1.00000	1.00000	1.16667	2.33333
5	2.45000	1.40000	1.00000	5.60000	1.16667
Cross efficiency	1.89000	1.19429	1.00000	2.88667	1.92222

2.4 Maximum Log Cross Efficiency

To address the non-uniqueness in cross efficiency, the idea of secondary goals was introduced, with the original proposal being to maximize or minimize the average appraisal of peers as indicated by A_k in Fig. 2.1. Specifically, A_k is the arithmetic average across the row k . However, due to the DEA model (CCR multiplier model) used, $E_{kj} = \frac{\sum_r \mu_{rk} y_{rj}}{\sum_i v_{ik} x_{ij}}$, where y_{rj} , ($r = 1, 2, \dots, s$) are outputs and x_{ij} , ($i = 1, 2, \dots, m$) are inputs for DMU_j , and μ_{rk} , v_{ik} are corresponding output and input weights chosen by DMU_k .

Thus, $A_k = \frac{1}{n} \sum_j E_{kj}$ appears in the form of a non-linear fractional problem that cannot be converted into linear format. To remedy this, Sexton et al. (1986), and Doyle and Green (1994) suggested the use of linear surrogates for the secondary goal in form of the numerators in E_{kj} minus the sum of the denominators, and modified ratios that can be converted into linear relations. However, due to the fact that these surrogates are not equivalent to the optimal values of A_k , the resulting cross efficiency scores are, at best, approximations of these optimal values.

While such approaches as those of Doyle and Green (1994), and those suggested by others (e.g., Liang et al. 2008b), help to reduce the variability of DEA optimal weights, these approaches all produce cross efficiency scores that differ from one another. Cook and Zhu (2014), on the other hand, propose to use multiplicative DEA models developed in Charnes et al. (1982) and Charnes et al. (1983) to obtain maximum (*and unique*) cross efficiency scores under the condition that each DMU's DEA efficiency score remains unchanged. To introduce the Cook and Zhu (2014) approach, we need first to present the multiplicative DEA models.

2.5 Multiplicative DEA Model

Charnes et al. (1982) introduce the following multiplicative DEA model when DMU_o is under evaluation

$$\begin{aligned} \max \quad & \frac{\prod_{r=1}^s y_{ro}^{\mu_r}}{\prod_{i=1}^m x_{io}^{v_i}} \\ \text{s.t.} \quad & \frac{\prod_{r=1}^s y_{rj}^{\mu_r}}{\prod_{i=1}^m x_{ij}^{v_i}} \leq 1, \quad j = 1, \dots, n \\ & \mu_r, v_i \geq 1 \end{aligned} \tag{2.8}$$

Taking logarithms (to any base), model (2.8) becomes

$$\max \sum_{r=1}^s \mu_r \hat{y}_{ro} - \sum_{i=1}^m v_i \hat{x}_{io}$$

subject to

$$\begin{aligned} \sum_{r=1}^s \mu_r \hat{y}_{rj} - \sum_{i=1}^m v_i \hat{x}_{ij} &\leq 0 \\ \mu_r, v_i &\geq 1 \end{aligned} \tag{2.9}$$

where (\wedge) denotes logarithms.

The dual to model (2.9) can be written as

$$\max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+$$

subject to

$$\begin{aligned} \sum_{j=1}^n \lambda_j \hat{x}_{ij} + s_i^- &= \hat{x}_{io} \quad i = 1, 2, \dots, m; \\ \sum_{j=1}^n \lambda_j \hat{y}_{rj} - s_r^+ &= \hat{y}_{ro} \quad r = 1, 2, \dots, s; \\ \lambda_j, s_i^-, s_r^+ &\geq 0 \end{aligned} \quad (2.10)$$

It can be seen that model (2.10) is actually the CRS additive model. We therefore call model (2.8) (and its equivalents) the CRS multiplicative DEA model.

Charnes et al. (1983) introduce the following multiplicative DEA model when DMU_o is under evaluation

$$\begin{aligned} \max \quad & \frac{e^\eta \prod_{r=1}^s y_{rk}^{\mu_r}}{e^\xi \prod_{i=1}^m x_{ik}^{v_i}} \\ \text{s.t.} \quad & \frac{e^\eta \prod_{r=1}^s y_{rj}^{\mu_r}}{e^\xi \prod_{i=1}^m x_{ij}^{v_i}} \leq 1, \quad j = 1, \dots, n \\ & \eta, \xi \geq 0, \mu_r, v_i \geq 1 \end{aligned} \quad (2.11)$$

Taking logarithms (to any base), model (2.11) becomes

$$\max \quad \eta - \xi + \sum_{r=1}^s \mu_r \hat{y}_{ro} - \sum_{i=1}^m v_i \hat{x}_{io}$$

subject to

$$\begin{aligned} \eta - \xi + \sum_{r=1}^s \mu_r \hat{y}_{rj} - \sum_{i=1}^m v_i \hat{x}_{ij} &\leq 0 \\ \eta, \xi &\geq 0 \\ \mu_r, v_i &\geq 1 \end{aligned} \quad (2.12)$$

where (\wedge) denotes logarithms.

The dual to model (2.12) is

$$\max \sum_{i=1}^m s_i^- + \sum_{r=1}^s s_r^+$$

subject to

$$\sum_{j=1}^n \lambda_j \hat{x}_{ij} + s_i^- = \hat{x}_{io} \quad i = 1, 2, \dots, m; \quad (2.13)$$

$$\sum_{j=1}^n \lambda_j \hat{y}_{rj} - s_r^+ = \hat{y}_{ro} \quad r = 1, 2, \dots, s;$$

$$\sum_{j=1}^n \lambda_j = 1$$

$$\lambda_j, s_i^-, s_r^+ \geq 0$$

Obviously, model (2.13) is the VRS version of the additive model. We therefore call model (2.11) (and its equivalent) the VRS multiplicative DEA model.

The above two multiplicative DEA models identify Cobb-Douglas production functions directly from observations (see Charnes et al. (1982, 1983) for more discussions.)

Model (2.8) or (2.9) yields the best efficiency score for DMU_o with a set of “weights” chosen by DMU_o . Denote an optimal set of weights by μ_{ro}^*, v_{io}^* , and the efficiency score from (2.8) as θ_o^* . Then cross efficiency of DMU_j using the weights that DMU_o has chosen, is given by

$$E_{oj} = \frac{\prod_{r=1}^s y_{rj}^{\mu_{ro}^*}}{\prod_{i=1}^m x_{ij}^{v_{io}^*}} \quad (2.14)$$

The efficiency score for DMU_o obtained from model (2.8) is $E_{oo} = \theta_o^*$

Then we define the following *geometric average* peer appraisal cross efficiency score as the CRS multiplicative cross efficiency score

$$\bar{E}_j = \left(\prod_{k=1}^n E_{kj} \right)^{1/n} = \left(\prod_{k=1}^n \frac{\prod_{r=1}^s y_{rk}^{\mu_{rk}^*}}{\prod_{i=1}^m x_{ik}^{v_{ik}^*}} \right)^{1/n} \quad (2.15)$$

The VRS multiplicative cross efficiency score can be defined in a similar manner. Specifically, for a DMU_k under evaluation of model (2.11), we have

$$E_{kj} = \frac{e^{\eta_k^*} \prod_{r=1}^s y_{rj}^{\mu_{rk}^*}}{e^{\xi_k^*} \prod_{i=1}^m x_{ij}^{v_{ik}^*}} \quad (2.16)$$

as the cross efficiency of DMU_j using the weights that DMU_k has chosen, where $\eta_k^*, \xi_k^*, \mu_{rk}^*, v_{ik}^*$ are optimal solutions from model (2.11) or (2.12).

Then we define the following *geometric average* peer appraisal VRS multiplicative cross efficiency score as

$$\bar{E}_j = \left(\prod_{k=1}^n E_{kj} \right)^{1/n} = \left(\prod_{k=1}^n \frac{e^{\eta_k^*} \prod_{r=1}^s y_{rj}^{\mu_{rk}^*}}{e^{\xi_k^*} \prod_{i=1}^m x_{ij}^{v_{ik}^*}} \right)^{1/n} \quad (2.17)$$

where E_{kk} is the optimal value to model (2.11) or (2.12).

2.6 Maximum Log Cross Efficiency

We now present the approach developed in Cook and Zhu (2014). These authors point out that one can maximize the average cross efficiency score \bar{E}_j (defined in (2.15)) subject to the condition that $E_{kk} = \theta_k^*$ for all $k = 1, \dots, n$. Specifically, for DMU_{j_0} we have

$$\begin{aligned} \max \quad & \left(\prod_{k=1}^n \frac{\prod_{r=1}^s y_{rj_0}^{\mu_{rk}^*}}{\prod_{i=1}^m x_{ij_0}^{v_{ik}^*}} \right)^{1/n} \\ \text{s.t.} \quad & \frac{\prod_{r=1}^s y_{rj}^{\mu_{rk}^*}}{\prod_{i=1}^m x_{ij}^{v_{ik}^*}} \leq 1, \quad j = 1, \dots, n, k = 1, \dots, n \\ & E_{kk} = \frac{\prod_{r=1}^s y_{rk}^{\mu_{rk}^*}}{\prod_{i=1}^m x_{ik}^{v_{ik}^*}} = \theta_k^*, k = 1, \dots, n \\ & \mu_{rk}, v_{ik} \geq 1, \quad k = 1, \dots, n; i = 1, \dots, m; r = 1, \dots, s \end{aligned} \quad (2.18)$$

Making logarithmic transformations in (2.18), we arrive at the following linear program

$$\begin{aligned} \max \quad & \frac{1}{n} \left(\sum_k \sum_r \mu_{rk} \hat{y}_{rj_0} - \sum_k \sum_i v_{ik} \hat{x}_{ij_0} \right) \\ \text{s.t.} \quad & \sum_{r=1}^s \mu_{rk} \hat{y}_{rj} - \sum_{i=1}^m v_{ik} \hat{x}_{ij} \leq 0, \quad j, k = 1, \dots, n \\ & \sum_{r=1}^s \mu_{rk} \hat{y}_{rk} - \sum_{i=1}^m v_{ik} \hat{x}_{ik} = \ln(\theta_k^*), \quad k = 1, \dots, n \\ & \mu_{rk}, v_{ik} \geq 1, \quad k = 1, \dots, n; i = 1, \dots, m; r = 1, \dots, s \end{aligned} \quad (2.19)$$

where “ \wedge ” denotes data in logarithmic form. Since logarithms are used in the process, we call this type of cross efficiency (the optimal value to model (2.19)) “Maximum Log Cross Efficiency”.

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