

Chapter 2

Notation and Basics

The aim of this chapter is to introduce basic notation and definitions, together with solvability theorems for Cauchy problems for DDEs and a remark on continuous dependence on the data. It is a preparatory work not only for the next theoretical chapters, but also for the numerical approaches, core of this monograph, presented in Part II and Part III. Finally, we introduce the definitions of stability of a given solution and the Principle of Linearized Stability in its generality. We refer to [70, 91, 121] for further details and for the proofs which are not given.

2.1 Notation

We denote the independent variable *time* by t , $t \in \mathbb{R}$, the *dimension* of the system (i.e., the number of equations) by d , $d \in \mathbb{N}$ with $d \geq 1$, and the dependent variable by the map $x : t \mapsto x(t) \in \mathbb{R}^d$, $x = (x_1, \dots, x_d)^T$. Let $\tau > 0$ be the *maximum delay* of the system. We denote by X the *state space* of continuous functions $C([-\tau, 0], \mathbb{R}^d)$, which is a Banach space when, as we choose here, it is equipped with the maximum norm $\|\varphi\|_X = \max_{\theta \in [-\tau, 0]} \|\varphi(\theta)\|_\infty$, $\varphi \in X$, where $\|\cdot\|_\infty$ is the infinity norm on $\mathbb{R}^d$. The relevant matrix induced norm, as well as induced norms for operators, are denoted simply by $\|\cdot\|$, unless differently specified.

We consider *nonlinear nonautonomous* DDEs of the form

$$x'(t) = F(t, x_t), \quad t \in \mathbb{I}, \quad (2.1)$$

for an interval $\mathbb{I} \subseteq \mathbb{R}$ unbounded on the right and a continuous function $F : \mathbb{I} \times X \rightarrow \mathbb{R}^d$. $x_t \in X$ is the *state* at time t defined, according to the standard Hale-Krasovsky notation [123], as

$$x_t(\theta) := x(t + \theta), \quad \theta \in [-\tau, 0]. \quad (2.2)$$

Notice that for DDEs the symbol $'$ denotes the right-hand derivative [91, p. 36]. When F does not depend explicitly on time, then $\mathbb{I} = \mathbb{R}$ and we call *autonomous* the resulting DDE:

$$x'(t) = F(x_t), \quad t \in \mathbb{R}. \quad (2.3)$$

Linear nonautonomous DDEs are described by

$$x'(t) = L(t)x_t, \quad t \in \mathbb{I}, \quad (2.4)$$

where $L(t) : X \rightarrow \mathbb{R}^d$, $t \in \mathbb{I}$, is a linear and bounded functional and the map $L(\cdot)\psi : \mathbb{I} \rightarrow \mathbb{R}^d$, $\psi \in X$, is continuous. L is representable as the Lebesgue–Stieltjes integral

$$L(t)\psi = \int_{-\tau}^0 d\eta(t, \theta)]\psi(\theta), \quad t \in \mathbb{I}, \quad \psi \in X, \quad (2.5)$$

where $\eta(t, \cdot)$, $t \in \mathbb{I}$, is a Normalized Bounded Variation (NBV) function [165, Riesz Representation Theorem]. By assuming that $t \mapsto \eta(t, \cdot)$ is continuous when $\text{NBV}([-\tau, 0], \mathbb{R}^{d \times d})$ is equipped with the total variation norm, the left-hand side of (2.5), i.e., the function $(t, \psi) \mapsto L(t)\psi$, is continuous on $\mathbb{I} \times X$. For a summary introduction on NBV functions and abstract integration see [70, Appendix I].

When there is $\omega > 0$ such that $L(t + \omega) = L(t)$ for all $t \in \mathbb{R}$, then $\mathbb{I} = \mathbb{R}$ and

$$x'(t) = L(t)x_t, \quad t \in \mathbb{R}, \quad (2.6)$$

is called *periodic*. When (2.4) is autonomous, i.e., $L = L(t)$ is independent of t , then $\mathbb{I} = \mathbb{R}$ and we simply write

$$x'(t) = Lx_t, \quad t \in \mathbb{R}. \quad (2.7)$$

Linear DDEs (2.4) arising in the applications of interest have the form

$$x'(t) = A(t)x(t) + \sum_{k=1}^p B_k(t)x(t - \tau_k) + \sum_{k=1}^p \int_{-\tau_k}^{-\tau_{k-1}} C_k(t, \theta)x(t + \theta)d\theta, \quad t \in \mathbb{I}, \quad (2.8)$$

where $0 < \tau_1 < \dots < \tau_p := \tau$ are p distinct delays (we also set $\tau_0 := 0$ for convenience), $A : \mathbb{I} \rightarrow \mathbb{R}^{d \times d}$, $B_k : \mathbb{I} \rightarrow \mathbb{R}^{d \times d}$ and $C_k : \mathbb{I} \times [-\tau_k, -\tau_{k-1}] \rightarrow \mathbb{R}^{d \times d}$ for $k = 1, \dots, p$. The above assumptions on η are fulfilled whenever the functions A and B_k are continuous, $C_k(\cdot, \theta)$ is continuous for all $\theta \in [-\tau_k, -\tau_{k-1}]$ and, for any compact interval I in $\mathbb{I}$, there exists $\gamma_k \in C([-\tau_k, -\tau_{k-1}], \mathbb{R})$ such that $\|C_k(t, \theta)\| \leq \gamma_k(\theta)$ for all $t \in I$ and $\theta \in [-\tau_k, -\tau_{k-1}]$. We call $A(t)x(t)$ the current time term and, for $k = 1, \dots, p$, $B_k(t)x(t - \tau_k)$ the k th discrete delay term and $\int_{-\tau_k}^{-\tau_{k-1}} C_k(t, \theta)x(t + \theta)d\theta$ the k th distributed delay term. When $p = 1$, we

simply have $\tau_1 = \tau$ and write $B_1 = B$ and $C_1 = C$. When (2.8) is autonomous, the dependence on time of A , B_k , and C_k is suppressed. Hereafter, we call (2.8) the *prototype DDE*.

2.2 The Cauchy Problem

This section is concerned with well-posedness of Cauchy problems for DDEs (2.1).

Definition 2.1 (*solution*) Given $t_0 \in \mathbb{I}$, a solution x of (2.1) on $[t_0 - \tau, t_f] \subseteq \mathbb{I}$ is a continuous function $x : [t_0 - \tau, t_f] \rightarrow \mathbb{R}^d$ which satisfies (2.1) on $[t_0, t_f]$.

To specify a solution we assign an initial condition, i.e., a function $\varphi \in X$ at a certain initial time t_0 . Given $(t_0, \varphi) \in \mathbb{I} \times X$, the Cauchy problem for (2.1) is defined as

$$\begin{cases} x'(t) = F(t, x_t), & t \geq t_0, \\ x(t_0 + \theta) = \varphi(\theta), & \theta \in [-\tau, 0]. \end{cases} \quad (2.9)$$

Definition 2.2 (*solution of the Cauchy problem*) Given $(t_0, \varphi) \in \mathbb{I} \times X$, a solution x of (2.9) on $[t_0 - \tau, t_f] \subseteq \mathbb{I}$ is a solution on $[t_0 - \tau, t_f]$ with initial function φ , i.e., $x_{t_0} = \varphi$. x is called *global* if it is defined on $[t_0 - \tau, +\infty)$.

To emphasize the dependence on t_0 and φ , we sometimes write $x(t; t_0, \varphi)$ for the solution at time t of (2.9). Various theorems on existence and uniqueness of solutions of (2.9) appear in the literature. The key ingredient is the Lipschitz continuity of F w.r.t. the state, either *locally*, i.e., for every $(t_0, \varphi) \in \mathbb{I} \times X$, there exists a neighborhood $\mathcal{U}$ of (t_0, φ) and a constant $\text{Lip}(F)$ such that $\|F(t, \psi_1) - F(t, \psi_2)\|_\infty \leq \text{Lip}(F)\|\psi_1 - \psi_2\|_X$ for all $(t, \psi_1), (t, \psi_2) \in \mathcal{U}$, or *globally*, i.e., there exists a constant $\text{Lip}(F)$ such that $\|F(t, \psi_1) - F(t, \psi_2)\|_\infty \leq \text{Lip}(F)\|\psi_1 - \psi_2\|_X$ for all $(t, \psi_1), (t, \psi_2) \in \mathbb{I} \times X$.

Theorem 2.1 (*local solution of the Cauchy problem*) *Let F be locally Lipschitz w.r.t. the state. Then, for every $(t_0, \varphi) \in \mathbb{I} \times X$, there exists $t_f > t_0$ and a unique solution x of (2.9) on $[t_0 - \tau, t_f]$. Moreover, x depends continuously on F , t_0 and φ .*

In general, the continuous dependence studies the effect on the solution of the errors in either F and φ [121, p.41], i.e., replacing F and φ in (2.9) by $\tilde{F}$ and $\tilde{\varphi}$ such that

- for any $t_1 \in [t_0, t_f]$ and $\varepsilon > 0$ there is $\delta > 0$ such that $\|\tilde{\varphi} - \varphi\|_X \leq \delta$ and $\|\tilde{F}(t, \psi) - F(t, \psi)\|_\infty \leq \delta, t \in [t_0, t_1], \|\psi - x_t\|_X \leq \varepsilon$;
- $\tilde{F}$ and $\tilde{\varphi}$ satisfy the assumptions of Theorem 2.1, so that the perturbed Cauchy problem has a unique solution $\tilde{x}$ on $[t_0 - \tau, \tilde{t}_f]$;

it implies $\sup_{t \in [t_0, \min\{t_f, \tilde{t}_f\})} \|\tilde{x}(t) - x(t)\|_\infty \leq \varepsilon$. Observe also that under the assumptions in Theorem 2.1, the solution of (2.9) belongs to $C([t_0 - \tau, t_f], \mathbb{R}^d) \cap C^1([t_0, t_f], \mathbb{R}^d)$. More regularity is ensured by further smoothness of F .

Theorem 2.2 (global solution of the Cauchy problem) *Let F be globally Lipschitz w.r.t. the state. Then, for every $(t_0, \varphi) \in \mathbb{I} \times X$, there exists a unique solution of (2.9) on $[t_0 - \tau, +\infty)$.*

For autonomous DDEs (2.3), it is not restrictive to assume $t_0 = 0$ (since $x(t; t_0, \varphi) = x(t + t_0; 0, \varphi)$) and we relax the notation $x(\cdot; 0, \varphi)$ to $x(\cdot; \varphi)$. For $\varphi \in X$, (2.9) reads

$$\begin{cases} x'(t) = F(x_t), & t \geq 0, \\ x_0 = \varphi. \end{cases} \quad (2.10)$$

2.3 Stability of Solutions

Stability, the crucial question addressed in the book, concerns the effects of small perturbations of φ w.r.t. a solution of interest $\bar{x}(\cdot; t_0, \varphi)$ given on $[t_0 - \tau, +\infty)$ under the hypothesis of Theorem 2.2. By considering $y(t) = x(t; t_0, \psi) - \bar{x}(t; t_0, \varphi)$ and the DDE $y'(t) = F(t, y_t + \bar{x}_t) - F(t, \bar{x}_t)$ corresponding to the zero initial function, the definitions of stability and hence the stability analysis of $\bar{x}$ reduce to the stability of the zero solution. Here we prefer to give all the stability definitions directly for $\bar{x}$.

Definition 2.3 (*stable/unstable solution*) The solution $\bar{x}(\cdot; t_0, \varphi)$ of (2.9) is called stable if for any $\varepsilon > 0$ there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x(t; t_0, \psi) - \bar{x}(t; t_0, \varphi)\|_\infty \leq \varepsilon$ for all $t \geq t_0$ and for any ψ such that $\|\psi - \varphi\|_X \leq \delta$. $\bar{x}$ is called uniformly stable when δ is independent of t_0 and unstable when it is not stable.

Definition 2.4 (*asymptotically stable solution*) The solution $\bar{x}(\cdot; t_0, \varphi)$ of (2.9) is called asymptotically stable if it is stable and, in addition, there exists $\delta = \delta(t_0) > 0$ such that $\|x(t; t_0, \psi) - \bar{x}(t; t_0, \varphi)\|_\infty \rightarrow 0$ as $t \rightarrow +\infty$ for any ψ such that $\|\psi - \varphi\|_X \leq \delta$. $\bar{x}$ is called uniformly asymptotically stable when it is uniformly stable and there is $r > 0$ such that for every $\gamma > 0$ there is $t_f(\gamma) > t_0$ such that $\|x(t; t_0, \psi) - \bar{x}(t; t_0, \varphi)\|_\infty \leq \gamma$ for any $t_0 \in \mathbb{R}$, $t \geq t_f(\gamma)$ and ψ such that $\|\psi - \varphi\|_X \leq r$.

Uniform asymptotic stability means that $\|x(t; t_0, \psi) - \bar{x}(t; t_0, \varphi)\|_\infty \rightarrow 0$ uniformly w.r.t ψ whenever $\|\psi - \varphi\|_X \leq r$ and, in general, is stronger than stability. For some DDEs they coincide, e.g., for autonomous and periodic ones [91, Chap. 5, Lemma 1.1]. The same holds also for asymptotic stability. We recall a further definition: $\bar{x}(t; t_0, \varphi)$ is called locally *exponentially* stable if there are positive constants α_1 , α_2 and β such that $\|x(t; t_0, \psi) - \bar{x}(t; t_0, \varphi)\|_\infty \leq \alpha_1 e^{-\alpha_2(t-t_0)}$ for all $t \geq t_0$ and for any ψ such that $\|\psi - \varphi\|_X \leq \beta$. In any case, all definitions reflect the local nature of stability: if we slightly perturb the initial function, then the perturbed solution stays in the neighborhood of $\bar{x}$ or, for asymptotic stability, returns to it.

Now, assume that F is continuously differentiable w.r.t. its second argument and denote by DF its Fréchet derivative [8]. To examine the stability of a specific solution $\bar{x}(\cdot; t_0, \varphi)$ of (2.9), we apply the *Principle of Linearized Stability*: the study of the stability of a solution $\bar{x}$ is reduced to the study of the system linearized at $\bar{x}$, i.e.,

$$x'(t) = DF(t, \bar{x}_t)x_t, \quad t \geq t_0. \quad (2.11)$$

System (2.11), also called *variational* in the literature, can be viewed as a first-order approximation of (2.1). In the linearization approach, the starting point is the study of the behavior of the zero solution of linear DDEs, which is complemented with suitable theorems asserting that the local behavior of the solutions close to $\bar{x}$ is determined, to the first order, by the behavior of the solutions of (2.11). It is important to underline that, in some critical situations, one cannot conclude anything without investigating higher order terms [70, Chap. IX]. According to this principle, the analysis of linear DDEs is essential: we present the theoretical aspects of the stability of linear autonomous DDEs in Chap. 3 and of linear periodic DDEs in Chap. 4, laying the foundation for the numerical methods of Part II.

Stability of Linear Delay Differential Equations

A Numerical Approach with MATLAB

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