Chapter 2
Interpolation of Operators: A Multiplier Theorem

In this chapter, we shall first study two basic results in interpolation of operators in L^p spaces, the Riesz–Thorin theorem and the Marcinkiewicz interpolation theorem (diagonal case). As a consequence of the former we shall prove the Hardy–Littlewood–Sobolev theorem for Riesz potentials. In this regard, we need to introduce one of the fundamental tools in harmonic analysis, the Hardy–Littlewood maximal function. In Section 2.4, we shall prove the Mihlin multiplier theorem.

The results deduced in this chapter are used frequently in these notes. In particular, in Chapter 4 the proof of Theorem 4.2 is based on the Riesz–Thorin theorem and the Hardy–Littlewood–Sobolev theorem.

2.1 The Riesz–Thorin Convexity Theorem

Let $(X, \mathcal{A}, \mu)$ be a measurable space (i.e., X is a set, $\mathcal{A}$ denotes a σ-algebra of subsets of X, and μ is a measure defined on $\mathcal{A}$). $L^p = L^p(X, \mathcal{A}, \mu)$, $1 \le p < \infty$ denotes the space of complex-valued functions f that are μ-measurable such that

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu \right)^{1/p} < \infty.$$

Functions in $L^p(X, \mathcal{A}, \mu)$ are defined almost everywhere with respect to μ. Similarly, we have $L^\infty(X, \mathcal{A}, \mu)$ the space of functions f that are μ-measurable, complex valued and essentially μ-bounded, with $\|f\|_\infty$ the essential supremum of f. The Riesz–Thorin convexity theorem can be obtained as a consequence of a version of the Hadamard three circles theorem, a result of the Phragmen–Lindelöf theorem, known as the *three lines theorem*.

Lemma 2.1. *Let F be a continuous and bounded function defined on*

$$S = \{z = x + iy : 0 \le x \le 1\}$$

which is also analytic in the interior of S. If for each $y \in \mathbb{R}$,

$$|F(iy)| \le M_0 \quad \text{and} \quad |F(1 + iy)| \le M_1,$$

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then for any $z = x + iy \in S$

$$|F(x+iy)| \le M_0^{1-x} M_1^x.$$

In other words, the function $\phi(x) = \log\, k_x$ *is convex, where* $k_x = \sup\{|F(x+iy)| : y \in \mathbb{R}\}$ *for* $x \in [0,1]$.

Proof. Without loss of generality one can assume that M_0, $M_1 > 0$. Moreover, considering the function $F(z)/M_0^{1-z}M_1^z$, the proof reduces to the case $M_0 = M_1 = 1$. Thus, we have that

$$|F(iy)| \le 1 \quad \text{and} \quad |F(1+iy)| \le 1 \quad \text{for any } y \in \mathbb{R},$$

and we want to show that $|F(z)| \le 1$ for any $z \in S$. If

$$\lim_{|y|\to\infty} F(x+iy) = 0 \quad \text{uniformly on } 0 \le x \le 1,$$

the result follows from the maximum principle. In this case, there exists $y_0 > 0$ such that $|F(x+iy)| \le 1$ for $|y| \ge y_0$ and $|F(z)| \le 1$ in the boundary of the rectangle with corners

$$iy_0, 1+iy_0, -iy_0, 1-iy_0.$$

The maximum principle guarantees the same estimate in the interior of the rectangle.

In the general case, we consider the function:

$$F_n(z) = F(z)e^{(z^2-1)/n}, \quad n \in \mathbb{Z}^+.$$

Since

$$\begin{aligned} |F_n(z)| &= |F(x+iy)|e^{-y^2/n}\, e^{(x^2-1)/n} \\ &\le |F(x+iy)|e^{-y^2/n} \to 0 \quad \text{as} \quad |y| \to \infty, \end{aligned}$$

uniformly on $0 \le x \le 1$, with $|F_n(iy)| \le 1$ and $|F_n(1+iy)| \le 1$, the previous argument proves that $|F_n(z)| \le 1$ for any $n \in \mathbb{Z}^+$. Letting $n \to \infty$, we obtain the desired estimate. □

Let T be a linear operator from $L^p(X)$ to $L^q(Y)$. If T is continuous or bounded, i.e.,

$$|||T||| = \sup_{f\neq 0} \frac{\|Tf\|_q}{\|f\|_p} < \infty, \tag{2.1}$$

we call the number $|||T|||$ the *norm of the operator* T.

Theorem 2.1 (Riesz–Thorin). *Let* $p_0 \neq p_1$, $q_0 \neq q_1$. *Let* T *be a bounded linear operator from* $L^{p_0}(X, \mathcal{A}, \mu)$ *to* $L^{q_0}(Y, \mathcal{B}, \nu)$ *with norm* M_0 *and from* $L^{p_1}(X, \mathcal{A}, \mu)$ *to* $L^{q_1}(Y, \mathcal{B}, \nu)$ *with norm* M_1. *Then,* T *is bounded from* $L^{p_\theta}(X, \mathcal{A}, \mu)$ *in* $L^{q_\theta}(Y, \mathcal{B}, \nu)$ *with norm* M_θ *such that*

$$M_\theta \le M_0^{1-\theta} M_1^\theta,$$

with

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0,1). \tag{2.2}$$

Proof. *(Thorin)*. Combining the notation

$$\langle h, g\rangle = \int_Y h(y)g(y)\,d\nu(y)$$

and a duality argument it follows that

$$\|h\|_q = \sup\{|\langle h, g\rangle| : \|g\|_{q'} = 1\}$$

and

$$M_{pq} \equiv \sup\{|\langle Tf, g\rangle| : \|f\|_p = \|g\|_{q'} = 1\},$$

where $1/p + 1/p' = 1/q + 1/q' = 1$. Since $p < \infty$ and $q' < \infty$, we can assume that $f,\ g$ are simple functions with compact support. Thus,

$$f(x) = \sum_j a_j \chi_{A_j}(x) \ \text{ and } \ g(y) = \sum_k b_k \chi_{B_k}(y).$$

For $0 \le \mathcal{R}e\ z \le 1$, we define

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \quad \frac{1}{q'(z)} = \frac{1-z}{q_0'} + \frac{z}{q_1'},$$
$$\varphi(z) = \varphi(x,z) = \sum_j |a_j|^{p_\theta/p(z)} e^{i\arg(a_j)} \chi_{A_j}(x),$$

and

$$\psi(z) = \psi(y,z) = \sum_k |b_k|^{q_\theta'/q'(z)} e^{i\arg(b_k)} \chi_{B_k}(y).$$

Thus, $\varphi(z) \in L^{p_j}$, $\psi(z) \in L^{q_j'}$, and $T\varphi(z) \in L^{q_j}, j = 0,1$. Also, $\varphi'(z) \in L^{p_j}$, $\psi'(z) \in L^{q_j'}$, and $(T\varphi)'(z) \in L^{q_j}$, $j = 0,1$ for $0 < \mathcal{R}e\ z < 1$. Therefore, the function

$$F(z) = \langle T\varphi(z), \psi(z)\rangle$$

is bounded and continuous on $0 \le \mathcal{R}e\ z \le 1$ and analytic in the interior. Moreover,

$$\|\varphi(it)\|_{p_0} = \||f|^{p_\theta/p_0}\|_{p_0} = \|f\|_{p_\theta}^{p_\theta/p_0} = 1$$

and

$$\|\varphi(1+it)\|_{p_1} = \||f|^{p_\theta/p_1}\|_{p_1} = \|f\|_{p_\theta}^{p_\theta/p_1} = 1.$$

Similarly, $\|\psi(it)\|_{q_0'} = \|\psi(1+it)\|_{q_1'} = 1$.

From the hypotheses it follows that

$$|F(it)| \le \|T\varphi(it)\|_{q_0} \ \|\psi(it)\|_{q_0'} \le M_0$$

and

$$|F(1+it)| \le \|T\varphi(1+it)\|_{q_1} \ \|\psi(1+it)\|_{q_1'} \le M_1.$$

Since $\varphi(\theta) = f$, $\psi(\theta) = g$, and $F(\theta) = \langle Tf, g\rangle$, by the three lines theorem we obtain $|\langle Tf, g\rangle| \le M_0^{1-\theta} M_1^{\theta}$. This completes the proof. $\square$

Definition 2.1. An operator T is said to be *sublinear* if $T(f+g)$ is determined by the values of Tf, Tg, and

$$|T(f+g)| \le |Tf| + |Tg|.$$

We shall say that a linear or sublinear operator T is of (strong) *type* (p,q) with constant M_{pq} if $\|Tf\|_q \le M_{pq}\|f\|_p$ for any $f \in L^p$.

With this definition we can rephrase the statement of the Riesz–Thorin theorem.

Let $p_0 \ne p_1$, $q_0 \ne q_1$, and T be a linear operator of type (p_0, q_0) with norm M_0 and of type (p_1, q_1) with norm M_1. Then T is of type (p,q) with

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \theta \in (0,1),$$

with norm

$$M \le M_0^{1-\theta} M_1^{\theta}.$$

2.1.1 Applications

Next we use the Riesz–Thorin theorem to establish some properties of the Fourier transform and the convolution operator. We fix $X = Y = \mathbb{R}^n$ and $\mu = \nu = dx$ the Lebesgue measure.

Theorem 2.2 (Young's inequality). *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, $1 \le p, q \le \infty$ with $\dfrac{1}{p} + \dfrac{1}{q} \ge 1$. Then $f * g \in L^r(\mathbb{R}^n)$, where $\dfrac{1}{r} = \dfrac{1}{p} + \dfrac{1}{q} - 1$. Moreover,*

$$\|f * g\|_r \le \|f\|_p \|g\|_q. \tag{2.3}$$

Proof. For $g \in L^q(\mathbb{R}^n)$, we define the operator

$$Tf(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy = (f * g)(x).$$

The Minkowski integral inequality shows

$$\|Tf\|_q \le \|g\|_q \|f\|_1.$$

On the other hand, using Hölder's inequality one sees that

$$\|Tf\|_\infty \le \|g\|_q \|f\|_{q'}.$$

Thus, T is of type $(1, q)$ and (q', ∞) with norm bounded by $\|g\|_q$. Hence, Theorem 2.1 (Riesz–Thorin) guarantees that T is of type (p, r), where

$$\frac{1}{p} = \frac{(1-\theta)}{1} + \frac{\theta}{q'} = 1 - \frac{\theta}{q}$$

and

$$\frac{1}{r} = \frac{(1-\theta)}{q} + 0 = \frac{1}{q} + \left(1 - \frac{\theta}{q}\right) - 1 = \frac{1}{q} + \frac{1}{p} - 1,$$

with norm less than $\|g\|_q$. □

Theorem 2.3 (Hausdorff–Young's inequality). *Let $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$. Then $\widehat{f} \in L^{p'}(\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and*

$$\|\widehat{f}\|_{p'} \le \|f\|_p. \tag{2.4}$$

Proof. From (1.2) and (1.11) it follows that the Fourier transform is of type $(1, \infty)$ and $(2, 2)$ with norm 1. Hence, Theorem 2.1 tells us that it is also of type (p, q) with

$$\frac{1}{p} = \frac{(1-\theta)}{1} + \frac{\theta}{2} = 1 - \frac{\theta}{2} \quad \text{and} \quad \frac{1}{q} = 0 + \frac{\theta}{2} = 1 - \frac{1}{p} = \frac{1}{p'}$$

with norm $M \le 1^{(1-\theta)}\, 1^{\theta} = 1$. □

This estimate is the best possible when $p = 1$ or 2. This is not the case for $1 < p < 2$. Beckner [B] found the best constant for the Hausdorff–Young inequality. He showed that if $f \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then

$$\|\widehat{f}\,\|_{p'} \le (A_p)^n \|f\|_p, \quad \text{where} \quad A_p = \left(\frac{p^{1/p}}{p'^{1/p'}}\right)^{1/2}.$$

2.2 Marcinkiewicz Interpolation Theorem (Diagonal Case)

Let $(X, \mathcal{A}, \mu)$ be a measurable space.

Definition 2.2. For a measurable function $f : X \to \mathbb{C}$, we define its distribution function as:

$$m(\lambda, f) = \mu(\{x \in X : |f(x)| > \lambda\}) = \mu(E_f^\lambda).$$

Thus, $m(\lambda, f)$ as a function of $\lambda \in [0, \infty]$ is well defined and takes values in $[0, \infty)$. Moreover, it is nonincreasing and continuous from the right.

Proposition 2.1. *For any measurable function $f : X \to \mathbb{C}$ and for any $\lambda \geq 0$ it follows that*

1. *(Tchebychev)*

$$m(\lambda, f) \leq \lambda^{-p} \int\limits_{E_f^\lambda} |f(x)|^p \, d\mu(x) \leq \lambda^{-p} \|f\|_p^p.$$

2. *If* $1 \leq p < \infty$,

$$\|f\|_p^p = -\int\limits_0^\infty \lambda^p \, dm(\lambda, f) = p \int\limits_0^\infty \lambda^{p-1} m(\lambda, f) \, d\lambda.$$

If $p = \infty$,

$$\|f\|_\infty = \inf\{\lambda : m(\lambda, f) = 0\}.$$

3. $m(\lambda, f + g) \leq m(\lambda/2, f) + m(\lambda/2, g)$.

Proof. It is left as an exercise. □

Definition 2.3. For $1 \leq p < \infty$, we denote by $L^{p*}(X, \mathcal{A}, \mu)$ (*weak* L^p*-spaces*) the space of all measurable functions $f : X \to \mathbb{C}$ such that

$$\|f\|_p^* = \sup_{\lambda > 0} \lambda (m(\lambda, f))^{1/p} < \infty.$$

Observe that $L^{\infty *} = L^\infty$.

Proposition 2.2. *If* $1 \leq p < \infty$, *then*

1. $L^p(\mathbb{R}^n) \subsetneq L^{p*}(\mathbb{R}^n)$.
2. $\|f + g\|_p^* \leq 2(\|f\|_p^* + \|g\|_p^*)$.

Proof. It is left as an exercise. □

Therefore, $L^{p*}(X, \mathcal{A}, \mu)$ is a *quasinormed vector space*

$$\|f + g\| \leq k(\|f\| + \|g\|)$$

with $k = 2$, i.e., it only satisfies a quasitriangular inequality. The spaces L^p and L^{p*} are particular cases of the *Lorentz spaces* $L^{p,q}$ (see [BeL]).

Definition 2.4. Let $(X_j, \mathcal{A}_j, \mu_j)$, $j = 1, 2$, be two measurable spaces. Let $M(X_2)$ be the space of complex-valued, measurable functions defined on X_2. A linear or sublinear operator $T : L^p(X_1) \to M(X_2)$ with $1 \leq p < \infty$ is said to be of *weak type* (p, q) if there exists a constant $c > 0$ such that for any $f \in L^p(X_1)$

$$\|Tf\|_q^* \leq c\|f\|_p.$$

If $q = \infty$, type (p, ∞) and weak type (p, ∞) agree. Tchebychev's inequality shows that if T is of type (p, q), then it is of weak type (p, q).

In the rest of this chapter, we shall consider $X_j = \mathbb{R}^n,\ j = 1, 2$.

Theorem 2.4 (Marcinkiewicz). *Let* $1 < r \le \infty$ *and*

$$T : L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n) \to M(\mathbb{R}^n)$$

be a sublinear operator (see Definition 2.1). If T is of weak type $(1,1)$ and of weak type (r,r), then T is of (strong) type (p,p) for any $p \in (1,r)$.

Proof. First we consider the case $r = \infty$. Changing the operator T by $\|T\|^{-1}T$ one can assume that

$$\|Tf\|_\infty \le \|f\|_\infty.$$

Given $f \in L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$, for each $\lambda \in \mathbb{R}^+$ we define

$$f_1^\lambda(x) = \begin{cases} f(x), & \text{if } |f(x)| \ge \lambda/2 \\ 0, & \text{if } |f(x)| < \lambda/2 \end{cases}$$

and $f_2^\lambda(x) = f(x) - f_1^\lambda(x)$. Therefore,

$$|Tf(x)| \le |Tf_1^\lambda(x)| + \lambda/2,$$

and

$$\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\} \subseteq \{x \in \mathbb{R}^n : |Tf_1^\lambda(x)| > \lambda/2\}.$$

Since T is of weak type $(1,1)$, it follows that

$$\begin{aligned} |\{x \in \mathbb{R}^n : |Tf_1^\lambda(x)| > \lambda/2\}| &\le c\left(\frac{\lambda}{2}\right)^{-1} \int_{\mathbb{R}^n} |f_1^\lambda(x)|\, dx \\ &= 2c\lambda^{-1} \int_{|f|>\lambda/2} |f(x)|\, dx, \end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure. Combining this estimate, part (2) of Proposition 2.1, and a change in the order of integration, one has:

$$\begin{aligned} \int_{\mathbb{R}^n} |Tf(x)|^p\, dx &= p\int_0^\infty \lambda^{p-1} |\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}|\, d\lambda \\ &\le p\int_0^\infty \lambda^{p-1} \left(2c\lambda^{-1} \int_{|f|>\lambda/2} |f(x)|\, dx\right) d\lambda \\ &= 2cp\int_0^\infty \lambda^{p-2} \left(\int_{|f|>\lambda/2} |f(x)|\, dx\right) d\lambda \\ &= 2cp\int_{\mathbb{R}^n} \left(\int_0^{2|f(x)|} \lambda^{p-2}\, d\lambda\right) |f(x)|\, dx = \frac{2^p cp}{p-1}\|f\|_p^p, \end{aligned}$$

which yields the result for the case $r = \infty$.

In the case $r < \infty$, we have

$$\begin{aligned} m(\lambda, Tf) &= |\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \\ &\leq m(\lambda/2, Tf_1^\lambda) + m(\lambda/2, Tf_2^\lambda) \\ &\leq c_1 \left(\frac{\lambda}{2}\right)^{-1} \int\limits_{\mathbb{R}^n} |f_1^\lambda(x)|\, dx + c_r^r \left(\frac{\lambda}{2}\right)^{-r} \int\limits_{\mathbb{R}^n} |f_2^\lambda(x)|^r\, dx \\ &= 2c_1\lambda^{-1} \int\limits_{|f| \geq \lambda/2} |f(x)|\, dx + (2c_r)^r \lambda^{-r} \int\limits_{|f| < \lambda/2} |f(x)|^r\, dx. \end{aligned}$$

As in the proof of the case $r = \infty$, we have that

$$\int\limits_0^\infty \lambda^{p-2} \left(\int\limits_{|f| \geq \lambda/2} |f(x)|\, dx \right) d\lambda = \frac{2^{p-1}}{p-1} \|f\|_p^p.$$

A similar argument shows that

$$\int\limits_0^\infty \lambda^{p-1-r} \left(\int\limits_{|f| < \lambda/2} |f(x)|^r\, dx \right) d\lambda = \frac{2^{p-r}}{r-p} \|f\|_p^p.$$

Combining these inequalities and part (2) of Proposition 2.1, we find that

$$\|Tf\|_p \leq c_p \|f\|_p, \quad \text{with} \quad c_p = 2\sqrt[p]{p} \left(\frac{c_1}{p-1} + \frac{c_r^r}{r-p} \right)^{1/p}.$$

□

2.2.1 Applications

We shall use the Marcinkiewicz interpolation theorem to study some basic properties of the Hardy–Littlewood maximal function. First, we introduce some notation.

We denote by $L^1_{\text{loc}}(\mathbb{R}^n)$ the spaces of functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_K |f| dx < \infty$ for any compact $K \subseteq \mathbb{R}^n$. The volume of the unit ball in $\mathbb{R}^n$ will be denoted by ω_n and $B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}$ is the ball of center x and radius r.

Definition 2.5. For a given $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, we define $\mathcal{M}f(x)$, the *Hardy–Littlewood maximal function* associated to f, as:

$$\begin{aligned} \mathcal{M}f(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f(y)|\, dy = \sup_{r>0} \frac{1}{\omega_n} \int\limits_{B_1(0)} |f(x - ry)|\, dy \\ &= \sup_{r>0} \left(|f| * \frac{1}{|B_r(0)|} \chi_{B_r(0)} \right)(x). \end{aligned}$$

Proposition 2.3.

1. *$\mathcal{M}$ defines a sublinear operator, i.e.,*

$$|\mathcal{M}(f+g)(x)| \leq |\mathcal{M}f(x)| + |\mathcal{M}g(x)|, \qquad x \in \mathbb{R}^n.$$

2. *If $f \in L^\infty(\mathbb{R}^n)$, then*

$$\|\mathcal{M}f\|_\infty \leq \|f\|_\infty. \tag{2.5}$$

Proof. It is left as an exercise. □

Part (2) of Proposition 2.3 tells us that $\mathcal{M}$ is of type (∞, ∞). Next, we show that $\mathcal{M}$ is of weak type $(1,1)$. For this purpose, we need the following result.

Lemma 2.2 (Vitali's covering lemma). *Let $E \subseteq \mathbb{R}^n$ be a measurable set such that $E \subseteq \cup_\alpha B_{r_\alpha}(x_\alpha)$ with the family of open balls $\{B_{r_\alpha}(x_\alpha)\}_\alpha$ satisfying $\sup\limits_\alpha r_\alpha = c_0 < \infty$. Then there exists a subfamily $\{B_{r_j}(x_j)\}_j$ disjoint and numerable such that*

$$|E| \leq 5^n \sum_{j=1}^\infty |B_{r_j}(x_j)|.$$

Proof. Choose $B_{r_1}(x_1)$ such that $r_1 \geq c_0/2$. For $j \geq 2$, take $B_{r_j}(x_j)$ such that $B_{r_j}(x_j) \cap \bigcup\limits_{k=1}^{j-1} B_{r_k}(x_k) = \emptyset$ and

$$r_j > \frac{1}{2} \sup\{r_\alpha \ : \ B_{r_\alpha}(x_\alpha) \cap B_{r_k}(x_k) = \emptyset \ \text{ for } \ k = 1, \ldots, j-1\}.$$

It is clear that the $B_{r_j}(x_j)$ are disjoint. If $\sum |B_{r_j}(x_j)| = \infty$, we have completed the proof. In the case $\sum |B_{r_j}(x_j)| < \infty$ (hence, $\lim\limits_{j\to\infty} r_j = 0$), it will suffice to show that

$$B_{r_\alpha}(x_\alpha) \subseteq \bigcup_j B_{5r_j}(x_j), \quad \text{for any } \alpha.$$

If $B_{r_\alpha}(x_\alpha) = B_{r_j}(x_j)$ for some j, there is nothing to prove. Thus, we assume that $B_{r_\alpha}(x_\alpha) \neq B_{r_j}(x_j)$ for any j. Define j_α as the smallest j such that $r_j < r_\alpha/2$. By the construction of $B_{r_j}(x_j)$, there exists $j \in \{1, \ldots, j_\alpha - 1\}$ such that $B_{r_\alpha}(x_\alpha) \cap B_{r_j}(x_j) \neq \emptyset$. Denoting by j^* this index it follows that $B_{r_\alpha}(x_\alpha) \subseteq B_{5r_{j^*}}(x_{j^*})$ since $r_{j^*} \geq r_\alpha/2$. □

Theorem 2.5 (Hardy–Littlewood). *Let $1 < p \leq \infty$. Then $\mathcal{M}$ is a sublinear operator of type (p,p), i.e., there exists c_p such that*

$$\|\mathcal{M}f\|_p \leq c_p \|f\|_p, \quad \text{for any } f \in L^p(\mathbb{R}^n). \tag{2.6}$$

Proof. We first show that $\mathcal{M}$ is of weak type $(1,1)$, that is, there exists a constant c_1 such that for any $f \in L^1(\mathbb{R}^n)$

$$\sup_{\lambda > 0} \lambda \, m(\lambda, \mathcal{M}f) \leq c_1 \|f\|_1. \tag{2.7}$$

Once (2.7) has been established, a combination of (2.5), (2.7), and the Marcinkiewicz theorem yields (2.6).

To obtain (2.7), we define $E_f^\lambda = \{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}$ for any $\lambda > 0$. Thus, if $x \in E_f^\lambda$, then there exists $B_{r_x}(x)$ such that

$$\int_{B_{r_x}(x)} |f(y)|dy > \lambda |B_{r_x}(x)|.$$

Clearly, we have that

$$E_f^\lambda \subseteq \bigcup_{x \in E_f^\lambda} B_{r_x}(x),$$

then the Vitali covering lemma guarantees the existence of a countable, disjoint subfamily $\{B_{r_{x_j}}(x_j)\}_{j\in\mathbb{Z}^+}$ such that

$$|E_f^\lambda| \leq 5^n \sum_{j=1}^{\infty} |B_{r_{x_j}}(x_j)| \leq 5^n \lambda^{-1} \sum_{j=1}^{\infty} \int_{B_{r_{x_j}}(x_j)} |f(y)|\, dy \leq 5^n \lambda^{-1} \|f\|_1,$$

which implies (2.7). □

Next, we extend the estimates (2.6) and (2.7) to a large class of kernels.

Proposition 2.4. *Let $\varphi \in L^1(\mathbb{R}^n)$ be a radial, positive, and nonincreasing function of $r = \|x\| \in [0, \infty)$. Then*

$$\sup_{t>0} |\varphi_t * f(x)| = \sup_{t>0} \left| \int_{\mathbb{R}^n} \frac{\varphi(t^{-1}(x-y))}{t^n} f(y)\, dy \right| \leq \|\varphi\|_1 \mathcal{M}f(x). \tag{2.8}$$

Proof. First, we assume that, in addition to the hypotheses, φ is a simple function

$$\varphi(x) = \sum_k a_k \chi_{B_{r_k}(0)}(x), \quad \text{with} \quad a_k > 0.$$

Hence,

$$\varphi * f(x) = \sum_k a_k |B_{r_k}(0)| \frac{1}{|B_{r_k}(0)|} \chi_{B_{r_k}(0)} * f(x) \leq \|\varphi\|_1 \mathcal{M}f(x).$$

(observe that $\|\varphi\|_1 = \sum_k a_k |B_{r_k}(0)|$).

In the general case, we approximate φ by an increasing sequence of simple functions satisfying the hypotheses. Since dilations of φ satisfy the same hypotheses and preserve the L^1-norm, they verify (2.8). Finally, passing to the limit we obtain the desired result. □

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