Chapter 2
Homotopy Theory of Simplicial Sets

We begin with a brief description of the homotopy theory of simplicial sets, and a first take on the homotopy theory of diagrams of simplicial sets. This is done to establish notation, and to recall some basic constructions and well-known lines of argument. A much more detailed presentation of this theory can be found in [32].

The description of the model structure for simplicial sets which appears in the second section is a bit unusual, in that we first show in Theorem 2.13 that there is a model structure for which the cofibrations are the monomorphisms and the weak equivalences are defined by topological realization. The proof of the fact that the fibrations for this theory are the Kan fibrations (Theorem 2.19) then becomes a somewhat delicate result whose proof is only sketched. For the argument which is presented here, it is critical to know Quillen's theorem [88] that the realization of a Kan fibration is a Serre fibration.

The proof of Theorem 2.13 uses the "bounded monomorphism property" for simplicial sets of Lemma 2.16. This is a rather powerful principle which recurs in various guises throughout the book.

The model structure for diagrams of simplicial sets (here called the projective model structure) appears in the third section, in Proposition 2.22. This result was first observed by Bousfield and Kan [14] and has a simple proof with a standard method of attack. In the context of subsequent chapters, Proposition 2.22 gives a preliminary, essentially non-local model structure for all simplicial presheaf categories.

2.1 Simplicial Sets

The finite ordinal number $\mathbf{n}$ is the set of counting numbers

$$\mathbf{n} = \{0, 1, \dots, n\}.$$

There is an obvious ordering on this set which gives it the structure of a poset, and hence a category. In general, if C is a category then the functors $\alpha : \mathbf{n} \to C$ can be identified with strings of arrows

$$\alpha(0) \to \alpha(1) \to \cdots \to \alpha(n)$$

J. F. Jardine, *Local Homotopy Theory,* Springer Monographs in Mathematics,
DOI 10.1007/978-1-4939-2300-7_2

of length n. The collection of all finite ordinal numbers and all order-preserving functions between them (or poset morphisms, or functors) form the *ordinal number category* $\boldsymbol{\Delta}$.

Example 2.1 The ordinal number monomorphisms $d^i : \mathbf{n-1} \to \mathbf{n}$ are defined by the strings of relations

$$0 \leq 1 \leq \cdots \leq i-1 \leq i+1 \leq \cdots \leq n$$

for $0 \leq i \leq n$. These morphisms are called *cofaces*.

Example 2.2 The ordinal number epimorphisms $s^j : \mathbf{n+1} \to \mathbf{n}$ are defined by the strings

$$0 \leq 1 \leq \cdots \leq j \leq j \leq \cdots \leq n$$

for $0 \leq j \leq n$. These are the *codegeneracies*.

The cofaces and codegeneracies together satisfy the following relations

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \text{ if } i < j, \\ s^j s^i &= s^i s^{j+1} \text{ if } i \leq j \\ s^j d^i &= \begin{cases} d^i s^{j-1} & \text{if } i < j, \\ 1 & \text{if } i = j, j+1, \\ d^{i-1} s^j & \text{if } i > j+1. \end{cases} \end{aligned} \tag{2.1}$$

The ordinal number category $\boldsymbol{\Delta}$ is generated by the cofaces and codegeneracies, subject to the *cosimplicial identities* (2.1) [76]. In effect, every ordinal number morphism has a unique epi-monic factorization, and has a canonical form defined in terms of strings of codegeneracies and strings of cofaces.

A *simplicial set* is a functor $X : \boldsymbol{\Delta}^{op} \to \mathbf{Set}$, or a contravariant set-valued functor on the ordinal number category $\boldsymbol{\Delta}$. Such things are often written as $\mathbf{n} \mapsto X_n$, and X_n is called the set of *n-simplices* of X. A *map of simplicial sets* $f : X \to Y$ is a natural transformation of such functors. The simplicial sets and simplicial set maps form the category of simplicial sets, which will be denoted by $s\mathbf{Set}$.

A simplicial set is a simplicial object in the set category. Generally, $s\mathbf{A}$ denotes the category of simplicial objects $\boldsymbol{\Delta}^{op} \to \mathbf{A}$ in a category $\mathbf{A}$. Examples include the categories $s\mathbf{Gr}$ of simplicial groups, $s(R-\mathbf{Mod})$ of simplicial R-modules, $s(s\mathbf{Set}) = s^2\mathbf{Set}$ of bisimplicial sets, and so on.

Example 2.3 The topological standard n-simplex is the space

$$|\Delta^n| := \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid 0 \leq t_i \leq 1, \sum_{i=0}^{n} t_i = 1\}$$

The assignment $\mathbf{n} \mapsto |\Delta^n|$ is a cosimplicial space, or a cosimplicial object in spaces. A covariant functor $\boldsymbol{\Delta} \to \mathbf{A}$ is a *cosimplicial object* in the category $\mathbf{A}$.

If X is a topological space, then the *singular set* or *singular complex* $S(X)$ is the simplicial set which is defined by

$$S(X)_n = \hom(|\Delta^n|, X).$$

The assignment $X \mapsto S(X)$ defines a functor

$$S : \mathbf{CGHaus} \to s\mathbf{Set},$$

and this functor is called the *singular functor*. Here, **CGHaus** is the category of compactly generated Hausdorff spaces, which is the usual category of spaces for homotopy theory [32, I.2.4].

Example 2.4 The ordinal number $\mathbf{n}$ represents a contravariant functor

$$\Delta^n = \hom_{\mathbf{\Delta}}(\ ,\mathbf{n}),$$

which is called the *standard n-simplex*. Write

$$\iota_n = 1_{\mathbf{n}} \in \hom_{\mathbf{\Delta}}(\mathbf{n},\mathbf{n}).$$

The n-simplex ι_n is often called the *classifying n-simplex*, because the Yoneda Lemma implies that there is a natural bijection

$$\hom(\Delta^n, Y) \cong Y_n$$

that is defined by sending the simplicial set map $\sigma : \Delta^n \to Y$ to the element $\sigma(\iota_n) \in Y_n$. One usually says that a simplicial set map $\Delta^n \to Y$ is an n-simplex of Y.

In general, if $\sigma : \Delta^n \to X$ is a simplex of X, then the i^{th} *face* $d_i(\sigma)$ is the composite

$$\Delta^{n-1} \xrightarrow{d^i} \Delta^n \xrightarrow{\sigma} X,$$

while the j^{th} *degeneracy* $s_j(\sigma)$ is the composite

$$\Delta^{n+1} \xrightarrow{s^j} \Delta^n \xrightarrow{\sigma} X.$$

Example 2.5 The simplicial set $\partial\Delta^n$ is the subobject of Δ^n which is generated by the $(n-1)$-simplices d^i, $0 \leq i \leq n$, and Λ^n_k is the subobject of $\partial\Delta^n$ which is generated by the simplices d^i, $i \neq k$. The object $\partial\Delta^n$ is called the *boundary* of Δ^n, and Λ^n_k is called the k^{th} *horn*.

The faces $d^i : \Delta^{n-1} \to \Delta^n$ determine a covering

$$\bigsqcup_{i=0}^{n} \Delta^{n-1} \to \partial\Delta^n,$$

and for each $i < j$, there are pullback diagrams

$$\begin{tikzcd} \Delta^{n-2} \arrow[r, "d^{j-1}"] \arrow[d, "d^i"'] & \Delta^{n-1} \arrow[d, "d^i"] \\ \Delta^{n-1} \arrow[r, "d^j"'] & \Delta^n. \end{tikzcd}$$

It follows that there is a coequalizer

$$\bigsqcup_{i<j, 0 \leq i,j \leq n} \Delta^{n-2} \rightrightarrows \bigsqcup_{0 \leq i \leq n} \Delta^{n-1} \to \partial \Delta^n$$

in $s\mathbf{Set}$. Similarly, there is a coequalizer

$$\bigsqcup_{i<j, i,j \neq k} \Delta^{n-2} \rightrightarrows \bigsqcup_{0 \leq i \leq n, i \neq k} \Delta^{n-1} \to \Lambda^n_k.$$

Example 2.6 Suppose that a category C is *small* in the sense that the morphisms Mor (C) of C form a set. Examples of such things include all finite ordinal numbers $\mathbf{n}$, all monoids (small categories having one object), and all groups.

If C is a small category then there is a simplicial set BC with

$$BC_n = \hom(\mathbf{n}, C),$$

meaning the functors $\mathbf{n} \to C$. The simplicial structure on BC is defined by precomposition with ordinal number maps. The object BC is called, variously, the *classifying space* or *nerve* of the category C.

Note that the standard n-simplex Δ^n is the classifying space $B\mathbf{n}$ in this notation.

Example 2.7 Suppose that I is a small category, and that $X : I \to \mathbf{Set}$ is a set-valued functor. The *translation category*, or *category of elements*

$$*/X = E_I(X)$$

associated to X has for objects all pairs (i, x) with $x \in X(i)$, or equivalently all functions

$$* \xrightarrow{x} X(i).$$

A morphism $\alpha : (i, x) \to (j, y)$ is a morphism $\alpha : i \to j$ of I such that $\alpha_*(x) = y$, or equivalently a commutative diagram

$$\begin{tikzcd} & X(i) \arrow[dd, "\alpha_*"] \\ * \arrow[ur, "x"] \arrow[dr, "y"'] & \\ & X(j) \end{tikzcd}$$

The simplicial set $B(E_I X)$ is often called the *homotopy colimit* for the functor X, and one writes

$$\underrightarrow{\mathrm{holim}}_I\, X = B(E_I X).$$

There is a canonical functor $E_I X \to I$ which is defined by the assignment $(i, x) \mapsto i$, and induces a canonical simplicial set map

$$\pi : B(E_I X) = \underrightarrow{\mathrm{holim}}_I\, X \to BI.$$

The functors $\mathbf{n} \to E_I X$ can be identified with strings

$$(i_0, x_0) \xrightarrow{\alpha_1} (i_1, x_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (i_n, x_n).$$

Such a string is uniquely specified by the underlying string $i_0 \to \cdots \to i_n$ in the index category Y and $x_0 \in X(i_0)$. It follows that there is an identification

$$(\underrightarrow{\mathrm{holim}}_I\, X)_n = B(E_I X)_n = \bigsqcup_{i_0 \to \cdots \to i_n} X(i_0).$$

This construction is functorial with respect to natural transformations in X. Thus, a diagram $Y : I \to s\mathbf{Set}$ in simplicial sets determines a bisimplicial set with (n, m) simplices

$$B(E_I Y)_m = \bigsqcup_{i_0 \to \cdots \to i_n} Y(i_0)_m.$$

The *diagonal* $d(Z)$ of a bisimplicial set Z is the simplicial set with n-simplices $Z_{n,n}$. Equivalently, $d(Z)$ is the composite functor

$$\boldsymbol{\Delta}^{op} \xrightarrow{\Delta} \boldsymbol{\Delta}^{op} \times \boldsymbol{\Delta}^{op} \xrightarrow{Z} \mathbf{Set}$$

where Δ is the diagonal functor.

The diagonal $dB(E_I Y)$ of the bisimplicial set $B(E_I Y)$ is the *homotopy colimit* $\underrightarrow{\mathrm{holim}}_I\, Y$ of the diagram $Y : I \to s\mathbf{Set}$ in simplicial sets. There is a natural simplicial set map

$$\pi : \underrightarrow{\mathrm{holim}}_I\, Y \to BI.$$

Example 2.8 Suppose that X and Y are simplicial sets. There is a simplicial set $\mathbf{hom}(X, Y)$ with n-simplices

$$\mathbf{hom}(X, Y)_n = \hom(X \times \Delta^n, Y),$$

called the *function complex*.

There is a natural simplicial set map

$$ev : X \times \mathbf{hom}(X, Y) \to Y,$$

which sends the pair $(x, f : X \times \Delta^n \to Y)$ to the simplex $f(x, \iota_n)$. Suppose that K is another simplicial set. The function

$$ev_* : \hom(K, \mathbf{hom}(X, Y)) \to \hom(X \times K, Y),$$

which is defined by sending the map $g : K \to \mathbf{hom}(X, Y)$ to the composite

$$X \times K \xrightarrow{1 \times g} X \times \mathbf{hom}(X, Y) \xrightarrow{ev} Y,$$

is a natural bijection, giving the *exponential law*

$$\hom(K, \mathbf{hom}(X, Y)) \cong \hom(X \times K, Y).$$

This natural isomorphism gives $s\mathbf{Set}$ the structure of a cartesian closed category. The function complexes also give $s\mathbf{Set}$ the structure of a category enriched in simplicial sets.

The *simplex category* $\mathbf{\Delta}/X$ for a simplicial set X has for objects all simplices $\Delta^n \to X$. Its morphisms are the incidence relations between the simplices (meaning all commutative diagrams).

$$\begin{tikzcd} \Delta^m \arrow[d, "\theta"'] \arrow[dr, "\tau"] & \\ \Delta^n \arrow[r, "\sigma"'] & X \end{tikzcd}$$

The *realization* $|X|$ of a simplicial set X is defined by

$$|X| = \varinjlim_{\Delta^n \to X} |\Delta^n|,$$

where the colimit is defined for the functor $\mathbf{\Delta}/X \to \mathbf{CGHaus}$, which takes a simplex $\Delta^n \to X$ to the space $|\Delta^n|$.

The space $|X|$ is constructed by glueing together copies of the topological standard simplices of Example 2.3 along the incidence relations of the simplices of X.

The assignment $X \mapsto |X|$ defines a functor

$$|\ | : s\mathbf{Set} \to \mathbf{CGHaus}.$$

The proof of the following lemma is an exercise:

Lemma 2.9 *The realization functor $|\ |$ is left adjoint to the singular functor S.*

Example 2.10 The realization $|\Delta^n|$ of the standard n-simplex is the space $|\Delta^n|$ described in Example 2.3, since the simplex category $\mathbf{\Delta}/\Delta^n$ has a terminal object, namely $1 : \Delta^n \to \Delta^n$.

Example 2.11 The realization $|\partial\Delta^n|$ of the simplicial set $\partial\Delta^n$ is the topological boundary $\partial|\Delta^n|$ of the space $|\Delta^n|$. The space $|\Lambda^n_k|$ is the part of the boundary $\partial|\Delta^n|$

with the face opposite the vertex k removed. To see this, observe that the realization functor is a left adjoint and therefore preserves coequalizers and coproducts.

The *n-skeleton* $\mathrm{sk}_n X$ of a simplicial set X is the subobject generated by the simplices X_i, $0 \leq i \leq n$. The ascending sequence of subcomplexes

$$\mathrm{sk}_0 X \subset \mathrm{sk}_1 X \subset \mathrm{sk}_2 X \subset \dots$$

defines a filtration of X, and there are pushout diagrams.

$$\begin{tikzcd} \bigsqcup_{x \in NX_n} \partial \Delta^n \arrow[r] \arrow[d] & \mathrm{sk}_{n-1} X \arrow[d] \\ \bigsqcup_{x \in NX_n} \Delta^n \arrow[r] & \mathrm{sk}_n X \end{tikzcd}$$

Here, NX_n denotes the set of non-degenerate n-simplices of X. An n-simplex x of X is non-degenerate if it is not of the form $s_i y$ for some $(n-1)$-simplex y.

It follows that the realization of a simplicial set is a CW-complex. Every monomorphism $A \to B$ of simplicial sets induces a cofibration $|A| \to |B|$ of spaces, since $|B|$ is constructed from $|A|$ by attaching cells.

The realization functor preserves colimits (is right exact) because it has a right adjoint. The realization functor, when interpreted as taking values in compactly generated Hausdorff spaces, also has a fundamental left exactness property:

Lemma 2.12 *The realization functor*

$$|\ | : s\mathbf{Set} \to \mathbf{CGHaus}.$$

preserves finite limits. Equivalently, it preserves finite products and equalizers.

This result is proved in [30].

2.2 Model Structure for Simplicial Sets

This section summarizes material which is presented in some detail in [32].

Say that a map $f : X \to Y$ of simplicial sets is a *weak equivalence* if the induced map $f_* : |X| \to |Y|$ is a weak equivalence of **CGHaus**. A map $i : A \to B$ of simplicial sets is a *cofibration* if and only if it is a monomorphism, meaning that all functions $i : A_n \to B_n$ are injective. A simplicial set map $p : X \to Y$ is a *fibration* if and only if it has the right lifting property with respect to all trivial cofibrations.

As usual, a *trivial cofibration* (respectively *trivial fibration*) is a cofibration (respectively fibration) which is also a weak equivalence.

In all that follows, a *closed model category* will be a category $\mathbf{M}$ equipped with three classes of maps, called cofibrations, fibrations and weak equivalences such that the following axioms are satisfied:

CM1 The category **M** has all finite limits and colimits.

CM2 Suppose given a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & Z & \end{tikzcd}$$

in **M**. If any two of the maps f, g and h are weak equivalences, then so is the third.

CM3 If a map f us a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4 Suppose given a commutative solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where i is a cofibration and p is a fibration. Then the dotted arrow exists, making the diagram commute, if either i or p is a weak equivalence.

CM5 Every map $f : X \to Y$ has factorizations $f = p \cdot i$ and $f = q \cdot j$, in which i is a cofibration and a weak equivalence and p is a fibration, and j is a cofibration and q is a fibration and a weak equivalence.

The definition of closed model category which is displayed here is the traditional one, which is due to Quillen [86]. There are variants in the literature, which involve either removing the finiteness condition from the limits and colimits in **CM1**, or insisting that the factorizations of **CM5** are functorial. These conditions almost always hold in practice, and in particular they hold for all model structures that we use.

There are common adjectives which decorate closed model structures. For example, one says that the model structure on **M** is *simplicial* if the category can be enriched in simplicial sets in a way that behaves well with respect to cofibrations and fibrations, and the model structure is *proper* if weak equivalences are preserved by pullback along fibrations and pushout along cofibrations. A model structure is *cofibrantly generated* if its classes of cofibrations and trivial cofibrations are generated by sets of maps in a suitable sense. The factorizations of **CM5** can be constructed functorially in a cofibrantly generated model structure, with a *small object argument*. Much more detail can be found in [32] or [44].

Theorem 2.13 *With the definitions given above of weak equivalence, cofibration and fibration, the category s**Set** of simplicial sets satisfies the axioms for a closed model category.*

Here are the basic ingredients of the proof:

Lemma 2.14 *A map $p : X \to Y$ is a trivial fibration if and only if it has the right lifting property with respect to all inclusions $\partial \Delta^n \subset \Delta^n$, $n \geq 0$.*

The proof of Lemma 2.14 is formal. If p has the right lifting property with respect to all inclusions $\partial\Delta^n \subset \Delta^n$ then it is a homotopy equivalence. Conversely, p has a factorization $p = q \cdot j$, where j is a cofibration and q has the right lifting property with respect to all maps $\partial\Delta^n \subset \Delta^n$, so that j is a trivial cofibration, and then p is a retract of q by a standard argument.

The following result can be proved with simplicial approximation techniques [62].

Lemma 2.15 *Suppose that a simplicial set X has at most countably many non-degenerate simplices. Then the set of path components $\pi_0|X|$ and all homotopy groups $\pi_n(|X|, x)$ are countable.*

The following *bounded monomorphism property* for simplicial sets is a consequence.

Lemma 2.16 *Suppose that $i : X \to Y$ is a trivial cofibration and that $A \subset Y$ is a countable subcomplex. Then there is a countable subcomplex $B \subset Y$ with $A \subset B$ such that the map $B \cap X \to B$ is a trivial cofibration.*

Lemma 2.16 implies that the set of countable trivial cofibrations generates the class of all trivial cofibrations, while Lemma 2.14 implies that the set of all inclusions $\partial\Delta^n \subset \Delta^n$ generates the class of all cofibrations. Theorem 2.13 then follows from small object arguments.

A *Kan fibration* is a map $p : X \to Y$ of simplicial sets which has the right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$. A *Kan complex* is a simplicial set X for which the canonical map $X \to *$ is a Kan fibration.

Every fibration is a Kan fibration, and every fibrant simplicial set is a Kan complex.

Kan complexes Y have combinatorially defined homotopy groups: if $x \in Y_0$ is a vertex of Y, then

$$\pi_n(Y, x) = \pi((\Delta^n, \partial\Delta^n), (Y, x))$$

where $\pi(\ ,\ )$ denotes simplicial homotopy classes of maps and pairs. The path components of any simplicial set X are defined by the coequalizer

$$X_1 \rightrightarrows X_0 \to \pi_0 X,$$

where the maps $X_1 \to X_0$ are the face maps d_0, d_1. Say that a map $f : Y \to Y'$ of Kan complexes is a *combinatorial weak equivalence* if it induces isomorphisms

$$\pi_n(Y, x) \xrightarrow{\cong} \pi_n(Y', f(x))$$

for all $x \in Y_0$, and

$$\pi_0(Y) \xrightarrow{\cong} \pi_0(Y').$$

Going further requires the following major theorem, due to Quillen [88],[32]:

Theorem 2.17 *The realization of a Kan fibration is a Serre fibration.*

The proof of this result requires much of the classical homotopy theory of Kan complexes (in particular the theory of minimal fibrations), and will not be discussed here—see [32].

Here are the consequences:

Theorem 2.18 *[Milnor theorem] Suppose that Y is a Kan complex and that $\eta : Y \to S(|Y|)$ is the adjunction homomorphism. Then η is a combinatorial weak equivalence.*

It follows that the combinatorial homotopy groups of $\pi_n(Y, x)$ coincide up to natural isomorphism with the ordinary homotopy groups $\pi_n(|Y|, x)$ of the realization, for all Kan complexes Y. The proof of Theorem 2.18 is a long exact sequence argument that is based on the path-loop fibre sequences in simplicial sets. These are Kan fibre sequences, and the key is to know, from Theorem 2.17 and Lemma 2.12, that their realizations are fibre sequences.

Theorem 2.19 *Every Kan fibration is a fibration.*

Proof [Sketch] The key step in the proof is to show, using Theorem 2.18, that every map $p : X \to Y$ which is a Kan fibration and a weak equivalence has the right lifting property with respect to all inclusions $\partial \Delta^n \subset \Delta^n$. This is true if Y is a Kan complex, since p is then a combinatorial weak equivalence by Theorem 2.18. Maps which are weak equivalences and Kan fibrations are stable under pullback by Theorem 2.17 and Lemma 2.12. It follows from Theorem 2.18 that all fibres of the Kan fibration p are contractible. It also follows, by taking suitable pullbacks, that it suffices to assume that p has the form $p : X \to \Delta^k$. If F is the fibre of p over the vertex 0, then the Kan fibration p is fibrewise homotopy equivalent to the projection $F \times \Delta^k \to \Delta^k$ [32, I.10.6]. This projection has the desired right lifting property, as does any other Kan fibration in its fibre homotopy equivalence class—see [32, I.7.10]. □

Remark 2.20 Theorem 2.19 implies that the model structure of Theorem 2.13 consists of cofibrations, Kan fibrations and weak equivalences. This is the standard, classical model structure for simplicial sets. The identification of the fibrations with Kan fibrations is the interesting part of this line of argument.

The realization functor preserves cofibrations and weak equivalences, and it follows that the adjoint pair

$$| \ | : s\mathbf{Set} \leftrightarrows \mathbf{CGHaus} : S,$$

is a *Quillen adjunction*. The following is a consequence of Theorem 2.18:

Theorem 2.21 *The adjunction maps $\eta : X \to S(|X|)$ and $\epsilon : |S(Y)| \to Y$ are weak equivalences, for all simplicial sets X and spaces Y, respectively.*

In particular, the standard model structures on the categories $s\mathbf{Set}$ of simplicial sets and **CGHaus** of compactly generated Hausdorff spaces are Quillen equivalent.

http://www.springer.com/978-1-4939-2299-4

Local Homotopy Theory
Jardine, J.
2015, IX, 508 p. 514 illus., Hardcover
ISBN: 978-1-4939-2299-4