

# Chapter 2

## Structural Modeling

### 2.1 Introduction

Aircraft have thin-walled, built-up structures for high strength-to-weight ratio and stiffness. The transverse and longitudinal members share the external loads with the outer skin panels in a semimonocoque construction. It would appear that an analysis of such a structure requires a detailed model of each component, which specifies the exact manner in which it is connected to the other members. Such a modeling would be a daunting task, requiring enormous computational resources. Fortunately, although a detailed analysis of individual structural components is indeed necessary for structural design, it is not required for aeroelastic purposes where some simplifying approximations can be made. Many aircraft components—such as the wings and the fuselage—are designed to be slender and streamlined for a high lift-to-drag ratio. The associated structures thus have small thicknesses and can be often idealized as either solid beams or plates. Furthermore, the necessity of preserving aerodynamic shape results in a much higher bending stiffness in the transverse (chordwise or radial) direction, compared with that in the longitudinal (lengthwise or spanwise) direction, which is achieved in practice by closely spaced ribs and frames. The resulting assumption of chordwise rigidity is quite valuable in reducing the degrees of freedom for a structural model. However, such a model would be inaccurate for wings with very small aspect ratio where chordwise and spanwise bending stiffnesses would be comparable. Another major simplification is the fact that any inelastic deformation leading to buckling of skin panels under design loads is unacceptable from aerodynamic design viewpoint. Therefore, an elastic stress–strain behavior is necessary, and results in a linear load–displacement relationship—a valuable model from aeroelastic perspective. However, post-buckling behavior of skin panels requires nonlinear structural modeling, which is excluded from usual aeroelastic design and analysis.

The main emphasis of an aeroelastic model is upon wing-like structures, which are quite thin in comparison with the chord and span. This offers a valuable modeling simplicity, which combined with the chordwise rigidity of high aspect-ratio wings, results in plane cross-sections remaining essentially plane due to a free warping of the structure under twisting loads. Conversely, a bending load would not produce any twisting deformation due to the same reason. Hence by Saint-Venant's theory [68],

one can decouple bending and torsion, thereby leading to the very useful concepts of shear center and elastic axis. Of course, such a decoupling is not possible for either short beams, or plate-like structures where sectional warping invariably causes some normal (bending) stresses. Furthermore, if shear deformations can be neglected due to an essentially thin beam, the bending deformations can be treated by a simple Euler–Bernoulli beam theory.

## 2.2 Static Load Deflection Model

Consider an elastic wing with an unloaded and undeformed mean surface defined by  $z = s(x, y)$  and generated by smoothly joining the wing's chord lines, camber lines, or any other centroidal features of the cross-sections. If a concentrated load,  $\mathbf{P}$ , is now applied at a point,  $(\xi, \eta)$ , located on the original mean surface, it will cause a structural deflection,  $\delta(x, y)$  such that a new equilibrium is achieved in the deformed configuration given by the deformed surface,  $z = s'(x, y)$ . The deflection vector,  $\delta$ , can be regarded as a change of location of an original point on the surface,  $(x, y, z)$ , to its new position on the deformed surface,  $(x', y', z')$ , and is given by

$$\delta(x, y) = \begin{Bmatrix} x' - x \\ y' - y \\ s'(x', y') - s(x, y) \end{Bmatrix} \quad (2.1)$$

The deflection vector is thus based upon a one-to-one mapping of all points in the closed set constituting the original surface to those on the deformed surface:

$$\begin{aligned} &\{(x, y, z) : x_1 \leq x \leq x_2; y_1 \leq y \leq y_2; z = s(x, y)\} \\ &\rightarrow \{(x', y', z') : x'_1 \leq x' \leq x'_2; y'_1 \leq y' \leq y'_2; z' = s(x', y')\} \end{aligned} \quad (2.2)$$

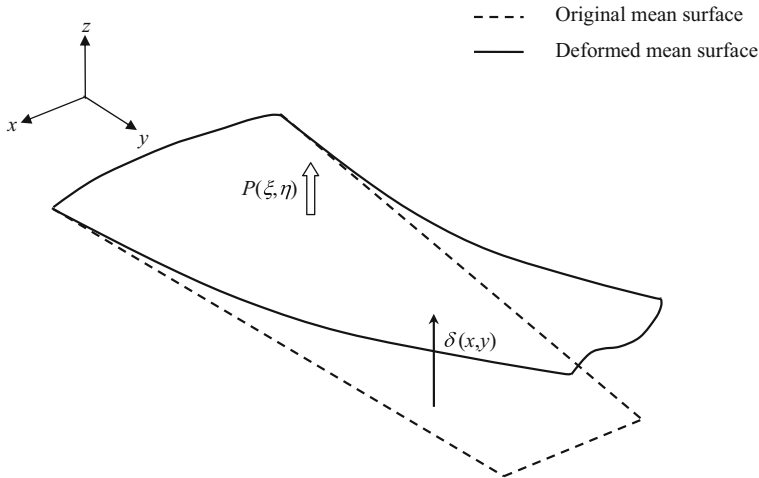
Such a map can be geometrically represented by the transformation

$$\begin{Bmatrix} x' \\ y' \\ s'(x', y') \end{Bmatrix} = \mathbf{T}(x', y', z' : x, y, z) \begin{Bmatrix} x \\ y \\ s(x, y) \end{Bmatrix} \quad (2.3)$$

Since the mean surface of most wing-like structures is essentially flat, the deflection at each point can be approximated by the displacement normal to the original surface,

$$\delta(x, y) = z' - s(x, y), \quad (2.4)$$

as shown in Fig. 2.1. In such a case, the deformed mean surface may not turn out to be flat, but can have a local curvature due to shear, twist, and spanwise and chordwise bending.



**Fig. 2.1** Deformation of a wing-like structure under a concentrated load

The transformation matrix,  $\mathbf{T}$ , in Eq.(2.3) produced by a general loading must obey the material properties called *constitutive relationships*, and is also subject to the geometric constraints (also called kinematical relationships, compatibility requirements, or boundary conditions) on the structure. For a linearly elastic structure, the constitutive relationships take the form of a linear stress–strain behavior, such as

$$\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\epsilon}, \quad (2.5)$$

where  $\boldsymbol{\tau}$  and  $\boldsymbol{\epsilon}$  are the stress and strain vectors, respectively, experienced by an infinitesimal structural element, and  $\mathbf{C}$  denotes the matrix comprising the material properties. The linear stress–strain behavior of the material produces a linear relationship between the load and displacement of the structure, given by:

$$\delta(x, y) = \mathbf{R}(x, y : \xi, \eta) \mathbf{P}(\xi, \eta), \quad (2.6)$$

where  $\mathbf{R}(x, y : \xi, \eta)$  is the matrix of *flexibility influence-coefficient* functions (also called *Green's functions*). By applying linear superposition, Eq. (2.6) can be extended for the case of a continuously distributed load per unit area,  $\mathbf{p}(\xi, \eta)$ , as follows:

$$\delta(x, y) = \iint_s \mathbf{R}(x, y : \xi, \eta) \mathbf{p}(\xi, \eta) d\xi d\eta. \quad (2.7)$$

Unfortunately, the flexibility influence-coefficient functions are rarely available in a closed form for any but the most simple structures. Therefore, numerical approximations must be made for the integral relationship given by Eq. (2.7). Such approximations are based upon a discretization of Eq. (2.7), whereby a continuous structure with infinitely many degrees of freedom is converted into an equivalent finite dimensional form. For example, the mean surface can be approximated by

$n$  flat elemental panels of individual dimensions,  $(\Delta\xi_i, \Delta\eta_i), i = 1 \dots n$ . The load distribution on the  $j$ th panel is then approximated by an average generalized load,  $\mathbf{P}_j = \mathbf{p}(\xi, \eta)\Delta\xi_j \Delta\eta_j$  acting at a given *load point* (such as the panel centroid) in each panel. Similarly, the displacement,  $\delta(x, y)$ , averaged over the  $i$ th panel is taken as the average deflection vector,  $\delta_i$ , at a given *collocation point*,  $(x_i, y_i), i = 1 \dots n$ . The discretized load–displacement relationship is then given for the  $i$ th panel as follows:

$$\delta_i = \sum_{j=1}^n \mathbf{R}_{ij} \mathbf{P}_j ; i = 1 \dots n . \quad (2.8)$$

Often, the individual elements of the displacement and load vectors are identified as *generalized displacements*,

$$\delta_i = \begin{Bmatrix} q_{1i} \\ q_{2i} \\ q_{3i} \end{Bmatrix} , \quad (2.9)$$

and *generalized loads*,

$$\mathbf{P}_i = \begin{Bmatrix} Q_{1i} \\ Q_{2i} \\ Q_{3i} \end{Bmatrix} , \quad (2.10)$$

respectively, corresponding to the individual degrees of freedom at each point. Then Eq. (2.8) collected for all points takes the following vector-matrix form:

$$\mathbf{q} = \mathbf{R}\mathbf{Q} , \quad (2.11)$$

where the  $3n \times 3n$  influence-coefficient matrix,  $\mathbf{R}$ , consists of  $\mathbf{R}_{ij}$  as its elements. Here, we note that  $\mathbf{P}_i$  denotes the vector of all generalized loads ( $Q_{1i}, Q_{2i}, Q_{3i}$ ) acting on the  $i$ th panel, while  $\mathbf{Q}$  is the generalized loads vector for the entire structure derived by collecting all the generalized loads acting on all the panels. Similarly, the generalized displacement vector on the  $i$ th panel,  $\delta_i$ , is to be distinguished from the overall generalized displacement vector of the structure,  $\mathbf{q}$ .

By making simplifying assumptions for a typical aircraft, the number of generalized coordinates can be significantly reduced. Such idealizations include chordwise rigidity, plate or beam-shaft approximations, and negligible streamwise loading on the structure. Due to their smaller dimensions and high stiffnesses, control surfaces are modeled simply by rotating angles about rigid hinge axes. Various definitions of the generalized loads and generalized displacements are possible, depending upon the idealizations and constitutive relationships used in deriving Eq. (2.11). There are also various alternative techniques for deriving the load–displacement relationship of Eq. (2.11) from constitutive relations and structural constraints. These include the *lumped parameters approximation*, the *finite-element method* (FEM) (or *Galerkin*

method), the *assumed modes* (or *Rayleigh–Ritz*) method, and the *boundary-element method*. Of these, the FEM is the most commonly employed due to its ease of implementation and modeling efficiency. We shall have the occasion to consider some examples of the FEM modeling a little later.

The influence-coefficient matrix,

$$\mathbf{R} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1N} \\ R_{21} & R_{22} & \cdots & R_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ R_{N1} & R_{N2} & \cdots & R_{NN} \end{pmatrix}, \quad (2.12)$$

with  $N = 3n$ , consists of influence coefficients,  $R_{ij}$ , which are defined as the  $i$ th generalized *virtual coordinate*,  $\delta q_{ij}$ , produced by an isolated generalized load at a given point,  $Q_j$ ,

$$\delta q_{ij} = R_{ij} P_j. \quad (2.13)$$

A virtual coordinate is an arbitrary, infinitesimal deflection in any of the three possible directions at a given point due to an isolated generalized load, and must be compatible with any kinematical constraints of the structure. The actual generalized coordinate,  $q_i$ , is a sum of all the virtual coordinates,  $\delta q_{ij}$ , caused by the generalized load,  $Q_j$ , and is given by the  $i$ th row of Eq. (2.11),

$$q_i = \sum_{j=1}^N \delta q_{ij} = \sum_{j=1}^N R_{ij} Q_j. \quad (2.14)$$

Therefore, the discrete influence coefficient,  $R_{ij}$ , can be understood as the  $i$ th virtual generalized coordinate due to the  $j$ th *unit* generalized load. The *reciprocal principle* of a linear structure requires that the  $i$ th virtual coordinate due to the  $j$ th unit load is the same as the  $j$ th virtual coordinate caused by the  $i$ th generalized load, i.e.,

$$R_{ij} = R_{ji}, \quad (2.15)$$

which implies that the matrix  $\mathbf{R}$  is symmetric.

In order to determine the generalized loads from the generalized displacements they actually produce an inversion of Eq. (2.11) is required as follows:

$$\mathbf{P} = \mathbf{K}\mathbf{q}, \quad (2.16)$$

where  $\mathbf{K} = \mathbf{R}^{-1}$  is the generalized *stiffness matrix* of the structure. Both  $\mathbf{R}$  and  $\mathbf{K}$  must be nonsingular and symmetric matrices. The element of  $[\mathbf{K}]$ ,  $k_{ij}$ —called the *stiffness coefficient*—is the  $i$ th generalized *virtual load* due to the  $j$ th *unit* generalized displacement. The work done by a generalized virtual load,

$$\delta Q_i = k_{ij} q_j, \quad (2.17)$$

in producing a generalized displacement,  $q_j$ , is given by

$$U_{ij} = \int \delta Q_i dq_j = \int k_{ij} q_j dq_j = \frac{1}{2} k_{ij} q_j^2 = \frac{1}{2} \delta Q_i q_j. \quad (2.18)$$

When summed over all points on the structure, the net work done by all the static forces is the total *strain energy* stored in the structure, given by

$$U = \sum_{i=1}^N \sum_{j=1}^N U_{ij} = \frac{1}{2} \mathbf{Q}^T \mathbf{q} = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q}. \quad (2.19)$$

The strain energy is the potential energy responsible for restoring the structure to its original shape once the loading is removed, and its quadratic form given by Eq. (2.19) is an important consequence of the linear elastic behavior. Since the external forces must be balanced by equal and opposite internal forces for a static equilibrium, one can regard  $U$  as the net work done by the internal, restoring (or conservative) forces.

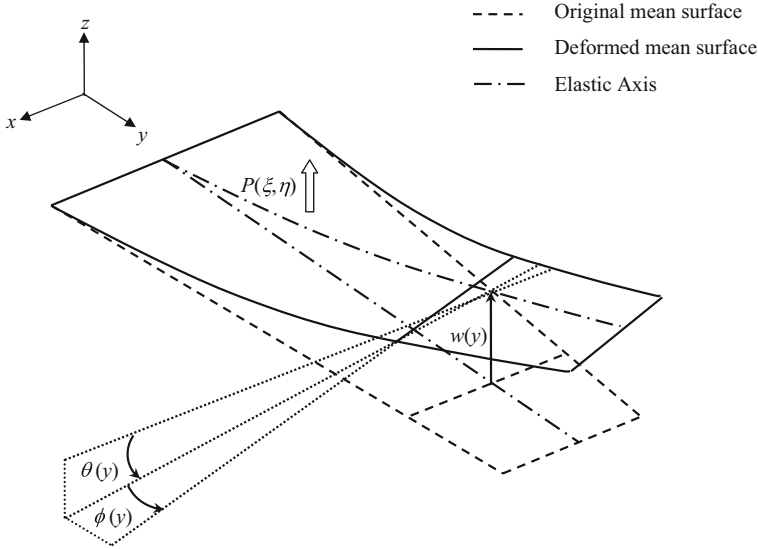
### 2.3 Beam-Shaft Idealization

Consider a thin, high aspect-ratio wing with an essentially flat mean surface. Let  $(x, y, z)$  be Cartesian coordinates, such that  $x$  is in the chordwise direction measured from the elastic axis,  $y$  in the spanwise direction along the elastic axis on the mean plane, and  $z$  is normal to the mean plane. Saint-Venant's theory [68] postulates that a point exists at each cross-section of a slender beam about which a twisting load will produce a pure twist and a free warping, but no bending deformation. Such a point is called the *shear center*. Conversely, if a vertical load  $P$  is applied directly at the shear center, it will only cause pure bending deformation without any twisting or warping of the beam. The *elastic axis* is the line joining the shear centers at all spanwise locations. Assuming there is no bending in the chordwise ( $x$ ) direction, and that plane cross-sections remain plane in the deformed configuration, the resulting structural displacement at any given point from the original (undeformed) shape can be represented by a combination of the normal deflection of the elastic axis at the given spanwise station,  $w(y)$ , the twist angle of the section about the elastic axis,  $\theta(y)$ , and the in-plane warp angle,  $\phi(y)$ , as depicted in Fig. 2.2. The net vertical deflection at location  $(x, y)$  is thus given by

$$\delta(x, y) = w(y) + x \tan \theta(y), \quad (2.20)$$

whereas the in-plane deformation due to warping is merely  $x \tan \phi(y)$ . Since the angles  $\theta, \phi$  are small, one can apply the approximation  $\tan \theta \simeq \theta$  and  $\tan \phi \simeq \phi$ , leading to the linear relationship

$$\delta(x, y) = w(y) + x\theta(y). \quad (2.21)$$



**Fig. 2.2** Deformation of a thin wing of large aspect ratio under a concentrated normal load

The warp angle  $\phi$  is inconsequential for aerodynamic loading, thus we have no need to model it any further. Since the structure is assumed to be linearly elastic, the load and displacement are linearly related by

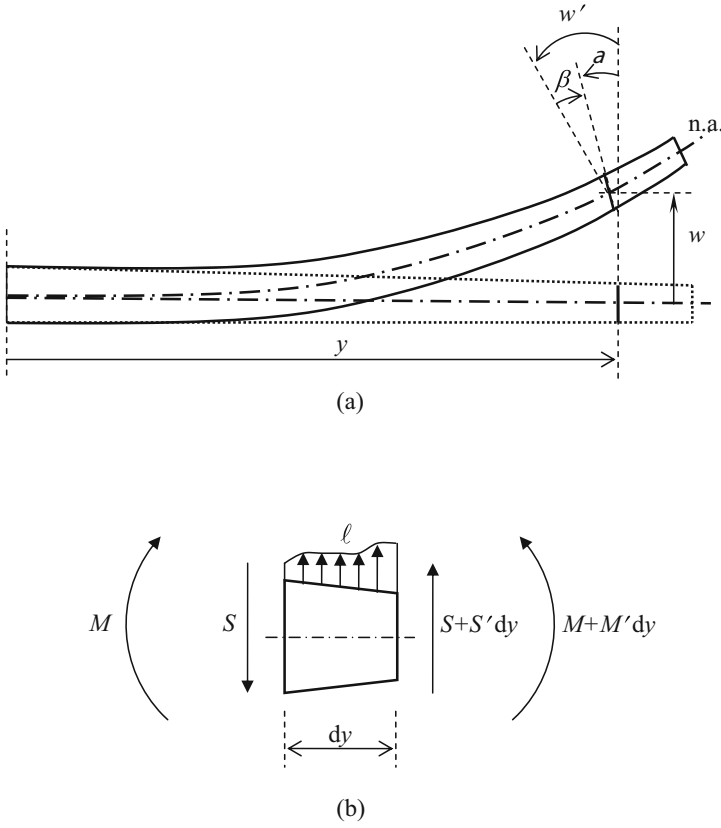
$$\delta(x, y) = R(x, y : \xi, \eta) P(\xi, \eta), \quad (2.22)$$

where  $R(x, y : \xi, \eta)$  is the flexibility influence-coefficient function (Green's function). By applying linear superposition, Eq. (2.22) can be extended for the case of a continuously distributed, normal load per unit area (pressure),  $p(\xi, \eta)$ , as follows:

$$\delta(x, y) = \iint_s R(x, y : \xi, \eta) p(\xi, \eta) d\xi d\eta. \quad (2.23)$$

The constitutive relations of the bending and twisting deformations are separately derived by considering a segment of the structure. For this purpose, the spanwise direction  $y$  is taken along the elastic axis, and bending deflection,  $w(y)$  measured normal to the mean surface (called *neutral axis*) as shown in Fig. 2.3a. The kinematics (or compatibility) of the bending and shearing deformations is based upon the assumption that an originally plane section normal to the neutral axis must remain plane after deformation. However, this section can undergo a rotation due to shear deformation (shearing strain),  $\beta(y)$ , such that the section is no longer normal to the neutral axis. The bending slope,  $w'(y)$ , and the rotation angle due to shear,  $\beta(y)$ , thus collectively produce the net rotation,  $\alpha(y)$ , according to the following kinematical relationship:

$$\alpha = w' - \beta. \quad (2.24)$$



**Fig. 2.3** Beam deflection geometry and static equilibrium of a beam segment

Equilibrium of a beam segment of infinitesimal length,  $dy$ , with a static lift load per unit span,  $\ell(y)$ , requires an internal shear force,  $S(y)$ , and bending moment,  $M(y)$ , in order to balance the external load,  $\ell(y)$ , as shown in Fig. 2.3b. Neglecting second and higher order terms of  $dy$  results in the following linear equilibrium equations:

$$-S' = \ell, \quad (2.25)$$

$$S + M' = 0. \quad (2.26)$$

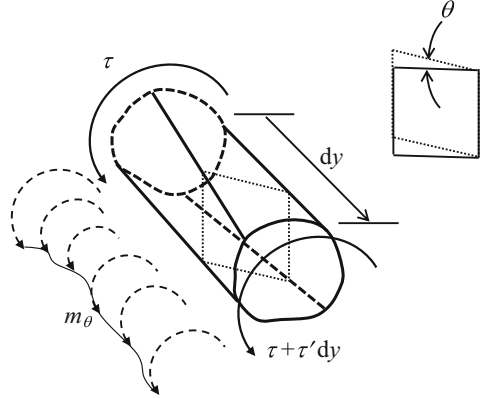
The bending and shear constitutive relationships of a material with Young's modulus of elasticity,  $E$ , and shear modulus,  $G$ , are the following:

$$\alpha' = \frac{M}{EI}, \quad (2.27)$$

$$\beta = \frac{S}{GK}, \quad (2.28)$$



**Fig. 2.4** Twisting of a slender shaft at a spanwise station,  $y$



where  $I(y)$  is the area moment of inertia and  $K(y)$  the shearing constant of the beam cross-section. The quantity  $EI(y)$  is called the bending stiffness and  $GK(y)$  the shearing stiffness of the local beam cross-section. Substitution of Eqs. (2.27), (2.28), and (2.24) into Eqs. (2.25) and (2.26) results in the following differential equations for the beam:

$$(EI\alpha')'' = \ell, \quad (2.29)$$

$$(EI\alpha')' + GK(\alpha - w') = 0, \quad (2.30)$$

which must be solved for  $\alpha(y)$  and  $w(y)$ , subject to the boundary conditions of the structure. The net strain energy of the wing semispan idealized as a beam of length  $b/2$  is then given by

$$U = \frac{1}{2} \int_0^{b/2} EI (\alpha')^2 dy + \frac{1}{2} \int_0^{b/2} GK (\alpha - w')^2 dy. \quad (2.31)$$

For thin, slender structures, the shear deformation,  $\beta(y)$ , can be neglected in comparison with the bending slope,  $w'(y)$ , leading to the following *Euler–Bernoulli* beam equation:

$$(EIw'')'' = \ell, \quad (2.32)$$

which must be solved for bending deflection,  $w(y)$ , subject to the boundary conditions. The net bending strain energy is now simply the following:

$$U = \frac{1}{2} \int_0^{b/2} EI (w'')^2 dy. \quad (2.33)$$

Wherever possible to apply, Euler–Bernoulli assumptions are extremely valuable due to the simplicity of the resulting model.

For a slender, shaft-like structure (Fig. 2.4), the twisting deformation,  $\theta(y)$ , by shear of an originally straight edged element, is related to the local twisting moment,  $\tau(y)$ , about the elastic axis by Saint-Venant's theory [68] as follows:

$$\tau = GJ\theta', \quad (2.34)$$

where  $G$  is the shear modulus and  $J(y)$  the local torsional constant (polar moment of inertia) of the cross-section. Due to the linear proportionality of the local torque,  $\tau(y)$ , with the twisting slope,  $\theta'(y)$ , the factor  $GJ(y)$  is termed torsional stiffness of the structure. Equilibrium of a segment of infinitesimal length,  $dy$ , under a distributed torque (pitching moment) load per unit length,  $m_\theta(y)$ , results in the following differential equation:

$$(GJ\theta')' + m_\theta = 0, \quad (2.35)$$

which must be solved for twist angle,  $\theta(y)$ , subject to the boundary conditions on the shaft. The net twisting strain energy of the wing semispan is then given by

$$U = \frac{1}{2} \int_0^{b/2} GJ (\theta')^2 dy. \quad (2.36)$$

When Euler–Bernoulli assumption of negligible shear deformation is applied along with that of a slender shaft, the net vertical deflection at a point,  $\delta(x, y)$ , is given by Eq. (2.21), which implies

$$\frac{\partial \delta}{\partial x} = \theta(y). \quad (2.37)$$

## 2.4 Dynamics

Aerodynamic loads on a vibrating structure are time dependent. Therefore, in order to construct a dynamic model, it is necessary to consider not only the strain (potential) energy,  $U$ , but also the net *kinetic energy*,  $T$ , and the work done upon the structure by nonconservative forces,  $W_n$ . For a thin wing, the linear load–displacement relationship can be used to derive the strain energy in terms of the vertical deflection,  $\delta(x, y, t)$ ,

$$U = \frac{1}{2} \iint_s \delta(x, y, t) \iint_s k(x, y : \xi, \eta) \delta(\xi, \eta, t) d\xi d\eta dx dy, \quad (2.38)$$

where  $k(x, y : \xi, \eta)$  is the stiffness influence function of the displacement produced at point  $(\xi, \eta)$  by a load applied at the point  $(x, y)$ . Similarly, the net kinetic energy of the structure is given by

$$T = \frac{1}{2} \iint_s \rho(x, y) \dot{\delta}^2(x, y, t) dx dy, \quad (2.39)$$

where  $\rho(x, y)$  is the structural mass per unit area and  $\dot{\delta}(x, y, t)$ , the local vertical velocity.

An aircraft wing is a thin structure essentially cantilevered at the root, with the span ( $b$ ) generally much larger than the average chord (Fig. 2.5). For an ASE (aeroservoelasticity) model, one is primarily concerned with wings of large aspect ratio that

Aeroservoelasticity

Modeling and Control

Tewari, A.

2015, XI, 318 p. 135 illus., 1 illus. in color., Hardcover

ISBN: 978-1-4939-2367-0