Chapter 2
Energetic rate-independent systems

To keep the connection with continuum mechanics, cf. also Section 1.3.2, we consider the basic *state space* split to two spaces

$$\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}, \tag{2.0.1}$$

where the fast component y and the slow component z of the state $q = (y, z)$ live. Whenever possible, however, we will write q instead of (y, z) to shorten the notation. The splitting is done such that the evolution of z in time involves dissipation, whereas that of y does not. The state space $\mathcal{Q}$ is equipped with a Hausdorff topology $\mathcal{T}_{\mathcal{Q}} = \mathcal{T}_{\mathcal{Y}} \times \mathcal{T}_{\mathcal{Z}}$, and we denote by $q_k \overset{\mathcal{Q}}{\to} q$, $y_k \overset{\mathcal{Y}}{\to} y$, and $z_k \overset{\mathcal{Z}}{\to} z$ the corresponding convergence of sequences. Throughout, it will be sufficient to consider sequential closedness, compactness, and continuity. For notational convenience, we will not write this explicitly.

Let us recall the abbreviations RIS and ERIS, which we will subsequently often use for *rate-independent systems* and *energetic* RIS, respectively. This chapter focuses exclusively on ERIS. One of the main features in this chapter is the interplay between the full ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ using the energy-storage functional $\mathcal{E}$ and its reduced version $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$, where $\mathcal{I} : [0, T] \times \mathcal{Z} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$, defined in (1.3.8) by

$$\mathcal{I}(t, z) := \inf \Big\{ \mathcal{E}(t, \tilde{y}, z) \;\Big|\; \tilde{y} \in \mathcal{Y} \Big\}, \tag{2.0.2}$$

is called the *reduced functional*. We will define energetic solutions for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ and $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ in such a way that each solution $q = (y, z)$ for the former system gives rise to a solution z for the latter. Conversely, each solution z for $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ can be made into a solution $q = (y, z)$ by a suitable choice of y. It is important to realize that it is not enough to choose an arbitrary $y(t) \in \{ \tilde{y} \in \mathcal{Y} \mid \mathcal{E}(t, \tilde{y}, z(t)) = \min \mathcal{E}(t, \cdot, z(t)) \}$; further restrictions are necessary.

A. Mielke, T. Roubíček, *Rate-Independent Systems: Theory and Application*,
Applied Mathematical Sciences 193, DOI 10.1007/978-1-4939-2706-7_2

At first glance, it might seem reasonable to consider first the reduced system $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ and establish an existence theory there, and then establish the desired existence result for the full problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$, as given in the applications. However, it turns out that in the reduction process, certain natural properties (such as differentiability in t) are lost. To compensate for that, a stronger assumption would have been necessary, which can be avoided by working on the full system instead. Thus we present the existence theory for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ first, although sometimes, we have to allude to the reduced system. In Section 2.2.1, we will then give methods to approach the reduced system $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ directly.

2.1 The main existence result

This section provides a detailed proof of the existence of energetic solutions for ERIS in the most general cases; see Theorem 2.1.6. We also discuss variants of the assumptions and the proof. We emphasize that the present proof has a long development, which began in two independent areas, namely the study of fracture in brittle materials, see [152, 196, 198], and the analysis of rate-independent models for shape-memory alloys in [373, 416, 426]. While the former series of work was restricted to unidirectional rate-independent processes (sometimes called *irreversible quasistatic evolution*),[1] the latter needed convexity properties in the y-variable. A crucial step was taken in [195], where abstract versions of important techniques from [149] were made available. See also Section 4.2.4.1 for a more detailed description of the specific techniques.

2.1.1 *Abstract setup of the problem*

The first ingredient of the energetic formulation is the *dissipation distance* $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty]$, which is an extended quasidistance. Here "extended" means that the value ∞ is allowed, and "quasi" means that we do not ask for symmetry. The following three conditions will be the main assumptions:

Extended quasidistance:

$$\text{(i)}\quad \forall\, z_1, z_2, z_3 \in \mathcal{Z}: \ \mathcal{D}(z_1, z_3) \le \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3), \tag{D1}$$
$$\text{(ii)}\quad \forall\, z_1, z_2 \in \mathcal{Z}: \ \mathcal{D}(z_1, z_2) = 0 \iff z_1 = z_2;$$

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is } \textit{lower semicontinuous}. \tag{D2}$$

[1] In thermodynamics, the adjective "irreversible" often has a different, wider meaning indicating dissipation of energy implying irreversibility of time. Thus, nonunidirectional processes are still irreversible in this sense; in particular, this applies to all RIS considered here.

Here (D1) says that $\mathcal{D}$ is a distance except for the symmetry and the fact that the value ∞ is allowed. Relation (i) is the triangle inequality, and (ii) is the positivity. The asymmetry is needed in many applications such as elastoplasticity and damage.

One major point of the theory is the interplay between the topology $\mathcal{T}_{\mathcal{Z}}$ and the dissipation distance. To have a typical nontrivial, but still simple, application in mind, one may consider $\mathcal{Z} = \{ z \in \mathrm{L}^1(\Omega, \mathbb{R}^k) \mid \|z\|_{\mathrm{L}^\infty} \leq 1 \}$ equipped with the weak L^1-topology and the dissipation distance $\mathcal{D}(z_1, z_2) = \|z_1 - z_2\|_{\mathrm{L}^1}$.

For a given curve $z : [0, T] \to \mathcal{Z}$, we define the *total dissipation* on $[s, t]$ via

$$\mathrm{Diss}_{\mathcal{D}}(z; [s, t]) := \sup \left\{ \sum_{j=1}^{N} \mathcal{D}(z(t_{j-1}), z(t_j)) \,\middle|\, N \in \mathbb{N},\ s = t_0 < t_1 < \cdots < t_N = t \right\}. \tag{2.1.1}$$

The functions are defined everywhere, and changing them at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\mathrm{Diss}_{\mathcal{D}}(z; [r, t]) = \mathrm{Diss}_{\mathcal{D}}(z; [r, s]) + \mathrm{Diss}_{\mathcal{D}}(z; [s, t]) \quad \text{for all } r < s < t. \tag{2.1.2}$$

Later on, we will sometimes use the notation $\mathcal{D}(q_0, q_1)$ instead of $\mathcal{D}(z_0, z_1)$, where $q_j = (y_j, z_j)$. This slight abuse of notation will never lead to confusion, since $\mathcal{D}$ as a function on $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ still satisfies all assumptions, but one has to keep in mind that $\mathcal{D}$ satisfies the positivity (D1) only on $\mathcal{Z}$ and not on $\mathcal{Q}$.

The second ingredient is the stored-energy functional $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$. Here $t \in [0, T]$ plays the role of a (very slow) process time that changes the underlying system by changing loading conditions. The following conditions form the basic assumptions on $\mathcal{E}$. In Section 2.2.1, we will discuss generalizations. As usual, we will denote by “Dom” the *domain*, i.e., the set of arguments that makes the functional in question finite. Thus $\mathrm{Dom}\,\mathcal{E} := \{ (t, q) \in [0, T] \times \mathcal{Q} \mid \mathcal{E}(t, q) < \infty \}$ and $\mathrm{Dom}\,\mathcal{E}(t, \cdot) := \{ q \in \mathcal{Q} \mid \mathcal{E}(t, q) < \infty \}$. Our basic qualification of $\mathcal{E}$ is then:

Compactness of sublevels:
$$\forall\, t \in [0, T] : \ \mathcal{E}(t, \cdot) : \mathcal{Q} \to \mathbb{R}_\infty \text{ has compact sublevels;} \tag{E1}$$

energetic control of power:
$$\begin{aligned} &\mathrm{Dom}\,\mathcal{E} = [0, T] \times \mathrm{Dom}\,\mathcal{E}(0, \cdot), \\ &\exists\, c_{\mathcal{E}} \in \mathbb{R},\ \lambda_{\mathcal{E}} \in \mathrm{L}^1(0, T),\ N_{\mathcal{E}} \subset [0, T] \text{ with } \mathcal{L}^1(N_{\mathcal{E}}) = 0 \\ &\forall\, q \in \mathrm{Dom}\,\mathcal{E}(0, \cdot) : \ \ \mathcal{E}(\cdot, q) \in \mathrm{W}^{1,1}(0, T), \\ &\qquad\qquad \partial_t \mathcal{E}(t, q) \text{ exists for } t \in [0, T] \backslash N_{\mathcal{E}} \text{ and satisfies} \\ &\qquad\qquad \left| \partial_t \mathcal{E}(t, q) \right| \leq \lambda_{\mathcal{E}}(t) \big(\mathcal{E}(t, q) + c_{\mathcal{E}} \big). \end{aligned} \tag{E2}$$

The Cartesian structure $\mathrm{Dom}\,\mathcal{E} = [0, T] \times \mathrm{Dom}\,\mathcal{E}(0, \cdot)$ assumed in (E2) makes $\mathrm{Dom}\,\mathcal{E}(t, \cdot)$ independent of t, which represents a certain structural assumption on possible constraints involved in $\mathcal{E}$. Gronwall’s inequality and (E2) easily give

$$\mathcal{E}(t,q)+c_{\mathcal{E}} \le \big(\mathcal{E}(s,q)+c_{\mathcal{E}}\big)\,\mathrm{e}^{|\Lambda(t)-\Lambda(s)|} \quad \text{with} \quad \Lambda(t) := \int_0^t \lambda_{\mathcal{E}}(\tau)\,\mathrm{d}\tau, \tag{2.1.3a}$$

$$\big|\partial_t\mathcal{E}(t,q)\big| \le \lambda_{\mathcal{E}}(t)\big(\mathcal{E}(s,q)+c_{\mathcal{E}}\big)\,\mathrm{e}^{|\Lambda(t)-\Lambda(s)|}, \tag{2.1.3b}$$

which implies absolute continuity of $t \mapsto \mathcal{E}(t,q)$. The notion of *self-controlling models* in [117, 118] corresponds closely to our condition (E2).

Furthermore, we denote by Lev_α a *sublevel set* of a function with respect to the threshold α, i.e., in particular,

$$\mathrm{Lev}_\alpha\mathcal{E} := \Big\{(t,q)\in[0,T]\times\mathcal{Q} \;\big|\; \mathcal{E}(t,q) \le \alpha\Big\}, \quad \mathrm{Lev}_\beta\mathcal{E}(t,\cdot) := \Big\{q\in\mathcal{Q} \;\big|\; \mathcal{E}(t,q) \le \beta\Big\}.$$

Compactness of all nonempty $\mathrm{Lev}_\beta\mathcal{E}(t,\cdot)$, $\alpha\in\mathbb{R}$, implies lower semicontinuity of $\mathcal{E}(t,\cdot):\mathcal{Q}\to\mathbb{R}_\infty$.

Lemma 2.1.1. *If* (E2) *holds, then the sets* $\mathrm{Lev}_\alpha\mathcal{E}$*,* $\alpha\in\mathbb{R}$*, are compact in* $[0,T]\times\mathcal{Q}$ *if and only if the sets* $\mathrm{Lev}_\beta\mathcal{E}(t,\cdot)$*,* $\beta\in\mathbb{R}$*,* $t\in[0,T]$*, are compact in* $\mathcal{Q}$*.*

Proof. For the $\Rightarrow$ implication, we use $\mathrm{Lev}_\beta\mathcal{E}(t,\cdot) = \mathrm{Lev}_\beta\mathcal{E}\cap\big(\{t\}\times\mathcal{Q}\big)$. Hence, the right-hand side is compact, since it is the intersection of a compact and a closed set.

In the other direction, take a sequence with $(t_n,q_n)\in\mathrm{Lev}_\alpha\mathcal{E}$. By taking a subsequence if necessary, we may assume $t_n\to t_*$ and $\mathcal{E}(t_*,q_n)\to\beta := \liminf_j \mathcal{E}(t_*,q_j)$. By (2.1.3a) (applied once with $s=t_*$ and $t=t_n$ and once with $s=t_n$ and $t=t_*$), we have $\mathcal{E}(t_n,q_n)\to\beta$, which implies $\beta\le\alpha$. Since $\mathrm{Lev}_\beta\mathcal{E}(t_*,\cdot)$ is compact, there exists a subsequence $(q_{n_l})_l$ such that $n_l\to\infty$, $q_{n_l}\to q_*$, and $q_*\in\mathrm{Lev}_\beta\mathcal{E}(t_*,\cdot)$. By $\beta\le\alpha$, we conclude that $(t_{n_l},q_{n_l})\to(t_*,q_*)\in\mathrm{Lev}_\alpha\mathcal{E}$, and we are done. □

Most typically, $\mathcal{Q}$ will be a closed, convex, and bounded subset of a reflexive Banach space (such as the Sobolev space $\mathrm{W}^{1,p}(\Omega,\mathbb{R}^m)$ or a Lebesgue space $\mathrm{L}^p(\Omega,\mathbb{R}^m)$ with $p\in(1,\infty)$) equipped with $\mathcal{T}_\mathcal{Q}$ as its weak topology. Then lower semicontinuity of $\mathcal{E}$ and $\mathcal{D}$ in $(\mathcal{Q},\mathcal{T}_\mathcal{Q})$ is the same as the classical weak lower semicontinuity in the calculus of variations; see [140].

Definition 2.1.2. A function $q=(y,z):[0,T]\to\mathcal{Q}=\mathcal{Y}\times\mathcal{Z}$ is called an *energetic solution* of the rate-independent system $(\mathcal{Q},\mathcal{E},\mathcal{D})$ if $t\mapsto\partial_t\mathcal{E}(t,q(t))$ is integrable and if the *global stability (S)* and the *energy balance (E)* hold for all $t\in[0,T]$:

(S) $\quad \forall\,\hat{q}=(\hat{y},\hat{z})\in\mathcal{Q}: \qquad \mathcal{E}(t,q(t)) \le \mathcal{E}(t,\hat{q}) + \mathcal{D}(z(t),\hat{z}).$

(E) $\quad \mathcal{E}(t,q(t)) + \mathrm{Diss}_\mathcal{D}(z;[0,t]) = \mathcal{E}(0,q(0)) + \int_0^t \partial_t\mathcal{E}(\tau,q(\tau))\,\mathrm{d}\tau.$

For subintervals $[t_0,t_1]\subset[0,T]$, we call $q:[t_0,t_1]\to\mathcal{Q}$ an energetic solution of $(\mathcal{Q},\mathcal{E},\mathcal{D})$ if (S) and (E) hold for all $t\in[t_0,t_1]$, where the initial time 0 in (E) is replaced by t_0.

The stability condition (S) is global in all of $\mathcal{Q}$, and it can be rephrased by defining the *set* $\mathcal{S}(t)$ *of stable states* at time t via

$$\mathcal{S}(t) := \Big\{ q \in \mathcal{Q} \;\big|\; \mathcal{E}(t,q) < \infty,\ \mathcal{E}(t,q) \le \mathcal{E}(t,\hat{q}) + \mathcal{D}(q,\hat{q}) \text{ for all } \hat{q} \in \mathcal{Q} \Big\},$$
$$\mathcal{S}_{[0,T]} := \Big\{ (t,q) \in [0,T]\times\mathcal{Q} \;\big|\; q \in \mathcal{S}(t) \Big\} = \textstyle\bigcup_{t\in[0,T]} (t, \mathcal{S}(t)). \tag{2.1.4}$$

We call $\mathcal{S}(t)$ the *stability set* at time t for short. Then (S) simply means that $q(t) \in \mathcal{S}(t)$ for all $t \in [0,T]$. The properties of the stability sets turn out to be crucial for deriving existence results.

Rate-independence in the sense of Definition 1.2.1 manifests itself by the fact that the problem has no intrinsic time scale. It is easy to show that q is a solution for $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ if and only if the reparameterized curve $\tilde{q} : t \mapsto q(\alpha(t))$, where $\dot{\alpha} > 0$, is a solution for $(\mathcal{D}, \tilde{\mathcal{E}})$ with $\tilde{\mathcal{E}}(t,q) = \mathcal{E}(\alpha(t),q)$. In particular, the stability (S) is a static concept, and the energy balance (E) is rate-independent, since the dissipation defined via (2.1.1) is invariant under rescaling, like the length of a curve.

Moreover, energetic solutions satisfy the *restriction property* and the *concatenation property*; see Definition 1.2.1. In fact, our ERIS are often input–output systems in the form that $\mathcal{E}(t,q) = \mathcal{F}(\ell(t),q)$ for a given functional $\mathcal{F} : \mathcal{X}\times\mathcal{Q} \to \mathbb{R}_\infty$ and inputs or loadings $\ell : [0,T] \to \mathcal{X}$ in a suitable set $\mathrm{F}_0([0,T];\mathcal{X})$ of functions. For general energetic solutions in the sense of Definition 2.1.2, the restriction property simply means that for every energetic solution $q : [t_0,t_1] \to \mathcal{Q}$ and subinterval $[s_0,s_1] \subset [t_0,t_1]$, the restriction $q|_{[s_0,s_1]} : [s_0,s_1] \to \mathcal{Q}$ is an energetic solution as well. The concatenation property states that for $0 \le t_1 < t_2 < t_3 \le T$ and all energetic solutions $\hat{q} : [t_1,t_2] \to \mathcal{Q}$ and $\tilde{q} : [t_2,t_3] \to \mathcal{Q}$ with $\hat{q}(t_2) = \tilde{q}(t_2)$, the concatenation $\hat{q} \bowtie \tilde{q} : [t_1,t_3] \to \mathcal{Q}$ defined in Definition 1.2.1(iii) is an energetic solution as well. This is obvious for the static condition (S). To see that (E) also shares these conditions, we define

$$\mathfrak{E}_{t_0}(t) = \mathcal{E}(t,q(t)) + \mathrm{Diss}_{\mathcal{D}}(q;[t_0,t]) - \int_{t_0}^{t} \partial_s \mathcal{E}(s,q(s))\,\mathrm{d}s.$$

Then (E) simply states that the function $\mathfrak{E}_{t_0}$ is equal to the constant value $\mathfrak{E}_{t_0}(t_0)$ on the whole interval. Using the additivity of $\mathrm{Diss}_{\mathcal{D}}$ and the integral, we obtain, for $r < s < t$, the relation

$$\mathfrak{E}_r(t) = \mathfrak{E}_r(s) - \mathcal{E}(s,q(s)) + \mathfrak{E}_s(t).$$

Thus, the constancy certainly remains true after restriction. When we concatenate two solutions $\hat{q}$ and $\tilde{q}$ with $\hat{q}(s) = \tilde{q}(s) = q_*$, we can use $\mathcal{E}(s,q_*) = \mathfrak{E}_s(s)$, which guarantees that the two constants are the same.

Before discussing the question of existence of solutions, we want to point out that the concept of energetic solutions provides a priori bounds on the solutions. For the time-continuous problem, these bounds are easy to derive, and the main structure becomes more transparent. Of course, similar estimates will be crucial in the time-discrete setting. Using the assumption (E2), the energy balance (E) gives

$$\mathscr{E}(t,q(t))+\mathrm{Diss}_{\mathscr{D}}(z;[0,t]) \leq \mathscr{E}(0,q(0))+\int_0^t \lambda_{\mathscr{E}}(s)\big(\mathscr{E}(s,q(s))+c_{\mathscr{E}}\big)\,\mathrm{d}s. \tag{2.1.5a}$$

Omitting the dissipation and adding $c_{\mathscr{E}}$ on both sides allows for an application of Gronwall's inequality, and we obtain

$$\mathscr{E}(t,q(t)) \leq \big(\mathscr{E}(0,q(0))+c_{\mathscr{E}}\big)\mathrm{e}^{\Lambda(t)} - c_{\mathscr{E}}. \tag{2.1.5b}$$

Inserting this again into (2.1.5), we can also estimate the dissipation via

$$\mathrm{Diss}_{\mathscr{D}}(z;[0,T]) \leq \big(\mathscr{E}(0,q(0))+c_{\mathscr{E}}\big)\mathrm{e}^{\Lambda(T)}, \tag{2.1.5c}$$

since $\mathscr{E}(t,q(t)) \geq -c_{\mathscr{E}}$ by (E2).

The application of the theory of ERIS to models in brittle fracture (cf. Section 4.2.4.1) and the following example highlight that there are important examples of ERIS in which the state space $\mathcal{Q}$ does not have the Cartesian product form $\mathcal{Y}\times\mathcal{Z}$ and does not have any underlying linear Banach-space structure.

Example 2.1.3 (Wetting and dewetting for moving drops). In [9], a quasistatically moving liquid drop occupying a domain $E(t) \subset \Omega \subset \mathbb{R}^3$ is considered. The fixed domain Ω is a bounded container whose exterior $\mathbb{R}^3 \setminus \Omega$ is considered solid, while $\Omega \setminus E(t)$ is filled with vapor. The boundary ∂E of E is thus decomposed into the solid–liquid interface Γ^E_{SL} and the liquid–vapor interface Γ^E_{LV}. With this, one can define an energy functional $\mathscr{E}$ on the state space

$$\mathcal{Q} := \Big\{E \subset \Omega \;\Big|\; \mathscr{H}^2(\partial E) < \infty\Big\},$$

which is defined via the contact energies and a potential energy given in terms of $V_{\mathrm{ext}}(t,x)$:

$$\mathscr{E}(t,E) = \sigma_{\mathrm{SL}}\mathscr{H}^2(\Gamma^E_{\mathrm{SL}}) + \sigma_{\mathrm{SV}}\mathscr{H}^2(\partial\Omega\backslash\Gamma^E_{\mathrm{SV}}) + \sigma_{\mathrm{LV}}\mathscr{H}^2(\Gamma^E_{\mathrm{LV}}) + \int_E V_{\mathrm{ext}}(t,x)\,\mathrm{d}x.$$

The activation energy (i.e., dissipation) is given in terms of wetting and dewetting, i.e., if the solid–liquid interface increases or decreases, respectively:

$$\mathscr{D}(E_0,E_1) = \mu_{\mathrm{wett}}\mathscr{H}^2(\Gamma^{E_1}_{\mathrm{SL}}\backslash\Gamma^{E_0}_{\mathrm{SL}}) + \mu_{\mathrm{dew}}(\Gamma^{E_0}_{\mathrm{SL}}\backslash\Gamma^{E_1}_{\mathrm{SL}})$$

with the specific activation energies $\mu_{\mathrm{wett}} > 0$ and $\mu_{\mathrm{dew}} > 0$. We refer to [9, Sect. 3] for the exact details to obtain existence of energetic solutions.

2.1.2 The time-incremental minimization problem

The most natural approach to solving (S)&(E) is via time discretization using the fact that incremental problems exist that have a variational structure, i.e., are minimization problems. It is then possible to find their solutions as minimizers of certain lower semicontinuous functionals on $\mathcal{Q}$. For this, we make use of the lower semicontinuity assumptions (D2) and (E1).

For the general, not necessarily equidistant, time-discretization, we will use the notation $\mathrm{Part}([r,s])$ for all finite partitions of the interval $[r,s] \subset \mathbb{R}$, i.e.,

$$\mathrm{Part}([r,s]) := \Big\{ (t_0, t_1, \ldots, t_N) \;\Big|\; r = t_0 < t_1 < \ldots < t_N = r \Big\}. \tag{2.1.6}$$

For a partition $\Pi \in \mathrm{Part}([r,s])$, we define its number of subintervals and its *fineness* as the length of its largest interval, i.e.,

$$N_\Pi := N \qquad \text{for a partition } \;\Pi = (t_0, t_1, \ldots, t_N), \tag{2.1.7a}$$

$$\varnothing(\Pi) := \max \Big\{ t_k - t_{k-1} \;\Big|\; k = 1, \ldots, N_\Pi \Big\}. \tag{2.1.7b}$$

Note that $\varnothing(\Pi) = 2\max_{t\in[0,T]} \mathrm{dist}(t,\Pi)$. In particular, we always have $\mathrm{dist}(t,\Pi) \leq \varnothing(\Pi)$. Having fixed a partition $\Pi = (t_0, t_1, \ldots, t_N) \in \mathrm{Part}([0,T])$, we look for q_k, $k = 1, \ldots, N_\Pi$, that approximate the solution q at t_k, i.e., $q_k \approx q(t_k)$.

Our energetic approach has the major advantage that the values q_k can be found incrementally via minimization problems. In concrete function spaces, this approach is referred to as the *direct method in the calculus of variations*. Since the methods of the calculus of variations are especially suited for applications in material modeling (cf. [125, 140, 442, 520]), this will enable us to treat a variety of quite different applications; see Chapter 4.

In our general setting, the *incremental minimization problem* takes the following form:

$$(\mathrm{IMP}^\Pi) \quad \begin{array}{l} \text{For given } q_0 \in \mathcal{S}(0) \subset \mathcal{Q}, \text{ find } q_1, \ldots, q_N \in \mathcal{Q} \text{ such that} \\ q_k \text{ minimizes } q \mapsto \mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q) \;\text{ for } k = 1, \ldots, N_\Pi. \end{array} \tag{2.1.8}$$

For brevity, we will write $q_k \in \mathrm{Arg\,min}_{\mathcal{Q}}\, \mathcal{E}(t_k,\cdot) + \mathcal{D}(q_{k-1},\cdot)$, where the operator $\mathrm{Arg\,min}$ denotes the set of all minimizers of a functional, i.e.,

$$\mathrm{Arg\,min}_{\mathcal{Q}}\, \mathfrak{J} := \Big\{ q \in \mathcal{Q} \;\Big|\; \mathfrak{J}(q) = \inf_{\mathcal{Q}} \mathfrak{J} \Big\} \subset \mathcal{Q}.$$

Depending on the context, we will also use the more explicit notation

$$\mathrm{Arg\,min}_{\mathcal{Q}}\, \mathfrak{J} = \mathrm{Arg\,min}\, \mathfrak{J} = \mathop{\mathrm{Arg\,min}}_{q\in\mathcal{Q}} \mathfrak{J}(q) = \mathrm{Arg\,min} \Big\{ \mathfrak{J}(q) \;\Big|\; q \in \mathcal{Q} \Big\}.$$

The following result shows that (IMP^{Π}) is intrinsically linked to (S)&(E). Without any smallness assumptions on the time steps, the solutions of (IMP^{Π}) satisfy properties that are closely related to (S)&(E).

Proposition 2.1.4. *Let* (D1) *and* (E2) *hold. Every solution of* (IMP^{Π}) *from* (2.1.8) *satisfies the following properties:*

(i) For $k = 1, \ldots, N_\Pi$ *we have that* q_k *is stable at time* t_k*, i.e.,* $q_k \in \mathcal{S}(t_k)$*.*
(ii) For $k = 1, \ldots, N_\Pi$*, we have*

$$\int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s \le e_k - e_{k-1} + \delta_k \le \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s, \tag{2.1.9}$$

where $e_j = \mathcal{E}(t_j, q_j)$ *and* $\delta_k = \mathcal{D}(z_{k-1}, z_k)$*.*
(iii) If (D2) *and* (E1) *hold additionally, then solutions of* (IMP^{Π}) *exist.*

Proof. Part (i). Stability follows from minimization properties of the solutions and the triangle inequality. For all $\hat{q} \in \mathcal{Q}$, we have

$$\begin{aligned}\mathcal{E}(t_k, \hat{q}) + \mathcal{D}(z_k, \hat{z}) &= \mathcal{E}(t_k, \hat{q}) + \mathcal{D}(z_{k-1}, \hat{z}) + \mathcal{D}(z_k, \hat{z}) - \mathcal{D}(z_{k-1}, \hat{z}) \\ &\ge \mathcal{E}(t_k, q_k) + \mathcal{D}(z_{k-1}, z_k) + \mathcal{D}(z_k, \hat{z}) - \mathcal{D}(z_{k-1}, \hat{z}) \ \ge\ \mathcal{E}(t_k, q_k).\end{aligned}$$

Part (ii). The first estimate is deduced from $q_{k-1} \in \mathcal{S}(t_{k-1})$ as follows:

$$\begin{aligned}&\mathcal{E}(t_k, q_k) + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, q_{k-1}) = \\ &= \mathcal{E}(t_{k-1}, q_k) + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s + \mathcal{D}(z_{k-1}, z_k) - \mathcal{E}(t_{k-1}, q_{k-1}) \ge \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_k)\,\mathrm{d}s.\end{aligned}$$

Since $q_k \in \mathrm{Arg\,min}\{\, \mathcal{E}(t_k, q) + \mathcal{D}(z_{k-1}, z) \mid q \in \mathcal{Q}\,\}$, the second estimate follows via

$$\begin{aligned}&\mathcal{E}(t_k, q_k) - \mathcal{E}(t_{k-1}, q_{k-1}) + \mathcal{D}(z_{k-1}, z_k) \\ &\le \mathcal{E}(t_k, q_{k-1}) - \mathcal{E}(t_{k-1}, q_{k-1}) + \mathcal{D}(z_{k-1}, z_{k-1}) \ = \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q_{k-1})\,\mathrm{d}s.\end{aligned}$$

Part (iii). The minimizers are constructed inductively. In the kth step, q_{k-1} is known, and every minimizer y has to satisfy $\mathfrak{J}_k(y) := \mathcal{E}(t_k, q) + \mathcal{D}(z_{k-1}, z) \le \mathcal{E}(t_k, q_{k-1}) = \mathfrak{J}_k(q_{k-1})$, since $q = q_{k-1}$ is a candidate. Using $\mathcal{D} \ge 0$, it suffices to minimize the lower semicontinuous functional $\mathfrak{J}_k$ on the compact sublevel $\mathcal{E}(t_k, \cdot) \le \mathcal{E}(t_k, q_{k-1})$. Hence, the Bolzano–Weierstrass theorem, Theorem A.2.1, provides the existence of a minimizer q_k. ☐

Now we use assumption (E2) to obtain a priori bounds on the energy and the dissipation for solutions of (IMP^{Π}), reflecting the bounds (2.1.5) for the time-continuous case. Combining (E2), (2.1.3), and the upper estimate of Proposition 2.1.4(ii) gives

$$e_k + \delta_k \le e_{k-1} + \big(c_{\mathscr{E}} + e_{k-1}\big)\big(\mathrm{e}^{\Lambda(t_k)-\Lambda(t_{k-1})} - 1\big) \tag{2.1.10a}$$

$$= \big(c_{\mathscr{E}} + e_{k-1}\big)\,\mathrm{e}^{\Lambda(t_k)-\Lambda(t_{k-1})} - c_{\mathscr{E}}. \tag{2.1.10b}$$

Using $\delta_k \ge 0$ and (2.1.10b), induction over k leads to

$$c_{\mathscr{E}} + e_k \le (c_{\mathscr{E}} + e_0) \prod_{j=1}^{k} \mathrm{e}^{\Lambda(t_j)-\Lambda(t_{j-1})} = (c_{\mathscr{E}} + e_0)\mathrm{e}^{\Lambda(t_k)} \text{ for } k = 1, \ldots, N_\Pi. \tag{2.1.11}$$

Summing (2.1.10a) from 1 to k, we obtain, after some cancellation and employing (2.1.11), the estimate

$$\begin{aligned} e_k + c_{\mathscr{E}} + \sum_{j=1}^{k} \delta_j &\le e_0 + c_{\mathscr{E}} + \sum_{j=1}^{k} \big(c_{\mathscr{E}} + e_{j-1}\big)\big(\mathrm{e}^{\Lambda(t_j)-\Lambda(t_{j-1})} - 1\big) \\ &\le \big(c_{\mathscr{E}} + e_0\big) + \big(c_{\mathscr{E}} + e_0\big) \sum_{j=1}^{k} (\mathrm{e}^{\Lambda(t_j)} - \mathrm{e}^{\Lambda(t_{j-1})}) = \big(c_{\mathscr{E}} + e_0\big)\mathrm{e}^{\Lambda(t_k)}. \end{aligned}$$

For each incremental solution $(q_k)_{k=1,\ldots,N}$ of (IMP$^\Pi$) associated with a partition $\Pi = (t_0, t_1, \ldots, t_N) \in \mathrm{Part}([0,T])$, we define the right-continuous *piecewise constant interpolant*, $\underline{q}^\Pi$ with

$$\underline{q}^\Pi(t) := q_{k-1} \quad \text{for } t \in [t_{k-1}, t_k), \text{ where } k = 1, \ldots, N, \quad \text{and} \quad \underline{q}^\Pi(T) := q_N, \tag{2.1.12}$$

which is continuous from the right. Occasionally, we will also use a left-continuous piecewise constant interpolant $\overline{q}^\Pi$ defined by

$$\overline{q}^\Pi(t) := q_k \quad \text{for } t \in (t_{k-1}, t_k], \text{ where } k = 1, \ldots, N, \quad \text{and} \quad \overline{q}^\Pi(0) = q_0. \tag{2.1.13}$$

Theorem 2.1.5. *Assume that* (D1) *and* (E2) *hold and let* $\Pi \in \mathrm{Part}([0,T])$. *Then for every solution* $(q_k)_{k=0,\ldots,N_\Pi}$ *of (IMP$^\Pi$) the interpolant* $\underline{q}^\Pi = (\underline{y}^\Pi, \underline{z}^\Pi) : [0,T] \to \mathcal{Y} \times \mathcal{Z}$ *satisfies the following three properties:*

(i) For $t \in \Pi$, *we have* $\quad \underline{q}^\Pi(t) = \overline{q}^\Pi(t) \in \mathcal{S}(t);$ $\quad$ *(S)$_{discr}$*

(ii) For $s, t \in \Pi$ *with* $s < t$, *we have the two-sided energy estimate*

$$\begin{aligned} &\int_s^t \partial_\tau \mathscr{E}(\tau, \overline{q}^\Pi(\tau))\,\mathrm{d}\tau \le \\ &\qquad \le \mathscr{E}(t, \underline{q}^\Pi(t)) + \mathrm{Diss}_{\mathscr{D}}(\underline{z}^\Pi; [s,t]) - \mathscr{E}(s, \underline{q}^\Pi(s)) \\ &\qquad\qquad \le \int_s^t \partial_\tau \mathscr{E}(\tau, \underline{q}^\Pi(\tau))\,\mathrm{d}\tau; \end{aligned} \tag*{(E)$_{discr}$}$$

(iii) For all $t \in [0,T]$, *we have the a priori estimate*

$$\mathscr{E}(t, \underline{q}^\Pi(t)) + c_{\mathscr{E}} + \mathrm{Diss}_{\mathscr{D}}(\underline{z}^\Pi; [0,t]) \le \mathrm{e}^{\Lambda(t)}\big(\mathscr{E}(0, q_0) + c_{\mathscr{E}}\big).$$

2.1.3 Statement of the main existence result

The existence theory developed below will build on the incremental minimization problem (IMP$^{\Pi}$) and the a priori estimates derived above. Choosing a sequence of partitions whose fineness tends to 0, we obtain a sequence of approximations, and we first need to extract a suitable subsequence that converges. This can be done for the z-component only, since the dissipation provides an a priori estimate of BV type. A suitable version of Helly's selection principle is stated in Theorem 2.1.24 and proved in Appendix B.5 in a more general variant; cf. Theorem B.5.13.

Since the y-component allows for no control of the temporal oscillations, it has to be handled differently. One can use a technique developed in [149, 195] involving choosing additional subsequences for each $t \in [0, T]$ and thus relying on the axiom of choice; cf. Remark 2.1.8 below. Here, however, we use a different technique based on a slightly stronger assumption of metrizability of the underlying topology, but it leads to simpler assumptions and guarantees the existence of solutions measurable in time. This idea uses the fact that for stable states $q = (y, z)$, the energy $\mathscr{E}(t, \cdot)$ in fact depends only on the component z, i.e., we can define the *reduced functional*

$$\mathscr{I}(t, z) := \min \Big\{ \mathscr{E}(t, y, z) \;\Big|\; y \in \mathcal{Y} \Big\}, \tag{2.1.14}$$

and we have

$$\forall (y, z) \in \mathcal{S}(t) : \qquad \mathscr{I}(t, z) = \mathscr{E}(t, y, z). \tag{2.1.15}$$

We also define a *reduced power* via

$$\mathcal{P}_{\mathrm{red}}(t, z) := \sup \Big\{ \partial_t \mathscr{E}(t, y, z) \;\Big|\; y \in \operatorname{Arg\,min}_{\mathcal{Y}} \mathscr{E}(t, \cdot, z) \Big\}, \tag{2.1.16}$$

and the crucial observation is that along every energetic solution, this reduced power is realized; see (2.1.19).

After having identified a subsequence and a limit function, it is necessary to show that this limit is an energetic solution. For this, we need further conditions on the functionals $\mathscr{E}$ and $\mathscr{D}$ expressing a certain compatibility of these two functionals, whereas the conditions (D1)–(D2) and (E1)–(E2) depend solely on $\mathscr{D}$ and $\mathscr{E}$, respectively. To define these conditions, we introduce the notion of a *stable sequence* $(t_m, q_m)_{m \in \mathbb{N}}$ via

$$\sup_{m \in \mathbb{N}} \mathscr{E}(t_m, q_m) < \infty \quad \text{and} \quad \forall\, m \in \mathbb{N} : \; q_m \in \mathcal{S}(t_m). \tag{2.1.17}$$

Note that this concept intrinsically links the type of convergence to the properties of $\mathscr{E}$ and $\mathscr{D}$. In particular, this generates a topology that is derived from the functionals. For instance, let us consider $\mathcal{Q} = \mathcal{Z} = \mathrm{L}^2(\Omega)$ and $\mathscr{D}(z_0, z_1) = \|z_1 - z_0\|_{\mathrm{L}^1}$ and the coercivity $\mathcal{E}(t,q) \geq c\|q\|^{\alpha}_{\mathrm{H}^1(\Omega)} - C$. Then stable sequences are bounded in $\mathrm{H}^1(\Omega)$, and the intrinsic convergence turns out to be the weak convergence in $\mathrm{H}^1(\Omega)$.

The *compatibility conditions* between $\mathcal{E}$ and $\mathcal{D}$ rely on convergent stable sequences:

$$\forall \text{ stable sequences } (t_m,q_m)_{m\in\mathbb{N}} \text{ with } (t_m,q_m) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) \text{ we have:}$$

$$t\in[0,T]\backslash N_{\mathcal{E}} \text{ with } N_{\mathcal{E}} \text{ from (E2)} \implies \partial_t\mathcal{E}(t,q) = \lim_{m\to\infty}\partial_t\mathcal{E}(t,q_m), \tag{C1}$$

$$q \in \mathcal{S}(t). \tag{C2}$$

Condition (C1) is called *conditioned continuity of the power of the external forces*. Note that the time is fixed to t in the limit in (C1), although $q_m \in \mathcal{S}(t_m)$. Condition (C2) is called the *closedness of the stability set* $\mathcal{S}_{[0,T]}$. These central conditions will be discussed in more detail in Section 2.1.5 after the statement of the main result, whose proof is given on pp. 72–75. For a first impression of the structure of the lengthy proof, we refer to the much simpler Hilbert-space case with a quadratic energy stated in Theorem 3.5.2, where a full and independent proof is given along the same lines as the general proof developed here.

Theorem 2.1.6 (Main existence result). *Assume that $\mathcal{E}$ and $\mathcal{D}$ satisfy the assumptions* (D1)–(D2)*,* (E1)–(E2)*, and the compatibility conditions* (C1) *and* (C2)*. Further, assume that*

$$\textit{the topology of } \mathcal{Q} \textit{ restricted to compact sets is separable and metrizable.} \tag{2.1.18}$$

Then:

(i) For each $q_0 \in \mathcal{S}(0)$, there exists an energetic solution $q = (y,z) : [0,T] \to \mathcal{Q}$ to the initial-value problem $(\mathcal{Q},\mathcal{E},\mathcal{D},q_0)$. Moreover, $q : [0,T] \to \mathcal{Q}$ is measurable and

$$\partial_t\mathcal{E}(t,q(t)) = \partial_t\mathcal{E}(t,y(t),z(t)) = \mathcal{P}_{\mathrm{red}}(t,z(t)) \text{ for a.e. } t\in[0,T]. \tag{2.1.19}$$

(ii) If $\Pi^l \in \mathrm{Part}([0,T])$ is a sequence of partitions with fineness $\varnothing(\Pi^l) \to 0$ for $l \to \infty$, and $\underline{q}^{\Pi_l}$ is the interpolant of a solution of the associated (IMP$^{\Pi^l}$), then there exist a subsequence $q_k = \underline{q}^{\Pi_{l_k}}$ and an energetic solution $\tilde{q} = (\tilde{y},\tilde{z})$ to the initial-value problem $(\mathcal{Q},\mathcal{E},\mathcal{D},q_0)$ such that the following holds:

$$\forall\, t \in [0,T]: \quad z_k(t) \overset{\mathcal{Z}}{\to} \tilde{z}(t); \tag{2.1.20a}$$

$$\forall\, t \in [0,T]: \quad \mathrm{Diss}_{\mathcal{D}}(z_k;[0,t]) \to \mathrm{Diss}_{\mathcal{D}}(\tilde{z};[0,t]); \tag{2.1.20b}$$

$$\forall\, t \in [0,T]: \quad \mathcal{E}(t,q_k(t)) \to \mathcal{E}(t,\tilde{q}(t)); \tag{2.1.20c}$$

$$\forall_{a.a.}\, t \in [0,T]: \quad \partial_t\mathcal{E}(t,q_k(t)) \to \partial_t\mathcal{E}(t,\tilde{q}(t)). \tag{2.1.20d}$$

Moreover, (E2) *and* (2.1.20d) *imply $\partial_t\mathcal{E}(\cdot,q_k(\cdot)) \to \partial_t\mathcal{E}(\cdot,\tilde{q}(\cdot))$ in $\mathrm{L}^1(0,T)$.*

(iii) If additionally, the functional $\mathscr{E}$ is such that for each stable point $q = (y, z) \in \mathcal{S}(t)$, the functional $\mathscr{E}(t, \cdot, z)$ has a unique minimizer y, then taking $\tilde{y}(t) = \arg\min \mathscr{E}(t, \cdot, \tilde{z}(t))$, the convergence in (2.1.20a) can be improved to

$$q_k(t) \xrightarrow{\mathcal{Q}} \tilde{q}(t). \tag{2.1.20e}$$

Before discussing the assumptions and going into the details of the proofs, we present a standard example in the Banach-space setting. This guiding example should be considered the first nontrivial case that on the one hand, provides a nontrivial application of the above theorem and on the other hand, provides a first intuition about the main structure of the assumptions. We refer to Chapter 3 for a detailed analysis of ERIS in Banach spaces.

Example 2.1.7 (The basic Banach-space case). We consider two separable and reflexive Banach spaces $\boldsymbol{U}$ and $\boldsymbol{Z}$ and choose as the topological space $\mathcal{Q}$ the Banach space $\boldsymbol{Q} := \boldsymbol{U} \times \boldsymbol{Z}$ equipped with the weak topology, which implies the assumption (2.1.18). Next we consider a functional $\mathscr{E} : [0, T]\times\boldsymbol{Q} \to \mathbb{R}_\infty$ and a dissipation distance $\mathscr{D} : \boldsymbol{Z}\times\boldsymbol{Z} \to [0, \infty)$ with

$$\mathscr{E}(t, q) = \mathcal{E}(q) - \big\langle \ell(t), q\big\rangle_{\boldsymbol{Q}^*\times\boldsymbol{Q}} \quad \text{with } \ell \in \mathrm{W}^{1,1}(0, T; \boldsymbol{Q}^*), \tag{2.1.21a}$$

$$\mathcal{E} : \boldsymbol{Q} \to \mathbb{R}_\infty \text{ is weakly lower semicontinuous, with} \tag{2.1.21b}$$

$$\text{superlinear growth: } \exists\, C_1, \alpha > 1\ \forall q\in\boldsymbol{Q} :\ \mathcal{E}(q) \geq \frac{1}{C_1}\|q\|_{\boldsymbol{Q}}^{\alpha} - C_1, \tag{2.1.21c}$$

$$\mathscr{D} \text{ satisfies (D1) and is weakly continuous.} \tag{2.1.21d}$$

Then Theorem 2.1.6 is applicable, and in particular, for all stable initial states there exists an energetic solution.

Indeed, the conditions (E1) and (E2) easily hold with $\mathrm{Dom}\mathscr{E} = [0, T]\times\mathrm{Dom}\mathcal{E}$ and $\lambda_{\mathscr{E}}(t) = C_2\|\dot{\ell}(t)\|_{\boldsymbol{Q}^*}$ for a constant $C_2 > 0$ depending on C_1, α and $\|\ell\|_{\mathrm{L}^\infty(0,T;\boldsymbol{Q}^*)}$ only. Moreover, with $N_{\mathscr{E}} = \{\, t \in [0, T] \mid \ell \text{ is not differentiable at } t\,\}$, the compatibility condition (C1) follows from the weak continuity of $q \mapsto \langle\dot{\ell}(t), q\rangle$.

For the dissipation distance, the conditions (D1) and (D2) are satisfied, so it remains to establish the closedness of the stability set $\mathcal{S}_{[0,T]}$, i.e., (C2). For a given stable sequence (t_k, q_k) with $t_k \to t$ and $q_k \rightharpoonup q$ and an arbitrary test state $\hat{q}$, we have

$$\mathscr{E}(t_k, q_k) = \mathcal{E}(q_k) - \big\langle\ell(t_k), q_k\big\rangle \leq \mathscr{E}(t_k, \hat{q}) + \mathscr{D}(q_k, \hat{q}).$$

Using the strong continuity of ℓ, the weak lower semicontinuity of $\mathcal{E}$ in (2.1.21b), and the weak continuity of $\mathscr{D}$ in (2.1.21d), we can pass to the limit in all terms and obtain the stability $\mathscr{E}(t, q) \leq \mathscr{E}(t, \hat{q}) + \mathscr{D}(q, \hat{q})$, because $\hat{q}$ was arbitrary.

Remark 2.1.8 (Convergence of $(y_k)_{k\in\mathbb{N}}$: the general case). For the general case, this theorem does not claim any convergence of the the nondissipative component $y_k(t)$

to $\tilde{y}(t)$. In fact, since we do not have any control of the temporal oscillations of the y's, we have no selection criterion. This is because it is more delicate in the general case, where one can say that for every $t \in [0,T]$, there is another subsequence $(q_{k_n^t}(t))_{n\in\mathbb{N}}$ selected from that in Theorem 2.1.6 such that

$$\forall t \in [0,T]: \qquad y_{k_n^t}(t) \xrightarrow{\mathcal{Y}} \tilde{y}(t). \tag{2.1.22}$$

In fact, such $\tilde{y}$ complies with (2.1.16) and forms an energetic solution as in Theorem 2.1.6(ii). Alternatively, employing a general topological concept of nets, we can say that there is a finer net $(q_{k_\xi})_{\xi\in\varXi}$ with $\varXi$ a suitable directed set[2] such that

$$\forall t \in [0,T]: \qquad \lim_{\xi\in\varXi} y_{k_\xi}(t) = \tilde{y}(t), \tag{2.1.23}$$

where the limit is meant in the topology of $\mathcal{Y}$ in the sense of *Moore–Smith convergence*; cf. p. 581.[3] This finer net keeps the convergence $\lim_{\xi\in\varXi} z_{k_\xi}(t) = \tilde{z}(t)$ with the original $\tilde{z}$ and also (2.1.20a,b). By (2.1.23), we have now $\lim_{\xi\in\varXi} q_{k_\xi}(t) = \tilde{q}(t)$. By lower semicontinuity of $\mathscr{E}(t,\cdot)$, also (2.1.20c) is preserved in the sense that $\lim_{\xi\in\varXi} \mathscr{E}(t, q_{k_\xi}(t)) \geq \mathscr{E}(t, \tilde{q}(t))$. Here we used the metrizability (2.1.18), which allows us to "translate" the sequential semicontinuity to the topological one.

By the conditioned continuity (C1) of $\partial_t\mathscr{E}(t,\cdot)$ and by (2.1.20d), also (2.1.20c) is preserved in the sense that $\lim_{\xi\in\varXi} \partial_t\mathscr{E}(t, q_{k_\xi}(t)) = \partial_t\mathscr{E}(t, \tilde{q}(t))$ for a.a. t. Realizing that (C1) and (C2), although formulated only sequentially, hold also topologically thanks to the metrizability (2.1.18), one can prove that $\tilde{q}(t) \in \mathcal{S}(t)$ and $\lim_{\xi\in\varXi} \mathscr{E}(t, q_{k_\xi}(t)) \geq \mathscr{E}(t, \tilde{q}(t))$ for all t. Hence, $\tilde{q}$ is again an energetic solution, but it is not measurable in general, and thus it may be different from $\tilde{q}$ in Theorem 2.1.6(ii). In other words, every such accumulation point in $\mathcal{Q}^{[0,T]}$ of the subsequence $(q_k)_{k\in\mathbb{N}}$ satisfying (2.1.20a,b) is an energetic solution.

Remark 2.1.9 (Convergence of $(y_k)_{k\in\mathbb{N}}$: a special case). The statement of Theorem 2.1.6(iii) relates to the "nonbuckling condition" used in [416, Eqn. (3.18)]:

$$\left.\begin{array}{r} q_1 = (y_1, z_1) \in \mathcal{S}(t), \\ q_2 = (y_2, z_2) \in \mathcal{S}(t), \\ z_1 = z_2 \end{array}\right\} \implies \quad y_1 = y_2. \tag{2.1.24}$$

[2]Here, using the metrizability (2.1.18), one takes $\varXi = \mathrm{Part}([0,T])$, or more precisely $\varXi = \mathfrak{F}([0,T])$ directed by inclusion. This set can be used to index a neighborhood basis of the compact topology for $\{\, q \in \mathcal{Q} \mid \mathscr{E}(0,q) \leq E^* \,\}^{[0,T]}$, where $E^* < \infty$ is sufficiently large; cf. Sect. A.2.

[3]This relies on the Tikhonov theorem, Theorem A.2.2, and thus on the axiom of choice, about a uncountable (and thus nonmetrizable) product of compact spaces ($= \mathrm{Lev}_\beta\mathcal{E}(t,\cdot)$) indexed by $t \in [0,T]$.

Then the y-component can be controlled more precisely. In [373], the slightly stronger assumption

$$q_k = (y_k, z_k) \in \mathcal{S}(t) \ \text{ and } \ z_k \xrightarrow{\mathcal{Z}} z \quad \Longrightarrow \quad y_k \xrightarrow{\mathcal{Y}} y \tag{2.1.25}$$

is used to conclude the stronger result $y_k(t) \xrightarrow{\mathcal{Y}} y(t)$ for all $t \in [0,T]$ involved in (2.1.20a) as well as continuity of $t \mapsto y(t) \in \mathcal{Q}$ for all t except at the (at most countable) jump points of $\mathrm{Diss}_{\mathcal{D}}(z;[0,T])$.

Remark 2.1.10 (Guaranteed solution-attribute of z). In fact, from the proof of Theorem 2.1.6, we can read even a bit more: every $\tilde{z}$ obtained as a limit satisfying (2.1.20a,b) forms an energetic solution in the sense that there exists some $\tilde{y}$ such that $\tilde{q} = (\tilde{y}, \tilde{z})$ is an energetic solution that also satisfies (2.1.20c,d) and $\partial_t \mathcal{E}(t, \tilde{q}(t)) = \mathcal{P}_{\mathrm{red}}(t, \tilde{z}(t))$.

We close this section with the formulation of an easily applicable version of the existence result, where we strengthen the assumptions considerably but still allow for a large variety of a applications. By making $\mathcal{D}$ continuous on $\mathcal{Z}$, it is possible to decouple the assumptions on $\mathcal{E}$ and $\mathcal{D}$ completely and the compatibility conditions (C1) and (C2) can easily be established. In contrast to the simple Banach-space case considered in Example 2.1.7, we allow here still a state space $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ in the form of a general topological space without linear structure.

Theorem 2.1.11 (Easy existence result). *Assume that* $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ *satisfy* (D1), (E1), (E2), (2.1.18) *as well as the following conditions:*

$$\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0,\infty) \textit{ is continuous on sublevels of } \mathcal{E}(0,\cdot); \tag{2.1.26}$$

$$\begin{aligned} &\forall\, E^* > 0 \ \forall\, \varepsilon > 0 \ \forall\, t \in [0,T] \backslash N_{\mathcal{E}} \ \exists\, \delta > 0 \ \forall\, \tau \in [0,T] \ \forall q \in \mathcal{Q} : \\ &\Big(\mathcal{E}(0,q) \le E^* \ \textit{ and } \ 0 < |\tau - t| \le \delta\Big) \implies \left| \frac{\mathcal{E}(\tau,q) - \mathcal{E}(t,q)}{\tau - t} - \partial_t \mathcal{E}(t,q) \right| \le \varepsilon, \end{aligned} \tag{2.1.27}$$

where again $N_{\mathcal{E}}$ *is from* (E2)*. Then all assumptions of the main existence result, Theorem 2.1.6, are fulfilled, and hence the existence of energetic solutions for* $(\mathcal{Q}, \mathcal{E}, \mathcal{D}, q_0)$ *is guaranteed for each* $q_0 \in \mathcal{S}(0)$.

Proof (Philosophy of the proof). Clearly, (2.1.26) implies (D2), and it remains to establish the compatibility conditions. But this is a consequence of Corollary 2.1.19 below. Note that (2.1.39) is slightly weaker than (2.1.21d) and is simply a rephrasing of (2.1.26). □

2.1.4 *Properties of energetic solutions*

Here we discuss some basic properties of solutions $q : [0,T] \to \mathcal{Q}$ for the ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$. First we exploit the energy balance to show that q satisfies simple a priori estimates for the energy and the dissipation. For this, we use that (E) holds for all intervals $[s,t]$. Omitting the nonnegative dissipation in (E) and employing (2.1.3b) in the power term gives

$$\mathcal{E}(t, q(t)) + c_{\mathcal{E}} \leq \big(\mathcal{E}(s, q(s)) + c_{\mathcal{E}}\big)\, \mathrm{e}^{\Lambda(t)-\Lambda(s)} \quad \text{for all } s, t \in [0,T] \text{ with } s < t. \tag{2.1.28}$$

Inserting this into the right-hand side of the energy balance gives

$$\mathcal{E}(t, q(t)) + c_{\mathcal{E}} + \mathrm{Diss}_{\mathcal{D}}(q; [s,t]) \leq \big(\mathcal{E}(s, q(s)) + c_{\mathcal{E}}\big)\, \mathrm{e}^{\Lambda(t)-\Lambda(s)}. \tag{2.1.29}$$

Second, we derive a simple lemma which will imply continuity of the z component almost everywhere.

Lemma 2.1.12. *Assume that* (D1) *and* (D2) *hold. Let* $K \subset \mathcal{Z}$ *be compact and* $(z_k)_{k\in\mathbb{N}}$ *contained in* K *and* $z \in \mathcal{Z}$. *Then*

$$\min\{\mathcal{D}(z_k, z), \mathcal{D}(z, z_k)\} \to 0 \quad \Longrightarrow \quad z_k \overset{\mathcal{Z}}{\to} z. \tag{2.1.30}$$

Proof. By compactness, we have a subsequence $(z_{k_n})_{n\in\mathbb{N}}$ and $\tilde{z}$ such that $z_{k_n} \overset{\mathcal{Z}}{\to} \tilde{z}$ and $\mathcal{D}(z_{k_n}, z) \to 0$ or $\mathcal{D}(z, z_{k_n}) \to 0$. Without loss of generality, we assume the latter case. Using (2.1.30) and the lower semicontinuity (D2), we have $\mathcal{D}(z, \tilde{z}) \leq \liminf_{n\to\infty} \mathcal{D}(z, z_{k_n}) = 0$. Thus, (D1) implies $\tilde{z} = z$, and we conclude the convergence of the whole sequence, since every subsequence has the same limit z, i.e., $z_k \overset{\mathcal{Z}}{\to} z$, as desired. $\square$

The following general properties of total variations with respect to the dissipation distance $\mathcal{D}$ will be used below:

$$\mathrm{Diss}_{\mathcal{D}}(z; [r,t]) = \mathrm{Diss}_{\mathcal{D}}(z; [r,s]) + \mathrm{Diss}_{\mathcal{D}}(z; [s,t]), \tag{2.1.31}$$

$$\lim_{\tau\to t^+} \mathrm{Diss}_{\mathcal{D}}(z; [t,\tau]) = \mathcal{D}(z(t), z(t^+)), \tag{2.1.32}$$

$$\lim_{\tau\to t^-} \mathrm{Diss}_{\mathcal{D}}(z; [\tau,t]) = \mathcal{D}(z(t^-), z(t)). \tag{2.1.33}$$

We continue to use the notation

$$f(b^+) := \lim_{a\to b^+} f(a) \quad \text{and} \quad f(b^-) := \lim_{a\to b^-} f(a) \tag{2.1.34}$$

for *one-sided limits*. In the first case, this means that only $a > b$ is considered, whereas in the second case, only $a < b$ is considered.

We analyze the behavior at jump points of z. First, note that $\mathrm{Diss}_{\mathscr{D}}(z;[0,T])<\infty$ implies that $\delta : t\mapsto \mathrm{Diss}_{\mathscr{D}}(z;[0,t])$ has at most a countable number of jump points. At a continuity point of δ, we have

$$\mathscr{D}(z(t-\varepsilon),z(t))+\mathscr{D}(z(t),z(t+\varepsilon))\le \mathrm{Diss}_{\mathscr{D}}(z;[t-\varepsilon,t+\varepsilon])\to 0 \text{ for } \varepsilon\to 0.$$

Hence, using Lemma 2.1.12, we conclude that $z:[0,T]\to\mathcal{Z}$ is continuous at every continuity point of δ.

Moreover, at every jump point of δ, we may define left-hand and right-hand limits $z(t^-)=\lim_{\tau\to t^-}z(\tau)$ and $z(t^+)=\lim_{\tau\to t^+}z(\tau)$, respectively. In general, the three values $z(t^-)$, $z(t)$, and $z(t^+)$ may be different. Here, we simply set $z(0^-)=z(0)$ and $z(T^+)=z(T)$.

Lemma 2.1.13. *Assume (D1), (D2), (E1), (E2), and (C2). Let $q=(y,z):[0,T]\to\mathcal{Q}$ be an energetic solution for $(\mathcal{Q},\mathscr{E},\mathcal{Z})$. Then for all $t\in[0,T]$, we have the relations*

$$\begin{aligned}
&\mathscr{I}(t,z(t))+\mathscr{D}(z(t^-),z(t))=\mathscr{I}(t,z(t^-)),\\
&\mathscr{I}(t,z(t^+))+\mathscr{D}(z(t),z(t^+))=\mathscr{I}(t,z(t)),\\
&\mathscr{I}(t,z(t^-))=\lim_{\tau\to t^-}\mathscr{I}(\tau,z(\tau)),\quad \mathscr{I}(t,z(t^+))=\lim_{\tau\to t^+}\mathscr{I}(\tau,z(\tau)),\\
&\mathscr{D}(z(t^-),z(t))+\mathscr{D}(z(t),z(t^+))=\mathscr{D}(z(t^-),z(t^+)).
\end{aligned}\tag{2.1.35}$$

Moreover, we have $z(t^-),z(t),z(t^+)\in\hat{\mathcal{S}}(t)\subset\mathcal{Z}$ for all $t\in[0,T]$, where the reduced stability set $\hat{\mathcal{S}}(t)$ is defined in (2.1.48).

Proof. We consider only the first statement for $t>0$, since the second works analogously for $t<T$. We subtract the energy balance for $\tau<t$ from that of t and use (2.1.31) to obtain

$$\mathscr{I}(t,z(t))+\mathrm{Diss}_{\mathscr{D}}(z;[\tau,t])=\mathscr{E}(\tau,z(\tau))+\int_\tau^t \mathcal{P}_{\mathrm{red}}(s,z(s))\,\mathrm{d}s.$$

In passing to the limit $\tau\to t^-$, the last term disappears, and using (2.1.33), we find that $\mathscr{I}(t,z(t))+\mathscr{D}(z(t^-),z(t))=\lim_{\tau\to t^-}\mathscr{I}(\tau,z(\tau))$.

We now claim that $\mathscr{I}(t,z(t^-))=\lim_{\tau\to t^-}\mathscr{I}(\tau,z(\tau))$. By the lower semicontinuity (E1), we immediately have $\mathscr{I}(t,z(t^-))\le\liminf_{\tau\to t^-}\mathscr{I}(\tau,z(\tau))$ as $z(\tau)\xrightarrow{\mathcal{Z}}z(t^-)$. The opposite inequality follows from the stability of $z(\tau)$ with respect to $z(t^-)$, namely

$$\mathscr{I}(\tau,z(\tau))\le\mathscr{I}(\tau,z(t^-))+\mathscr{D}(z(\tau),z(t^-)).$$

Using (E2), we obtain $\limsup_{\tau\to t^-}\mathscr{I}(\tau,z(\tau))\le\mathscr{I}(t,z(t^-))+0$, and the first three lines in (2.1.35) are established.

To prove the last assertion, fix $t \in (0,T]$ and consider $q_n = q(t-\frac{1}{n}) \in \mathcal{S}(t-\frac{1}{n})$. Using (E1) and (C2), there exists a convergent subsequence such that $q_{n_m} \overset{\mathcal{Q}}{\to} (\tilde{y}, z(t^-)) \in \mathcal{S}(t)$, i.e., $z(t^-) \in \hat{\mathcal{S}}(t)$. Analogously, we show that $z(t^+) \in \hat{\mathcal{S}}(t)$.

To establish the last identity in (2.1.35) it suffices to prove the relation $\leq$, since the triangle inequality (D1) implies $\geq$. For this, we use $z(t^-) \in \hat{\mathcal{S}}(t)$ and test with $z(t^+)$ to obtain $\mathscr{I}(t, z(t^-)) \leq \mathscr{I}(t, z(t^+)) + \mathscr{D}(z(t^-), z(t^+))$. From inserting the first two identities in (2.1.35), the desired estimate follows. □

2.1.5 On the compatibility conditions (C1) and (C2)

Before we provide the proof of the main existence result in Section 2.1.6, we discuss the compatibility conditions in a more detail.

The major condition that makes the whole theory work is the conditioned closedness of the stability set $\mathcal{S}_{[0,T]}$ in the form (C2). For this condition, the interplay of the chosen topology and the properties of $\mathscr{E}$ and $\mathscr{D}$ are essential. The main philosophy of this condition is that stable sequences behave better than usual sequences. For instance, in many cases, it can be shown that for convergent stable sequences, we have the energy convergence $\mathscr{E}(t_k, q_k) \to \mathscr{E}(t_*, q_*)$. Using this, it is often possible to improve the convergence in Banach spaces from weak to strong or to conclude directly that the Gâteaux derivatives $\mathrm{D}\mathscr{E}(t_k, q_k)$ weakly converge to $\mathrm{D}\mathscr{E}(t_*, q_*)$. We will continue to discuss these ideas on an abstract level.

At the end of this section, we also discuss the conditioned continuity of the power. In many applications, the compatibility condition (C1) is really a condition on $\mathscr{E}$ alone, namely that $\partial_t \mathscr{E} : \mathrm{Lev}_\alpha \mathscr{E} \to \mathbb{R}$ be continuous. Such cases typically occur if the space $\mathcal{Q}$ is a reflexive Banach space equipped with the weak topology and if the loading of the problem is of lower order or even linear. However, there are also important applications in which the full generality of (C1) is needed, e.g., in finite-strain elastoplasticity; see Section 4.2.1.

Lemma 2.1.14. *Compatibility condition (C2) is equivalent to*

$$\begin{aligned} &\forall \text{ stab.seq. } (t_l, q_l) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) \ \forall\, \tilde{q} \in \mathcal{Q}\ \exists\, (\tilde{q}_l)_{l\in\mathbb{N}} : \\ &\limsup_{l\to\infty} \big(\mathscr{E}(t_l, \tilde{q}_l) + \mathscr{D}(q_l, \tilde{q}_l) - \mathscr{E}(t_l, q_l)\big) \leq \mathscr{E}(t, \tilde{q}) + \mathscr{D}(q, \tilde{q}) - \mathscr{E}(t,q). \end{aligned} \tag{2.1.36}$$

Proof. For brevity, we set $\mathcal{H}(t,q,\tilde{q}) := \mathscr{E}(t,\tilde{q}) + \mathscr{D}(q,\tilde{q}) - \mathscr{E}(t,q)$. Then $q \in \mathcal{S}(t)$ is equivalent to $\mathcal{H}(t,q,\tilde{q}) \geq 0$ for all $\tilde{q} \in \mathcal{Q}$.

The implication (C2) ⇒ (2.1.36) follows immediately by taking the sequence $\tilde{q}_l = q_l$. Then (2.1.36) holds, since $\mathcal{H}(t_l, q_l, \tilde{q}_l) = 0$ and (C2) implies $\mathcal{H}(t,q,\tilde{q}) \geq 0$.

The opposite implication (2.1.36) ⇒ (C2) is seen as follows. For arbitrary $\tilde{q}$, we choose a sequence $(\tilde{q}_l)_{l\in\mathbb{N}}$ according to (2.1.36). Using $q_l \in \mathcal{S}(t_l)$, we have

$\mathcal{H}(t_l, q_l, \tilde{q}_l) \geq 0$. Taking $\limsup_{l\to\infty}$ and employing (2.1.36), we conclude that $\mathcal{H}(t, q, \tilde{q}) \geq 0$. Since $\tilde{q} \in \mathcal{Q}$ was arbitrary, this gives $q \in \mathcal{S}(t)$. □

Condition (2.1.36) does not ask for $\tilde{q}_l \xrightarrow{\mathcal{Q}} \tilde{q}$; hence $(\tilde{q}_l)_{l\in\mathbb{N}}$ is not a recovery sequence in the sense of Γ-limits. In fact, the inequality in (2.1.36) has the property that the right-hand side depends on $\tilde{q}$ but not on $(\tilde{q}_l)_{l\in\mathbb{N}}$, while the left-hand side is independent of $\tilde{q}$. Nevertheless, the condition is useful in choosing a suitable sequence $(\tilde{q}_l)_{l\in\mathbb{N}}$ with $\tilde{q}_l \xrightarrow{\mathcal{Q}} \tilde{q}$ such that $\mathcal{E}(t_l, \tilde{q}_l)+\mathcal{D}(q_{k_l}, \tilde{q}_l)-\mathcal{E}(t_l, q_l) \to \mathcal{E}(t, \tilde{q})+\mathcal{D}(q, \tilde{q})-\mathcal{E}(t, q)$. For later use, we display such a slight strengthening of (2.1.36):

$$\begin{aligned}&\forall \text{ stab.seq. } (t_l, q_l) \xrightarrow{[0,T]\times\mathcal{Q}} (t, q)\ \forall\, \tilde{q} \in \mathcal{Q}\ \exists\, \tilde{q}_l \xrightarrow{\mathcal{Q}} \tilde{q} :\\ &\limsup_{l\to\infty} \big(\mathcal{E}(t_l, \tilde{q}_l)+\mathcal{D}(q_l, \tilde{q}_l)-\mathcal{E}(t_l, q_l)\big) \leq \mathcal{E}(t, \tilde{q})+\mathcal{D}(q, \tilde{q})-\mathcal{E}(t, q).\end{aligned} \tag{2.1.37}$$

We call $(\tilde{q}_l)_{l\in\mathbb{N}}$ in (2.1.37) a *mutual recovery sequence* for (t_l, q_l) and $\tilde{q}$; cf. [420].

We provide two more conditions that are stronger than (2.1.37) and hence can be used to establish the crucial closedness (C2) of the stability set. The weaker of these two conditions is based on the existence of a mutual recovery sequence and reads

$$\begin{aligned}&\forall \text{ stab.seq. } (t_l, q_l) \xrightarrow{[0,T]\times\mathcal{Q}} (t, q)\ \forall\, \tilde{q} \in \mathcal{Q}\ \exists\, \tilde{q}_l \xrightarrow{\mathcal{Q}} \tilde{q} :\\ &\limsup_{l\to\infty} \big(\mathcal{E}(t_l, \tilde{q}_l)+\mathcal{D}(q_l, \tilde{q}_l)\big) \leq \mathcal{E}(t, \tilde{q})+\mathcal{D}(q, \tilde{q}).\end{aligned} \tag{2.1.38}$$

The stronger of these two conditions uses the continuity of the dissipation distance on sublevels of $\mathcal{E}$, namely

$$\left.\begin{aligned}&q_k \xrightarrow{\mathcal{Q}} q,\ \tilde{q}_k \xrightarrow{\mathcal{Q}} \tilde{q} \text{ and}\\ &\sup_{k\in\mathbb{N}} \big(\mathcal{E}(t, q_k)+\mathcal{E}(t, \tilde{q}_k)\big) < \infty\end{aligned}\right\} \implies \mathcal{D}(q_k, \tilde{q}_k) \to \mathcal{D}(q, \tilde{q}). \tag{2.1.39}$$

Proposition 2.1.15 (Conditions guaranteeing (C2)). *Assume that (E1) holds.*

(i) If for each stable sequence (t_l, q_l) converging to (t, q), there exists a sequence $(\tilde{q}_l)_{l\in\mathbb{N}}$ such that $\limsup_{l\to\infty} \mathcal{E}(t_l, \tilde{q}_l)+\mathcal{D}(q_l, \tilde{q}_l) \leq \mathcal{E}(t, q)$, then the energy converges along the stable sequences, i.e.,

$$\forall \textit{ stable sequence } (t_l, q_l) \xrightarrow{[0,T]\times\mathcal{Q}} (t, q) : \quad \mathcal{E}(t_l, q_l) \to \mathcal{E}(t, q). \tag{2.1.40}$$

In particular, (2.1.38) implies (2.1.40).

(ii) We have the following implications:

$$(2.1.39) \implies (2.1.38) \implies (2.1.37) \implies (2.1.36) \iff (C2).$$

(iii) If (E2) holds additionally, then the conditions (2.1.36), (2.1.37), and (2.1.38) remain the same if $\mathcal{E}(t_l, \cdot)$ is replaced by $\mathcal{E}(t, \cdot)$.

Proof. Part (i). By (E1), we have $\mathscr{E}(t,q) \le \liminf_{l\to\infty} \mathscr{E}(t_l,q_l)$. Using that $(q_l)_{l\in\mathbb{N}}$ is a stable sequence, we immediately obtain

$$\limsup_{l\to\infty} \mathscr{E}(t_l,q_l) \le \limsup_{l\to\infty} \mathscr{E}(t_l,\tilde{q}_l) + \mathscr{D}(q_l,\tilde{q}_l) \le \mathscr{E}(t,q),$$

where the last inequality uses (2.1.38) by specifying $\tilde{q} = q$. This proves (2.1.40).
Part (ii). For the first implication, we begin with a stable sequence $(t_l,q_l) \to (t,q)$ and a general $\tilde{q}$. Employing (2.1.39), we then obtain $\limsup_{l\to\infty} \mathscr{E}(t_l,\tilde{q}_l)+\mathscr{D}(q_l,\tilde{q}) \le \mathscr{E}(t,\tilde{q})+\mathscr{D}(q,\tilde{q})$, which is the desired result (2.1.38) with $\tilde{q}_l = \tilde{q}$.

For (2.1.38) $\Rightarrow$ (2.1.37), note that (E1) implies $\limsup_{l\to\infty} \big(-\mathscr{E}(t_l,q_{k_l})\big) \le -\mathscr{E}(t,q)$ whenever $(t_l,q_{k_l}) \xrightarrow{[0,T]\times\mathcal{Q}} (t,q)$. Adding this to (2.1.38), we easily obtain the desired result (2.1.37).

The next implication follows directly from the definition, since the requirement $\tilde{q}_{k_l} \xrightarrow{\mathcal{Q}} \tilde{q}$ is dropped. The final equivalence was already shown in Lemma 2.1.14.
Part (iii). For this, we simply use that $\sup_{l\in\mathbb{N}} \mathscr{E}(t_l,q_l)$ and $\sup_{l\in\mathbb{N}} \mathscr{E}(t_l,\tilde{q}_l)$ are finite and apply (2.1.3a). ☐

In [373, Thm. 5.2], the condition $\liminf_{l\to\infty} \mathscr{E}(t_l,q_l)-\mathscr{D}(q_j,\tilde{q}) \ge \mathscr{E}(t,q) - \mathscr{D}(q,\tilde{q})$ for all $\tilde{q} \in \mathcal{Q}$ is proved to be sufficient for (C2). In fact, this condition is stronger than (2.1.37), since we may simply choose $\tilde{q}_l = \tilde{q}$ there. The sufficient condition given in [373, Thm. 5.3] turns out to be equivalent to the present condition (2.1.38).

Example 2.1.16 (Different mutual recovery conditions). We show that (2.1.37) does not imply (2.1.38) by considering a typical situation for material modeling. On $\mathcal{Q} = \mathrm{L}^2(\Omega)$ equipped with its weak topology, consider the functionals $\mathscr{E}(t,q) = \int_\Omega \frac{1}{2}q(x)^2 - f(t,x)q(x)\,\mathrm{d}x$ with $f \in \mathrm{C}^1([0,T];\mathrm{L}^2(\Omega))$ and $\mathscr{D}(q_0,q_1) = \|q_1-q_0\|_{\mathrm{L}^1}$. We have $\mathcal{S}(t) = \{\, q \in \mathrm{L}^2(\Omega) \mid \|q-f(t,\cdot)\|_{\mathrm{L}^\infty} \le 1\,\}$, and by convexity and strong closedness, we have weak closedness, and (C2) holds. Moreover, we want to establish the sufficient condition (2.1.37). For this, we choose the mutual-recovery sequence $\tilde{q}_l = \tilde{q} - q + q_l$, whence $\tilde{q}_l \rightharpoonup \tilde{q}$. Moreover, $\mathscr{D}(q_l,\tilde{q}_l) = \|\tilde{q}-q\|_{\mathrm{L}^1} = \mathscr{D}(q,\tilde{q})$ and

$$\begin{aligned}\mathscr{E}(t,\tilde{q}_l) - \mathscr{E}(t,q_l) &= \big\langle \tfrac{1}{2}(\tilde{q}_l+q_l) - f(t), \tilde{q}_l-q_l\big\rangle_{\mathrm{L}^2} = \big\langle \tfrac{1}{2}(\tilde{q}-q+2q_l) - f(t), \tilde{q}-q\big\rangle_{\mathrm{L}^2}\\ &\to \big\langle \tfrac{1}{2}(\tilde{q}+q) - f(t), \tilde{q}-q\big\rangle_{\mathrm{L}^2} = \mathscr{E}(t,\tilde{q}) - \mathscr{E}(t,q),\end{aligned}$$

which proves (2.1.37) with equality. We call this argument the *quadratic trick* and formalize it in Lemma 3.5.3. It will be used. e.g., in classical linearized elastoplasticity with hardening (cf. Sect. 4.3.1.1) and for evolutionary Γ-convergence in dimension reduction or two-scale homogenization (cf. Sections 4.3.1.4 and 4.3.1.5, respectively).

To show that (2.1.38) does not hold, we consider $t = 0$ and a stable sequence q_l with $|q_l-f(0,\cdot)| \equiv 1$ but $q_l \rightharpoonup q = f(0,\cdot)$. Moreover, let $\tilde{q} = q$, so that the right-hand side in (2.1.38) takes the value $-\frac{1}{2}\|q\|^2_{\mathrm{L}^2}$. Writing the mutual recovery sequence $\tilde{q}_l$ in the form $\tilde{q}_l = q_l + w_l$, we need $w_l \rightharpoonup 0$, and the left-hand side in (2.1.38) gives

$$\begin{aligned}\mathcal{E}(0,\tilde{q}_l) + \mathcal{D}(q_l,\tilde{q}_l) &= \int_\Omega \frac{1}{2}\big(q_l{+}w_l{-}q\big)^2 - \frac{1}{2}|q|^2 + |w_l|\,\mathrm{d}x \\ &\geq \int_\Omega \frac{1}{2} - \frac{1}{2}|q|^2\,\mathrm{d}x \;>\; -\frac{1}{2}\|q\|^2_{\mathrm{L}^2} = \mathcal{E}(0,q){+}\mathcal{D}(q,q),\end{aligned}$$

where we used $|q_l{-}q| \equiv 1$ and minimized with respect to w_l. Thus, we have shown that (2.1.38) cannot hold.

Concerning the conditioned continuity of the power, we often consider the case that $\mathcal{Q}$ is a weakly closed subset of a reflexive Banach space $\boldsymbol{Q}$ equipped with the weak topology. Moreover, the energy takes the form $\mathcal{E}(t,q) = \mathcal{E}(q) - \langle \ell(t), q\rangle$, where $\ell \in \mathrm{W}^{1,1}(0,T;\boldsymbol{Q}^*)$. Then it is easy to establish (C1) using $\partial_t\mathcal{E}(t,q) = -\langle\dot{\ell}(t),q\rangle$ even without using the stability; cf. Example 2.1.7 and Corollary 3.1.2 below.

The following abstract result establishes the continuity of the power (C1) under much more general conditions. It relies purely on semicontinuity properties and is independent of a linear structure. The motivation for this approach stems from the need to treat time-dependent boundary conditions via a suitable translation in the function space, e.g., for a functional $\mathcal{E}(t,y,z) = \mathcal{E}(y{-}y_{\mathrm{D}}(t),z)$ with a general lower semicontinuous functional $\mathcal{E}$, the time derivative becomes troublesome. For such treatments of time-dependent Dirichlet conditions, we refer to Example 2.1.20 below and to Section 4.2.1 in finite-strain elastoplasticity. This method was first developed in [149] for fracture problems and generalized to an abstract and largely simplified setting in [195].

Proposition 2.1.17. *If $\mathcal{E}$ satisfies (E1)–(E2) and (2.1.27), then for all $t \in [0,T]\backslash N_{\mathcal{E}}$, where $N_{\mathcal{E}}$ is from (E2), we have*

$$\left.\begin{array}{l}(t_m,q_m) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) \quad \text{and} \\ \mathcal{E}(t_m,q_m) \to \mathcal{E}(t,q) < \infty\end{array}\right\} \quad\Longrightarrow\quad \partial_t\mathcal{E}(t,q_m) \to \partial_t\mathcal{E}(t,q). \tag{2.1.41}$$

In condition (2.1.27), the convergence of the difference quotients to the derivative is uniform in $q \in \mathrm{Lev}_{E_*}\mathcal{E}(0,\cdot)$ but may depend on the time $t \in [0,T]\backslash N_{\mathcal{E}}$. Thus, the condition is invariant under rescaling time by absolutely continuous diffeomorphisms.

Proof. Let $E_0, h_0 > 0$, and $t \in (0,T)\backslash N_{\mathcal{E}}$ be such that $t \pm h_0 \in [0,T]$ and $\mathcal{E}(t,q_m), \mathcal{E}(t,q) \leq E_0$ for sufficiently large m. Then condition (2.1.27) implies the existence of a *modulus of continuity* $\omega_0 : [0,h_0] \to [0,\infty)$ (i.e., ω_0 is monotonically increasing, and $\omega_0(h) \to 0$ for $h \to 0^+$), so that

$$\left|\frac{\mathcal{E}(t{\pm}h,q_m){-}\mathcal{E}(t,q_m)}{h} \mp \partial_t\mathcal{E}(t,q_m)\right| \leq \omega_0(h) \;\text{ for all }\; h \in (0,h_0). \tag{2.1.42}$$

The same estimate also holds for q. Using $h > 0$, (E1) (i.e., $\mathscr{E}(t\pm h,\cdot)$ lower semicontinuous), and the assumed convergence of the energy, we obtain

$$\liminf_{m\to\infty} \frac{\mathscr{E}(t\pm h, q_m)-\mathscr{E}(t, q_m)}{h} \geq \frac{\mathscr{E}(t\pm h, q)-\mathscr{E}(t, q)}{h}.$$

Combining the case "+" with (2.1.42), we obtain

$$\begin{aligned}\liminf_{m\to\infty} \partial_t\mathscr{E}(t, q_m) &\geq \liminf_{m\to\infty} \frac{\mathscr{E}(t+h, q_m)-\mathscr{E}(t, q_m)}{h} - \omega_0(h) \\ &\geq \frac{\mathscr{E}(t+h, q)-\mathscr{E}(t, q)}{h} - \omega_0(h) \geq \partial_t\mathscr{E}(t, q) - 2\omega_0(h).\end{aligned}$$

Similarly, the case "−" gives $\limsup_{m\to\infty} \partial_t\mathscr{E}(t, q_m) \leq \partial_t\mathscr{E}(t, q) + 2\omega_0(h)$. Since h can be made arbitrarily small, the assertion is proved. □

In some situations, it is easier to prove a slightly strengthened version of (2.1.27), where we use uniform continuity on sublevels in the form

$$\begin{aligned}&\forall\, E^* > 0\ \forall\, \varepsilon > 0\ \exists \delta > 0\ \forall\, t_1, t_2 \in [0, T]\ \ \forall q \in \mathcal{Q} :\\ &\Big(\mathscr{E}(0, q) \leq E^* \text{ and } 0 < |t_1 - t_2| \leq \delta\Big) \implies \big|\partial_t\mathscr{E}(t_1, q) - \partial_t\mathscr{E}(t_2, q)\big| \leq \varepsilon.\end{aligned} \tag{2.1.43}$$

Using $\frac{1}{\tau-t}(\mathscr{E}(\tau, q) - \mathscr{E}(t, q)) = \partial_t\mathscr{E}(s, q)$ for some s between τ and t, it is easy to see that (2.1.27) holds with $N_{\mathscr{E}} = \emptyset$.

In some applications, it is possible to exploit the Cartesian structure $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z}$ and a certain regularization property of $\mathscr{E}(t, \cdot, z)$, since we need the continuity of the power (C1) only conditioned to stable sequences.

Proposition 2.1.18. *Let $\mathcal{T}'_{\mathcal{Y}}$ be a finer topology on $\mathcal{Y}$ such that for all $t \in [0, T]\backslash N_{\mathscr{E}}$,*

$$\partial_t\mathscr{E}(t, \cdot, \cdot) : \mathcal{Y}\times\mathcal{Z} \to \mathbb{R} \text{ continuous with respect to } \mathcal{T}'_{\mathcal{Y}} \times \mathcal{T}_{\mathcal{Z}}, \tag{2.1.44a}$$

$$\left.\begin{array}{l} y_m \overset{\mathcal{Y}}{\to} y,\quad z_m \overset{\mathcal{Z}}{\to} z, \\ y_m \text{ minimizes } \mathscr{E}(t, \cdot, z_m) \end{array}\right\} \implies y_m \overset{\mathcal{T}'_{\mathcal{Y}}}{\to} y. \tag{2.1.44b}$$

Then $\partial_t\mathscr{E}(t, q_m) \to \partial_t\mathscr{E}(t, q)$ whenever $q_m = (y_m, z_m) \overset{\mathcal{Q}}{\to} q$ and y_m minimizes $\mathscr{E}(t, \cdot, z_m)$, and in particular, (C1) holds.

The proof of Proposition 2.1.18 is obvious. Together with Proposition 2.1.15, we obtain the following result.

Corollary 2.1.19. *Assume that (D1), (D2), (E1), and (E2) hold, and one of the following two assumptions is satisfied:*

(a) (2.1.38) *and* (2.1.27) *are valid;*
(b) (2.1.36) *and* (2.1.44) *hold for some topology $\mathcal{T}'_{\mathcal{Y}}$. Then both compatibility conditions* (C1) *and* (C2) *are satisfied.*

Proof. Assume first (a). Then (2.1.38) implies (C2) by Proposition 2.1.15(ii). Moreover, (C1) follows from Proposition 2.1.17, where we use Proposition 2.1.15(i), which guarantees, again relying on (2.1.38), that the energy along stable sequences is continuous, i.e., (2.1.41) is applicable.

Assuming (b), we know that (2.1.36) implies (C2) by Proposition 2.1.15(ii), whereas (C1) follows from Proposition 2.1.18. □

The following example highlights the difference between the condition (2.1.43) as a sufficient condition for (2.1.27) and the weaker condition (2.1.44). The example is in fact close to the complete-damage model treated in Section 4.3.2.2.

Example 2.1.20 (Conditioned continuity of the power). Consider $\mathcal{Y} = \mathrm{W}^{1,p}(\Omega)$ and $\mathcal{Z} = \mathrm{W}^{1,2}(\Omega)$ equipped with the weak topologies, and a "damage-type" stored energy

$$\mathcal{E}(t, y, z) = \int_\Omega \frac{1}{p} a(z)|\nabla(y - y_\mathrm{D}(t))|^p + \frac{1}{2}|\nabla z|^2 \,\mathrm{d}x$$

with $a : \mathbb{R} \to \mathbb{R}$ continuous, bounded, and $\inf a(\cdot) > 0$. Obviously, we have

$$\partial_t \mathcal{E}(t, y, z) = \int_\Omega a(z)|\nabla(y_\mathrm{D}(t) - y)|^{p-2}\nabla(y_\mathrm{D}(t) - y)\cdot\nabla\dot{y}_\mathrm{D}(t) \,\mathrm{d}x,$$

and for $p \neq 2$, the power $\partial_t\mathcal{E}(t, \cdot, z)$ is not weakly continuous; hence $\partial_t\mathcal{E}(t, q) = \lim_{k\to\infty} \partial_t\mathcal{E}(t, q_k)$ in (C1) is not automatic for arbitrary $q_k \rightharpoonup q$. On the one hand, (2.1.43), which guarantees (C1), follows from $\dot{y}_\mathrm{D} \in \mathrm{C_w}([0, T]; \mathrm{W}^{1,p}(\Omega))$. On the other hand, $\partial_t\mathcal{E}(t, \cdot, z)$ is strongly continuous, so we can use Proposition 2.1.18 with the $\mathcal{T}'_\mathcal{Y}$ = norm topology and (2.1.44b) based on the so-called (S_+)-property; cf. Example 3.4.11 below.[4] Hence, (2.1.44) already holds under the weaker assumption $\dot{y}_\mathrm{D} \in \mathrm{L}^1(0, T; \mathrm{W}^{1,p}(\Omega))$.

[4]In more detail, (3.4.19) can be used by estimating (where $\underline{a} = \inf_\Omega a$)

$$\begin{aligned}
&\underline{a}\big(\|\nabla y_k\|_{\mathrm{L}^p}^{p-1} - \|\nabla y\|_{\mathrm{L}^p}^{p-1}\big)\big(\|\nabla y_k\|_{\mathrm{L}^p} - \|\nabla y\|_{\mathrm{L}^p}\big)\\
&\quad\le \underline{a}\int_\Omega \big(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y\big)\cdot\nabla(y_k - y)\,\mathrm{d}x\\
&\qquad\le \int_\Omega a(z_k)\big(|\nabla y_k|^{p-2}\nabla y_k - |\nabla y|^{p-2}\nabla y\big)\cdot\nabla(y_k - y)\,\mathrm{d}x\\
&\qquad\quad\le \int_\Omega \big(a(z) - a(z_k)\big)|\nabla y|^{p-2}\nabla y\big)\cdot\nabla(y_k - y)\,\mathrm{d}x \to 0,
\end{aligned}$$

because $(a(z) - a(z_k))|\nabla y|^{p-2}\nabla y \to 0$ in $L^{p'}(\Omega; \mathbb{R}^d)$, while $\nabla(y_k - y) \to 0$ weakly in $L^p(\Omega; \mathbb{R}^d)$.

2.1.6 *Proof of Theorem 2.1.6*

Before going into the details of the proof, we advise the less-experienced reader to acquire some familiarity with the proof of Theorem 3.5.2, which is a much simpler case of the present general situation. The simplified proof follows the same steps as the general proof; see Table 2.1 on p. 72.

The proof follows the theory developed in [195]; however, it includes a new argument, namely the characterization of the power in (2.1.19) and Proposition 2.1.23. This approach allows us to simply the assumptions considerably in the case that $\mathcal{Q}$ satisfies the metrizability condition (2.1.18).

The first lemma of this section concerns the approximation of the Lebesgue integrals $\int_r^s \partial_t\mathcal{E}(t,q(t))\,\mathrm{d}t$ by Riemann sums for a given $q : [0,T] \to \mathcal{Q}$. For a partition $\Pi = \{r = t_0^\Pi < t_1^\Pi < \cdots < t_{N_\Pi}^\Pi = s\} \in \mathrm{Part}([r,s])$, we define the discrete values $q_k^\Pi := q(t_k^\Pi)$ and the left-continuous approximants $\overline{q}^\Pi$ via

$$\overline{q}^\Pi(t) := q(\overline{\tau}^\Pi(t)) \quad \text{with} \quad \overline{\tau}^\Pi(t) := \min\left\{ t_k^\Pi \in \Pi \;\middle|\; t_k^\Pi \geq t \right\}. \tag{2.1.45}$$

Note that the left-continuous interpolants $\overline{q}^\Pi$ in (2.1.13) are defined for solutions of the incremental minimization problems, whereas here a measurable $q : [0,T] \to \mathcal{Q}$ is given a priori. Clearly, we have $t \leq \overline{\tau}^\Pi(t) \leq t+\varnothing(\Pi)$ for all t, where the fineness $\varnothing(\Pi)$ of Π is the length of its largest subinterval; see (2.1.7b).

Lemma 2.1.21. *Let the conditions* (E2), (C1) *and the metrizability condition* (2.1.18) *hold. Moreover, assume that* $q : [0,T] \to \mathcal{Q}$ *is measurable and that there exists* $C > 0$ *such that for all* $t \in [0,T]$*, we have* $\mathcal{E}(t,q(t)) \leq C$ *and* $q(t) \in \mathcal{S}(t)$*. Then for all* $r,s \in [0,T]$ *with* $r < s$*, we have*

$$\sup_{\Pi\in\mathrm{Part}([r,s])} \int_r^s \partial_t\mathcal{E}(t,\overline{q}^\Pi(t))\,\mathrm{d}t \geq \int_r^s \partial_\tau\mathcal{E}(t,q(t))\,\mathrm{d}t.$$

Proof. Note that each function $t \mapsto \mathcal{E}(t+h,q(t))$ is measurable. Hence, the power $\tau \mapsto \partial_\tau\mathcal{E}(\tau,q(\tau))$ is measurable as well, since it is a pointwise limit of measurable difference quotients. Moreover, there is a constant $c_0 > 0$ such that $|\partial_t\mathcal{E}(t,\overline{q}^\Pi(t))| \leq c_0\lambda_{\mathcal{E}}(t)$.

Using (2.1.18) we may apply Lusin's theorem, Theorem B.3.7, to q, which takes values in a compact set, since the energy is bounded. For arbitrary $\varepsilon > 0$, we obtain a compact set $K \subset [r,s]$ such that

$$\int_{[r,s]\setminus K} c_0\lambda(t)\,\mathrm{d}t < \varepsilon \quad \text{and} \quad q|_K : K \to \mathcal{Q} \text{ is continuous.} \tag{2.1.46}$$

The first property in (2.1.46) implies $\int_r^s \partial_t\mathcal{E}(t,\overline{q}^\Pi(t))\,\mathrm{d}t \geq \int_K \partial_t\mathcal{E}(t,\overline{q}^\Pi(t))\,\mathrm{d}t - \varepsilon$ for all partitions Π. We now construct a suitable sequence of partitions $(\Pi_n)_n$ that allows us to prove the assertion. For given $n \in \mathbb{N}$, let $t_0^n = r$. The other points are defined inductively, namely, as long as $t_j^n < s$, we set

$$t_{j+1}^n = \begin{cases} \max\left\{ t \in K \mid t_j < t \le t_j^n + \frac{1}{n} \right\} & \text{if } K \cap \left(t_j, t_j + \frac{1}{n}\right] \neq \emptyset, \\ \min\left\{t_j + \frac{1}{n}, s\right\} & \text{otherwise.} \end{cases}$$

On the one hand, there cannot be two adjacent intervals that are small: if $t_{j+1} < t_j + \frac{1}{n}$, then $K \cap (t_{j+1}, t_j + \frac{1}{n}]$ is empty. Now if $t_{j+1} < s$, then t_{j+2} exceeds $\min\{t_j + \frac{1}{n}, s\}$. Hence, Π_n has at most $2(s-r)n+1$ intervals, and by construction, the fineness satisfies $\varnothing(\Pi_n) \le \frac{1}{n}$.

On the other hand, the choice of the nodes in Π_n is such that for $t \in K$, we always have $\overline{\tau}^{\Pi}(t) \in K$ as well. Indeed, $t_{j+1} \in \Pi \backslash K$ occurs only if $(t_j, t_{j+1}]$ has empty intersection with K. Thus, we have shown that

$$\forall\, t \in K : \quad \overline{\tau}^{\Pi_n}(t) \in K \ \text{ and } \ \overline{\tau}^{\Pi_n}(t) \to t^+ \ \text{ for } \ n \to \infty.$$

Now we recall $\overline{q}^{\Pi_n}(t) = q(\overline{\tau}^{\Pi_n}(t))$, and using the stability of q, we conclude that $(\overline{\tau}^{\Pi_n}(t), \overline{q}^{\Pi_n}(t))_{n\in\mathbb{N}}$ is a stable sequence converging to $(t, q(t))$ because of (2.1.46). Hence exploiting (C1), we conclude that $\partial_t \mathscr{E}(t, \overline{q}^{\Pi_n}(t)) \to \partial_t \mathscr{E}(t, q(t))$, and Lebesgue's dominated convergence theorem gives

$$\lim_{n\to\infty} \int_K \partial_t \mathscr{E}(t, \overline{q}^{\Pi_n}(t))\,\mathrm{d}t = \int_K \partial_t \mathscr{E}(t, q(t))\,\mathrm{d}t, \tag{2.1.47}$$

where a lower estimate "$\ge$" would be sufficient to proceed. In summary, we have

$$\begin{aligned} \sup_{\Pi} \int_r^s \partial_t \mathscr{E}(t, \overline{q}^{\Pi}(t))\,\mathrm{d}t &\ge \limsup_{n\to\infty} \int_r^s \partial_t \mathscr{E}(t, \overline{q}^{\Pi_n}(t))\,\mathrm{d}t \\ &\ge -\varepsilon + \limsup_{n\to\infty} \int_K \partial_t \mathscr{E}(t, \overline{q}^{\Pi_n}(t))\,\mathrm{d}t \\ &= -\varepsilon + \int_K \partial_t \mathscr{E}(t, q(t))\,\mathrm{d}t \ge -2\varepsilon + \int_r^s \partial_t \mathscr{E}(t, q(t))\,\mathrm{d}t. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this is the desired result. $\square$

The proof of Theorem 2.1.6 will make extensive use of the *reduced energy*

$$\mathscr{I} : [0,T]\times\mathcal{Z} \to \mathbb{R}_\infty;\ (t, z) \mapsto \min\left\{ \mathscr{E}(t, \tilde{y}, z) \;\middle|\; \tilde{y} \in \mathcal{Y} \right\}.$$

The point is that stability and energy balance can be formulated easily for the *reduced rate-independent system* $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$. For this, we define the *reduced stability sets*

$$\hat{\mathcal{S}}(t) := \Big\{ z \in \mathcal{Z} \,\Big|\, \mathscr{I}(t,z) < \infty,\ \forall\, \tilde{z} \in \mathcal{Z}:\ \mathscr{I}(t,z) \le \mathscr{I}(t,\tilde{z}) + \mathscr{D}(z,\tilde{z}) \Big\}. \tag{2.1.48}$$

Clearly, we have $q = (y,z) \in \mathcal{S}(t)$ if and only if $z \in \hat{\mathcal{S}}(t)$ and $y \in \operatorname{Arg\,min} \mathscr{E}(t,\cdot,z)$. The only difficulty in reducing from $\mathcal{Q}$ to $\mathcal{Z}$ is that in general, $t \mapsto \mathscr{I}(t,z)$ is no longer differentiable; see the discussion in Section 2.2.1. Thus, we define energetic solutions $z : [0,T] \to \mathcal{Z}$ of the reduced ERIS $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$ via the reduced power $\mathcal{P}_{\mathrm{red}}$ defined in (2.1.16) as follows:

$$\begin{aligned} &(\mathrm{S})_{\mathrm{red}} \quad z(t) \in \hat{\mathcal{S}}(t), \\ &(\mathrm{E})_{\mathrm{red}} \quad \mathscr{I}(t,z(t)) + \mathrm{Diss}_{\mathscr{D}}(z;[0,t]) = \mathscr{I}(0,z(0)) + \int_0^t \mathcal{P}_{\mathrm{red}}(s,z(s))\,\mathrm{d}s. \end{aligned} \tag{2.1.49}$$

Obviously, each energetic solution $q = (y,z) : [0,T] \to \mathcal{Q}$ generates a reduced energetic solution $z : [0,T] \to \mathcal{Z}$ for $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$. The next lemma shows that the opposite is also true: each solution z for $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$ can be made into a full solution $q = (y,z)$ for the ERIS $(\mathcal{Q}, \mathscr{E}, \mathscr{D})$ by selection of a suitable measurable y-component. In Section 2.2.1, we address the question of producing a direct existence theory for the reduced system $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$, but the theory there would lead to a slightly stronger assumption for the full problem. That is why we deal with the full system first.

Lemma 2.1.22. *Assume* (D1), (D2), (E1), (E2), (C1), (C2), *and* (2.1.18). *Let* $z : [0,T] \to \mathcal{Z}$ *be measurable with* $\mathrm{Diss}_{\mathscr{D}}(z;[0,T]) + \sup_{t\in[0,T]} \mathscr{I}(t,z(t)) < \infty$ *and assume* $z(t) \in \hat{\mathcal{S}}(t)$ *for all* $t \in [0,T]$. *Then with* $N_{\mathscr{E}}$ *from* (E2), *there exists a measurable function* $y : [0,T] \to \mathcal{Y}$ *such that*

$$\begin{aligned} &(y(t),z(t)) \in \mathcal{S}(t) \quad \textit{for all} \quad t \in [0,T] \quad \textit{and} \\ &\mathcal{P}_{\mathrm{red}}(t,z(t)) = \partial_t \mathscr{E}(t,y(t),z(t)) \quad \textit{for all} \quad t \in [0,T] \backslash N_{\mathscr{E}}. \end{aligned}$$

Proof. Our proof will be based on a variant of Filippov's selection result provided in Proposition B.1.2 on p. 593, which we will use here with the complete measure space $(S, \mathfrak{S}, \mu)$ as $S = [0,T]$ with $\mathfrak{S}$ equal to the σ-algebra of the Lebesgue-measurable subsets and $\mu = \mathscr{L}^1(\cdot)$ the one-dimensional Lebesgue measure.

For given $(t,z) \in [0,T] \times \mathcal{Z}$, we define $M(t,z) := \operatorname{Arg\,min}\{ \mathscr{E}(t,\tilde{y},z) \mid \tilde{y} \in \mathcal{Y} \}$ and compose a set-valued mapping $G : [0,T] \rightrightarrows \mathcal{Y}$ via

$$G(t) := M(t,z(t^-)) \cup M(t,z(t)) \cup M(t,z(t^+)) \subset Y \subset \mathcal{Y},$$

where Y is a compact subset of $\mathcal{Y}$, which exists due to (E1). By assumption (2.1.18), the topology on Y is complete, separable, and metrizable. Using (E1), each $M(t,z)$ is nonempty, and hence each $G(t)$ is nonempty.

Employing (C2), we will show that the graph $\mathrm{Gr}(G) = \{ (t,y) \mid y \in G(t) \}$ is closed in $[0,T] \times Y$ and hence is measurable; cf. the definition on p. 592. Indeed, consider $(t_k, y_k) \in \mathrm{Gr}(G)$ with $t_k \to t_*$ and $y_k \to y_*$. Then there exists $z_k = z(t_k^\nu)$ with $t_k^\nu \in \{t_k, t_k^-, t_k^+\}$ such that $y_k \in M(t_k, z_k)$. Using Lemma 2.1.13, we conclude that $(y_k, z_k) \in \mathcal{S}(t_k)$. After taking a subsequence (not relabeled), we may assume $z_k \xrightarrow{\mathcal{Z}} z_*$, and (C2) provides $(y_*, z_*) \in \mathcal{S}(t_*)$, which implies $y_* \in M(t_*, z_*)$. Moreover, (t_*, z_*) lies in the closure of $\mathrm{Gr}(z) \subset [0,T]\times\mathcal{Z}$, which means that $z_* = z(t_*^\nu)$. Thus, we have established $y_* \in G(t_*)$, as desired.

We now define the set-valued mapping $F : [0,T] \rightrightarrows Y$ via $F(t) = M(t, z(t))$. Clearly, $F(t)$ is nonempty and closed for each t. Since z is continuous outside an at most countable set $N_{\mathrm{jump}} \subset [0,T]$, we have $F(t) = G(t)$ for $t \in [0,T]\backslash N_{\mathrm{jump}}$. Thus, F is a measurable set-valued mapping as well.

Referring again to $N_{\mathcal{E}}$ from (E2), we now define the function $g : \mathrm{Gr}(F) \to \mathbb{R}$ via

$$g(t,y) := \begin{cases} \partial_t \mathcal{E}(t, y, z(t)) - \mathcal{P}_{\mathrm{red}}(t, z(t)) & \text{for } t \in [0,T]\backslash N_{\mathcal{E}} \text{ and } y \in F(t), \\ 0 & \text{otherwise.} \end{cases}$$

For fixed $t \in [0,T]$, the function $g(t,\cdot) : F(t) \to \mathbb{R}$ is continuous using (C1). Since $\mathcal{E}$ is has compact sublevels by (E1), it is Borel-measurable. Moreover, $z : [0,T] \to \mathcal{Z}$ is Borel-measurable, since it is continuous except for a countable number of points. Thus, for each $h > 0$, the function $\gamma_h : [0, T-h]\times Y \to \mathbb{R}; (t,y) \mapsto \mathcal{E}(t+h, y, z(t))$ is $(\mathfrak{L} \otimes \mathfrak{B}(Y), \mathfrak{B}(\mathbb{R}))$-measurable. Now, on $([0,T[\backslash N_{\mathcal{E}}) \times Y$, the function g is the pointwise limit of the measurable difference quotients $\frac{1}{h}(\gamma_h - \gamma_0)$. So is $N_{\mathcal{E}} \cup \{T\}$ has measure 0, $g : [0,T]\times Y \to \mathbb{R}$ is measurable as well, and hence its restriction to $\mathrm{Gr}(F)$.

Next, we show that for each $t \in [0,T]\backslash N_{\mathcal{E}}$, there exists $y \in F(t)$ with $g(t,y) = 0$. Indeed, by (E1), the set $M(t,z) = \mathrm{Arg\,min}\, \mathcal{E}(t,\cdot,z)$ is a nonempty compact set. Choose a sequence $(y_m)_m$ approaching the supremum in the definition (2.1.16) of the reduced power, that is, $\mathcal{P}_{\mathrm{red}}(t,z) = \sup\{\, \partial_t\mathcal{E}(t, \tilde{y}, z(t)) \mid \tilde{y} \in M(t,z)\,\}$. Taking a subsequence, we may assume $y_{m_n} \xrightarrow{\mathcal{Y}} y_* \in M(t,z)$. Since $(y_{m_n}, z) \in \mathcal{S}(t)$, we have a stable sequence, and (C1) gives $\mathcal{P}_{\mathrm{red}}(t, z(t)) = \lim_{n\to\infty} \partial_t\mathcal{E}(t, y_{m_n}, z) = \partial_t\mathcal{E}(t, y_*, z)$, as desired.

Since for $t \in N_{\mathcal{E}}$, the relation $g(t,y) = 0$ always holds, we are able to apply Proposition B.1.2 and obtain the desired measurable selection $y : [0,T] \to Y$ with $y(t) \in F(t)$ and $g(t, y(t)) = 0$. □

We next present a lower energy estimate that is valid for all stable processes. Based on this general lower energy estimate, one sees that for the following result, the compatibility condition (C1) could be replaced by the weaker one-sided condition $\partial_t\mathcal{E}(t,q) \le \limsup \partial_t\mathcal{E}(t, q_k)$; see the text after (2.1.47) and compare to the weaker versions in (2.4.11i) or (RC1). The following lower energy estimate, which is based on Lemma 2.1.21, can be seen as a degenerate version of a chain-rule inequality in the sense of [413, Prop. 2.4].

The fact that stability implies such a lower energy estimate was first observed in [425]. Here we use a stronger version that replaces the work of the external forces on the right-hand side by the integral of the reduced power $\mathcal{P}_{\mathrm{red}}(\cdot, z(\cdot))$. Since the left-hand side in (2.1.50) does not depend on the y-component of the stable process, it is clear that the lower bound on the right-hand side should also be expressible in terms of z alone. It is this seemingly simple observation that allowed us to simplify the assumptions of the main existence result.

Proposition 2.1.23. *Assume that* (D1), (D2), (E1), (E2), (C1), (C2), *and* (2.1.18) *hold. Let* $q = (y, z) : [0, T] \to \mathcal{Q}$ *be measurable and satisfy* $\sup_{t\in[0,T]} \mathcal{E}(t, q(t)) < \infty$, $\mathrm{Diss}_{\mathcal{D}}(z; [0, T]) < \infty$, *and* $q(t) \in \mathcal{S}(t)$ *for all* $t \in [0, T]$. *Then for all* $0 \le r < s \le T$, *we have the lower energy inequality*

$$\mathcal{E}(s, q(s)) + \mathrm{Diss}_{\mathcal{D}}(z; [r, s]) - \mathcal{E}(r, q(r)) \ge \int_r^s \mathcal{P}_{\mathrm{red}}(t, z(t))\,\mathrm{d}t \ge \int_r^s \partial_t \mathcal{E}(t, q(t))\,\mathrm{d}t. \tag{2.1.50}$$

Proof. First of all, we use that the left-hand side is independent of the y-component, that is, (2.1.15), by the stability of $q(r)$ and $q(s)$, can be written as $\mathcal{I}(s, z(s)) + \mathrm{Diss}_{\mathcal{D}}(z; [r, s]) - \mathcal{I}(r, z(r))$. Thus, it suffices to prove the first inequality in (2.1.50), since the second one follows directly from the definition (2.1.16) of the reduced power, namely $\partial_t\mathcal{E}(t, q(t)) \le \mathcal{P}_{\mathrm{red}}(t, z(t))$ for all $t \in [0, T]$. By Lemma 2.1.22, we may choose y such that we have equality.

Hence we assume now that q satisfies this equality, i.e., $\partial_t\mathcal{E}(t, q(t)) = \mathcal{P}_{\mathrm{red}}(t, z(t))$. Take any partition $\Pi \in \mathrm{Part}([r, s])$. For each $t_{j-1} \in \Pi$, we use $q(t_{j-1}) \in \mathcal{S}(t_{j-1})$ to obtain $\mathcal{E}(t_{j-1}, q(t_{j-1})) \le \mathcal{E}(t_{j-1}, q(t_j)) + \mathcal{D}(q(t_{j-1}), q(t_j))$, which is the same as

$$\mathcal{E}(t_j, q(t_j)) + \mathcal{D}(z(t_{j-1}), z(t_j)) - \mathcal{E}(t_{j-1}, q(t_{j-1})) \ge \mathcal{E}(t_j, q(t_j)) - \mathcal{E}(t_{j-1}, q(t_j)).$$

Summing over $j \in \{1, \dots, N\}$, $N = N_\Pi$, we obtain

$$\begin{aligned}
\mathcal{E}(s, q(s)) &+ \mathrm{Diss}_{\mathcal{D}}(z; [r, s]) - \mathcal{E}(r, q(r)) \\
&\ge \mathcal{E}(s, q(s)) + \sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j)) - \mathcal{E}(r, q(r)) \\
&\ge \sum_{j=1}^N \Big(\mathcal{E}(t_j, q(t_j)) - \mathcal{E}(t_{j-1}, q(t_j))\Big) \\
&= \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \partial_t\mathcal{E}(t, q(t_j))\,\mathrm{d}t = \int_r^s \partial_t \mathcal{E}(t, \bar{q}^\Pi(t))\,\mathrm{d}t.
\end{aligned} \tag{2.1.51}$$

Since the partition Π was arbitrary, Lemma 2.1.21 yields the desired result. □

The existence theory developed below will build on the (IMP$^\Pi$) and the standard a priori estimates. The general strategy for constructing solutions to (S)&(E) is to choose a sequence of partitions Π^m with fineness $\varnothing(\Pi^m)$ tending to 0, extract a convergent subsequence of $(z^l)_{l\in\mathbb{N}}$ of $(z^{\Pi^m})_{m\in\mathbb{N}}$, and then show that the limit $z : [0, T] \to \mathcal{Z}$ solves (S)&(E). The existence of a convergent subsequence is guaranteed by the following version of Helly's selection principle, which is proved in Appendix B.5 in a more general variant; cf. Theorem B.5.13.

Theorem 2.1.24 (Generalized version of Helly's selection principle). *Let* $\mathscr{D}:\mathcal{Z}\times\mathcal{Z}\to[0,\infty]$ *satisfy* (D1) *and* (D2). *Moreover, let* K *be a compact subset of* $\mathcal{Z}$. *Then for every sequence* $(z^l)_{l\in\mathbb{N}}$ *with* $z^l:[0,T]\to K$ *and bounded dissipation, i.e.,*

$$\sup_{l\in\mathbb{N}}\mathrm{Diss}_{\mathscr{D}}\big(z^l;[0,T]\big)\le C<\infty,$$

there exist a subsequence $(z^{l_n})_{n\in\mathbb{N}}$, *a function* $z_\infty:[0,T]\to K$, *and a function* $\delta_\infty:[0,T]\to[0,C]$ *such that the following hold:*

(i) $\delta_{l_n}(t):=\mathrm{Diss}_{\mathscr{D}}(z^{l_n};[0,t])\to\delta_\infty(t)$ *for all* $t\in[0,T]$,
(ii) $z_{l_n}(t)\overset{\mathcal{Z}}{\to}z_\infty(t)$ *for all* $t\in[0,T]$,
(iii) $\mathrm{Diss}_{\mathscr{D}}(z_\infty;[t_0,t_1])\le\delta_\infty(t_1)-\delta_\infty(t_0)$ *for all* $0\le t_0<t_1\le T$.

Proof of Theorem 2.1.6. For the reader's convenience, we will divide the proof into six steps, see steps no.1–6 in Table 2.1, which will be used consistently throughout this and the following chapters.

Table 2.1 General scheme of proving existence of energetic solutions. In the rate-independent cases in Chapters 2–4, Step 0 is usually simple and separated from the main proof, while Step 7 is irrelevant.

Step no.	Action
(0)	Construction of approximate solutions
1	A priori estimates
2	Selection of convergent subsequences
3	Stability of the limit
4	Upper energy estimate
5	Lower energy estimate
6	Improved convergence
(7)	Limit passage in possible other (rate-dependent) parts (Chap. 5)

Step 1: A priori estimates. We choose an arbitrary sequence of partitions Π^m whose fineness $\varnothing(\Pi^m)$ tends to 0. The time-incremental minimization problems (IMP$^\Pi$) are solvable, and the piecewise constant interpolants $\underline{q}^m=(\underline{y}^m,\underline{z}^m):[0,T]\to\mathcal{Q}$ defined in (2.1.12) satisfy the a priori estimates

$$\mathrm{Diss}_{\mathscr{D}}(\underline{z}^m;[0,T])\le C_{\mathscr{D}}\quad\text{and}\quad\forall\,t\in[0,T]:\ \mathscr{E}(t,\underline{q}^m(t))\le C_{\mathscr{E}},$$

where $C_{\mathscr{D}}$ and $C_{\mathscr{E}}$ are given explicitly in Theorem 2.1.5.

Step 2: Selection of subsequences. Our version of Helly's selection principle in Theorem 2.1.24 allows us to select a subsequence of $(\underline{z}^m)_{m\in\mathbb{N}}$ that converges pointwise and that makes the dissipation converge as well. Moreover, the functions $P^m:t\mapsto\partial_t\mathscr{E}(t,\underline{q}^m(t))$ form an equibounded sequence in $\mathrm{L}^1(0,T)$. Thus, by choosing a further subsequence $(\underline{q}^{m_k})_{k\in\mathbb{N}}$, we may assume the following convergence properties for $k\to\infty$, where we write q_k as shorthand for $\underline{q}^{m_k}$ and p_k for P^{m_k}:

$$\forall\, t\in[0,T]:\ \delta_k(t) := \mathrm{Diss}_{\mathscr{D}}(z_k;[0,t]) \to \delta(t) \quad \text{and} \quad z_k(t) \overset{\mathcal{Z}}{\to} z(t);$$
$$p_k \rightharpoonup p_{\mathrm{weak}} \quad \text{in } \mathrm{L}^1(0,T);$$

i.e., (2.1.20a) holds. Since the limit $z:[0,T]\to\mathcal{Z}$ satisfies $\mathrm{Diss}_{\mathscr{D}}(z;[0,T]) \le \delta(T) \le C_{\mathscr{D}} < \infty$, we know that z is measurable and satisfies the energetic bound $\mathscr{I}(t,z(t)) \le C_{\mathscr{E}}$. Thus, Lemma 2.1.22 provides a measurable function $y:[0,T]\to\mathcal{Y}$ with

$$\begin{aligned} &y(t) \in \operatorname{Arg\,min} \mathscr{E}(t,\cdot,z(t)) \quad \text{for } t\in(0,T] \text{ and}\\ &\partial_t\mathscr{E}(y(t),z(t)) = \mathcal{P}_{\mathrm{red}}(t,z(t)) \quad \text{for } t\in(0,T]\setminus N_{\mathscr{E}}, \end{aligned} \tag{2.1.52}$$

and $y(0)=y_0$, where $q_0=(y_0,z_0)$ is the given initial value with $q_0\in\mathcal{S}(0)$. Note that by construction, $z(0)=z^m(0)=z_0$ such that $y(0)=y_0$ is an admissible choice satisfying the first relation in (2.1.52) but not necessarily the second.

The aim of the next steps is to show that $q=(y,z)$ is an energetic solution of $(\mathcal{Q},\mathscr{E},\mathscr{D},q_0)$.

Step 3: Stability of the limit function. We will use compatibility condition (C2). For fixed $t\in[0,T]$ we define τ_k to be the largest value in $\Pi^{m_k}\cap[0,t]$ giving $q_k(t)=q_k(\tau_k)$. Then

$$q_k(t)\in\mathcal{S}(\tau_k), \quad \tau_k\le t, \text{ and } \tau_k\to t. \tag{2.1.53}$$

By choosing a further subsequence if necessary, we obtain $q_{k_l}(t) \overset{\mathcal{Q}}{\to} \tilde{q} = (\tilde{y},z(t))$ for a suitable $\tilde{y}$. In particular, $(\tau_{k_l},q_{k_l}(t))_{l\in\mathbb{N}}$ forms a convergent stable sequence. Now (C2) yields $\tilde{q}\in\mathcal{S}(t)$. However, this also implies $q(t)=(y(t),z(t))\in\mathcal{S}(t)$, since for all $\hat{q}=(\hat{y},\hat{z})\in\mathcal{Q}$, we have

$$\mathscr{E}(t,q(t)) = \mathscr{I}(t,z(t)) = \mathscr{E}(t,\tilde{q}) \le \mathscr{E}(t,\hat{q}) + \mathscr{D}(z(t),\hat{z}).$$

Step 4: Upper energy estimate. We define the functions

$$\begin{aligned} &e_k(t) := \mathscr{E}(t,q_k(t)), \quad \delta_k(t) := \mathrm{Diss}_{\mathscr{D}}(z_k;[0,t]), \quad e_\infty(t) := \liminf_{k\to\infty} e_k(t),\\ &E(t) := \mathscr{E}(t,q(t)), \quad \Delta(t) := \mathrm{Diss}_{\mathscr{D}}(z;[0,t]), \quad \delta_\infty(t) := \lim_{k\to\infty} \delta_k(t),\\ &w_k(t) := \int_0^t \partial_s\mathscr{E}(s,q_k(s))\,\mathrm{d}s = \int_0^t p_k(s)\,\mathrm{d}s,\\ &W(t) := \int_0^t \partial_s\mathscr{E}(s,q(s))\,\mathrm{d}s = \int_0^t \mathcal{P}_{\mathrm{red}}(s,z(s))\,\mathrm{d}s, \end{aligned}$$

where by construction, $e_k(0) = E(0) = e_\infty(0)$. Employing Theorem 2.1.5(ii) and the boundedness of $\partial_t\mathcal{E}$ by a $C_1\lambda_{\mathcal{E}}(\cdot)$ (use (E2) and Step 1) gives

$$e_k(t) + \delta_k(t) \leq E(0) + w_k(t) + C_1\omega_\Lambda(\varnothing(\Pi^{m_k})), \tag{2.1.54}$$

where ω_Λ is a modulus of continuity of Λ. Since $\mathcal{E}$ and $\mathrm{Diss}_{\mathcal{D}}$ are lower semicontinuous (see Theorem 2.1.24(iii) for the latter) and since by weak convergence, we have $w_\infty(t) = \lim_{k\to\infty} w_k(t) = \int_0^t p_{\mathrm{weak}}(s)\,\mathrm{d}s$, the limit $k\to\infty$ leads to

$$E(t)+\Delta(t) \leq e_\infty(t)+\delta_\infty(t) \leq E(0)+w_\infty(t) = E(0)+\int_0^t p_{\mathrm{weak}}(s)\,\mathrm{d}s. \tag{2.1.55}$$

The next step is now to relate p_{weak} and $\mathcal{P}_{\mathrm{red}}(\cdot, z(\cdot))$ using the compatibility condition for the power (C1). As in Step 3, we may choose a subsequence of $(q_k(t))_k$ such that

$$\mathcal{S}(\tau_{k_l}) \ni q_{k_l}(t) \xrightarrow{\mathcal{Q}} \tilde{q} \quad \text{and} \quad p_{k_l}(t) \to p_{\sup}(t) = \limsup_{k\to\infty} p_k(t).$$

Thus, (C1) is applicable, and we obtain

$$p_{k_l}(t) = \partial_t\mathcal{E}(t, q_{k_l}(t)) \to \mathcal{E}(t, \tilde{q}) = p_{\sup}(t) \leq \mathcal{P}_{\mathrm{red}}(t, z(t)),$$

where the latter estimate follows from $\tilde{q} = (\tilde{y}, z(t)) \in \mathcal{S}(t)$ and the definition of $\mathcal{P}_{\mathrm{red}}$. Since by Fatou's lemma, we know that $w_\infty(t) \leq \int_0^t p_{\sup}(s)\,\mathrm{d}s$, we conclude that $w_\infty(t) \leq W(t)$. Thus, we have derived the upper energy estimate

$$E(t) + \Delta(t) \leq e_\infty(t) + \delta_\infty(t) \leq E(0) + w_\infty(t) \leq E(0) + W(t).$$

Step 5: Lower energy estimate. Because of our construction of the function $q : [0,T] \to \mathcal{Q}$, we are able to apply Proposition 2.1.23 and obtain the lower energy estimate

$$\begin{aligned} E(t) + \Delta(t) &= \mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{D}}(z; [0,t]) \\ &\geq \mathcal{E}(0, z(0)) + \int_0^t \partial_s\mathcal{E}(s, q(s))\,\mathrm{d}s = E(0) + W(t). \end{aligned}$$

Thus, we have shown that the limit function $q : [0,T] \to \mathcal{Q}$ satisfies stability and energy balance for all times, whence it is an energetic solution.

Step 6: Improved convergence. Finally, we show that the remaining convergences (2.1.20b-e) hold. The lower and upper energy estimates imply

$$\begin{aligned} E(0)+W(t) &\leq E(t) + \Delta(t) \leq e_\infty(t)+\delta_\infty(t) \\ &\leq E(0)+\int_0^t p_{\mathrm{weak}}\,\mathrm{d}s \leq E(0)+\int_0^t p_{\sup}\,\mathrm{d}s \leq E(0)+W(t). \end{aligned}$$

Hence, all inequalities are in fact equalities. Using the inequalities $E(t) \le e_\infty(t)$, $\varDelta(t) \le \delta_\infty(t)$, and $p_{\mathrm{weak}} \le p_{\mathrm{sup}} \le \mathcal{P}_{\mathrm{red}}$, we conclude that $\varDelta(t) = \delta_\infty(t)$ and $E(t) = e_\infty(t)$, which proves the convergence statements (2.1.20b) and (2.1.20c). Moreover, we also find that $p_{\mathrm{weak}}(t) = p_{\mathrm{sup}}(t) = \mathcal{P}_{\mathrm{red}}(t, z(t))$ a.e. in $[0, T]$. This shows that the weak limit and the pointwise lim sup of the sequence p_k coincide, which implies the desired pointwise convergence (2.1.20d); cf. [195, Prop. A2]. Also, (2.1.20d), and the existence even of an L^∞-majorant due to (2.1.54) and the energy control imply, by Lebesgue's theorem B.3.2, even $\partial_t\mathcal{E}(\cdot, q_k(\cdot)) \to \partial_t\mathcal{E}(\cdot, \tilde{q}(\cdot))$ in $\mathrm{L}^1(0, T)$.

Finally, if $\mathcal{E}$ is qualified as in Theorem 2.1.6(iii), then $(y_k(t))_{k\in\mathbb{N}}$ has just one cluster point, i.e., it converges, and thus $y_k(t) \to y(t)$ in $\mathcal{Y}$, so that also (2.1.20e) holds.

Thus, Theorem 2.1.6 is proved. □

2.2 Generalizations

Here we collect some variants of the above method of construction for energetic solutions that might be advantageous in certain applications.

2.2.1 Direct treatment of the reduced RIS

As indicated in Section 2.1.6, it may be of interest to address the reduced ERIS $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ directly. Of course, we may assume the same assumption as in Section 2.1 by simply taking $\mathcal{Y} = \{0\}$. However, the task of this section is to find conditions on $\mathcal{I}$ that are more general. In particular, these conditions should be such that they can be applied also in such cases where $\mathcal{I}$ is obtained via a reduction procedure as given in (2.1.14). The problem is that after minimizing with respect to $y \in \mathcal{Y}$ in $\mathcal{E}(t, y, z)$, we lose in general the differentiability of $\mathcal{I}$ with respect to t. In Proposition 2.2.4, we will explain more precisely how the subsequent assumptions can be derived from the assumption on $\mathcal{E}$ in Section 2.1.

We consider a Hausdorff topological space $\mathcal{Z}$ and denote by $\stackrel{\mathcal{Z}}{\to}$ the convergence, where all topological notions are understood in the sequential sense. The dissipation distance $\mathcal{D} : \mathcal{Z}\times\mathcal{Z} \to [0, \infty]$ satisfies the two conditions (D1)–(D2) from Section 2.1.1. For the *reduced stored-energy functional* $\mathcal{I} : [0, T]\times\mathcal{Z} \to \mathbb{R}_\infty$, we make the following assumptions:

Compactness of sublevels:

$$\forall\, t \in [0, T] : \qquad \mathcal{I}(t, \cdot) : \mathcal{Z} \to \mathbb{R}_\infty \text{ has compact sublevels;} \tag{I1}$$

energetic control of power:

$$
\begin{aligned}
&\mathrm{Dom}\,\mathcal{I} = [0,T]\times\mathrm{Dom}\,\mathcal{I}(0,\cdot)\\
&\exists\, c_{\mathcal{I}}\in\mathbb{R},\ \lambda_{\mathcal{I}}\in\mathrm{L}^1(0,T),\ N_{\mathcal{I}}\subset[0,T],\ \mathcal{L}^1(N_{\mathcal{I}})=0\ \ \forall z\in\mathrm{Dom}\,\mathcal{I}(0,\cdot):\\
&\mathcal{I}(\cdot,z)\in\mathrm{W}^{1,1}(0,T),\qquad \partial_t^-\mathcal{I}(t,z)\ \text{ exists for } t\in[0,T]\backslash N_{\mathcal{I}}\\
&\text{and satisfies } |\partial_t^-\mathcal{I}(t,z)|\le\lambda_{\mathcal{I}}(t)\big(\mathcal{I}(t,z)+c_{\mathcal{I}}\big)\text{ for all } t\in[0,T]\backslash N_{\mathcal{I}};
\end{aligned}
\tag{I2}
$$

uniform "semidifferentiability" from the left:

$$
\begin{aligned}
\text{for } N_{\mathcal{I}} \text{ from (I2)}:\quad &\forall\, z\in\mathcal{Z}\ \forall\, E^*>0\ \forall\,\varepsilon>0\ \exists\,\delta>0:\\
&\mathcal{I}(t,z)\le E^* \text{ and } \max\{0,t-\delta\}\le s\le t\notin N_{\mathcal{I}}\\
&\Longrightarrow\ \partial_t^-\mathcal{I}(t,z)\le\frac{\mathcal{I}(t,z)-\mathcal{I}(s,z)}{t-s}+\varepsilon.
\end{aligned}
\tag{I3}
$$

Here $\partial_t^-\mathcal{I}(t,z)$ is the *left time derivative* $\partial_t^-\mathcal{I}(t,z) = \lim_{h\to 0^+}\frac{1}{h}(\mathcal{I}(t,z)-\mathcal{I}(t-h,z))$ for $t>0$ and $\partial_t^-\mathcal{I}(0,z):=0$. The one-sided limit $h\to 0^+$ is defined in (2.1.34).

Again (I1) implies that $\mathcal{I}(t,\cdot):\mathcal{Z}\to\mathbb{R}_\infty$ is lower semicontinuous. Condition (I2) says that the absolutely continuous function $\mathcal{I}(\cdot,z_*)$ is differentiable almost everywhere and the left derivative coincides with the classical derivative wherever the latter exists. Again, the set $[0,T]\backslash N_{\mathcal{I}}$ where the derivative exists must be independent of z. In particular, the fundamental theorem of calculus holds, namely $\mathcal{I}(t,z_*)-\mathcal{I}(s,z_*)=\int_s^t\partial_\tau^-\mathcal{I}(\tau,z_*)\,\mathrm{d}\tau$, and the Gronwall estimates (2.1.3) hold similarly.

Remark 2.2.1. Condition (I3) is quite strong in the sense that the choice of δ must also be uniform in $t\in[0,T]\backslash N_{\mathcal{I}}$, while in the analogous two-sided estimate (2.1.27) for $\mathcal{E}$, the choice of δ may depend on t. It is exactly this gap between the stronger condition (I3) and the weaker condition (C1) that forces us to prove the existence directly for $(\mathcal{Q},\mathcal{E},\mathcal{D})$ rather than for the conceptually simpler reduced system $(\mathcal{Z},\mathcal{I},\mathcal{D})$.

Solutions for the reduced ERIS $(\mathcal{Z},\mathcal{I},\mathcal{D})$ are defined via stability $(\mathrm{S})_{\mathrm{red}}$ and energy balance $(\mathrm{E})_{\mathrm{red}}$, where now $\partial_t^-\mathcal{I}$ plays the role of the power of the external forces:

$$
\begin{aligned}
&(\mathrm{S})_{\mathrm{red}}\quad z(t)\in\widehat{\mathcal{S}}(t),\\
&(\mathrm{E})_{\mathrm{red}}\quad \mathcal{I}(t,z(t))+\mathrm{Diss}_{\mathcal{D}}(z;[0,t])=\mathcal{I}(0,z(0))+\int_0^t\partial_s^-\mathcal{I}(s,z(s))\,\mathrm{d}s,
\end{aligned}
\tag{2.2.1}
$$

where $\hat{\mathbb{S}}(t)$ is defined in (2.1.48). As in the case of the full problem, we define *stable sequences* $(t_k, z_k)_{k\in\mathbb{N}}$ via

$$\sup_{k\in\mathbb{N}} \mathscr{I}(t_k, z_k) < \infty \quad \text{and} \quad \forall\, k \in \mathbb{N} : \; z_k \in \hat{\mathbb{S}}(t_k).$$

For the existence theory of solutions we need two compatibility conditions between the functionals $\mathscr{I}$ and $\mathscr{D}$. The *reduced compatibility conditions* for $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$ are direct modifications of (C1) and (C2):

reduced compatibility condition:

$\forall$ stable sequence $(t_k, z_k)_{k\in\mathbb{N}}$ with $(t_k, z_k) \overset{[0,T]\times\mathcal{Z}}{\longrightarrow} (t, z)$ we have:

$$t \in [0,T]\backslash N_{\mathscr{I}} \text{ with } N_{\mathscr{I}} \text{ from (I2)} \implies \partial_t^- \mathscr{I}(t,z) \geq \limsup_{k\to\infty} \partial_t^- \mathscr{I}(t, z_k), \tag{RC1}$$

$$z \in \hat{\mathbb{S}}(t). \tag{RC2}$$

Theorem 2.2.2. *Assume that $\mathscr{I}$ and $\mathscr{D}$ satisfy the assumptions* (D1)*,* (D2)*,* (I1)*,* (I2)*,* (I3)*, and the reduced compatibility conditions* (RC1) *and* (RC2)*. Then:*

(i) For each $z_0 \in \hat{\mathbb{S}}(0)$, there exists an energetic solution $z : [0,T] \to \mathcal{Z}$ for $(\mathcal{Z}, \mathscr{I}, \mathscr{D}, z_0)$.
(ii) If $\Pi^l \in \mathrm{Part}([0,T])$ is a sequence of partitions whose fineness $\varnothing(\Pi^l)$ tends to 0 and z^{Π_l} is the interpolant of any solution of the associated (IMP$^{\Pi^l}$), then there exist a subsequence $z_k = z^{\Pi_{l_k}}$ and a solution $\tilde{z} : [0,T] \to \mathcal{Z}$ for $(\mathcal{Z}, \mathscr{I}, \mathscr{D}, z_0)$ such that the following holds

$$\forall\, t \in [0,T] : \quad z_k(t) \overset{\mathcal{Z}}{\to} \tilde{z}(t); \tag{2.2.2a}$$

$$\forall\, t \in [0,T] : \quad \mathrm{Diss}_{\mathscr{D}}(z_k; [0,t]) \to \mathrm{Diss}_{\mathscr{D}}(\tilde{z}; [0,t]); \tag{2.2.2b}$$

$$\forall\, t \in [0,T] : \quad \mathscr{I}(t, z_k(t)) \to \mathscr{I}(t, \tilde{z}(t)); \tag{2.2.2c}$$

$$\partial_t^- \mathscr{I}(\cdot, z_k(\cdot)) \to \partial_t^- \mathscr{I}(\cdot, \tilde{z}(\cdot)) \;\; in \;\; \mathrm{L}^1(0,T). \tag{2.2.2d}$$

Proof. For the proof, we follow the six steps according to Table 2.1 on p. 72 and mention where differences occur.

Step 1. A priori estimates. This part can be executed exactly in the same way as before, since for the case of the left derivative $\partial_t^- \mathscr{I}$, the Gronwall estimates (2.1.3) work exactly in the same way because $\partial_t^- \mathscr{I}$ equals the two-sided partial derivative $\partial_t \mathscr{I}(t,z)$ if it exists. The point is that the latter exists almost everywhere, but the null set where it does not exist is allowed to depend on z, whereas the left derivative $\partial_t^- \mathscr{I}(t,z)$ must exist outside the null set $N_{\mathscr{I}}$. Thus, for a sequence of Π^m with $\varnothing(\Pi^m) \to 0$, we have an approximations $\underline{z}^m$ with m-independent bounds on $\mathscr{I}(t, \underline{z}^m(t))$ and $\mathrm{Diss}_{\mathscr{D}}(\underline{z}^m; [0,T])$.

Step 2: Selection of subsequences. The selection of a convergent subsequence $z_k = \underline{z}^{m_k}$ follows again by Helly's selection principle (Theorem 2.1.24), and we are done. Again we assume that $p_k = \partial_t^- \mathscr{I}(t, z_k(t))$ satisfies $p_k \rightharpoonup p_{\text{weak}}$.

Step 3: Stability of the limit. This is now a direct consequence of (RC2).

Step 4: Upper energy estimate. The upper estimate follows analogously, since (RC1) provides the necessary estimate $p_{\sup}(t) := \limsup_{k\to\infty} p_k(t) \leq p(t) := \partial_t^- \mathscr{I}(t, z(t))$. Integration and Fatou's lemma yield $\mathscr{I}(t, z(t)) + \text{Diss}_{\mathscr{D}}(z; [0, t]) \leq \mathscr{I}(0, z(0)) + \int_0^t p(t)\, \mathrm{d}t$, which is the desired upper estimate.

Step 5: Lower energy estimate. The main difference between this and the proof in Section 2.1.6 is this part. For a lower bound, the compatibility condition (RC1) cannot be used. Hence we employ the additional assumption (I3). As in (2.1.51), for each partition $\Pi = \{r = t_0 < t_1 < \cdots < t_N = s\}$ of $[r, s]$, we obtain

$$\mathscr{I}(s, z(s)) + \text{Diss}_{\mathscr{D}}(z; [r, s]) - \mathscr{I}(r, z(r)) \geq \sum_{j=1}^{N} \Big(\mathscr{I}(t_j, z(t_j)) - \mathscr{I}(t_{j-1}, z(t_j))\Big).$$

Since $t \mapsto \mathscr{I}(t, z(t))$ is bounded by E^*, (I3) implies the existence of a modulus of continuity ω with $\varepsilon \leq \omega(\delta)$ such that under the additional assumption $t_j \notin N_{\mathscr{I}}$ for $j = 1, 2, \ldots, N-1$, we have

$$\begin{aligned} &\mathscr{I}(s, z(s)) + \text{Diss}_{\mathscr{D}}(z; [r, s]) - \mathscr{I}(r, z(r)) \\ &\geq \sum_{j=1}^{N-1} \big(\partial_t^- \mathscr{I}(t_j, z(t_j)) - \varepsilon\big)(t_j - t_{j-1}) \; - (\Lambda(s) - \Lambda(t_{N-1}))\,(E^* + c_{\mathscr{I}}) \\ &= \text{Riem}(p, \Pi) - \omega(\varnothing(\Pi))(s-r) - (\Lambda(s) - \Lambda(s - \varnothing(\Pi)))\,(E^* + c_{\mathscr{I}}). \end{aligned}$$

Here Λ is defined as in (2.1.3a) with $\lambda_{\mathscr{E}}$ replaced by $\lambda_{\mathscr{I}}$, and $\text{Riem}(p, \Pi) = \sum_{j=1}^{N} p(t_j)(t_j - t_{j-1})$ is the Riemann sum for the integral $\int_r^s p(\tau)\, \mathrm{d}\tau$ with respect to the partition Π of $[s, t]$; cf. (B.5.2) on p. 604. If $s \in N_{\mathscr{I}}$ and the value $p(s) = p(t_N)$ is not defined, we set $p(s) = 0$, which explains the last term in the estimate above. By Theorem B.5.3, we can take a sequence of partitions Π^m with $\varnothing(\Pi^m) \to 0$ such that $\text{Riem}(p, \Pi^m) \to \int_r^s p(\tau)\, \mathrm{d}\tau$ for $m \to \infty$. Following the arguments in [149, 372] it is easy to show that we may choose all Π^m such that $\Pi^m \cap N_{\mathscr{I}} \subset \{r, s\}$. Thus the lower energy estimate $\mathscr{I}(t, z(t)) + \text{Diss}_{\mathscr{D}}(z; [0, t]) - \mathscr{I}(0, z(0)) \geq \int_0^t p(t)\, \mathrm{d}t$ is obtained.

Step 6: Improved convergence. This step works exactly as in Section 2.1.6. □

The following one-sided analogue to Proposition 2.1.17 is useful in establishing the compatibility condition (RC1). It is of interest that the condition (I3), which was used for the lower energy estimate in Step 4, is also sufficient here.

Proposition 2.2.3. *If $\mathscr{I}$ satisfies (I1) and (I3), then for all $t \in [0, T] \backslash N_{\mathscr{I}}$, the following implication holds:*

$$\left.\begin{array}{l} z_m \xrightarrow{\mathcal{Z}} z \text{ and} \\ \mathscr{I}(t, z_m) \to \mathscr{I}(t, z) < \infty \end{array}\right\} \quad \Longrightarrow \quad \partial_t^- \mathscr{I}(t, z) \geq \limsup_{m\to\infty} \partial_t^- \mathscr{I}(t, z_m). \tag{2.2.3}$$

Proof. We follow the proof of Proposition 2.1.17. For $t = 0$, we have $\partial_t^- \mathscr{I}(0, z) = 0$, and there is nothing to prove. Hence we assume $t > 0$ and $t \notin N_{\mathscr{I}}$ from now on. Let $E_0, h_0 > 0$ be such that $t-h_0 \in [0, T]$ and $\mathscr{I}(t, z_m), \mathscr{I}(t, z) \le E_0$ for sufficiently large m. Then for $h \in (0, h_0)$, condition (I3) implies

$$\partial_t^- \mathscr{I}(t, z_m) \le \frac{\mathscr{I}(t, z_m) - \mathscr{I}(t-h, z_m)}{h} + \omega_0(h).$$

The same estimate also holds for z. Using $h > 0$, the lower semicontinuity of $\mathscr{I}(t-h, \cdot)$ (following from (I1)), and the assumed convergence of the energy, we obtain

$$\limsup_{m\to\infty} \frac{\mathscr{I}(t, z_m) - \mathscr{I}(t-h, z_m)}{h} \le \frac{\mathscr{I}(t, z) - \mathscr{I}(t-h, z)}{h}.$$

Combining the above two estimates, we obtain, for $h \in (0, h_0)$,

$$\begin{aligned}\limsup_{m\to\infty} \partial_t^- \mathscr{I}(t, z_m) &\le \limsup_{m\to\infty} \frac{\mathscr{I}(t, z_m) - \mathscr{I}(t-h, z_m)}{h} + \omega_0(h) \\ &\le \frac{\mathscr{I}(t, z) - \mathscr{I}(t-h, z)}{h} + \omega_0(h).\end{aligned}$$

Taking the limit $h \to 0^+$, we obtain the desired result. □

Next we discuss how the properties that were assumed here for $\mathscr{I}$ can be derived from the properties imposed on $\mathscr{E}$ in Section 2.1. We refer to [302, 307] for related and more general results for this kind of reduction. In particular, under the stronger assumption that each $\mathscr{E}(\cdot, q)$ lies in $\mathrm{C}^1([0, T])$, one can prove continuity from the right $\partial_t^- \mathscr{I}(t_k, z) \to \partial_t^- \mathscr{I}(t, z)$ for $t_k \to t^+$, whereas from the left one has only lower semicontinuity $\partial_t^- \mathscr{I}(t, z) \le \liminf_{t_k\to t^-} \partial_t^- \mathscr{I}(t_k, z)$.

Proposition 2.2.4 (One-sided time derivatives [302, 307]). *Let $\mathscr{E} : [0, T]\times\mathcal{Y}\times\mathcal{Z} \to \mathbb{R}_\infty$ satisfy (E1), (E2), and (2.1.27) from Section 2.1. Then the reduced functional $\mathscr{I}$ defined in (2.1.14) satisfies (I1) and (I2). In particular, (I2) holds with $\lambda_{\mathscr{I}} = \lambda_{\mathscr{E}}$ and $c_{\mathscr{I}} = c_{\mathscr{E}}$. If $\mathscr{I}(s, z) < \infty$, then for all $t \in [0, T]\backslash N_{\mathscr{I}}$, both one-sided time derivatives $\partial_t^\pm \mathscr{I}(t, z)$ exist and satisfy*

$$\begin{aligned}\partial_t^+ \mathscr{I}(t, z) &= \min\Big\{ \partial_t \mathscr{E}(t, y, z) \;\Big|\; y \in \operatorname{Arg\,min} \mathscr{E}(t, \cdot, z) \Big\} \\ &\le \max\Big\{ \partial_t \mathscr{E}(t, y, z) \;\Big|\; y \in \operatorname{Arg\,min} \mathscr{E}(t, \cdot, z) \Big\} = \partial_t^- \mathscr{I}(t, z).\end{aligned} \tag{2.2.4}$$

Proof. Note that $\mathscr{I}(t, z) \le \alpha$ if and only if there exists y such that $\mathscr{E}(t, y, z) \le \alpha$. Thus, (sequential) compactness of the sublevels of $\mathscr{I}(t, \cdot)$ follows easily from that of $\mathscr{E}(t, \cdot, \cdot)$. This establishes (I1).

Further, we fix any z such that $\mathscr{I}(s,z) < \infty$ and set $Y(t) := \operatorname{Arg\,min} \mathscr{E}(t,\cdot,z) \subset \mathcal{Y}$. These are compact sets, and a simple argument shows that the set-valued mapping is upper semicontinuous, namely

$$t_m \to t_*,\ y_m \overset{\mathcal{Y}}{\to} y_*,\ y_m \in Y(t_m) \text{ for } m \in \mathbb{N} \quad \Longrightarrow \quad y_* \in Y(t_*). \tag{2.2.5}$$

Since condition (2.1.27) holds, Proposition 2.1.17 implies that

$$\begin{aligned} a^+(t) &:= \min \Big\{ \partial_t \mathscr{E}(t,y,z) \ \Big|\ y \in \operatorname{Arg\,min} \mathscr{E}(t,\cdot,z) \Big\} \\ &\le \max \Big\{ \partial_t \mathscr{E}(t,y,z) \ \Big|\ y \in \operatorname{Arg\,min} \mathscr{E}(t,\cdot,z) \Big\} =: a^-(t) \end{aligned}$$

are attained, i.e., there exist $y^\pm(t) \in \operatorname{Arg\,min} \mathscr{E}(t,\cdot,z)$ with $a^\pm(t) = \partial_t \mathscr{E}(t,y^\pm,z)$.

We now show that a^- is a left time derivative; the case for a^+ is analogous. On the one hand, for $t > h > 0$, take any $\hat{y} \in Y(t)$. Then $\frac{1}{h}\big(\mathscr{I}(t,z) - \mathscr{I}(t{-}h,z)\big) \ge \frac{1}{h}\big(\mathscr{E}(t,\hat{y},z) - \mathscr{E}(t{-}h,\hat{y},z)\big) = a^-(t)$. Taking the limit $h \to 0^+$ and using that $\hat{y}$ was arbitrary, we obtain

$$\liminf_{h\to 0+} \frac{\mathscr{I}(t,z) - \mathscr{I}(t{-}h,z)}{h} \ge \sup \Big\{ \partial_t \mathscr{E}(t,\hat{y},z) \ \Big|\ \hat{y} \in Y(t) \Big\} = a^-(t). \tag{2.2.6}$$

On the other hand, choosing $y_h \in Y(t{-}h)$ and using (2.1.27) yields

$$\frac{\mathscr{I}(t,z) - \mathscr{I}(t{-}h,z)}{h} \le \frac{\mathscr{E}(t,y_h,z) - \mathscr{E}(t{-}h,y_h,z)}{h} \le \partial_t \mathscr{E}(t,y_h,z) + \omega(h)$$

with $\omega(h) \to 0$ for $h \to 0^+$. Now choose a sequence $(h_m)_{m\in\mathbb{N}}$ such that $h_m \to 0^+$ and the left-hand side approaches its limit superior. By compactness, we assume that additionally, $y_{h_m} \overset{\mathcal{Y}}{\to} y_* \in Y(t)$ holds. Employing Proposition 2.1.17 once again, we obtain

$$\begin{aligned} \limsup_{h\to 0^+} \frac{\mathscr{I}(t,z) - \mathscr{I}(t{-}h,z)}{h} &= \lim_{m\to\infty} \frac{\mathscr{I}(t,z) - \mathscr{I}(t{-}h_m,z)}{h_m} \\ &\le \lim_{m\to\infty} \partial_t \mathscr{E}(t,y_{h_m},z) + \omega(h_m) = \partial_t \mathscr{E}(t,y_*,z) \le a^-(t). \end{aligned}$$

Together with (2.2.6), this shows that $\partial_t^- \mathscr{I}(t,z) = a^-(t)$, and (I2) is established. □

Example 2.2.5 (A nontrivial reduced ERIS). The following ERIS has a solution z for which $\partial_t^- \mathscr{I}(t,z(t))$ is different from $\partial_t^+ \mathscr{I}(t,z(t))$ for all times. We let $\mathcal{Y} = \{0,1\}$, $\mathcal{Z} = \mathbb{R}$,

$$\mathscr{E}(t,j,z) = \mathscr{I}_j(t,z) = \begin{cases} \frac{1}{2}z^2 - (1{+}t)z & \text{for } j = 0, \\ \frac{1}{2}(z{-}t)^2 - \frac{1}{2}(2{+}t)t & \text{for } j = 1, \end{cases}$$

and $\mathscr{D}(z_0,z_1) = |z_1-z_0|$. We have $\mathscr{I}(t,z) = \mathscr{I}_0(t,z)$ for $z \geq t$, and $\mathscr{I}(t,z) = \mathscr{I}_1(t,z)$ for $z \leq t$. The function z with $z(t) = t$ is an energetic solution, since an easy calculation shows that $\hat{\mathbb{S}}(t) = [t-1, t+2]$ and since the energy balance holds because of

$$\mathscr{I}(t,z(t)) = -(2+t)t/2, \quad \mathrm{Diss}_{\mathscr{D}}(z;[0,t]) = t, \quad \partial_t^- \mathscr{I}(t,t) = \partial_t^- \mathscr{I}_0(t,t) = -t.$$

Note that $\partial_t^+ \mathscr{I}(t,t) = \partial_t^- \mathscr{I}_1(t,t) = -t-1 < \partial_t^- \mathscr{I}(t,t)$. The point is that the solution is sliding along an edge of the potential all the time. (There are other solutions as well, namely $z(t) = t$ for $t \in [0,t_*]$, $z(t) = \max\{t_*, t-1\}$ for $t \geq t_*$ for any $t_* \geq 0$.)

2.2.2 Other nonsmooth behavior of $\mathscr{I}$

In the previous section, we discussed the reduced functional $\mathscr{I}$ under assumptions that are compatible with the reduction process $\mathscr{I}(t,z) = \min \mathscr{E}(t,\cdot,z)$. This implies in particular the relation $\partial_t^+ \mathscr{I} \leq \mathscr{I}_t^- \mathscr{I}$. There may be other cases in which exactly the opposite inequality holds. In [397], the case is considered that $\mathscr{I}$ is defined via a functional $\mathfrak{J} : \boldsymbol{X} \times \mathcal{Z} \to \mathbb{R}$ in

$$\mathscr{I}(t,z) = \mathfrak{J}(x(t),z), \quad \text{where } \mathfrak{J}(\cdot,z) : \boldsymbol{X} \to \mathbb{R} \text{ is convex,}$$

$\boldsymbol{X}$ is a Banach space, and $x \in \mathrm{W}^{1,1}(0,T;\boldsymbol{X})$. Since for $\mathfrak{J}(\cdot,z)$, the *directional derivatives*

$$\mathrm{D}_x\mathfrak{J}(x,z;v) := \lim_{h\to 0^+} \frac{\mathfrak{J}(x+hv,z) - \mathfrak{J}(x,z)}{h}$$

exist because of convexity and satisfy $\mathrm{D}_x\mathfrak{J}(x,z;v) \geq -\mathrm{D}_x\mathfrak{J}(x,z;-v)$, we have

$$\partial_t^- \mathscr{I}(t,z) = -\mathrm{D}_x\mathfrak{J}(x(t),z;-\dot{x}(t)) \leq \mathrm{D}_x\mathfrak{J}(x(t),z;\dot{x}(t)) = \partial_t^+ \mathscr{I}(t,z)$$

for all t where $x(\cdot)$ is differentiable.

The following example gives a very simple model having exactly this structure.

Example 2.2.6 (A nontrivial generalized energetic solution). We consider the case $\mathcal{Z} = \mathbb{R}$ with $\mathscr{D}(z_1,z_2) = |z_2-z_1|$ and $\mathscr{I}(t,z) = 2|z-\ell(t)| + \rho(t)z$ with $\rho, \ell \in \mathrm{C}^1([0,T])$ and $|\rho(t)| < 1$ for all t. It is easily seen that for all $t \in \mathbb{R}$, the stability set is a singleton, namely $\hat{\mathbb{S}}(t) = \{\ell(t)\}$. In particular, the incremental minimization problem $z_k \in \operatorname{Arg\,min}\big(\mathscr{I}(t_k,\cdot)+\mathscr{D}(z_{k-1},\cdot)\big)$ always leads to the solution $z_k = \ell(t_k)$. Thus, we would like to consider the unique limit $z : t \mapsto \ell(t)$ as an energetic solution for $(\mathcal{Z}, \mathscr{I}, \mathscr{D})$. However, along this solution z, we have

$$\partial_t^- \mathscr{I}(t,z(t)) = \dot{\rho}(t)\ell(t) - 2|\dot{\ell}(t)| < \dot{\rho}(t)\ell(t) + 2|\dot{\ell}(t)| = \partial_t^+ \mathscr{I}(t,z(t)).$$

We calculate the necessary power p of the external forces from the energy balance

$$\mathscr{I}(t,z(t)) + \mathrm{Diss}_{\mathscr{D}}(z;[r,t]) = \rho(t)\ell(t) + \int_r^t |\dot{\ell}(s)|\,\mathrm{d}s = \rho(r)\ell(r) + \int_r^t p(s)\,\mathrm{d}s,$$

where the last equality provides the desired energy balance. Thus, this balance holds only if $p(t) = \dot{\rho}(t)\ell(t)+\rho(t)\dot{\ell}(t)+|\dot{\ell}(t)|$ a.e. on $\mathbb{R}$. In particular, recalling $|\rho(t)| < 1$, this p lies strictly between $\partial_t^- \mathscr{I}(t,z(t))$ and $\partial_t^+ \mathscr{I}(t,z(t))$.

We now want to generalize the condition (I2) to handle such a situation. We begin again with the reduced energetic system $(\mathcal{Z},\mathscr{I},\mathscr{D})$ satisfying the assumption (D1), (D2), and (I1). Condition (I2) is strengthened by assuming that both one-sided derivatives exist:

$$\begin{aligned}
&\mathrm{Dom}\,\mathscr{I} = [0,T]\times \mathrm{Dom}\,\mathscr{I}(0,\cdot),\\
&\qquad\qquad \exists\, c_{\mathscr{I}} \in \mathbb{R},\ \lambda_{\mathscr{I}} \in \mathrm{L}^1(0,T),\ N_{\mathscr{I}} \subset [0,T],\ \mathscr{L}^1(N_{\mathscr{I}})=0\\
&\forall z \in \mathrm{Dom}\,\mathscr{I}(0,\cdot):\ \ \mathscr{I}(\cdot,z)\in \mathrm{W}^{1,1}(0,T),\quad \forall\, t\in[0,T]\backslash N_{\mathscr{I}}\ \exists\, \partial_t^{\pm}\mathscr{I}(t,z)\ \text{and satisfy}\\
&\qquad\qquad\qquad\qquad \big|\partial_t^{\pm}\mathscr{I}(t,z)\big| \le \lambda_{\mathscr{I}}(t)\big(\mathscr{I}(t,z)+c_{\mathscr{I}}\big)\ \text{for all } t\in[0,T]\backslash N_{\mathscr{I}}.
\end{aligned} \tag{2.2.7}$$

Example 2.2.6 motivates the following generalization of energetic solutions, where we use the abbreviation

$$\mathcal{P}_{\max}(t,z) := \max\big\{\partial_t^- \mathscr{I}(t,z),\, \partial_t^+ \mathscr{I}(t,z)\big\}.$$

Definition 2.2.7 (Generalized energetic solutions). Let the ERIS $(\mathcal{Z},\mathscr{I},\mathscr{D})$ satisfy the assumptions (D1), (D2), (I1), and (2.2.7). Then $z:[0,T]\to\mathcal{Z}$ is called a *generalized energetic solution* if there exists $p\in \mathrm{L}^1(0,T)$ such that for all $t\in[0,T]$, we have stability (S) and *weakened energy balance* (WE)

$$\text{(S)}\qquad \mathscr{I}(t,z(t)) \le \mathscr{I}(t,\tilde{z}) + \mathscr{D}(z(t),\tilde{z})\ \text{for all } \tilde{z}\in\mathcal{Z},$$

$$\text{(WE)}\qquad \mathscr{I}(t,z(t)) + \mathrm{Diss}_{\mathscr{D}}(z;[0,t]) = \mathscr{I}(0,z(0)) + \int_0^t p(s)\,\mathrm{d}s \ \text{with}$$

$$\partial_t^- \mathscr{I}(s,z(s)) \le p(s) \le \mathcal{P}_{\max}(s,z(s))\quad \text{for a.a. } s\in[0,T]\backslash N_{\mathscr{I}}.$$

Generalized energetic solutions are usual energetic solutions as soon as we know $\partial^+\mathscr{I} \le \partial_t^- \mathscr{I}$ as in the case of reduced energy functionals.

To find a suitable existence theory for generalized energetic solutions, one just has to check how upper and lower energy estimates are obtained. The latter was established above for all $[r,s]\subset[0,T]$; however, we now also need to derive the upper energy estimate on *all* intervals $[r,s]$ with $\mathcal{P}_{\max}$ on the right-hand side. Then

(WE) will follow. To obtain these upper energy estimates, we will assume that the energy is continuous on the stability sets $\hat{\mathcal{S}}(t)$.

The new conditions in the following existence result are exactly tailored to provide these conditions. A suitable variant could also be derived for the full ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$.

Theorem 2.2.8 (Existence of generalized energetic solutions). *Assume that the energetic system* $(\mathcal{Z}, \mathcal{I}, \mathcal{D})$ *satisfies* (D1), (D2), (I1), (I3), *and* (2.2.7). *Moreover, assume*

the generalized compatibility conditions:

$\forall$ *stable sequences* $(t_l, z_l)_{l\in\mathbb{N}}$ *with* $(t_l, z_l) \overset{[0T]\times\mathcal{Z}}{\longrightarrow} (t, z)$ *we have:*

$$\forall\, \tilde{z} \in \mathcal{Z}\ \exists\, \tilde{z}_l \overset{\mathcal{Z}}{\to} \tilde{z} :\ \limsup_{l\to\infty} \big(\mathcal{I}(t_l, \tilde{z}_l)+\mathcal{D}(z_l, \tilde{z}_l)\big) \le \mathcal{I}(t, \tilde{z})+\mathcal{D}(z, \tilde{z}), \tag{GC1}$$

$$t\in[0,T]\backslash N_{\mathcal{I}} \text{ with } N_{\mathcal{I}} \text{ from (I2)} \implies \mathcal{P}_{\max}(t, z) \ge \limsup_{l\to\infty} \partial_t^- \mathcal{I}(t, z_l). \tag{GC2}$$

Then for each $z_0 \in \hat{\mathcal{S}}(0)$, *the initial-value problem* $(\mathcal{Z}, \mathcal{I}, \mathcal{D}, z_0)$ *has a generalized energetic solution. Moreover, for incremental approximation, the same result holds as in Theorem* 2.2.2.

Proof. The proof follows again the six steps of Table 2.1 on p. 72.

Steps 1 and 2 work as above.

Step 3: Stability of the limit function. Because of condition (GC1), which corresponds to (2.1.38), we can apply Proposition 2.1.15(ii) and obtain the stability $z(t) \in \hat{\mathcal{S}}(t)$.

Step 4: Upper energy estimate. By Proposition 2.1.15(i), we know that (GC1) implies continuity of $\mathcal{I}$ on the stability set. Thus, in the discrete upper energy estimate

$$\mathcal{I}(t_n^m, z_m(t_n^m)) + \sum_{l=j+1}^{n} \mathcal{D}(z_m(t_{l-1}^m), z_m(t_l^m)) - \mathcal{I}(t_j^m, z_m(t_j^m)) \le \int_{t_j^m}^{t_n^m} \partial_\tau^- \mathcal{I}(\tau, z_m(\tau))\, \mathrm{d}\tau,$$

we may pass to the lim inf on the left-hand side and to the lim sup on the right-hand side. Assuming $t_j^m \to r$ and $t_n^m \to s$ and using (2.1.40) (now for $\mathcal{I}$) and (GC2), we obtain the desired upper estimate

$$\mathcal{I}(s, z(s)) + \mathrm{Diss}_{\mathcal{D}}(z; [r,s]) \le \mathcal{I}(r, z(r)) + \int_r^s \mathcal{P}_{\max}(\tau, z(\tau))\, \mathrm{d}\tau$$

for all $r, s \in [0, T]$ with $r < s$.

Steps 5 and 6 follow as above, since we have also assumed (I3). □

In the following example, we show that the notion of generalized energetic solution, which involves the weakened energy balance (WE) with the Clarke differential, is really necessary in cases in which the one-sided partial derivatives satisfy $\partial_t^- \mathscr{I}_0(t,z) < \partial_t^+ \mathscr{I}_0(t,z)$ at some points. In particular, it is impossible to make an a priori choice like $p(t) = \max\{\partial_t^{\mathrm{Cl}} \mathscr{I}_0(t,z(t))\}$, which worked in [307], since there, $\partial_t^- \mathscr{I}_0(t,z) \geq \partial_t^+ \mathscr{I}_0(t,z)$ holds.

Example 2.2.9 (Generalized energetic solutions as limit of energetic solutions). This example has a smooth energy $\mathscr{I}_\varepsilon$ such that $\partial_t \mathscr{I}_\varepsilon$ exists, while in the limit, $\mathscr{I}_0$ is only Lipschitz in t. We let $\mathcal{Z} = \mathbb{R}$ and $\mathscr{D}(z, \tilde{z}) = |\tilde{z}-z|$. The energy functional reads

$$\mathscr{I}_\varepsilon(t,z) = H_\varepsilon\big(z-\alpha(t)\big) \text{ and } \mathscr{I}_0(t,z) = 2|z-\alpha(t)|,$$

where $\alpha \in \mathrm{C}^1(0,T)$ is given and $H_\varepsilon(u) = 2u^2/\sqrt{\varepsilon^2+u^2}$. For the partial derivatives with respect to time, we have

$$\partial_t \mathscr{I}_\varepsilon(t,z) = -H_\varepsilon'(z-\alpha(t))\dot{\alpha}(t) \text{ and } \partial_t^{\mathrm{Cl}} \mathscr{I}_0(t,z) = -2|\dot{\alpha}(t)|\mathrm{Sign}(z-\alpha(t)).$$

Since $\mathscr{I}_\varepsilon(t,\cdot)$ is smooth and strictly convex, the energetic solutions for $(\mathbb{R}, \mathscr{I}_\varepsilon, \mathscr{D})$ are exactly the solutions of the doubly nonlinear equation (cf. [425])

$$\mathrm{Sign}(\dot{z}(t)) + H_\varepsilon'(z(t)-\alpha(t)) \ni 0.$$

For $\varepsilon > 0$, the system is smooth, while for $\varepsilon = 0$, we have $H_0(u) = 2|u|$ and set $\mathscr{I}_0(t,z) = H_0(z-\alpha(t))$. Consider the special case $\alpha(t) = t$ and $z_\varepsilon(0) = 0$. If β_ε is the unique solution of $H_\varepsilon'(\beta_\varepsilon) = 1$, then the unique energetic solution is $z_\varepsilon(t) = \max\{0, t-\beta_\varepsilon\}$. Using $0 < \beta_\varepsilon \to 0$, we obtain the limit solution $z(t) = t = \lim_{\varepsilon\to 0} z_\varepsilon(t)$. It is a generalized energetic solution if we use $p(t) = 1 \in [-2,2] = \partial_t^{\mathrm{Cl}} \mathscr{I}_0(t,t)$ in (WE).

2.2.3 The case of noncompact sublevels of $\mathscr{E}$

There are applications, e.g., in plasticity, where the sublevels of $\mathscr{E}$ are not compact, but merely closed. In that situation, it can be used that the a priori bounds in Theorem 2.1.5(3) or (2.1.29) are bounds on the sum of energy plus dissipation.

Corollary 2.2.10. *Let all the assumptions of the main existence result, Theorem 2.1.6, hold except for* (E1)*. Instead, we impose that $\mathscr{E}$ be lower semicontinuous and that for the initial value $q_0 \in \mathcal{S}(0)$, the functional $(t,q) \mapsto \mathscr{E}(t,q) + \mathscr{D}(q_0,q)$ have compact sublevels.*

Then all the assertions of Theorem 2.1.6 remain valid.

Proof. Using $\mathscr{D}(q_0, \underline{q}^\Pi(t)) \le \mathrm{Diss}_{\mathscr{D}}(\underline{q}^\Pi; [0,t])$ and Theorem 2.1.5(3), we see that all incremental approximations $\underline{q}^\Pi$ lie in a compact set, since $\mathscr{E}(t, \underline{q}^\Pi(t)) + \mathscr{D}(q_0, \underline{q}^\Pi(t))$ remains uniformly bounded in t and Π. All other steps of the proof of Theorem 2.1.6 remain the same. □

2.3 Semicontinuity of approximate incremental problems

A natural question arises as to how many energetic solutions exist in comparison to those that are obtained as limits of the incremental minimization problem (IMP$^\Pi$) in (2.1.8). As we have shown above, the approximations $\underline{z}^\Pi$ associated with (IMP$^\Pi$) can be considered upper semicontinuous in the sense that every limit point z obtained from a sequence $(\underline{z}^{\Pi_n})_{n\in\mathbb{N}}$ with $\varnothing(\Pi_n) \to 0$ is an energetic solution. The question is whether the opposite is true as well, namely that every energetic solution is such a limit point for a suitable sequence of partitions and corresponding incremental solutions. The following example shows that this is in general not the case.

Example 2.3.1 (Nonapproximable energetic solutions). We consider the state space $\mathcal{Z} = \{0\} \cup [1,2] \cup \{3\} \subset \mathbb{R}$ with dissipation distance $\mathscr{D}(z_0, z_1) = |z_1 - z_0|$ and the energy functional $\mathscr{I} : [0,3]\times\mathcal{Z} \to \mathbb{R}$ with

$$\mathscr{I}(t,0) = -2t, \quad \mathscr{I}(t,z) = \frac{1}{2}(z-t-2)^2 - \frac{1}{2} \;\text{ for }\; z\in[1,2], \quad\text{and}\quad \mathscr{I}(t,3) = -1.$$

We have two different energetic solutions starting in $z_0 = 1$:

$$z^{(1)}(t) = \begin{cases} t+1 & \text{for } t \in [0,1), \\ 3 & \text{for } t \in [1,2), \\ 0 & \text{for } t \in [2,3]; \end{cases} \qquad z^{(2)}(t) = \begin{cases} t+1 & \text{for } t \in [0,1), \\ 0 & \text{for } t \in [1,3]. \end{cases}$$

However, incremental solutions cannot approach solution $z^{(1)}$. Starting with $z_0 = 1$, we have $z_k = 1+t_k$ as long as $t_k < 1$. Now assume $t_{k+1} \ge 1$. Then $J(z) = \mathscr{I}(t_{k+1}, z) + |z-1-t_k|$ satisfies $J(3) = 1 - t_k > 0 > J(0) = 1 + t_k - 2t_{k+1}$ and $J(z) \ge J(2) = \frac{1}{2}t_{k+1}^2 - \frac{1}{2} - t_k \ge 0$ for $z \in [1,2]$. Hence, the global minimizer of J is $z_{k+1} = 0$, and all incremental solutions converge to $z^{(2)}$.

To avoid the necessity of lower semicontinuity, it is advantageous to introduce the concept of approximate incremental minimization problems. To this end, we define the *set of ε-approximate minimizers* of a functional $\mathfrak{J} : \mathcal{Q} \to \mathbb{R}_\infty$ as follows:

$$\operatorname{Arg\,min}_{\varepsilon,\mathcal{Q}} \mathfrak{J} := \Big\{ q \in \mathcal{Q} \;\Big|\; \mathfrak{J}(q) \le \varepsilon + \inf_{\hat q \in \mathcal{Q}} \mathfrak{J}(\hat q) \Big\}, \tag{2.3.1}$$

i.e., $\varepsilon > 0$ is the tolerance allowed for approximating the infimum. To a given partition $\Pi = (t_0 < t_1 < \cdots < t_N) \in \mathrm{Part}([0,T])$ we associate an N-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N) \in [0,\infty)^N$ of tolerances and define two types of approximate incremental problems.

Definition 2.3.2 (Approximate incremental problems). Given $(\mathcal{Q}, \mathscr{E}, \mathscr{D}, z_0)$, the approximate incremental problems ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) and ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$) consist in finding $(q_k)_{k=1,\ldots,N}$ in $\mathcal{Q}$ with

$$(\mathrm{AIP}^{\Pi}_{\varepsilon}) \qquad q_k \in \mathrm{Arg\,min}_{\varepsilon_k,\mathcal{Q}} \big(\mathscr{E}(t_k,\cdot) + \mathscr{D}(q_{k-1},\cdot)\big);$$

$$(\mathrm{SAIP}^{\Pi}_{\varepsilon}) \qquad \begin{cases} q_k \in \mathrm{Arg\,min}_{\varepsilon_k,\mathcal{Q}} \big(\mathscr{E}(t_k,\cdot) + \mathscr{D}(q_{k-1},\cdot)\big) \\ \text{and } \mathscr{E}(t_k,q_k) + \mathscr{D}(q_{k-1},q_k) \le \mathscr{E}(t_k,q_{k-1}), \end{cases}$$

where ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$) is called the *strengthened approximate incremental problem*.

If $\varepsilon = (0,\ldots,0)$, then ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) and ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$) just give the old incremental minimization problem (IMP^{Π}). If all ε_k are positive, then ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) always has a solution, even without any assumption on the lower semicontinuity of $\mathscr{I}$ and $\mathscr{D}$. This is even the case for the strengthened approximate incremental problem, where the existence of solutions follows easily if all ε_k are positive. Indeed, since $A_k = \mathrm{Arg\,min}_{\varepsilon_k,\mathcal{Q}} \big(\mathscr{E}(t_k,\cdot) + \mathscr{D}(q_{k-1},\cdot)\big)$ is never empty, we find that $\hat{q}_k \in A_k$. If $\mathscr{E}(t_k,\hat{q}_k) + \mathscr{D}(q_{k-1},\hat{q}_k) \le \mathscr{E}(t_k,q_{k-1})$, then we are done by choosing $q_k = \hat{q}_k$. If not, we can choose $q_k = q_{k-1}$, since the following estimate shows that $q_{k-1} \in A_k$:

$$\begin{aligned} \mathscr{E}(t_k,q_{k-1}) + \mathscr{D}(q_{k-1},q_{k-1}) &< \mathscr{E}(t_k,\hat{q}_k) + \mathscr{D}(q_{k-1},\hat{q}_k) && (2.3.2)\\ &\le \varepsilon_k + \inf\big(\mathscr{E}(t_k,\cdot) + \mathscr{D}(q_{k-1},\cdot)\big). && (2.3.3) \end{aligned}$$

Our aim is to give conditions that guarantee that solutions to ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) and ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$) have the same compactness conditions as the solutions to (IMP^{Π}). The final aim is to show that for a suitable subsequence, we have convergence of the z-component and that the limit gives rise to an energetic solution. The first result gives the discrete result on the a priori estimate, stability, and the energy estimates. In the second result, we provide convergence results. Since we minimize only approximately, we also need the *approximate-stability sets* defined via

$$\mathcal{S}^{\alpha}(t) := \Big\{ q \in \mathcal{Q} \;\Big|\; \mathscr{E}(t,q) < \infty,\ \forall\, \tilde{q} \in \mathcal{Q} : \mathscr{E}(t,q) \le \alpha + \mathscr{E}(t,\tilde{q}) + \mathscr{D}(q,\tilde{q}) \Big\}.$$

Of course, $\mathcal{S}^0(t) = \mathcal{S}(t)$ and $\alpha < \beta$ implies $\mathcal{S}^{\alpha}(t) \subset \mathcal{S}^{\beta}(t)$. The following result is a simple generalization of Proposition 2.1.4.

Proposition 2.3.3. *Assume that* $(\mathcal{Q}, \mathscr{E}, \mathscr{D})$ *satisfies* (D1) *and* (E2). *Then for each partition* $\Pi \in \mathrm{Part}([0,T])$ *and each* $\varepsilon \in (0,\infty)^N$ *with* $N = N_{\Pi}$, *the solutions* $(q_k)_{k=1,\ldots,N}$ *and* $(\hat{q}_k)_{k=1,\ldots,N}$ *to (*$\mathrm{AIP}^{\Pi}_{\varepsilon}$*) and (*$\mathrm{SAIP}^{\Pi}_{\varepsilon}$*), respectively, exist and have the following properties:*

(i) For $k = 1, \dots, N$ we have that q_k is ε_k-approximate, i.e., $q_k \in \mathcal{S}^{\varepsilon_k}(t_k)$;
(ii) With $\sigma = 1$ for (AIP$^{\Pi}_{\varepsilon}$) and $\sigma = 0$ for (SAIP$^{\Pi}_{\varepsilon}$) we have, for all $k = 1, \dots, N$:

$$\mathscr{E}(t_j, q_j) + \mathscr{D}(z_{k-1}, z_k) \le \sigma\varepsilon_k + \mathscr{E}(t_{j-1}, q_{j-1}) + \int_{t_{k-1}}^{t_k} \partial_s \mathscr{E}(s, q_{k-1})\,\mathrm{d}s. \tag{2.3.4}$$

As in Section 2.1.2, we now obtain a priori estimates by induction. In fact, if we define $e_k = \mathscr{E}(t_k, q_k) + \sum_{j=1}^{k} \varepsilon_j$ and $\delta_k = \mathscr{D}(z_{k-1}, z_k)$, then all the calculations done there remain valid with small modifications:

$$\mathscr{E}(t_k, q_k) + c_{\mathscr{E}} + \sum_{j=1}^{k} \mathscr{D}(z_{j-1}, z_j) \le \Big(\sigma \sum_{j=1}^{k} \varepsilon_j \mathrm{e}^{-\Lambda(t_j)} + \mathscr{E}(0, q_0) + c_{\mathscr{E}}\Big)\, \mathrm{e}^{\Lambda(t_k)}.$$

Subsequently, we use the following notation for the 1-norm and the ∞-norm:

$$|\varepsilon|_1 = \sum_{k=1}^{N} \varepsilon_k \quad \text{and} \quad |\varepsilon|_\infty = \max\Big\{\varepsilon_k \;\Big|\; k = 1, \dots N\Big\}.$$

Thus, for given a $R > 0$, we obtain uniform a priori bounds for the solutions $(q_n)_{n=1,\dots,N}$ for (AIP$^{\Pi}_{\varepsilon}$) for all $\Pi \in \mathrm{Part}^N([0,T])$, $N \in \mathbb{N}$, and $\varepsilon \in [0,\infty)^N$ with $|\varepsilon|_1 \le R$. For (SAIP$^{\Pi}_{\varepsilon}$), no restriction on ε is necessary, since $\sigma = 0$, and we have the same estimates as for (IMP$^{\Pi}$).

To pass to the limit, we need to strengthen the compatibility conditions (C1) and (C2), since they were based on stable sequences. We now introduce *approximately stable sequences*. The sequence $((t_k, q_k))_{k\in\mathbb{N}}$ is called *approximately stable* if there exists a sequence $(\alpha_k)_{k\in\mathbb{N}}$ such that

$$q_k \in \mathcal{S}^{\alpha_k}(t_k) \text{ for } k \in \mathbb{N}, \quad \sup_{k\in\mathbb{N}} \mathscr{E}(t_k, q_k) < \infty, \quad \alpha_k \to 0^+. \tag{2.3.5}$$

The compatibility condition for approximately stable sequences reads exactly like those for truly stable sequences. However, the condition is more restrictive, since there are far more approximately stable sequences than stable ones.

compatibility condition for approximately stable sequences:

$\forall$ approximately stable sequences $(t_k, q_k)_{k\in\mathbb{N}}$ with $(t_k, q_k) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t, q)$

$$t \in [0,T] \backslash N_{\mathscr{E}} \text{ with } N_{\mathscr{E}} \text{ from (E2)} \implies \partial_t \mathscr{E}(t, q) = \lim_{k\to\infty} \partial_t \mathscr{E}(t, q_k), \tag{AC1}$$

$$q \in \mathcal{S}(t). \tag{AC2}$$

Theorem 2.3.4 (Convergence for ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) and ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$)). *Assume that $\mathcal{E}$ and $\mathcal{D}$ satisfy the assumptions* (D1)–(D2), (E1)–(E2), *and the compatibility conditions* (AC1)–(AC2). *Consider a sequence of partitions $\Pi^l \in \mathrm{Part}^{N_l}([0,T])$ and $\varepsilon^{(l)} \in (0,\infty)^{N_l}$ with $\varnothing(\Pi^l) \to 0$ and $|\varepsilon^{(l)}|_\infty \to 0$ for $l \to \infty$ and let $(q_k^{(l)})_{k=1,\dots,N_l}$ be any solution of ($\mathrm{SAIP}^{\Pi^l}_{\varepsilon^{(l)}}$), and $\underline{q}^{\Pi^l}$ the associated interpolant. Then there exist a subsequence $q_k = q^{\Pi^{l_k}}$ and a solution $\tilde{q} = (\tilde{y}, \tilde{z})$ to the initial-value problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D}, q_0)$ such that the following hold (with $N_{\mathcal{E}}$ from* (E2)*):*

$$
\begin{aligned}
&\forall\, t \in [0,T]: &&z_k(t) \overset{\mathcal{Z}}{\rightharpoonup} \tilde{z}(t) \ \text{in } \mathcal{Z}; &&\text{(2.3.6a)}\\
&\forall\, t \in [0,T]: &&\mathrm{Diss}_{\mathcal{D}}(z_k;[0,t]) \to \mathrm{Diss}_{\mathcal{D}}(\tilde{z};[0,t]); &&\text{(2.3.6b)}\\
&\forall\, t \in [0,T]: &&\mathcal{E}(t,q_k(t)) \to \mathcal{E}(t,\tilde{q}(t)); &&\text{(2.3.6c)}\\
&\forall\, t \in [0,T]\backslash N_{\mathcal{E}}: &&\partial_t\mathcal{E}(t,q_k(t)) \to \partial_t\mathcal{E}(t,\tilde{q}(t)). &&\text{(2.3.6d)}
\end{aligned}
$$

In particular, also $\partial_t\mathcal{E}(\cdot,q_k(\cdot)) \to \partial_t\mathcal{E}(\cdot,\tilde{q}(\cdot))$ in $\mathrm{L}^1(0,T)$. If additionally $|\varepsilon^{(l)}|_1 \to 0$, then the same holds for the solutions of ($\mathrm{AIP}^{\Pi^l}_{\varepsilon^{(l)}}$).

Proof. We again follow the six steps of Table 2.1 on p. 72.

Steps 1 and 2 are identical, since we have uniform a priori bounds.

Step 3: Stability of the limit function. This follows directly from the approximate stability of the approximate solutions and from $|\varepsilon^{(l)}|_\infty \to 0$ by exploiting (AC2).

Step 4: Upper energy estimate. The upper estimate is the same as for (IMP^{Π}) in the case of ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$), since $\sigma = 0$. For ($\mathrm{AIP}^{\Pi}_{\varepsilon}$), we have additional terms from ε that arise by summing (2.3.4) in Proposition 2.3.3(ii), namely

$$
\mathcal{E}(t_n^l, \underline{q}^{\Pi^l}(t_n^l)) + \mathrm{Diss}_{\mathcal{D}}(\underline{q}^{\Pi^l};[t_m^l,t_n^l]) \le \sum_{m+1}^{n} \varepsilon_j + \mathcal{E}(t_n^l, \underline{q}^{\Pi^l}(t_n^l)) + \int_{t_m^l}^{t_n^l} \partial_s\mathcal{E}(s,\underline{q}^{\Pi^l}(s))\,\mathrm{d}s.
$$

Now, using the additionally assumption $|\varepsilon^{(l)}|_1 \to 0$ for ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) gives the desired upper energy estimate.

Step 5: Lower energy estimate. Here we deal only with the limit that is exactly stable. Hence ε does not matter.

Step 6 works as usual. □

Now we follow [408] and show that for every energetic solution $q : [0,T] \to \mathcal{Q}$ of the initial-value problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D}, q_0)$, there exist approximate incremental solutions $(q_k)_{k=1,\dots,N}$ for ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) if ε is not too small. The obvious choice, which we will also use, is given by $q_k = q(t_k^{\Pi})$, and the task is to find out for which ε this is an approximate solution. As such, this result is not too surprising, but certain variants have impact in the existence theory for optimal controls of RIS; see [506, 507].

Theorem 2.3.5 (Reverse approximation of energetic solutions). *Assume that the ERIS $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ satisfies (D1) and (E2). Let $q : [0,T] \to \mathcal{Q}$ be an energetic solution. Then for every partition $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$, the $(N{+}1)$-tuple $(q_k)_{k=0,1,\ldots,N}$ with $q_k = q(t_k)$ solves (AIP$^{\Pi}_{\varepsilon}$) whenever $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_N)$ satisfies*

$$\varepsilon_k > 2E^*\big(\Lambda(t_k){-}\Lambda(t_{k-1})\big) \quad \textit{for } k = 1, \ldots, N, \tag{2.3.7}$$

where $E^* = \sup\{\, \mathcal{E}(r, q(t)) \mid r, t \in [0,T]\,\}$.

Proof. Let $\delta > 0$ be arbitrary. For $k = 1, \ldots, N$, we let $\alpha_k = \inf\{\, \mathcal{E}(t_k, \tilde{q}) + \mathcal{D}(q_{k-1}, \tilde{q}) \mid \tilde{q} \in \mathcal{Q}\,\}$. Thus there exists $\hat{q}_k$ with $\mathcal{E}(t_k, \hat{q}_k) + \mathcal{D}(q_{k-1}, \hat{q}_k) \le \alpha_k + \delta$. Arguing as we did following the definition of (SAIP$^{\Pi}_{\varepsilon}$), we may also assume $\mathcal{E}(t_k, \hat{q}_k) \le \mathcal{E}(t_k, q_{k-1})$.

Obviously, (q_k) solves (AIP$^{\Pi}_{\varepsilon}$) if we set ε_k equal to $\tilde{\varepsilon}_k = \mathcal{E}(t_k, q_k) + \mathcal{D}(q_{k-1}, q_k) - \alpha_k$ or equal to any upper bound. Using the energy balance of the solution q, we obtain

$$\tilde{\varepsilon}_k \le \mathcal{E}(t_k, q_k) + \mathrm{Diss}_{\mathcal{D}}(q; [t_{k-1}, t_k]) - \alpha_k \le \mathcal{E}(t_{k-1}, q_{k-1}) + \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, q(s))\,\mathrm{d}s - \alpha_k.$$

Since $q_{k-1} \in \mathcal{S}(t_{k-1})$, we may test with $\hat{q}_k$ and obtain

$$\mathcal{E}(t_{k-1}, q_{k-1}) \le \mathcal{E}(t_{k-1}, \hat{q}_k) + \mathcal{D}(q_{k-1}, \hat{q}_k) \le \alpha_k + \delta - \int_{t_{k-1}}^{t_k} \partial_s \mathcal{E}(s, \hat{q}_k)\,\mathrm{d}s.$$

Inserting this into the previous estimate and applying (2.1.3b), we arrive at

$$\begin{aligned} \tilde{\varepsilon}_k &\le \delta + \int_{t_{k-1}}^{t_k} \big(\partial_s \mathcal{E}(s, q(s)) - \partial_s \mathcal{E}(s, \hat{q}_k)\big)\,\mathrm{d}s \\ &\le \delta + \int_{t_{k-1}}^{t_k} \lambda_{\mathcal{E}}\mathcal{E}(t)\big((\mathcal{E}(s, q(s)){+}c_{\mathcal{E}}) + (\mathcal{E}(s, \hat{q}_k){+}c_{\mathcal{E}})\big)\,\mathrm{d}s. \end{aligned} \tag{2.3.8}$$

Thus, we obtain $\tilde{\varepsilon}_k \le \delta + 2E^*\big(\Lambda(t_k){-}\Lambda(t_{k-1})\big)$. Since $\delta > 0$ was arbitrary, the desired estimate (2.3.7) is established. □

Overall, the results are not yet satisfactory, since for the upper semicontinuity of Theorem 2.3.4, we need $|\varepsilon|_\infty \to 0$ if we deal with (SAIP$^{\Pi}_{\varepsilon}$). However, for (AIP$^{\Pi}_{\varepsilon}$), we need the stronger condition $|\varepsilon|_1 \to 0$. The opposite result in Theorem 2.3.5 applies only to (AIP$^{\Pi}_{\varepsilon}$) and provides a lower bound on ε that is compatible with $|\varepsilon|_\infty \to 0$ but enforces a lower bound on $|\varepsilon|_1$, namely $|\varepsilon|_1 \ge 2E^*\Lambda(T)$. We refer to the discussion in [507, Sect. 3.3] for a more careful analysis.

In the following example, we show that in general, it is not possible to reduce the linear order of the lower bound (2.3.7). Only in very good cases (smooth and uniformly convex) we might be able to show that $\hat{q}_k$ is sufficiently close to $q_k = q(t_k)$ that it would be possible to take advantage of a cancellation in the integral in (2.3.8).

Example 2.3.6 (Example 2.3.1 continued). We may compare the lower bound on ε derived here with the one we need in the above example. In fact, we have see there that $J(3) = 1-t_k > J(0) = 1+t_k-2t_{k+1}$ holds. Thus, to jump to $z = 3$ in that step, which is certainly the best moment, we need $\varepsilon_k = J(3) - J(0) = 2(t_{k-1}-t_k)$. Thus, we have to allow for error levels ε_k that are of the order of the loading increment.

2.4 Evolutionary Γ-convergence for sequences of ERIS

We will now consider sequences of rate-independent systems $((\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k))_{k\in\mathbb{N}}$ and study the question under what conditions energetic solutions $q_k : [0,T] \to \mathcal{Q}$ converge to a limit solution that is again an energetic solution for a limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. Surprisingly, as already revealed in [420], this theory is still very close to the existence theory for energetic solutions above, so our proof will follow the six-step scenario as in Table 2.1 on page 72.

The evolutionary Γ-convergence for ERIS has found numerous applications in, e.g., fracture [223], two-scale homogenization [260, 400, 429], numerical approximation [222, 328, 418], and delamination [421, 545]. We will address evolutionary Γ-convergence in the following chapters, for instance in a numerical convergence theory in Section 3.6.1 and in the derivation of material models with microstructure; see Sections 4.2.2.2 and 4.4.1.2. A special Γ-convergence theory for quadratic ERIS in Hilbert spaces will be developed in Theorem 3.5.14 and applied in linearized elastoplasticity for dimension reduction and homogenization; see Sections 4.3.1.4 and 4.3.1.5, respectively.

2.4.1 Basics on static Γ-convergence

The notion of *Γ-convergence*, introduced by De Giorgi [155, 156], exclusively applies to functionals, i.e., static problems without time-dependence. It is sometimes also called *variational convergence* or *epigraph convergence*. We refer to [28, 30, 94, 95, 141] for the full theory and applications. Here we just give a brief outline that is sufficient for our purposes.

We consider a metrizable topological space $\mathcal{Q}$, which means for our application that we will restrict to a compact sublevel and use the metrizability assumption (2.1.18). For a sequence $(\mathfrak{J}_k)_{k\in\mathbb{N}}$ of functionals $\mathfrak{J}_k : \mathcal{Q} \to \mathbb{R}_\infty$, one is interested in the behavior for $k \to \infty$, which reflects the behavior of minimizers. In particular, one defines the limit $\mathfrak{J}_\infty$ in such a way that if q_k minimizes $\mathfrak{J}_k$ and $q_k \overset{\mathcal{Q}}{\to} q_\infty$, then q_∞ minimizes $\mathfrak{J}_\infty$.

For a sequence $(\mathfrak{J}_k)_{k\in\mathbb{N}}$, the $\varGamma$-lim inf and the $\varGamma$-lim sup are defined as follows:

$$\varGamma\text{-}\liminf_{k\to\infty} \mathfrak{J}_k : q \mapsto \inf\Big\{ \liminf_{n\to\infty} \mathfrak{J}_n(q_n) \,\Big|\, q_n \xrightarrow{\mathfrak{Q}} q \Big\} \quad \text{and} \tag{2.4.1a}$$

$$\varGamma\text{-}\limsup_{k\to\infty} \mathfrak{J}_k : q \mapsto \inf\Big\{ \limsup_{n\to\infty} \mathfrak{J}_n(q_n) \,\Big|\, q_n \xrightarrow{\mathfrak{Q}} q \Big\}. \tag{2.4.1b}$$

By definition, we say that $\mathfrak{J}_k$ $\varGamma$-converges to $\mathfrak{J}$ if $\varGamma\text{-}\liminf_{k\to\infty} \mathfrak{J}_k = \mathfrak{J} = \varGamma\text{-}\limsup_{k\to\infty} \mathfrak{J}_k$. The following definition makes this a little more explicit.

Definition 2.4.1 ($\varGamma$-convergence and Mosco convergence). Let $(\mathfrak{J}_k)_{k\in\mathbb{N}}$ be a sequence of functionals on the metrizable topological space $\mathfrak{Q}$. Then $\mathfrak{J}_k$ *$\varGamma$-converges to* $\mathfrak{J} : \mathfrak{Q} \to \mathbb{R}_\infty$, written $\mathfrak{J} = \varGamma\text{-}\lim_{k\to\infty} \mathfrak{J}_k$ or for even greater brevity $\mathfrak{J}_k \xrightarrow{\varGamma} \mathfrak{J}$, if

($\varGamma$ inf) *$\varGamma$-lim inf estimate*:

$$q_k \xrightarrow{\mathfrak{Q}} q \implies \mathfrak{J}(q) \le \liminf_{k\to\infty} \mathfrak{J}_k(q_k), \tag{2.4.2a}$$

($\varGamma$ sup) *$\varGamma$-lim sup estimate or "existence of recovery sequences":*

$$\forall\, \hat{q} \in \mathfrak{Q}\ \exists\, (\hat{q}_k)_{k\in\mathbb{N}} \text{ with } \hat{q}_k \xrightarrow{\mathfrak{Q}} \hat{q} : \quad \mathfrak{J}(\hat{q}) \ge \limsup_{k\to\infty} \mathfrak{J}_k(\hat{q}_k). \tag{2.4.2b}$$

If the underlying space $\mathfrak{Q}$ is a Banach space $\boldsymbol{Q}$, we can define *weak $\varGamma$-convergence* and *strong $\varGamma$-convergence*, by equipping $\boldsymbol{Q}$ with the weak or the strong topology, respectively. Moreover, we say that $\mathfrak{J}_k$ *Mosco converges* to $\mathfrak{J}$ and write $\mathfrak{J}_k \xrightarrow{\mathrm{M}} \mathfrak{J}$ if we have weak and strong $\varGamma$-convergence. In particular, this means that ($\varGamma$ inf) holds in the weak topology, while ($\varGamma$ sup) holds in the strong topology.

Here the sequence $(\hat{q}_k)_{k\in\mathbb{N}}$ is called a *recovery sequence* for the limit $\hat{q}$, since ($\varGamma$ inf) and ($\varGamma$ sup) imply $\mathfrak{J}_k(\hat{q}_k) \to \mathfrak{J}(\hat{q})$, i.e., $\hat{q}_k$ recovers the correct energy level.

The following results are fundamental in the theory of $\varGamma$-convergence.

Proposition 2.4.2. *Under the above assumptions we have the following:*

(i) $\mathfrak{J}_{\inf} = \varGamma\text{-}\liminf_{k\to\infty} \mathfrak{J}_k$ *is always lower semicontinuous. In particular, if* $\mathfrak{J} = \varGamma\text{-}\lim \mathfrak{J}_k$ *exists, then it is lower semicontinuous.*

(ii) For $\mathfrak{J}, \mathfrak{J}_k : \mathfrak{Q} \to \mathbb{R}_\infty$ *with* $\mathfrak{J} = \varGamma\text{-}\lim_{k\to\infty} \mathfrak{J}_k$ *set* $\alpha = \inf_{\mathfrak{Q}} \mathfrak{J}$ *and* $\alpha_k = \inf_{\mathfrak{Q}} \mathfrak{J}_k$. *Assume* $\alpha \in \mathbb{R}$ *and that there exist* $\delta > 0$ *and a compact set* $C \subset \mathfrak{Q}$ *such that all sublevels* $\{ q \mid \mathfrak{J}_k(q) \le \alpha+\delta \}$ *are contained in* C. *Then* $\alpha_k \to \alpha$ *and for each sequence* q_k *with* $q_k \to \tilde{q}$ *and* $\limsup_{k\to\infty} \mathfrak{J}_k(\tilde{q}_k) = \lim_{k\to\infty} \alpha_k = \alpha$, *we have* $\mathfrak{J}(\tilde{q}) = \alpha$, *i.e.,* $\tilde{q}$ *is a minimizer of* $\mathfrak{J}$. *In particular, if* q_k *are minimizers of* $\mathfrak{J}_k$, *we conclude that all accumulation points of* $(q_k)_k$ *are minimizers of* $\mathfrak{J} = \varGamma\text{-}\lim_{k\to\infty} \mathfrak{J}_k$.

Example 2.4.3 (Monotone approximation from above). Let us consider a decreasing pointwise-convergent sequence $(\mathfrak{J}_k)_{k\in\mathbb{N}}$, i.e.,

$$\mathfrak{J}_k(q) \ge \mathfrak{J}_{k+1}(q) \ge \mathfrak{J}(q) \ \text{ and } \ \lim_{k\to\infty} \mathfrak{J}_k(q) = \mathfrak{J}(q) \quad \text{for all } q \in \mathfrak{Q}, \tag{2.4.3}$$

and let $\mathcal{J}$ be lower semicontinuous. Then $\Gamma\text{-}\lim_{k\to\infty}\mathcal{J}_k = \mathcal{J}$. Indeed, for every $q_k \to q$, we have $\mathcal{J}(q) \le \liminf_{k\to\infty}\mathcal{J}(q_k) \le \liminf_{k\to\infty}\mathcal{J}_k(q_k)$, so (2.4.2a) obviously holds. Taking the recovery sequence constant, i.e., $\tilde{q}_k := \tilde{q}$, makes (2.4.2b) satisfied as a consequence of the pointwise convergence $\mathcal{J}_k \to \mathcal{J}$.

Example 2.4.4 (Abstract "numerical" approximation). Let us consider the case

$$\mathcal{Q}_k \subset \mathcal{Q}_{k+1} \subset \mathcal{Q} \text{ and } \mathcal{J}_k := \mathcal{J} + \boldsymbol{\delta}_{\mathcal{Q}_k}. \tag{2.4.4}$$

Further, suppose that for some topology $\mathcal{T}$ finer than (and not necessarily identical to) the topology of $\mathcal{Q}$ considered for existence of energetic solutions, $\mathcal{J}$ is $\mathcal{T}$-continuous and $\bigcup_{k\in\mathbb{N}}\mathcal{Q}_k$ is $\mathcal{T}$-dense in $\mathcal{Q}$. Typically, such occurs in various numerical approximations where the $\mathcal{Q}_k$ are finite-dimensional manifolds. Then $\Gamma\text{-}\lim_{k\to\infty}\mathcal{J}_k = \mathcal{J}$. Indeed, the ($\Gamma$ inf)-condition holds because again, $\mathcal{J}_k \ge \mathcal{J}_{k+1} \ge \mathcal{J}$ as in Example 2.4.3. For every $\hat{q} \in Q$, there is $\hat{q}_k \in \mathcal{Q}_k$ such that $\hat{q}_k \stackrel{\mathcal{T}}{\to} \hat{q}$. Then $\lim_{k\to\infty}\mathcal{J}_k(\hat{q}_k) = \lim_{k\to\infty}\mathcal{J}(\hat{q}_k) = \mathcal{J}(\hat{q})$ and also $\lim_{k\to\infty}\hat{q}_k = q$ in $\mathcal{Q}$, so that $\{\hat{q}_k\}_{k\in\mathbb{N}}$ is a recovery sequence for (2.4.2b). Note that lower semicontinuity of $\mathcal{J}$ would not be sufficient. A simple counterexample is $\mathcal{J} := \boldsymbol{\delta}_{\{q\}}$ with some $q \in \mathcal{Q}\setminus\bigcup_{k\in\mathbb{N}}\mathcal{Q}_k$, where $\mathcal{J}_k \equiv \infty$ obviously does not Γ-converge to $\mathcal{J}$.

Example 2.4.5 (Monotone approximation from below: penalty function). The situation $\mathcal{J}_k \le \mathcal{J}_{k+1} \le \mathcal{J}_\infty$ can be illustrated by considering $A \subset \mathcal{Q}$ closed, $\mathcal{Q}$ equipped with a metric d inducing a topology finer than (but not necessarily identical to) the topology of $\mathcal{Q}$, $\mathcal{J}$ lower semicontinuous, $\mathcal{J}_\infty := \mathcal{J} + \boldsymbol{\delta}_A$, and

$$\mathcal{J}_k(q) := \mathcal{J}(q) + k\,\mathrm{dist}(q,A)^\alpha := \mathcal{J}(q) + k\inf_{\tilde{q}\in A} d(q,\tilde{q})^\alpha \tag{2.4.5}$$

with $\alpha > 0$, where the last term is called a *penalty function* for the constraint $q \in A$. Then $\Gamma\text{-}\lim_{k\to\infty}\mathcal{J}_k = \mathcal{J}_\infty$. The ($\Gamma$ inf)-condition is trivial for $q \in A$, because then, $\mathcal{J}_\infty(q) = \mathcal{J}(q) \le \liminf_{k\to\infty}\mathcal{J}(q_k) \le \liminf_{k\to\infty}\mathcal{J}_k(q_k)$ if we use successively lower semicontinuity of $\mathcal{J}$ and that $\mathcal{J} \le \mathcal{J}_k$. If $q \notin A$, then $\mathrm{dist}(q,A) > 0$, because A is closed and there is k_0 such that $\mathrm{dist}(q_k,A) \ge \frac{1}{2}\mathrm{dist}(q,A) > 0$ for all $k \ge k_0$, and then

$$\begin{aligned}\liminf_{k\to\infty}\mathcal{J}_k(q_k) &= \liminf_{k\to\infty}\mathcal{J}(q_k) + \lim_{k\to\infty} k\,\mathrm{dist}(q,A)^\alpha \\ &\ge \mathcal{J}(q) + \lim_{k\to\infty} k\frac{\mathrm{dist}(q,A)^\alpha}{2^\alpha} = \infty = \mathcal{J}_\infty(q).\end{aligned}$$

The (Γ sup)-condition works for the constant recovery sequence $\hat{q}_k = \hat{q}$: if $\hat{q} \in A$, then $\mathcal{J}_k(\hat{q}_k) = \mathcal{J}(\hat{q}) = \mathcal{J}_\infty(\hat{q})$, and if $\hat{q} \notin A$, then $\mathcal{J}_\infty(\hat{q}) = \infty$, and (2.4.2b) holds trivially.

Merging Examples 2.4.4 and 2.4.5 has some use and is not entirely straightforward, because the recovery sequence in Example 2.4.5 no longer can be taken constant, and thus deserves a careful formulation and a detailed proof.

Proposition 2.4.6. *Let $\{\mathcal{Q}_k\}_{k\in\mathbb{N}}$ be as in (2.4.4), let $\mathcal{J}$ be bounded from below, let $A \subset \mathcal{Q}$ be compact, and assume that both $\mathcal{J}$ and the metric d determining* $\operatorname{dist}(\cdot, A)$ *are lower semicontinuous in the topology of $\mathcal{Q}$. Moreover, assume $A \cap \operatorname{Dom}\mathcal{J} \neq \emptyset$, and let there be a metrizable topology $\mathcal{T}$ on $\mathcal{Q}$ such that the metric d is $\mathcal{T}$-continuous, $\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k$ is $\mathcal{T}$-dense in $\mathcal{Q}$, and the functional $\mathcal{J}$ is $\mathcal{T}$-continuous. Furthermore, let*

$$\mathcal{J}_{\varepsilon k}(q) := \mathcal{J}(q) + \frac{1}{\varepsilon}\operatorname{dist}(q, A)^{\alpha} + \boldsymbol{\delta}_{\mathcal{Q}_k}(q) \tag{2.4.6}$$

with $\alpha > 0$. Then there exists $K : \mathbb{R}^+ \to \mathbb{N}$ such that:

(i) $\lim_{\varepsilon\to 0,\, k\to\infty,\, k\geq K(\varepsilon)} \min \mathcal{J}_{\varepsilon k} = \min \mathcal{J}_A$ *and* $(\varepsilon, k) \mapsto \operatorname{Arg\,min} \mathcal{J}_{\varepsilon k}$ *is upper semicontinuous conditioned to $k \geq K(\varepsilon)$ in the sense that every cluster point (in the topology of $\mathcal{Q}$) of every sequence $(q_{\varepsilon k})_{\varepsilon\to 0,\, k\to\infty,\, k\geq K(\varepsilon)}$ with $q_{\varepsilon k} \in$* $\operatorname{Arg\,min} \mathcal{J}_{\varepsilon k}$ *belongs to* $\operatorname{Arg\,min} \mathcal{J}_A$.

(ii) If additionally, $\mathcal{T}$ coincides with the topology of $\mathcal{Q}$, which is still assumed to be metrizable, then K can be chosen in such a way that even

$$\mathop{\varGamma\text{-}\lim}_{\substack{\varepsilon\to 0, k\to\infty \\ k\geq K(\varepsilon)}} \mathcal{J}_{\varepsilon k} = \mathcal{J}_A := \mathcal{J} + \boldsymbol{\delta}_A. \tag{2.4.7}$$

Proof.[5] As to (i), we can first use Example 2.4.5, exploiting now the topology $\mathcal{T}$ to show that $0 \leq \min \mathcal{J}_A - \min \mathcal{J}_\varepsilon \to 0$ for $\varepsilon \to 0$, where we have now defined $\mathcal{J}_\varepsilon := \mathcal{J} + \frac{1}{\varepsilon}\operatorname{dist}(\cdot, A)^\alpha$. Then for every $\varepsilon > 0$ fixed, we can use Example 2.4.4 and obtain some $K(\varepsilon) \in \mathbb{N}$ such that for every $k \geq K(\varepsilon)$, we have $0 \leq \min \mathcal{J}_{\varepsilon k} - \min \mathcal{J}_\varepsilon \leq \min \mathcal{J}_A - \min \mathcal{J}_\varepsilon$. Merging these two results yields

$$\begin{aligned} \big|\min \mathcal{J}_{\varepsilon k} - \min \mathcal{J}_A\big| &\leq \big(\min \mathcal{J}_{\varepsilon k} - \min \mathcal{J}_\varepsilon\big) + \big(\min \mathcal{J}_A - \min \mathcal{J}_\varepsilon\big) \\ &\overset{k\geq K(\varepsilon)}{\leq} \big(\min \mathcal{J}_A - \min \mathcal{J}_\varepsilon\big) \overset{\varepsilon\to 0}{\to} 0. \end{aligned} \tag{2.4.8}$$

Thus $\lim_{\varepsilon\to 0,\, k\to\infty,\, k\geq K(\varepsilon)} \min \mathcal{J}_{\varepsilon k} = \min \mathcal{J}_A$ is proved.

Then for every $q_{\varepsilon k} \overset{\mathcal{Q}}{\to} q$ with $q_{\varepsilon k} \in \operatorname{Arg\,min} \mathcal{J}_{\varepsilon k}$, we must have $q \in A$, because otherwise, $\liminf \operatorname{dist}(q_{\varepsilon k}, A) \geq \operatorname{dist}(q, A) > 0$, so that $\frac{1}{\varepsilon}\operatorname{dist}(q_{\varepsilon k}, A)^\alpha \to \infty$, which contradicts $\mathcal{J}_{\varepsilon k}(q_{\varepsilon k}) = \min \mathcal{J}_{\varepsilon k} \to \min \mathcal{J}_A < \infty$. Here the lower semicontinuity of d and hence also that of $\operatorname{dist}(\cdot, A)$ have been used. Using also the lower semicontinuity of $\mathcal{J}$, we have

$$\min \mathcal{J}_A = \lim_{\substack{\varepsilon\to 0, k\to\infty \\ k\geq K(\varepsilon)}} \mathcal{J}_{\varepsilon k}(q_{\varepsilon k}) = \lim_{\substack{\varepsilon\to 0, k\to\infty \\ k\geq K(\varepsilon)}} \mathcal{J}_{\varepsilon}(q_{\varepsilon k}) \geq \liminf_{\substack{\varepsilon\to 0, k\to\infty \\ k\geq K(\varepsilon)}} \mathcal{J}(q_{\varepsilon k}) \geq \mathcal{J}(q). \tag{2.4.9}$$

Thus $q \in \operatorname{Arg\,min} \mathcal{J}_A$, which proves (i).

[5] Cf. also [328, Proof of Prop. 5.6] or the older work [519].

As for (ii), let us first treat the Γ-lim inf estimate (2.4.2a). We may assume that $\liminf_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}_{\varepsilon k}(q_{\varepsilon k}) =: \gamma < \infty$, since otherwise, nothing is to be shown. By assumption, $\mathcal{J}(q) \geq -C > -\infty$, and thus we conclude that $\mathrm{dist}(q_{\varepsilon k}, A)^\alpha \leq (1+\gamma+C)\varepsilon$ for $(\varepsilon, 1/k)$ sufficiently close to $(0,0)$. Hence, $q_{\varepsilon k} \to q$ implies $q \in A$. Moreover, the convergence $q_{\varepsilon k} \to q$ implies $\lim_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}(q_{\varepsilon k}) \geq \mathcal{J}(q)$. Using $q \in A$, we obtain

$$\begin{aligned}\mathcal{J}_A(q) = \mathcal{J}(q) &\leq \liminf_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}(q_{\varepsilon k}) \\ &\leq \liminf_{\varepsilon\to 0,\, k\to\infty} \Big(\mathcal{J}(q_{\varepsilon k}) + \frac{1}{\varepsilon}\,\mathrm{dist}(q_{\varepsilon k}, A)^\alpha + \boldsymbol{\delta}_{\mathcal{Q}_k}(q_{\varepsilon k})\Big) = \liminf_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}_{\varepsilon k}(q_{\varepsilon k}),\end{aligned}$$

which is the desired Γ-lim inf estimate.

For the Γ-lim sup estimate (2.4.2b), we first consider the case $q \notin A$, which leads to $\mathcal{J}_A(q) = \infty$. Now every sequence $q_{\varepsilon k} \to q$ is a recovery sequence. The continuity of d yields $\mathrm{dist}(q_{\varepsilon k}, A) \to \mathrm{dist}(q, A) > 0$, so that $\frac{1}{\varepsilon}\mathrm{dist}(q_{\varepsilon k}, A) \to \infty$, which implies $\liminf_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}_{\varepsilon k}(q_{\varepsilon k}) = \infty = \mathcal{J}_A(q)$, as desired. (Here lower semicontinuity of d would suffice.)

Next, we consider the case $q \in A$. By the assumed density of $\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k$, there is a sequence $(q_k)_{k\in\mathbb{N}}$, $q_k \in \mathcal{Q}_k$ with $q_k \to q$.

Using the compactness and the metrizability of $\mathcal{Q}$, we can think about a metric, say ϱ, inducing this convergence. Due to the density of $\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k$ in $\mathcal{Q}$, for every $\delta > 0$, there is some $k_\delta \in \mathbb{N}$ sufficiently large such that for every $q \in A$, there is $\tilde{q} \in \mathcal{Q}_k$, $\varrho(q, \tilde{q}) \leq \delta$; the proof is by contradiction: if for every $k \in \mathbb{N}$, there were some $q_k \in A$ whose δ-neighborhood was disjoint from $\mathcal{Q}_k$, then by compactness of $\mathcal{Q}$, we would get a limit q of a subsequence of $(q_k)_{k\in\mathbb{N}}$ converging for $k \to \infty$ whose $\delta/2$-neighborhood would still be disjoint from $\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k$.

By the assumption, d is continuous with respect to the metric ϱ, i.e., for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that $\varrho(q, \tilde{q}) \leq \delta$ implies $d(q, \tilde{q}) \leq \varepsilon^{2/\alpha}$.

Merging these two results, we can see that for every $\varepsilon > 0$, there is $K(\varepsilon) > 0$ sufficiently large, namely $K(\varepsilon) = k_{\delta}(\varepsilon)$, such that for every $k \geq K(\varepsilon)$ and $q \in A$, there is $q_k \in \mathcal{Q}_k$ such that $d(q, q_k) \leq \varepsilon^{2/\alpha}$. Then we put $q_{\varepsilon k} = q_k$. In particular, $d(q, q_{\varepsilon k}) \leq \varepsilon^{2/\alpha}$ implies $\lim_{\varepsilon\to 0,\, k\to\infty} q_{\varepsilon k} = q$ with respect to the topology induced by d. Also note that $\frac{1}{\varepsilon}\mathrm{dist}(q_{\varepsilon k}, A)^\alpha = \frac{1}{\varepsilon}\mathrm{dist}(q_k, A)^\alpha \leq \frac{1}{\varepsilon} d(q_k, q)^\alpha \leq \frac{1}{\varepsilon}\varepsilon^2 = \varepsilon$. Therefore,

$$\limsup_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}_{\varepsilon k}(q_{\varepsilon k}) = \limsup_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}(q_k) + \frac{1}{\varepsilon}\mathrm{dist}(q_k, A)^\alpha \leq \lim_{\varepsilon\to 0,\, k\to\infty} \mathcal{J}(q_k) + \varepsilon = \mathcal{J}(q),$$

which proves (2.4.2b). Here we used that $\mathcal{J}$ is continuous with respect to the topology induced by d. $\square$

2.4.2 The main assumptions for evolutionary Γ-convergence

We first list the assumptions on the rate-independent systems $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$, $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, that are sufficient for our convergence theory. They are in complete analogy with the assumptions in the existence theory above. We just need to have certain uniformity assumption. Moreover, note that certain properties are needed only for the limiting system with $k = \infty$. Then we will present some results about the limits of energetic solutions.

Since we are already dealing with a sequence of problems and we have to choose subsequences several times, we need to adjust the notion of stable sequences; see (2.1.17). The stability sets $\mathcal{S}_k(t)$ are defined for $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ as in (2.1.4). A sequence $((t_l, q_{k_l}))_{l\in\mathbb{N}}$ is called a *stable sequence for the family* $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)_{k\in\mathbb{N}}$ (abbreviated as stab.seq.$^{\mathbb{N}}$ further on) if

$$q_{k_l} \in \mathcal{S}_{k_l}(t_l) \text{ for all } l \in \mathbb{N} \quad \text{and} \quad \sup_{l\in\mathbb{N}} \mathcal{E}_{k_l}(t_l, q_{k_l}) < \infty. \tag{2.4.10}$$

Note that $(q_{k_l})_{l\in\mathbb{N}}$ denotes a subsequence to indicate the index k_l for which we have stability. As in the previous sections, we say that $((t_l, \tilde{q}_l))_{l\in\mathbb{N}}$ is a stable sequence for $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ if $\tilde{q}_l \in \mathcal{S}_\infty(t_l)$, and for brevity, we write stab.seq.$^\infty$ in that case.

We now state our assumptions together and comment on them afterward:

Quasidistance: $\forall\, k \in \mathbb{N}_\infty\ \forall\, z, \tilde{z}, \hat{z} \in \mathcal{Z}$:
$$\mathcal{D}_k(z, \tilde{z}) = 0 \;\Leftrightarrow\; z = \tilde{z} \;\text{ and }\; \mathcal{D}_k(z, \hat{z}) \le \mathcal{D}_k(z, \tilde{z}) + \mathcal{D}_k(\tilde{z}, \hat{z}). \tag{2.4.11a}$$

Lower semicontinuity of $\mathcal{D}_k$:
$$\forall\, k \in \mathbb{N}_\infty : \quad \mathcal{D}_k : \mathcal{Z}\times\mathcal{Z} \to [0, \infty] \text{ is lower semicontinuous.} \tag{2.4.11b}$$

Lower Γ-limit for $\mathcal{D}_k$:
$$\forall \text{ stab.seq.}^{\mathbb{N}}\ (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t, q) \text{ and } (\tilde{t}_l, \tilde{q}_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (\tilde{t}, \tilde{q}) : \\ \mathcal{D}_\infty(q, \tilde{q}) \le \liminf_{l\to\infty} \mathcal{D}_{k_l}(q_{k_l}, \tilde{q}_{k_l}). \tag{2.4.11c}$$

Compactness of energy sublevels:

For all $t \in [0, T]$ and all $E \in \mathbb{R}$ we have

(i) $\forall\, k \in \mathbb{N}_\infty : \quad \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \le E \,\}$ is compact;

(ii) $\bigcup_{k=1}^{\infty} \{\, q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \le E \,\}$ is relatively compact. (2.4.11d)

Separability and metrizability: The topology restricted to sublevels of $\mathcal{E}(t, \cdot)$ is compact, separable and metrizable. (2.4.11e)

Uniform control of the power $\partial_t\mathcal{E}_k$:

$$\begin{aligned}&\mathrm{Dom}\,\mathcal{E}_k = [0,T]\times\mathrm{Dom}\,\mathcal{E}_k(0,\cdot) \ \text{ and}\\ &\exists\, c_{\mathcal{E}} \in \mathbb{R}\ \exists\, \lambda_{\mathcal{E}} \in \mathrm{L}^1(0,T)\ \exists\, N_{\mathcal{E}} \subset [0,T],\ \mathcal{L}^1(N_{\mathcal{E}})=0\\ &\forall\, k\in\mathbb{N}_\infty\ \forall\, q\in\mathrm{Dom}\,\mathcal{E}_k(0,\cdot):\ \mathcal{E}_k(\cdot,q)\in \mathrm{W}^{1,1}(0,T) \ \text{ and}\\ &\qquad\qquad\qquad\qquad \partial_t\mathcal{E}_k(t,q) \text{ exists for } t\in[0,T]\setminus N_{\mathcal{E}} \text{ with}\\ &\qquad\qquad\qquad\qquad |\partial_t\mathcal{E}_k(t,q)| \le \lambda_{\mathcal{E}}(t)(\mathcal{E}_k(s,q)+c_{\mathcal{E}}).\end{aligned} \tag{2.4.11f}$$

Lower Γ-limit for $\mathcal{E}_k$:

$$\forall\ \text{stab.seq.}^{\mathbb{N}}\ (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q):\ \ \mathcal{E}_\infty(t,q) \le \liminf_{l\to\infty} \mathcal{E}_{k_l}(t_l,q_{k_l}). \tag{2.4.11g}$$

Conditioned semicontinuity of the power: $\forall\, t\in[0,T]\setminus N_{\mathcal{E}}$:

$$\forall\ \text{stab.seq.}^{\mathbb{N}}\ (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q):\ \ \partial_t\mathcal{E}_\infty(t,q) \ge \limsup_{l\to\infty} \partial_t\mathcal{E}_{k_l}(t,q_{k_l}), \tag{2.4.11h}$$

$$\forall\ \text{stab.seq.}^{\infty}\ (t_l,\tilde q_l) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q):\ \ \partial_t\mathcal{E}_\infty(t,q) \le \limsup_{l\to\infty} \partial_t\mathcal{E}_{\infty}(t,\tilde q_l). \tag{2.4.11i}$$

Conditioned upper semicontinuity of stability sets:

$$\forall\ \text{stab.seq.}^{\mathbb{N}}\ (t_l,q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q):\ \ \ q\in\mathcal{S}_\infty(t). \tag{2.4.11j}$$

Assumptions (2.4.11a-c) mainly concern the dissipation distances $\mathcal{D}_k$: the first correspond to the earlier conditions (D1) and (D2), whereas (2.4.11c) is the new Γ-lim inf condition. Assumptions (2.4.11d-g) are mainly on the stored-energy functionals $\mathcal{E}_k$: the first two correspond to the earlier (E1) and (E2), but now requiring uniformity with $c_{\mathcal{E}}$, $\lambda_{\mathcal{E}}$, and $N_{\mathcal{E}}$ independent of $k\in\mathbb{N}_\infty$, whereas (2.4.11g) is the new Γ-lim inf condition. Conditions (2.4.11j) and (2.4.11h,i) correspond to the compatibility conditions (C1) and (C2), respectively.

It may seem strange that we do not require the Γ-convergence of $\mathcal{D}_k \overset{\Gamma}{\to} \mathcal{D}_\infty$ and $\mathcal{E}_k(t,\cdot) \overset{\Gamma}{\to} \mathcal{E}_\infty$ for $k\to\infty$. In fact, we don't need this in general, but the compatibility conditions (2.4.11h-j) implicitly provide the Γ-lim sup estimates when restricted to the stability sets $\mathcal{S}_\infty(t)$. In fact, the sufficient conditions for (2.4.11j) we discuss below (cf. (2.4.13)–(2.4.14)) all involve the construction of a mutual-recovery sequence. In many practical applications, we will certainly have $\mathcal{D}_k \overset{\Gamma}{\to} \mathcal{D}_\infty$ and $\mathcal{E}_k(t,\cdot) \overset{\Gamma}{\to} \mathcal{E}_\infty$. The importance here is that we are automatically forced to consider Γ-convergence in the intrinsic topology, namely the one induced by convergence of stable sequences.

For a given function $z: [0,T]\to\mathcal{Z}$, $[r,s]\subset[0,T]$, and $k\in\mathbb{N}_\infty$, we use the abbreviation $\mathrm{Diss}_k(z;[r,s]) = \mathrm{Diss}_{\mathcal{D}_k}(z;[r,s])$.

Lemma 2.4.7 (Γ-lim inf estimate for Diss_k**).** *Assume that* (2.4.11a) *and* (2.4.11c) *hold. Let $q_k:[0,T]\to\mathcal{Q}$ be given such that for all $t\in[0,T]$, we have $q_k(t)\in\mathcal{S}_k(t)$ and $z_{k_l}(t)\overset{\mathcal{Z}}{\to} z(t)$ for $l\to\infty$ for a subsequence $(k_l)_l$. Then*

$$\mathrm{Diss}_\infty(z;[r,s]) \leq \liminf_{l\to\infty} \mathrm{Diss}_{k_l}(z_{k_l};[r,s]) \quad \textit{for all } [r,s]\subset[0,T]. \tag{2.4.12}$$

The same statement holds for piecewise constant interpolants $q_k = \underline{q}^{\Pi^k}$ *if* $q_k(t_j^k) \in \mathcal{S}_k(t_j^k)$ *for all* k *and all* $t_j^k \in \Pi^k$*, if* $\varnothing(\Pi^{k_l}) \to 0$*, and if* $z_{k_l}(t) \xrightarrow{\mathcal{Z}} z(t)$ *for all* t*.*

Proof. For arbitrary $\varepsilon > 0$, choose a finite partition $\Pi = \{r = \tau_0 < \cdots < \tau_N = s\}$ of $[r,s]$ with $\sum_{j=1}^N \mathcal{D}_\infty(z(\tau_{j-1}), z(\tau_j)) \geq \mathrm{Diss}_\infty(z;[r,s]) - \varepsilon$. Using (2.4.11c), we obtain

$$\mathrm{Diss}_\infty(z;[r,s]) - \varepsilon \;\leq\; \liminf_{l\to\infty} \sum_{j=1}^N \mathcal{D}_{k_l}(z_{k_l}(\tau_{j-1}), z_{k_l}(\tau_j)) \;\leq\; \liminf_{l\to\infty} \mathrm{Diss}_{k_l}(z_{k_l};[r,s]).$$

In (2.4.11c), also the convergence of the y-component is assumed. This is, however, irrelevant here, since $\mathcal{D}_k$ is independent of this. (In fact, we could choose even a further subsequence to make the y-components convergent as well on all $\tau_j \in \Pi$.) Since $\varepsilon > 0$ was arbitrary, this gives (2.4.12).

For the second case, we proceed similarly. For each $\tau_j \in \Pi$ and $k \in \mathbb{N}$, we choose $t_j^k \in \Pi^k$ with $|t_j^k - \tau_j| \leq \varnothing(\Pi^k)$ and $q_k(\tau_j) = q_l(t_j^k)$, where we use $q_k = \underline{q}^{\Pi^k}$. Then for each fixed $j = 0, \ldots, N$, the sequence $(t_j^{k_l}, q_{k_l}(\tau_j))_{l\in\mathbb{N}}$ is stable ($q_k(\tau_j) = q_l(t_j^k) \in \mathcal{S}_k(t_j^k)$) with $t_j^{k_l} \to \tau_j$ and $z_{k_l}(\tau_j) \xrightarrow{\mathcal{Z}} z(t)$. ◻

The major compactness result is a generalization of Helly's selection principle, which is proved in Section B.5. Using (2.4.11a-c), it is shown that every sequence of functions $z_k : [0,T] \to \mathcal{Z}$ for which $\mathrm{Diss}_k(z_k;[0,T])$ and $\mathcal{E}_k(\cdot, q_k(\cdot))$ are uniformly bounded and that is stable has a pointwise convergent subsequence.

The condition concerning the convergence of the power is split into two conditions, namely (i) the lim sup estimate (2.4.11h), which will be used to derive the upper energy estimate, and (ii) the lim inf estimate (2.4.11i) for the lower energy estimate. The latter is derived solely from the stability of the limit problem; hence it is a condition on the limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ only.

The major condition that makes the whole theory work is (2.4.11j). This condition couples the potentials $\mathcal{E}_k$ and $\mathcal{D}_k$ and provides a kind of upper Γ-limit estimate for $\mathcal{E}_k$ and $\mathcal{D}_k$ simultaneously. In [223], a similar condition is derived to study the Γ-convergence of the solutions in families of crack problems. There, our notion of stability is called the "unilateral minimality property," and our notion of upper semicontinuity of the stability sets is called "stability of the unilateral minimality property." In that paper, Theorems 7.2 and 8.3 provide what we call condition (2.4.11j).

As in Section 2.1.5, we have a hierarchy of conditions that imply (2.4.11j). The weakest condition involves the existence of suitable *mutual recovery sequences* [420]:

$$\forall \text{ stab.seq.}^{\mathbb{N}}\ (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q)\ \forall\, \tilde{q} \in \mathcal{Q}\ \exists\, \tilde{q}_{k_l} \overset{\mathcal{Q}}{\to} \tilde{q}:$$
$$\limsup_{l\to\infty} \mathcal{E}_{k_l}(t_l,\tilde{q}_{k_l}) + \mathcal{D}_{k_l}(q_{k_l},\tilde{q}_{k_l}) - \mathcal{E}_{k_l}(t_l,q_{k_l}) \le \mathcal{E}_\infty(t,\tilde{q}) + \mathcal{D}_\infty(q,\tilde{q}) - \mathcal{E}_\infty(t,q). \tag{2.4.13}$$

The next stronger conditions reads as follows:

$$\begin{aligned}&\forall \text{ stab.seq.}^{\mathbb{N}}\ (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q)\ \forall\, \tilde{q} \in \mathcal{Q}\ \exists\, \tilde{q}_{k_l} \overset{\mathcal{Q}}{\to} \tilde{q}:\\ &\limsup_{l\to\infty} \big(\mathcal{E}_{k_l}(t_l,\tilde{q}_{k_l}) + \mathcal{D}_{k_l}(q_{k_l},\tilde{q}_{k_l})\big) \le \mathcal{E}_\infty(t,\tilde{q}) + \mathcal{D}_\infty(q,\tilde{q}).\end{aligned} \tag{2.4.14}$$

The strongest condition requires $\mathcal{E}_k \overset{\Gamma}{\to} \mathcal{E}_\infty$ and continuous convergence of $\mathcal{D}_k$:

$$\begin{aligned}&(2.4.11\text{g}) \text{ holds and } \forall\, t\in[0,T]\ \forall\, \hat{q}\in\mathcal{Q}\\ &\exists\, (\hat{q}_k)_{k\in\mathbb{N}} \text{ with } \hat{q}_k \overset{\mathcal{Q}}{\to} \hat{q}:\ \limsup_{k\to\infty} \mathcal{E}_k(t,\hat{q}_k) \le \mathcal{E}_\infty(t,\hat{q}),\end{aligned} \tag{2.4.15a}$$

$$\left.\begin{array}{l} q_k \overset{\mathcal{Q}}{\to} q,\ \ \tilde{q}_k \overset{\mathcal{Q}}{\to} \tilde{q},\ \text{ and}\\ \sup_{k\in\mathbb{N}} \big(\mathcal{E}_k(t,q_k) + \mathcal{E}_k(t,\tilde{q}_k)\big) < \infty \end{array}\right\} \implies \mathcal{D}_k(q_k,\tilde{q}_k) \to \mathcal{D}_\infty(q,\tilde{q}). \tag{2.4.15b}$$

The following result is a direct analogue of Proposition 2.1.15, and the proof is the same after adding the subscripts $_{k_l}$ and $_\infty$ suitably.

Proposition 2.4.8. *Assume that* (2.4.11g) *holds.*

(i) If for each stab.seq.$^{\mathbb{N}}$ (t_l, q_{k_l}) *converging to* (t,q)*, there exists a sequence* $(\tilde{q}_l)_{l\in\mathbb{N}}$ *such that* $\limsup_{l\to\infty} \mathcal{E}_{k_l}(t_l,\tilde{q}_l) + \mathcal{D}_{k_l}(q_{k_l},\tilde{q}_l) \le \mathcal{E}_\infty(t,q)$*, then the energy converges along the stable sequences, i.e.,*

$$\forall \textit{ stab.seq.}^{\mathbb{N}}\ (t_l, q_{k_l}) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q):\quad \mathcal{E}_{k_l}(t_l,q_{k_l}) \to \mathcal{E}_\infty(t,q). \tag{2.4.16}$$

In particular, (2.4.14) *implies* (2.4.16).

(ii) We have the implications (2.4.15) $\implies$ (2.4.14) $\implies$ (2.4.13) $\implies$ (2.4.11j).

(iii) If additionally, (2.4.11f) *holds, then in all conditions,* $\mathcal{E}(t_l,\cdot)$ *can be replaced by* $\mathcal{E}(t,\cdot)$*, but* $q_{k_l} \in \mathcal{S}_{k_l}(t)$ *remains.*

The following examples show that the above implications cannot be reversed. It is easy to provide such examples taking $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ strictly lower than the corresponding Γ-limits. Our examples below are chosen such that equality between $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ and the corresponding Γ-limits hold. In particular, this means that (2.4.11g) and (2.4.15a) hold. For simplicity, we drop the dependence on the time $t \in [0, T]$, since the main emphasis of condition (2.4.11j) is on the convergence of q_k. Using the assumption (2.4.11f), it is then easy to obtain the more general version including $t_k \to t$.

Example 2.4.9 (Different mutual recovery conditions [420]).

(I) In Example 2.1.16, we proved (2.4.13) $\not\Rightarrow$ (2.4.14), even for constant sequences.

(II) To prove (2.4.13) $\not\Rightarrow$ (2.4.14) $\not\Rightarrow$ (2.4.15), we consider $\mathcal{Q} = \mathbb{R}$, $\mathcal{E}_k(q) = \frac{1}{2}(k^\alpha q)^2$, and $\mathcal{D}_k(q, \tilde{q}) = k^\beta |\tilde{q} - q|$. Here, $\alpha, \beta \geq 0$ are parameters. The corresponding stability sets are $\mathcal{S}_k = [-k^{\beta-\alpha}, k^{\beta-\alpha}]$. The Γ-limits are easily obtained, namely $\mathcal{E}_\infty = \mathcal{E}_1$ if $\alpha = 0$ and $\mathcal{E}_\infty = \boldsymbol{\delta}_{\{0\}}$ otherwise, and $\mathcal{D}_\infty(q, \tilde{q}) = |\tilde{q} - q|$ if $\beta = 0$ and $\mathcal{D}_\infty(q, \tilde{q}) = \boldsymbol{\delta}_{\{0\}}(\tilde{q} - q)$ otherwise, where $\boldsymbol{\delta}_A$ is the indicator function, i.e., $\boldsymbol{\delta}_A(b) = 0$ for $b \in A$ and ∞ otherwise; cf. (A.5.8).

The different conditions can be checked easily. In particular, (2.4.15b) holds if and only if $\alpha > \beta \geq 0$ or if $\alpha = \beta = 0$. Condition (2.4.14) holds if and only if $\alpha > \beta \geq 0$ or if $\alpha = 0$, which is a strictly larger set. Note that for $0 < \alpha \leq \beta$, the property (2.4.16) does not hold, and hence, by Proposition 2.4.8(i), condition (2.4.14) must be violated. Finally, condition (2.4.13) holds in all cases by choosing $\tilde{q}_{k_l} = q_{k_l} + \tilde{q} - q$.

(III) To prove (2.4.11j) $\not\Rightarrow$ (2.4.13), we let $\mathcal{E}_k(q) = \mathcal{E}(q) = \frac{1}{2}q^2$ for $k \in \mathbb{N}_\infty$ and choose $\mathcal{D}_k$ via $\mathcal{D}_k(q, \tilde{q}) = \left| \int_q^{\tilde{q}} m_k(p)\, \mathrm{d}p \right|$ with $m_k(p) = 1$ for $p \geq 0$ and $= k$ otherwise. The Γ-limit $\mathcal{D}_\infty$ reads $\mathcal{D}_\infty(q, \tilde{q}) = |\tilde{q} - q|$ for $q, \tilde{q} \geq 0$, $\mathcal{D}_\infty(q, \tilde{q}) = 0$ for $\tilde{q} = q < 0$, and $+\infty$ otherwise. Some computations give $\mathcal{S}_k = [-k, 1]$ and $\mathcal{S}_\infty = (-\infty, 1]$, and thus (2.4.11j) holds. The sequence $q_k = -1/k$ is a stable sequence converging to $q = 0$. For $\tilde{q} = 1$, every sequence $(\tilde{q}_k)_{k \in \mathbb{N}}$ with $\tilde{q}_k \to \tilde{q} = 1$ satisfies $\mathcal{D}_k(q_k, \tilde{q}_k) \to 2 < \mathcal{D}_\infty(q, \tilde{q}) = \mathcal{D}_\infty(0, 1) = 1$. Hence, since $\mathcal{E}$ is continuous, (2.4.13) cannot hold.

2.4.3 Convergence of energetic solutions

We present three different convergence results of increasing complexity. Our first result concerns the limit of a sequence of energetic solutions $q_k : [0, T] \to \mathcal{Q}$ of ERIS $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$. Under the above assumptions, pointwise convergence of the solutions is enough to guarantee that limits of solutions are energetic solutions of the limit system $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. The second result is concerned with time-incremental minimization problems (IMP$^{\Pi^k}$) for $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$. For brevity, let us denote it by (IMP)^k, with a given sequence of partitions $(\Pi^k)_{k \in \mathbb{N}}$ satisfying $\varnothing(\Pi^k) \to 0$.

Again, we are able to prove (partial[6]) convergence of the piecewise interpolants $\underline{q}^k : [0,T] \to \mathcal{Q}$. In the last case, we treat the situation in which the functionals $\mathcal{E}_k$ and $\mathcal{D}_k$ may not be lower semicontinuous. We replace the incremental minimization problem $(\mathrm{IMP})^k$ with an associated approximate incremental problem $(\mathrm{AIP}^{\Pi^k}_{\varepsilon^k})$ of the strengthened version $(\mathrm{SAIP}^{\Pi^k}_{\varepsilon^k})$; see Section 2.3. Generalizing the above conditions suitably, we again obtain that limits of approximate solutions solve the limit problem.

2.4.3.1 Solutions of $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ converge to solutions

The first result is concerned with exact energetic solutions of the initial-value problems $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k, q_k(0))$, and we already assume that these solutions converge. This is not a restrictive assumption, since from the proof, it will be clear that every sequence of solutions has a subsequence for which the z-component converges pointwise, and that is the only important assumption; for y-convergence, we refer to Remarks 2.1.8 and 2.1.9.

Theorem 2.4.10 (Evolutionary Γ-convergence). *Let assumptions* (2.4.11) *hold, and let $q_k : [0,T] \to \mathcal{Q}$ be energetic solutions of* $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$. *Further, assume that for all $t \in [0,T]$, we have $q_k(t) \overset{\mathcal{Q}}{\to} q(t)$ and $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q(0))$ for $k \to \infty$. Then $q : [0,T] \to \mathcal{Q}$ is an energetic solution of* $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. *With $N_{\mathcal{E}}$ from* (2.4.11f)*, we have*

$$\forall\, t \in [0,T]: \qquad \mathcal{E}_k(t, q_k(t)) \to \mathcal{E}_\infty(t, q(t)), \tag{2.4.17a}$$

$$\forall\, t \in [0,T]: \qquad \mathrm{Diss}_k(q_k; [0,t]) \to \mathrm{Diss}_\infty(q; [0,t]), \tag{2.4.17b}$$

$$\forall\, t \in [0,T]\setminus N_{\mathcal{E}}: \qquad \partial_t\mathcal{E}_k(t, q_k(t)) \to \partial_t\mathcal{E}_\infty(t, q(t)). \tag{2.4.17c}$$

Moreover, we also have $\partial_t\mathcal{E}_k(\cdot, q_k(\cdot)) \to \partial_t\mathcal{E}_\infty(\cdot, q(\cdot))$ in $\mathrm{L}^1(0,T)$.

Proof. First, we use $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q(0))$ and condition (2.4.11f) to show that $\mathcal{E}_k(t, q_k(t))$ is bounded uniformly in $t \in [0,T]$ and $k \in \mathbb{N}$ (use (2.1.29) for $s = 0$). Now condition (2.4.11j) gives the stability $q(t) \in \mathcal{S}_\infty(t)$.

On the one hand, we can employ $\mathcal{E}_k(0, q_k(0)) \to \mathcal{E}_\infty(0, q(0))$, (2.4.11h), and Lemma 2.4.7 to derive the upper energy estimate

$$\mathcal{E}_\infty(t, q(t)) + \mathrm{Diss}_\infty(q; [0,t]) \le \mathcal{E}_\infty(0, q(0)) + \int_0^t \partial_s\mathcal{E}_\infty(s, q(s))\,\mathrm{d}s \quad \text{for all } t \in [0,T].$$

[6] It means convergence in z-component only, while the y-component is again subject to the discussion in Remarks 2.1.8 and 2.1.9.

On the other hand, using (2.4.11a-c,e,g,i), we are able to apply Proposition 2.1.23 to the stable process q; note that condition (C1) there can be replaced by the weaker lower bound (2.4.11i). This supplies the lower energy estimate

$$\mathscr{E}_\infty(t,q(t)) + \mathrm{Diss}_\infty(q;[r,t]) \geq \mathscr{E}_\infty(r,q(r)) + \int_r^t \partial_s \mathscr{E}_\infty(s,q(s))\,\mathrm{d}s \quad \text{for all } [r,t] \subset [0,T].$$

Thus, we have the energy balance as well, and q is an energetic solution for $(\mathcal{Q}, \mathscr{E}_\infty, \mathscr{D}_\infty)$.

The convergences now follow as in Step 6 of the proof of Theorem 2.1.6. □

A similar convergence result was established in [569, Lemma 8.2] for the case that $\mathcal{Q}$ is a reflexive Banach space, all $\mathscr{E}_k(t,\cdot)$ are uniformly convex, and $\mathscr{D}_k(z,\tilde{z}) = \mathcal{R}_k(\tilde{z}-z)$. Here we give such a simple result by considering the fundamental Banach-space case as in Example 2.1.7. More advanced evolutionary Γ-convergence results will be treated in Section 3.5.4 for quadratic energies in Hilbert spaces $\boldsymbol{Q}$, in Section 3.6 for numerical approximations, and in Chapter 4 for various applications. We also refer to the survey [401] for a general approach to evolutionary Γ-convergence in the case of rate-dependent systems.

Corollary 2.4.11 (Evolutionary Γ-convergence in Banach spaces). *Consider a separable reflexive Banach space $\mathbf{Z}$ and a family $(\mathbf{Z}, \mathscr{I}_k, \mathscr{D}_k)_{k\in\mathbb{N}_\infty}$ of ERIS with $\mathscr{I}_k(t,z) = \mathfrak{I}_k(z) - \langle \ell_k(t), z\rangle$, where for each $k \in \mathbb{N}_\infty$, the assumptions (2.1.21) hold with constants C_1 and α independent of k. Moreover, assume the convergences*

$$\ell_k \to \ell_\infty \ \ \textit{in}\ \mathrm{W}^{1,1}(0,T;\mathbf{Z}^*), \tag{2.4.18a}$$

$$\mathfrak{I}_k \xrightarrow{\mathbf{Z}} \mathfrak{I}_\infty \quad (\Gamma\textit{-convergence in the weak topology of } \mathbf{Z}), \tag{2.4.18b}$$

$$z_k \rightharpoonup z \text{ and } \hat{z}_k \rightharpoonup \hat{z} \quad \Longrightarrow \quad \mathscr{D}_k(z_k,\hat{z}_k) \to \mathscr{D}_\infty(z,\hat{z}). \tag{2.4.18c}$$

Then for energetic solutions $z_k : [0,T] \to \mathbf{Z}$ of $(\mathbf{Z}, \mathscr{I}_k, \mathscr{D}_k)$ with

$$z_k(0) \rightharpoonup z^0 \ \ \text{and} \ \ \mathscr{I}_k(0,z_k(0)) \to \mathscr{I}_\infty(0,z^0),$$

there exist an energetic solution $z : [0,T] \to \mathbf{Z}$ of $(\mathbf{Z}, \mathscr{I}_\infty, \mathscr{D}_\infty, z^0)$ and a subsequence $(z_{k_m})_{m\in\mathbb{N}}$ with $k_m \to \infty$ and

$$\forall\, t \in [0,T]: \ \ z_{k_m}(t) \rightharpoonup z(t) \ \ \text{and} \ \ \mathscr{I}_{k_m}(t, z_{k_m}(t)) \to \mathscr{I}_\infty(t,z(t)) \ \ \text{for} \ \ k \to \infty.$$

Proof. We first choose a subsequence such that $(\dot{\ell}_{k_l})_{l\in\mathbb{N}}$ has an integrable majorant $\lambda_{\mathscr{I}} \in \mathrm{L}^1(0,T)$ and is pointwise convergent, that is,

$$\big\|\dot{\ell}_{k_l}(t)\big\|_{\mathbf{Z}^*} \leq \lambda_{\mathscr{I}}(t) < \infty \quad \text{and} \quad \dot{\ell}_k(t) \to \dot{\ell}_\infty(t) \ \ \text{for} \ \ t \in [0,T]\backslash N_{\mathscr{I}} \tag{2.4.19}$$

with $\mathscr{L}^1(N_{\mathscr{I}}) = 0$. By the standard a priori estimates, we may select a further subsequence (not relabeled) such that we also have $z_{k_l}(t) \rightharpoonup z(t)$ in $\boldsymbol{Z}$ for some function $z : [0,T] \to \boldsymbol{Z}$.

Now we simply apply Theorem 2.4.10 after checking the missing assumptions of (2.4.11), where the topological space is given by $\boldsymbol{Z}$ equipped with the weak topology. In particular, we see that (2.4.11e) holds. Moreover, the uniform assumptions (2.1.21) imply that the assumptions (2.4.11a,b,d,f) hold. The lower Γ-limits (2.4.11c+g) are simple consequences of (2.4.18), where we can use $\ell_k \to \ell_\infty$ in $\mathrm{C}^0([0,T];\boldsymbol{Z})$.

The conditioned semicontinuity of the power (2.4.11h+i) follows from (2.4.19) and the simple form of the power, namely $\partial_t \mathscr{I}_k(t,z) = -\langle \dot{\ell}_k(t), z\rangle$.

Thus, it remains to establish the closedness of the stable sets, namely (2.4.11j). For a given stable sequence (t_l, z_{k_l}) with $t_l \to t$ and $z_{k_l} \rightharpoonup z$ and a test state $\hat{z}$, we choose a mutual recovery sequence by taking a recovery sequence $\hat{z}_l \rightharpoonup \hat{z}$ with $\mathfrak{I}_{k_l}(\hat{z}_l) \to \mathfrak{I}_\infty(\hat{z})$, where we use $\mathfrak{I}_{k_l} \xrightharpoonup{\Gamma} \mathfrak{I}_\infty$. From the stability of z_{k_l}, we have

$$\mathscr{I}_{k_l}(t_l, z_{k_l}) = \mathfrak{I}_{k_l}(z_{k_l}) - \langle \ell_{k_l}(t_l), z_{k_l}\rangle \le \mathscr{I}_{k_l}(t_l, \hat{z}_l) + \mathscr{D}_{k_l}(z_{k_l}, \hat{z}_l).$$

Using $\ell_k \to \ell_\infty$ in $\mathrm{C}^0([0,T];\boldsymbol{Z})$ and (2.4.18c), we can pass to the limit in all terms and obtain the stability $z \in \mathcal{S}(t)$, because $\hat{z}$ was arbitrary. $\square$

In Section 3.5.4.2, we will study the case that $\mathcal{Q}$ is a Hilbert space, that the energy is quadratic in the form $\mathscr{E}_k(t,q) = \frac{1}{2}\langle A_k q, q\rangle - \langle \ell_k(t), q\rangle$, and that the dissipation is of the form $\mathscr{D}_k(z,\tilde{z}) = \mathcal{R}_k(\tilde{z}-z)$. Under the assumption that $\mathscr{E}_k(t,\cdot)$ converges to $\mathscr{E}_\infty(t,\cdot)$ in the sense of Mosco convergence (see Definition 2.4.1), we obtain a more precise convergence result; see Theorem 3.5.14.

The following counterexample shows that a mutual condition on the sequences $(\mathscr{E}_k)_{k\in\mathbb{N}}$ and $(\mathscr{D}_k)_{k\in\mathbb{N}}$ is needed to obtain the above convergence. In particular, the above result as well as the conclusion of Theorem 2.4.13 below may fail if we have merely the two independent Γ-convergences $\mathscr{E}_k \xrightarrow{\Gamma} \mathscr{E}_\infty$ and $\mathscr{D}_k \xrightarrow{\Gamma} \mathscr{D}_\infty$.

Example 2.4.12 (Evolutionary Γ-convergence may fail). Take $\mathcal{Q} = \mathbb{R}^2$, and for $\alpha > 0$ and $\beta \ge 0$, let

$$\mathscr{E}_k(t,q) = \frac{1}{2}q_1^2 + \frac{k^\alpha}{2}\Big(q_2 - \frac{1}{k}q_1\Big)^2 - tq_1 \quad \text{and} \quad \mathscr{D}_k(q,\tilde{q}) = |q_1 - \tilde{q}_1| + k^\beta |q_2 - \tilde{q}_2|.$$

Under the initial condition $q(0) = 0$, the explicit solution can be obtained from the subdifferential equation $0 \in \partial\mathcal{R}_k(\dot{q}) + A_k q - (t,0)^\mathsf{T}$, $q(0) = 0$; cf. [409, 425] for the equivalence to energetic solutions in this uniformly convex case. Here

$$A_k = \begin{pmatrix} 1+k^{\alpha-2} & -k^{\alpha-1} \\ -k^{\alpha-1} & k^\alpha \end{pmatrix}, \qquad \partial\mathcal{R}_k(v) = \mathrm{Sign}(v_1)\times\big(k^\beta\,\mathrm{Sign}(v_2)\big) \subset \mathbb{R}^2,$$

where Sign is the set-valued signum function. With $T(k) = 1 + k^{\beta-1} + k^{\beta+1-\alpha}$, we have the solutions $q_k : [0, \infty) \to \mathbb{R}^2$ with

$$q_k(t) = \begin{cases} (0,0)^\top & \text{for } t \in [0,1], \\ \big(\frac{t-1}{k^{\alpha-2}+1}, 0\big)^\top & \text{for } t \in [1, T(k)], \\ \big(t-1-k^{\beta-1}, \frac{t-T(k)}{k}\big)^\top & \text{for } t \geq T(k). \end{cases}$$

For all choices of α and β, the limit $q(t) = \lim_{k\to\infty} q_k(t)$ exists. For $t \in [0,1]$, we always have $q(t) = 0$, and for $t \geq 1$, we obtain

$$\lim_{k\to\infty} q_k(t) = \begin{cases} (\max\{0, t-1\}, 0)^\top & \text{for } \beta \in [0,1) \text{ or } \alpha \in (0,2), \\ \big(\max\{0, (t-1)/2, t-2\}, 0\big)^\top & \text{for } (\alpha,\beta) = (2,1), \\ (\max\{0, (t-1)/2\}, 0)^\top & \text{for } \alpha = 2 \text{ and } \beta > 1, \\ \big(\max\{0, t-2\}, 0\big)^\top & \text{for } \alpha > 2 \text{ and } \beta = 1, \\ (0,0)^\top & \text{for } \alpha > 2 \text{ and } \beta > 1. \end{cases}$$

It is easy to see that we have

$$\mathcal{E}_k(t,\cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty(t,\cdot) \colon q \mapsto \begin{cases} \frac{1}{2}q_1^2 - tq_1 & \text{for } q_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

For $\beta = 0$, we have $\mathcal{D}_\infty = \mathcal{D}_k$, and we conclude the continuous convergence (2.4.15b). Hence, (2.4.11j) holds. For $\beta > 0$, we have

$$\mathcal{D}_k \xrightarrow{\Gamma} \mathcal{D}_\infty : (q, \tilde{q}) \mapsto \begin{cases} |q_1 - \tilde{q}_1| & \text{for } q_2 = \tilde{q}_2 = 0, \\ \infty & \text{otherwise.} \end{cases}$$

The unique energetic solution of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty, 0)$ is $q(t) = (\max\{0, t-1\}, 0)^\top$. Thus, convergence of q_k to the limit solution holds if and only if $\alpha \in (0,2)$ or $\beta \in [0,1)$.

It is interesting to see that the crucial conditional upper semicontinuity of (2.4.11j) of the stability sets holds if and only if $\beta \in [0,1)$. To see this, note that $\mathcal{S}_\infty(t) = [t-1, t+1] \times \{0\}$ and that $\mathcal{S}_k(t)$ is the parallelogram defined by the corners $A_k^{-1}(t + \sigma_1, \sigma_2 k^\beta)^\top$ with $\sigma_1, \sigma_2 \in \{-1, 1\}$. The restriction $\sup \mathcal{E}_k(t, q_k) < \infty$ for stable sequences implies $q_k \cdot (0,1)^\top \to 0$. In fact, the stronger condition of *unconditioned* upper semicontinuity of the stability sets (i.e., (2.4.11j) without the boundedness of the energy in the definition of $\text{stab.seq.}^{\mathbb{N}}$) holds if and only if $0 \leq \beta < \min\{\alpha, 1\}$.

2.4.3.2 Basic homogenization via evolutionary Γ-convergence

As a first application of the abstract theory for evolutionary Γ-convergence of ERIS, we consider a simple problem on *homogenization*. The general theory for homogenization is a mathematical method applicable to *composite materials*, where the material composition varies on a microscopic length scale proportional to a small parameter $\varepsilon > 0$. The homogenized model to be obtained in the limit $\varepsilon \to 0$ reflects the phenomenon that macroscopical properties of composite materials may be very different from properties of particular components. We refer to Sections 3.5.5 and 4.3.1.5 for more homogenization results.

We consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ with $d \leq 3$, the function space $\boldsymbol{Z} = \mathrm{H}^1(\Omega)$, and the functionals

$$\mathscr{I}_\varepsilon(t,z) = \int_\Omega \frac{1}{2}\nabla z \cdot A\Big(\frac{x}{\varepsilon}\Big)\nabla z - \frac{1}{2}a\Big(\frac{x}{\varepsilon}\Big)z^2 + \frac{1}{4}b\Big(\frac{x}{\varepsilon}\Big)z^4 - \ell(t,x)z\,\mathrm{d}x$$

$$\text{and} \quad \mathscr{D}_\varepsilon(z_1,z_2) = \int_\Omega D\Big(\frac{x}{\varepsilon},z_1,z_2\Big)\,\mathrm{d}x,$$

where $A \in \mathrm{L}^\infty(\mathbb{R}^d;\mathbb{R}^{d\times d})$ and $a, b, D(\cdot,z_1,z_2) \in \mathrm{L}^\infty(\mathbb{R}^d)$, and $D(\cdot,z_1,z_2)$ are 1-periodic in the sense that $a(y{+}k) = a(y)$ for all $y \in \mathbb{R}^d$ and $k \in \mathbb{Z}^d$. Moreover, we assume the positivities $\xi \cdot A(y)\xi \geq c_0|\xi|^2$, $b(y) \geq c_0$, and $D(y,z_1,z_2) \geq c_0|z_2{-}z_1|$ for some $c_0 > 0$ and a.a. $y \in \mathbb{R}^d$.

This defines a family $(\boldsymbol{Z},\mathscr{I}_\varepsilon,\mathscr{D}_\varepsilon)$, $\varepsilon \in (0,1)$, of ERIS on $\mathcal{Z} = \boldsymbol{Z} = \mathrm{H}^1(\Omega)$ equipped with the weak topology. Thus, we are in the basic Banach-space case of Example (2.1.7) and Corollary 2.4.11 with uniform equicoercivity of $\mathscr{I}_\varepsilon(t,\cdot)$ in $\boldsymbol{Z}$.

The assumptions (2.1.21) hold by classical arguments, and (2.4.18a) is true with $\ell_\varepsilon = \ell$. Thus, we have to find the static Γ-limit of $\mathscr{I}_\varepsilon(t,\cdot)$ in the weak topology of $\boldsymbol{Z} = \mathrm{H}^1(\Omega)$ via classical homogenization theory; see, e.g., [95, 141]. We obtain

$$\mathscr{I}_\varepsilon(t,\cdot) \xrightharpoonup{\Gamma} \mathscr{I}_{\mathrm{hom}}(t,\cdot) : z \mapsto \int_\Omega \frac{1}{2}\nabla z \cdot A_{\mathrm{eff}}\nabla z - \frac{a_{\mathrm{av}}}{2}z^2 + \frac{b_{\mathrm{av}}}{4}z^4 - \ell(t,x)z\,\mathrm{d}x,$$

where the tensor $A_{\mathrm{eff}} \in \mathbb{R}^{d\times d}$ has to be calculated by a cell problem, whereas $a_{\mathrm{av}} = \int_{[0,1]^d} a(y)\,\mathrm{d}y$ and $b_{\mathrm{av}} = \int_{[0,1]^d} b(y)\,\mathrm{d}y$ are simple averages.

Using the compact embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$, one easily obtains

$$z_\varepsilon \overset{\mathrm{H}^1}{\rightharpoonup} z \text{ and } \hat{z}_\varepsilon \overset{\mathrm{H}^1}{\rightharpoonup} \hat{z} \implies \mathscr{D}_\varepsilon(z_\varepsilon,\hat{z}_\varepsilon) \to \mathscr{D}_{\mathrm{hom}}(z,\hat{z}) = \int_\Omega D_{\mathrm{av}}(z,\hat{z})\,\mathrm{d}x$$

with $D_{\mathrm{av}}(u,v) = \int_{[0,1]^d} D(y,u,v)\,\mathrm{d}y$.

Thus we have established evolutionary Γ-convergence of $(\boldsymbol{Z},\mathscr{I}_\varepsilon,\mathscr{D}_\varepsilon)$ to the homogenized ERIS $(\boldsymbol{Z},\mathscr{I}_{\mathrm{hom}},\mathscr{D}_{\mathrm{hom}})$.

2.4.3.3 Solutions of (IMP$^{\Pi}$) converge to solutions

The major result of this section is the fact that even incremental solutions of $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ for a given sequence $(\Pi^k)_{k\in\mathbb{N}}$ of partition with fineness $\varnothing(\Pi^k) \to 0$ have subsequences converging to solutions of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$. Thus, we do not need exact solutions of each $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ to guarantee that the limiting functions are solutions. This fact will be exploited in the case of space-time discretization in Section 3.6.

For the partition Π^k given by

$$\Pi^k = \big(t_0^k < t_1^k < \cdots < t_{N_k-1}^k < t_{N_k}^k\big) \in \mathrm{Part}([0,T]),$$

we use fully implicit time discretization to define the incremental minimization problem (IMP)^k via

(IMP)^k Given $q_0^k \in \mathcal{Q}$, for $j = 1, \ldots, N_k$ find $q_j^k \in \mathrm{Arg\,min}_{q\in\mathcal{Q}} \big(\mathcal{E}_k(t_j^k, q) + \mathcal{D}_k(q_{j-1}^k, q)\big)$.

As in (2.1.12), for each solution $((t_j^k, q_j^k))_{j=0,1,\ldots,N_k}$, we define the piecewise constant interpolants $\underline{q}^k : [0,T] \to \mathcal{Q}$, which are continuous from the right:

$$\underline{q}^k(t) = q_{j-1}^k \ \text{ for } t \in [t_{j-1}^k, t_j^k) \quad \text{and} \quad \underline{q}^k(T) = q_{N_k}^k. \tag{2.4.20}$$

Theorem 2.4.13. *Assume* (2.1.18) *and* (2.4.11). *Let the sequence of partitions* Π^k, $k \in \mathbb{N}$, *satisfy* $\varnothing(\Pi^k) \to 0$, *and let the initial conditions* q_0^k, $k \in \mathbb{N}$, *satisfy*

$$q_0^k \xrightarrow{\mathcal{Q}} q_0 \quad \text{and} \quad \mathcal{E}_k(0, q_0^k) \to \mathcal{E}_\infty(0, q_0) \in \mathbb{R}. \tag{2.4.21}$$

Then:

(i) Each (IMP)^k has at least one solution $\underline{q}^k : [0,T] \to \mathcal{Q}$, and there exist a subsequence $(\underline{q}^{k_l})_{l\in\mathbb{N}}$ and a measurable energetic solution $q : [0,T] \to \mathcal{Q}$ for the initial-value problem $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty, q_0)$ such that (i)–(iv) hold:

$$\forall\, t \in [0,T]: \quad \underline{z}^{k_l}(t) \xrightarrow{\mathcal{Z}} z(t); \tag{2.4.22a}$$

$$\forall\, t \in [0,T]: \quad \mathrm{Diss}_{k_l}(\underline{q}^{k_l}; [0,t]) \to \mathrm{Diss}_\infty(q; [0,t]), \tag{2.4.22b}$$

$$\forall\, t \in [0,T]: \quad \mathcal{E}_{k_l}(t, \underline{q}^{k_l}(t)) \to \mathcal{E}_\infty(t, q(t)); \tag{2.4.22c}$$

$$\forall_{a.a.}\, t \in [0,T]: \quad \partial_t\mathcal{E}_{k_l}(t, \underline{q}^{k_l}(t)) \to \partial_t\mathcal{E}_\infty(t, q(t)). \tag{2.4.22d}$$

In particular, also $\partial_t\mathcal{E}_{k_l}(\cdot, \underline{q}^{k_l}(\cdot)) \to \partial_t\mathcal{E}_\infty(\cdot, q(\cdot))$ *in* $\mathrm{L}^1(0,T)$.

(ii) If additionally, the functional $\mathscr{E}$ is such that for each stable point $q = (y, z) \in \mathcal{S}(t)$, the functional $\mathscr{E}(t, \cdot, z)$ has the unique minimizer y, then taking $\tilde{y}(t) = \arg\min \mathscr{E}(t, \cdot, \tilde{z}(t))$, the convergence in (2.1.20a) can be improved to

$$q_k(t) \xrightarrow{\mathcal{Q}} \tilde{q}(t). \tag{2.4.22e}$$

(iii) Moreover, every $\tilde{q} : [0, T] \to \mathcal{Q}$ obtained as such a limit is an energetic solution of the ERIS $(\mathcal{Q}, \mathscr{E}_\infty, \mathscr{D}_\infty, q_0)$ if additionally $y(t) \in \operatorname{Arg\,min} \mathscr{E}(t, \cdot, z(t))$ for all t and $\partial_t \mathscr{E}_\infty(t, y(t), z(t)) = \mathcal{P}^\infty_{\text{red}}(t, z(t))$ a.e. in $[0, T]$.

Proof. We follow the six steps according to Table 2.1 on p. 72.

Step 1: A priori estimates. Using the uniform control of the power in terms of the energy and the assumption (2.4.21) on the initial energies, the a priori estimates for the incremental problems $(\text{IMP})^k$ in Theorem 2.1.5 hold uniformly.

Step 2: Selection of subsequences. We use the generalized version of Helly's selection principle as discussed in Appendix B.5 as Theorem B.5.13. It guarantees the existence of a subsequence $(k_l)_{l\in\mathbb{N}}$, an increasing function $\delta^\infty : [0, T] \to [0, \infty]$, and $z : [0, T] \to \mathcal{Z}$ such that for all $t \in [0, T$, we have

$$z^{k_l}(t) \xrightarrow{\mathcal{Z}} z(t) \quad \text{and} \quad \text{Diss}_{k_l}(z^{k_l}; [0, t]) \to \delta^\infty(t).$$

Moreover, Lemma 2.4.7 gives $\text{Diss}_\infty(z; [r, s]) \le \delta^\infty(s) - \delta^\infty(r)$ for all $[r, s] \subset [0, T]$. The selection result of Lemma 2.1.22 yields a measurable $y : [0, T] \to \mathcal{Y}$ with

$$y(t) \in \operatorname{Arg\,min} \mathscr{E}(t, \cdot, z(t)) \text{ for all } t \text{ and } \partial_t \mathscr{E}_\infty(t, y(t), z(t)) = \mathcal{P}^\infty_{\text{red}}(t, z(t)) \text{ a.e. in } [0, T].$$

Step 3: Stability of the limit function. By construction, we have $\underline{q}^{k_l}(t_j^{k_l}) \in \mathcal{S}_{k_l}(t_j^{k_l})$, and using $\varnothing(\Pi^{k_l}) \to 0$, we can employ (2.4.11j) to conclude that $q(t) \in \mathcal{S}_\infty(t)$ for all $t \in [0, T]$.

Step 4: Upper energy estimate. The upper energy estimate follows from

$$\mathscr{E}_{k_l}(t_n^{k_l}, q^{k_l}(t_n^{k_l})) + \sum_{j=1}^{n} \mathscr{D}_{k_l}(z^{k_l}(t_{j-1}^{k_l}), z^{k_l}(t_j^{k_l})) \le \mathscr{E}_{k_l}(0, q_0^{k_l}) + \int_0^{t_n^{k_l}} \partial_\tau \mathscr{E}_{k_l}(\tau, q^{k_l}(\tau))\, \mathrm{d}\tau$$

using (2.4.21) and (2.4.11h) on the right-hand side and (2.4.11g) and Step 2 on the left-hand side. For $t^{k_l} \to t$, we obtain

$$\mathscr{E}_\infty(t, q(t)) + \text{Diss}_\infty(q; [0, t]) \le \mathscr{E}_\infty(0, q_0) + \int_0^t \partial_\tau \mathscr{E}_\infty(\tau, q(\tau))\, \mathrm{d}\tau.$$

Step 5: Lower energy estimate. This is a consequence of Proposition 2.1.23 applied to $(\mathcal{Q}, \mathscr{E}_\infty, \mathscr{D}_\infty)$.

Step 6: Improved convergence. This follows in the usual way. $\square$

2.5 Relaxation of ERIS via Γ-convergence

Here we treat a question that is closely linked to evolutionary Γ-convergence as considered above, namely that of *relaxation*. For static problems, the theory of relaxation is well developed; see, e.g., [140, 520]. It is related to procedures that vary slightly and, in some natural way, modify an investigated problem to guarantee existence of a solution that the original problem for some rather natural reasons does not guarantee. Often, this can be done by constructing a lower semicontinuous envelope in a suitable topology. Here we want to address the analogous question for ERIS, i.e., for evolutionary systems given by two functionals $\mathcal{E}_1$ and $\mathcal{D}_1$ that do not fully satisfy the assumption on the existence theory in Section 2.1. Previous work on the relaxation of ERIS can be found in [135, 392, 403, 423, 592].

2.5.1 Relaxation of incremental minimization problems

To simply the notation, we will restrict our attention to the case of a single pair of a stored-energy functional $\mathcal{E}_1$ and a dissipation distance $\mathcal{D}_1$, though sequences in the sense of Section 2.4.2 could also be treated. However, we weaken the conditions (2.4.11b) and (2.4.11d)(i) in such a way that the incremental minimization problem (IMP$^\Pi$) need not have a solution. Instead, we will consider the approximate incremental problem (AIP$^\Pi_\varepsilon$) or the strengthened approximate incremental problem (SAIP$^\Pi_\varepsilon$) for $\mathcal{E}_1$ and $\mathcal{D}_1$, as discussed in Section 2.3. These problems always have solutions, and we provide mutual conditions on $\mathcal{E}_1$ and $\mathcal{D}_1$ and suitable relaxations $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ such that solutions $\underline{q}^k : [0,T] \to \mathcal{Q}$ of (AIP$^\Pi_\varepsilon$) or of (SAIP$^\Pi_\varepsilon$) converge to energetic solutions of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$.

As in Section 2.3, our assumptions on $\mathcal{E}_1 : [0,T]\times\mathcal{Q} \to \mathbb{R}_\infty$ and $\mathcal{D}_1 : \mathcal{Z}\times\mathcal{Z} \to \mathbb{R}_\infty$ need the notion of the approximate-stability sets $\mathcal{S}_1^\alpha(t)$. For $\alpha \geq 0$, we set

$$\mathcal{S}_1^\alpha(t) = \Big\{ q \in \mathcal{Q} \;\Big|\; \mathcal{E}_1(t,q) < \infty,\ \forall\, \tilde{q} \in \mathcal{Q} :\ \mathcal{E}_1(t,q) \leq \alpha + \mathcal{E}_1(t,\tilde{q}) + \mathcal{D}_1(q,\tilde{q}) \Big\}.$$

The points in $\mathcal{S}_1^\alpha(t)$ are called approximately stable. Following (2.3.5), a sequence $((t_k, q_k))_{k\in\mathbb{N}}$ is called approximately stable if there exists $(\alpha_k)_{k\in\mathbb{N}}$ with

$$q_k \in \mathcal{S}^{\alpha_k}(t_k) \ \text{ for } \ k\in\mathbb{N}, \qquad \sup_{k\in\mathbb{N}} \mathcal{E}(t_k, q_k) < \infty, \qquad \alpha_k \to 0^+.$$

Our conditions are the following:

Quasi-metric: $\quad \forall\, j \in \{1,\infty\}\ \forall\, z_1, z_2, z_3 \in \mathcal{Z}$:
$$\mathcal{D}_j(z_1,z_2) = 0 \ \Leftrightarrow\ z_1 = z_2, \ \text{and}\ \mathcal{D}_j(z_1,z_3) \leq \mathcal{D}_j(z_1,z_2)+\mathcal{D}_j(z_2,z_3). \tag{2.5.1a}$$

Lower semicontinuity of $\mathscr{D}_\infty$:
$$\mathscr{D}_\infty : \mathcal{Z}\times\mathcal{Z} \to [0,\infty] \text{ is lower semicontinuous.} \tag{2.5.1b}$$

Lower Γ-limit:
$$\forall \text{ approx. stab.seq. } (t_k, q_k)_k \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) \text{ and } (\tilde{t}_k, \tilde{q}_k)_k \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (\tilde{t}, \tilde{q}) : \quad \mathscr{D}_\infty(q,\tilde{q}) \leq \liminf_{k\to\infty} \mathscr{D}_1(q_k, \tilde{q}_k). \tag{2.5.1c}$$

Compactness of energy sublevels:
For all $t \in [0,T]$ and all $E \in \mathbb{R}$ we have
(i) $\{\, q \in \mathcal{Q} \mid \mathscr{E}_1(t,q) \leq E \,\}$ is compact;
(ii) $\{\, q \in \mathcal{Q} \mid \mathscr{E}_\infty(t,q) \leq E \,\}$ is compact. (2.5.1d)

Separability and metrizability: The topology restricted to sublevels of $\mathscr{E}_\infty(t,\cdot)$ is compact, separable, and metrizable. (2.5.1e)

Uniform control of the power $\partial_t \mathscr{E}_j$:
$$\begin{aligned}
&\mathrm{Dom}\,\mathscr{E}_j = [0,T]\times\mathrm{Dom}\,\mathscr{E}_j(0,\cdot) \ \text{ and}\\
&\exists\, c_{\mathscr{E}} \in \mathbb{R}\ \exists\, \lambda_{\mathscr{E}} \in \mathrm{L}^1(0,T)\ \exists\, N_{\mathscr{E}} \subset [0,T] \ \text{ with } \ \mathscr{L}^1(N_{\mathscr{E}}) = 0\\
&\forall\, j \in \{1,\infty\}\ \forall\, q \in \mathrm{Dom}\,\mathscr{E}_j(0,\cdot):\ \mathscr{E}_j(\cdot,q) \in \mathrm{W}^{1,1}(0,T) \ \text{ and}\\
&\qquad\qquad \partial_t \mathscr{E}_j(t,q) \text{ exists for } t \in [0,T]\setminus N_{\mathscr{E}} \text{ with}\\
&\qquad\qquad |\partial_t \mathscr{E}_j(t,q)| \leq \lambda_{\mathscr{E}}(t)(\mathscr{E}_j(s,q) + c_{\mathscr{E}}).
\end{aligned} \tag{2.5.1f}$$

Lower Γ-limit for $\mathscr{E}_j$:
$$\forall \text{ approx. stab.seq.}^{\mathbb{N}}\ (t_k, q_k) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) :\ \mathscr{E}_\infty(t,q) \leq \liminf_{k\to\infty} \mathscr{E}_1(t_k, q_k). \tag{2.5.1g}$$

Conditioned semicontinuity of the power: $\forall\, t \in [0,T]\setminus N_{\mathscr{E}}$:

$$\forall \text{ approx. stab.seq.}^{\mathbb{N}}\ (t_k, q_k) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) :\ \limsup_{k\to\infty} \partial_t \mathscr{E}_1(t, q_k) \leq \partial_t \mathscr{E}_\infty(t,q), \tag{2.5.1h}$$

$$\forall \text{ stab.seq.}^{\infty}\ (t_k, \tilde{q}_k) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) : \quad \liminf_{k\to\infty} \partial_t \mathscr{E}_\infty(t, \tilde{q}_k) \geq \partial_t \mathscr{E}_\infty(t,q). \tag{2.5.1i}$$

Conditioned upper semicontinuity of stability sets:
$$\forall \text{ approx. stab.seq.}^{\mathbb{N}}\ (t_k, q_k) \overset{[0,T]\times\mathcal{Q}}{\longrightarrow} (t,q) : \quad q \in \mathcal{S}_\infty(t), \tag{2.5.1j}$$

where again stab.seq.$^\infty$ in (2.5.1i) means that $((t_l, \tilde{q}_l))_{l\in\mathbb{N}}$ is a stable sequence for $(\mathcal{Q}, \mathscr{E}_\infty, \mathscr{D}_\infty)$.

As discussed in Section 2.1.5, the condition (2.5.1j) on the closedness of the stability set can be established via a hierarchy of several stronger conditions. Note that the conditions (2.5.1) are in complete analogy to (2.4.11), the only differences being that lower semicontinuity is required only for $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ and that the stable sequences are replaced by approximately stable sequences.

We recall that the approximate incremental problem ($\mathrm{AIP}^{\Pi}_{\varepsilon}$) and the strengthened version ($\mathrm{SAIP}^{\Pi}_{\varepsilon}$) are given by approximate minimizers in $\mathrm{Arg\,min}_{\varepsilon,\mathcal{Q}}$ (cf. (2.3.1)):

$$(\mathrm{AIP}^{\Pi}_{\varepsilon}) \qquad q_j \in \mathrm{Arg\,min}_{\varepsilon_j,\mathcal{Q}}\, \mathcal{E}_1(t_j,\cdot) + \mathcal{D}_1(q_{j-1},\cdot)$$

$$(\mathrm{SAIP}^{\Pi}_{\varepsilon}) \quad \begin{cases} q_j \in \mathrm{Arg\,min}_{\varepsilon_j,\mathcal{Q}}\, \mathcal{E}_1(t_j,\cdot) + \mathcal{D}_1(q_{j-1},\cdot) \\ \qquad\qquad \text{and } \mathcal{E}_1(t_j,q_j) + \mathcal{D}_1(q_{j-1},q_j) \le \mathcal{E}_1(t_j,q_{j-1}). \end{cases}$$

For every partition $\Pi = (t_0 < t_1 < \cdots < t_N) \in \mathrm{Part}([0,T])$ and vector $\varepsilon = (\varepsilon_1,\ldots,\varepsilon_N) \in (0,\infty)^N$, the existence of solutions was established in Proposition 2.3.3.

We will consider a sequence $(\Pi^k)_{k\in\mathbb{N}}$ of partitions, where Π^k has N_k intervals, and a sequence $(\varepsilon^{(k)})_{k\in\mathbb{N}}$ of error-level vectors $\varepsilon^{(k)} \in (0,\infty)^{N_k}$. For brevity, we will denote by $(\mathrm{AIP})^k$ and $(\mathrm{SAIP})^k$ the problems $(\mathrm{AIP}^{\Pi^k}_{\varepsilon^{(k)}})$ and $(\mathrm{SAIP}^{\Pi^k}_{\varepsilon^{(k)}})$, respectively. For a solution $(q^k_j)_{j=1,\ldots,N_k}$ of either of these problems, we denote the piecewise constant interpolant by $\underline{q}^k : [0,T] \to \mathcal{Q}$, defined in (2.4.20).

The following result shows that under the assumptions $\varnothing(\Pi^k) \to 0$ and $|\varepsilon^{(k)}|_\infty = \max_j \varepsilon^{(k)}_j \to 0$ for $(\mathrm{SAIP})^k$, and under the additional assumption $|\varepsilon|_1 = \sum_{j=1}^{N_k} \varepsilon^{(k)}_j \to 0$ also for $(\mathrm{AIP})^k$, suitably chosen subsequences $(\underline{q}_{k_l})$ converge to a limit process $q : [0,T] \to \mathcal{Q}$, which is an energetic solution for $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{D}_\infty)$.

Theorem 2.5.1. *Let $\mathcal{Q}$, $\mathcal{E}_1$, $\mathcal{E}_\infty$, $\mathcal{D}_1$, and $\mathcal{D}_\infty$ satisfy conditions* (2.5.1). *Let us choose arbitrary sequences $(\Pi^k)_{k\in\mathbb{N}}$ and $(\varepsilon^{(k)})_{k\in\mathbb{N}}$ with $\Pi^k \in \mathrm{Part}([0,T])$, $\varnothing(\Pi^k) \to 0$, $\varepsilon^{(k)} \in (0,\infty)^{N_k}$, and $|\varepsilon^{(k)}|_\infty \to 0$. Then for every sequence $(\underline{q}^k)_{k\in\mathbb{N}}$ of approximants constructed from $(SAIP)^k$ satisfying*

$$q^k(0) \to q_0 \qquad \textit{and} \qquad \mathcal{E}_1(0,q^k(0)) \to \mathcal{E}_\infty(0,q_0),$$

there exist a subsequence $(\underline{q}^{k_l})_{l\in\mathbb{N}}$ and an energetic solution $q : [0,T] \to \mathcal{Q}$ of the ERIS $(\mathcal{Q},\mathcal{E}_\infty,\mathcal{D}_\infty,q_0)$ such that we have the convergences (2.4.22) *stated in Theorem 2.4.13.*

If additionally $|\varepsilon^{(k)}|_1 \to 0$, then the same statements hold for solutions of $(AIP)^k$.

The proof is a simple combination of the proofs of Theorems 2.3.4 and 2.4.13.

2.5.2 An example of relaxation via evolutionary Γ-convergence

To illustrate the theory developed above, we consider a nonconvex functional that has a nontrivial lower semicontinuous envelope. We begin with the ERIS $(\mathcal{Q}, \mathcal{I}_1, \mathcal{D}_1)$ given via $\mathcal{Q} = \mathcal{Z} = \mathrm{W}^{1,4}(0,1)$ and the functionals

$$\mathcal{I}_1(t,z) = \int_0^1 W(z'(x)) + \frac{a}{2}z(x)^2 - f(t,x)z(x)\,\mathrm{d}x, \tag{2.5.2}$$

$$\mathcal{D}_1(z_0,z_1) = \int_0^1 \big|z_1(x) - z_0(x)\big|\,\mathrm{d}x = \big\|z_1 - z_0\big\|_{\mathrm{L}^1}, \tag{2.5.3}$$

where $W(e) = (e^2{-}1)^2$, $f \in \mathrm{C}^1([0,T];\mathrm{L}^1(0,1))$, and $a \geq 0$.

2.5.2.1 Coarse relaxation by lower semicontinuous envelope

We first apply the theory of the previous subsection to the simple lower semicontinuous envelopes in the weak topology of $\mathcal{Z} = \mathrm{W}^{1,4}(0,1)$. For this, we define the lower semicontinuous envelope $\mathcal{I}_\infty$ of $\mathcal{I}_1$ via

$$\mathcal{I}_\infty(t,z) = \int_0^1 W^{**}(z'(x)) + \frac{a}{2}z(x)^2 - f(t,x)z(x)\,\mathrm{d}x, \tag{2.5.4}$$

where W^{**} is the convexification of W, i.e.,

$$W^{**}(e) = \begin{cases} (e^2{-}1)^2 = W(e) & \text{for } |e| \geq 1, \\ 0 & \text{for } |e| \leq 1. \end{cases} \tag{2.5.5}$$

It is a well-known fact that $\mathcal{I}_1$ is not weakly lower semicontinuous on $\mathcal{Z}$ and that $\mathcal{I}_\infty$ is its relaxation on $\mathcal{Z}$. Thus, all conditions on $\mathcal{I}_1$ and $\mathcal{I}_\infty$ are easily proved to hold. Note that due to $W(s) \geq W^{**}(e) \geq \frac{1}{2}e^4 - 1$, we have the coercivity estimate

$$\begin{aligned} \mathcal{I}_1(t,z) \geq \mathcal{I}_\infty(t,z) &\geq \frac{1}{2}\big\|z'\big\|_{\mathrm{L}^4}^4 - 1 + \frac{a}{2}\big\|z\big\|_{\mathrm{L}^2}^2 - \big\|f(t)\big\|_{(\mathrm{W}^{1,4})^*}\big\|z\big\|_{\mathrm{W}^{1,4}} \\ &\geq \big\|z'\big\|_{\mathrm{L}^4}^2 - \frac{3}{2} + \frac{a}{2}\big\|z\big\|_{\mathrm{L}^2}^2 - \big\|f(t)\big\|_{(\mathrm{W}^{1,4})^*}\big\|z\big\|_{\mathrm{W}^{1,4}}, \end{aligned} \tag{2.5.6}$$

from which we can see that $\mathcal{I}_1(t,\cdot)$ has at least quadratic growth on $\mathcal{Z} = \mathrm{W}^{1,4}(0,1)$ uniformly in t if $a > 0$ (as we will suppose here).

Furthermore, for the dissipation, we choose $\mathcal{D}_\infty = \mathcal{D}_1$ with $\mathcal{D}_1$ defined in (2.5.3), and the assumptions on $\mathcal{D}_1$ and $\mathcal{D}_\infty$ follow easily.

The crucial assumption is the upper semicontinuity (2.5.1j) of the stability sets.

Lemma 2.5.2. *Let* $0 < \alpha_l \to 0$, $t_l \to t$, $z_l \rightharpoonup z$ *in* $\mathcal{Z}$, *and* $z_l \in \mathcal{S}^{\alpha_l}(t_l)$; *i.e.,* $\forall\, l \in \mathbb{N}$ $\forall\, \tilde{z} \in \mathcal{Z}:\ \mathcal{I}_1(t_l, z_l) \leq \alpha_l + \mathcal{I}_1(t_l, \tilde{z}) + \mathcal{D}_1(z_l, \tilde{z})$. *Then* $z \in \mathcal{S}_\infty(t)$.

Proof. Choose an arbitrary test function $\tilde{z} \in \mathcal{Z} = \mathrm{W}^{1,4}(0,1)$. Since $\mathcal{I}_\infty$ is the Γ-limit of $(\mathcal{I}_1)_{l\in\mathbb{N}}$, there is a recovery sequence $(\tilde{z}_l)_{l\in\mathbb{N}}$ such that $\tilde{z}_l \rightharpoonup \tilde{z}$ and $\mathcal{I}_1(t_l, \tilde{z}_l) \to \mathcal{I}_\infty(t, z)$. Now we have

$$\begin{aligned}\mathcal{I}_\infty(t,z) &\leq \liminf_{l\to\infty} \mathcal{I}_1(t_l, z_l)\\ &\leq \liminf_{l\to\infty} \big(\alpha_l + \mathcal{I}_1(t_l, \tilde{z}_l) + \big\|\tilde{z}_l - z_l\big\|_{\mathrm{L}^1}\big) = \mathcal{I}_\infty(t, \tilde{z}) + \big\|\tilde{z} - z\big\|_{\mathrm{L}^1},\end{aligned}$$

where we have used the weak-$\mathrm{W}^{1,4}$ continuity of the L^1 norm. Since $\tilde{z}$ was arbitrary, this proves the assertion. ◻

Now it is easy to check the remaining assumptions of Theorem 2.5.1, and we obtain the following convergence result.

Proposition 2.5.3. *Assume* $a > 0$, $f \in \mathrm{C}^1([0,T]; \mathrm{L}^1(0,1))$, $0 < \varepsilon_k \to 0$, *and* $\varnothing(\Pi_k) \to 0$ *for a sequence of partitions* $(\Pi_k)_{k\in\mathbb{N}} \subset \mathrm{Part}([0,T])$. *Choose* $z_0 \in \mathcal{S}_1(0) \subset \mathcal{Z}$ *and define the piecewise constant interpolants* $\underline{z}_k : [0,T] \to \mathcal{Z}$ *associated to some solution of the approximate incremental problem (AIP)$_k$ with initial value* $z_0^k = z_0$. *Then:*

(i) there exist a subsequence $(k_l)_{l\in\mathbb{N}}$ *and a limit function* $z : [0,T] \to \mathcal{Z}$ *such that for all* $t \in [0,T]$, *we have*

$$\begin{aligned}&z_{k_j}(t) \rightharpoonup z(t) \text{ in } \mathrm{W}^{1,4}(0,1), \quad \mathcal{I}_1(t, z_{k_j}(t)) \to \mathcal{I}_\infty(t, z(t)),\\ &\mathrm{Diss}_1(z_{k_j}; [0,t]) \to \mathrm{Diss}_\infty(z; [0,t]) = \textstyle\int_0^t \|\dot{z}(t)\|_{\mathrm{L}^1}\,\mathrm{d}t.\end{aligned}$$

Moreover, $z : [0,T] \to \mathcal{Z}$ *is an energetic solution for* $(\mathcal{Z}, \mathcal{I}_\infty, \mathcal{D}_\infty)$ *that satisfies* $z \in \mathrm{L}^\infty(0,T; \mathrm{W}^{1,4}(0,1)) \cap \mathrm{BV}([0,T]; \mathrm{L}^1(0,1))$.

(ii) Moreover, if $f \in \mathrm{C}^1([0,T]; \mathrm{L}^2(0,1))$, *then also* $z \in \mathrm{C}^{\mathrm{Lip}}([0,T]; \mathrm{L}^2(0,1))$.

The only new part in this result is the time regularity of z, namely $\dot{z} \in \mathrm{L}^\infty(0,T; \mathrm{L}^2(\Omega))$. This fact is a property of all energetic solutions, since by $a > 0$, the energy $\mathcal{I}_\infty$ is uniformly convex on $\mathrm{L}^2(0,1)$. The proof of this result follows from Corollary 3.4.6 by choosing $\boldsymbol{S} = \mathrm{L}^2(\Omega)$ or from [409, Lem. 3.3].

So far, we have been unable to prove that solutions associated with microstructure really occur as limits of solutions of (AIP)$_k$. In (S)$_\infty$&(E)$_\infty$, this simply means that solutions satisfy $|z'(t,x)| < 1$. However, it is easy to see that (S)$_\infty$&(E)$_\infty$ has solutions of this type. Consider the case $a = 1$, $f(t,x) = (1-t)x$, and $z_0(x) = x$. Then the function $z : [0,3] \to \mathrm{W}^{1,4}(0,1)$ with

$$z(t,x) = \begin{cases} x & \text{for } x \in [0, 1/(1+t)],\\ \frac{1}{2}\big((1-t)x + 1\big) & \text{for } x \in [1/(1+t), 1],\end{cases}$$

is a solution. It would be sufficient to show that this solution is unique. Then all accumulation points of solutions of $(\mathrm{AIP})_k$ would necessarily converge to this unique solution.

2.5.2.2 Regularization by singular perturbation and its Γ-limit

Instead of solving the $(\mathrm{AIP})_k$, we may also treat a regularized problem using the energies

$$\mathscr{I}_k(t,z) = \int_0^1 \frac{1}{2k}\big(z''(x)\big)^2 + W(z'(x)) + \frac{a}{2}z(x)^2 - f(t,x)z(x)\,\mathrm{d}x. \tag{2.5.7}$$

The added term $\frac{1}{2k}\big(z''(x)\big)^2$, intended to disappear in the limit $k \to \infty$, is called a *singular perturbation*.

We show that for this situation, the Γ-convergence results of Sections 2.4.3.1 and 2.4.3.3 are applicable. For this, we still keep the underlying space $\mathcal{Q} = \mathcal{Z} = \mathrm{W}^{1,4}(0,1)$ equipped with the weak topology. Now each $\mathscr{I}_k$ has compact sublevels, since they are closed and bounded in $\mathrm{H}^2(0,1)$, although not uniformly with respect to k; cf. condition (i) and (ii) in (2.4.11d). In particular, if we choose a smooth stable initial value z_0, the existence theory of Section 2.1.3 provides energetic solutions z_k for the initial-value problem $(\mathcal{Z}, \mathscr{I}_k, \mathscr{D}_k, z_0)$. In fact, they are solutions of the differential inclusion[7]

$$0 \in \mathrm{Sign}(\dot{z}) + \frac{1}{k}z'''' - \big(\partial W(z')\big)' + az - f(t,x) \quad \text{on } [0,T]\times\Omega, \tag{2.5.8a}$$

$$z(0,\cdot) = z_0 \quad \text{on } \Omega, \tag{2.5.8b}$$

with $z_k \in \mathrm{L}^\infty(0,T;\mathrm{H}^2(0,1)) \cap \mathrm{BV}([0,T];\mathrm{L}^1(0,1))$. We need to assume now $z_0 \in \mathrm{H}^2(0,1)$, for otherwise an approximation of z_0 would be needed. In $\mathrm{L}^\infty(0,T;\mathrm{H}^2(0,1))$, the norm will tend to ∞ with k, whereas in $\mathrm{L}^\infty(0,T;\mathrm{W}^{1,4}(0,1))$, there is a k-independent bound.

Hence, we may pass to the limit for $k \to \infty$, since it is well known that $\mathscr{I}_\infty$ is the Γ-limit of $\mathscr{I}_k$; see [94, 141].

Theorem 2.4.10 is applicable, and we conclude that convergent subsequences of $(z_k)_{k\in\mathbb{N}}$ exist and that their limit points are energetic solutions associated with the relaxed functionals $\mathscr{I}_\infty$ and $\mathscr{D}_\infty$. Moreover, Theorem 2.4.13 can be employed to show that the solutions of suitable incremental problems converge to solutions of $(\mathrm{S})_\infty\&(\mathrm{E})_\infty$ as well.

[7] If W is smooth, (2.5.8a) can equivalently be written as $0 \in \mathrm{Sign}(\dot{z}) + \frac{1}{k}z'''' - \partial^2 W(z')z'' + az - f(t,x)$.

2.5.2.3 Fine relaxation by extension in terms of Young measures

An alternative relaxation is based on so-called *Young measures* and a continuous extension of W. To be more specific, let

$$\mathcal{Q}_{\rm YM} := \Big\{\, q{=}(\nu,z){\in}\mathscr{Y}^4(0,1){\times}{\rm W}^{1,4}(0,1)\;\Big|\; \int_{\mathbb{R}} e\,\nu_x({\rm d}e){=}z'(x) \text{ for a.a. } x{\in}(0,1)\Big\},$$

where

$$\mathscr{Y}^4(0,1) := \Big\{\nu{=}(\nu_x)_{x\in(0,1)} \mid \nu_x \text{ is a probability measure on } \mathbb{R} \text{ for a.a. } x{\in}(0,1),\; \forall\, v{\in}{\rm C}_0(\mathbb{R}){:}\;\; x\mapsto\int_{\mathbb{R}} v(e)\nu_x({\rm d}e) \text{ is measurable},\; \int_0^1\!\!\int_{\mathbb{R}} e^4\nu_x({\rm d}e)\,{\rm d}x<\infty\Big\}$$

is the set of the L^4-Young measures; cf. Sect. C.3. Then it is natural to define

$$\mathscr{I}_1(t,z,\nu) = \begin{cases}\displaystyle\int_0^1 W(z'(x)){+}\frac{a}{2}z(x)^2{-}f(t,x)z(x)\,{\rm d}x & \text{if } \nu_x=\delta_{z'(x)} \text{ for a.a. } x{\in}(0,1),\\ \infty & \text{otherwise,}\end{cases} \tag{2.5.9}$$

while

$$\mathscr{I}_{\rm YM}(t,z,\nu) = \int_0^1\Big(\int_{\mathbb{R}} W(e)\nu_x({\rm d}e) + \frac{a}{2}z(x)^2{-}f(t,x)z(x)\Big)\,{\rm d}x. \tag{2.5.10}$$

The set $\mathcal{Q}_{\rm YM}$ can be considered a convex subset of the linear space ${\rm W}^{1,4}(0,1){\times}H^*$ with $H := {\rm C}([0,1])\otimes\{\, e\mapsto (1{+}e^4)v(e){+}\alpha e^4 \mid v\in{\rm C}_0(\mathbb{R}),\ \alpha\in\mathbb{R}\,\}$ under the natural embedding

$$(z,\nu)\mapsto\Big(z, h\mapsto\int_0^1\!\!\int_{\mathbb{R}} h(x,e)\nu_x({\rm d}e)\,{\rm d}x\Big)$$

for $h\in H$, i.e., $h(x,e)=g(x)(1{+}e^4)v(e){+}\alpha(x)e^4$ with $g,\alpha\in{\rm C}([0,1])$. Note that $1\otimes W$, as well as $1\otimes{\rm id}$, belongs to H, where ${\rm id}:\mathbb{R}\to\mathbb{R}$ denotes the identity. This linear space H^* is standardly topologized by the weak* topology,[8] which makes $\mathscr{I}_{\rm YM}(t,\cdot)$ the Γ-limit of $\mathscr{I}_1(t,\cdot)$. Let us remark that the space ${\rm W}^{1,4}(0,1)$ itself is embedded into $\mathcal{Q}$ by $z\mapsto(z,(\delta_{z'(x)})_{x\in(0,1)})$, and thus also into the closure of $\mathcal{Q}$ in ${\rm W}^{1,4}(0,1){\times}H^*$, which is a convex locally compact metrizable envelope of

[8] In fact, in talking about the dual space H^*, we must specify some topology on H that itself is a linear space. A universal one can be induced by the norm of ${\rm Car}^4((0,1);\mathbb{R})$, cf. (C.2.2) on p. 618.

$\mathrm{W}^{1,4}(0,1)$ into which $\mathrm{W}^{1,4}(0,1)$ is embedded (norm, weak*)-homeomorphically.[9] More specifically,

$$\mathcal{Q} := \Big\{ (\eta, z) \in Y_H^4(0,1)\times \mathrm{W}^{1,4}(0,1) \;\Big|\; \mathrm{id}\bullet\eta = z' \Big\}, \tag{2.5.11}$$

with $Y_H^4(0,1)$ referring to (C.2.4) with $p = 4$ and $\mathrm{id}\bullet\eta \in \mathrm{L}^4(0,1)$ defined by $\int_0^1 [\mathrm{id}\bullet\eta](x)g(x)\,\mathrm{d}x = \langle \eta, g \otimes \mathrm{id}\rangle$ for $g \in \mathrm{C}([0,1])$; cf. also (C.1.5) on p. 616. Then we define $\mathcal{I}_\infty : [0,T]\times\mathcal{Q} \to \mathbb{R}$ by

$$\mathcal{I}_\infty(t,z,\eta) := \big\langle \eta, 1 \otimes W\big\rangle + \int_0^1 \frac{a}{2} z^2(x) - f(t,x)z(x)\,\mathrm{d}x. \tag{2.5.12}$$

In fact, $\mathcal{I}_\infty(t,\cdot,\cdot)$ is a continuous extension of $\mathcal{I}_1(t,\cdot)$ from (2.5.9) from $\mathrm{W}^{1,4}(0,1)$ to its locally compact convex envelope $\mathcal{Q}$ from (2.5.11).

Again, the theory of Section 2.5.1 is applicable. This shows that piecewise constant interpolants of the solutions of the approximate incremental problem $(\mathrm{AIP}_\varepsilon^\Pi)$ associated with $\mathcal{I}_1$ and $\mathcal{D}_1$ have subsequences that converge to energetic solutions associated with $(\mathcal{Q}, \mathcal{I}_\infty, \mathcal{D}_\infty)$.

Alternatively, we can pose the *relaxed problem* by setting $\mathcal{Q} = \mathcal{Y}\times\mathcal{Z}$ and $\mathcal{I}_\infty : [0,T]\times\mathcal{Y}\times\mathcal{Z} \to \mathbb{R}_\infty$ defined by

$$\mathcal{Y} := Y_H^4(0,1), \qquad \mathcal{Z} := \mathrm{W}^{1,4}(0,1), \tag{2.5.13a}$$

$$\mathcal{I}_\infty(t,\eta,z) := \begin{cases} \big\langle \eta, 1 \otimes W\big\rangle + \int_0^1 \frac{a}{2} z^2 - f(t,\cdot)z\,\mathrm{d}x & \text{if } \mathrm{id}\bullet\eta = z' \text{ on } (0,1), \\ \infty & \text{otherwise.} \end{cases} \tag{2.5.13b}$$

For an energetic solution (η, z) to $(\mathcal{Q}, \mathcal{I}_\infty, \mathcal{D}_\infty)$ at any time $t \in [0,T]$, $\eta(t)$ minimizes $\mathcal{I}_\infty(t,\cdot,z(t))$ on $\mathcal{Y}$, and realizing that W is coercive with growth greater than linear, we can see that $\eta(t)$ has a Young-measure representation $\nu(t)$; cf. Section C.3.[10] Thus we return to an energetic solution (ν, z) for $(\mathcal{Q}_{\mathrm{YM}}, \mathcal{I}_{\mathrm{YM}}, \mathcal{D}_\infty)$.

[9]This could be shown by adapting Proposition C.2.1(a) to the case of the Sobolev space $\mathrm{W}^{1,4}(0,1)$ instead of $\mathrm{L}^4(0,1)$, which can unfortunately be performed only in one dimension; cf. also [520, Chap.5].

[10]In fact, every stable state $q = (\eta, z)$ prevents concentrations and thus has a Young-measure representation. Indeed, if such were not the case, its nonconcentrating modification $\tilde{q} = (\mathring{\eta}, z)$ would satisfy $\mathrm{id}\bullet\mathring{\eta} = \mathrm{id}\bullet\eta = z'$, but by Proposition C.3.1(iv), it would yield $\mathcal{I}_\infty(t,\eta,z) > \mathcal{I}_\infty(t,\mathring{\eta},z) + \mathcal{D}(z,\tilde{z})$ if $\mathring{\eta} \neq \eta$, contradicting the stability.

In the vectorial multidimensional case, a more sophisticated Young measure relaxation in the rate-independent setting is given in [328]. Related evolutionary systems for Young measures, also in the rate-dependent case, are discussed in [96, 145, 146, 388, 392, 403, 416, 591]. We will return to relaxation methods in the multidimensional case in Section 4.2.2.2 and also 4.4.1.2. For $p = 2$, uniform integrability that is uniform also in time has been derived in [188, Lemma 7.3] using sophisticated quasiminimizer-based arguments.

http://www.springer.com/978-1-4939-2705-0

Rate-Independent Systems
Theory and Application
Mielke, A.; Roubicek, T.
2015, XXII, 660 p. 75 illus., 13 illus. in color., Hardcover
ISBN: 978-1-4939-2705-0